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Invariant manifolds for random dynamical systems with slow and fast variables

Björn Schmalfuß [1] , Klaus R. Schneider[2]

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[1] Institute for Mathematics
University of Paderborn
Warburger Straße 100
05251 Paderborn
Germany
E-Mail: schmalfu@math.upb.de

[2] present address:
Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: schneider@wias-berlin.de

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Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider random dynamical systems with slow and fast variables driven by two independent metric dynamical systems modelling stochastic noise. We establish the existence of a random inertial manifold eliminating the fast variables. If the scaling parameter tends to zero, the inertial manifold tends to another manifold which is called the slow manifold. We achieve our results by means of a fixed point technique based on a random graph transform. To apply this technique we need an asymptotic gap condition.

1 Introduction

Mathematical modelling of continuous spatially homogeneous deterministic processes with different time scales leads in general to systems of singularly perturbed differential equations

$$\begin{aligned} \varepsilon \frac{dx}{dt} &= f(x, y, t, t/\varepsilon, \varepsilon), \\ \frac{dy}{dt} &= g(x, y, t, t/\varepsilon, \varepsilon) \end{aligned} \tag{1}$$

or, using the scaling $t \to \varepsilon t$, to slow-fast systems of the type

$$\begin{aligned} \frac{dx}{dt} &= f(x, y, \varepsilon t, t, \varepsilon), \\ \frac{dy}{dt} &= \varepsilon g(x, y, \varepsilon t, t, \varepsilon). \end{aligned} \tag{2}$$

Here, ε is a small positive parameter, $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$. A variety of perturbation methods has been developed to investigate systems with different time-scales: matched asymptotic expansion [8, 17], method of multiple scales [15], boundary layer function method [4], averaging [19], renormalization group theory [18].

The theory of invariant manifolds [10, 23] is another approach for the qualitative analysis of dynamical systems with different time-scales. Its goal consists in establishing an invariant manifold M_ε of dimension $m < d_1 + d_2$ permitting a reduction of the dimension of the state space by eliminating fast variables. In applications,

M_ε is exponentially attracting and has in general the dimension d_1 characterizing the number of slow variables such that it can be used to justify the so-called quasi-steady state assumption [13, 21, 22]. In that case, M_ε is referred to as slow invariant manifold.

Generically, the invariant manifold M_ε persists under small perturbations [9, 24] of system (1) or (2) which can be interpreted as multiplicative or additive deterministic noise. This property inspires the question for the influence of stochastic noise on the persistence of the slow manifold in slow-fast systems. A first answer in this direction has been given by N. Berglund and B. Gentz [3]. They considered the following singularly perturbed autonomous system under the influence of noise taking into account different time scales

$$\begin{aligned} dx &= \frac{1}{\varepsilon} f(x, y, \varepsilon)dt + \frac{\sigma}{\sqrt{\varepsilon}}\ F(x, y, \varepsilon)\ dW_t, \\ dy &= g(x, y, \varepsilon)dt + \sigma'\ G(x, y, \varepsilon)\ dW_t. \end{aligned} \tag{3}$$

Here, W_t denotes a standard Brownian motion, ε, σ and σ' are small parameters, where σ and σ' depend on ε. The authors do not introduce the concept of an invariant manifold for a stochastic differential system. Their main goal is to estimate quantitatively the noise-induced spreading of typical paths as well as the probability of exceptional paths. They show that the sample paths of (3) are concentrated in some neighborhood of the slow/adiabatic invariant manifold of the corresponding deterministic system up to some time with certain probability.

In this paper we model the stochastic noise by a metric dynamical system $\Theta = (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ and describe the influence of noise on our process under consideration by a random dynamical system. Especially, we consider the following random dynamical system with two time scales

$$\begin{aligned} \frac{dx}{dt} &= f(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2, x, y) := A(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2)x + F(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2, x, y), \\ \frac{dy}{dt} &= \varepsilon g(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2, x, y) := \varepsilon B(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2)y + \varepsilon G(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2, x, y). \end{aligned} \tag{4}$$

Our goal is, for sufficiently small ε, to establish the existence of a random invariant manifold M_ε for system (4) that eliminates the fast variables. Our approach consists in deriving an appropriate random graph transform for random dynamical systems. Such a transform was introduced for very special random dynamical systems, see [7], [20]. Here, we describe this transform for general random dynamical systems and apply this technique to slow–fast random dynamical systems. Assuming a gap condition for the slow–fast system we can establish a random fixed point of that transform whose graph yields the wanted invariant manifold. From that point of view our approach is indeed geometric.

We also refer to the book by Kabanov and Pergamenshchikov [14] which deals with the behavior of two-scale stochastic differential equations. However, our ansatz differs from that ansatz. In particular, the limit behavior in [14] is determined by the *zeros* of the time dependent drift coefficient of the fast system. In contrast to that system we only use the stationary regime given by the dynamics of our equation which allows us to introduce a (stationary) slow manifold.

The paper is organized as follows. In Section 2 we recall some basic facts from the theory of random dynamical systems. Section 3 is devoted to random dynamical systems with two time scales. In Section 4 we establish an inertial manifold M_ε for a random dynamical system with two time-scales, Section 5 shows that M_ε converges to the slow invariant manifold of system (4) with $\varepsilon = 0$.

2 Preliminaries on Random dynamical Systems

In this section we are going to introduce the main tools we need to find inertial manifolds for systems of differential equations driven by random perturbations. These tools stem from the theory of random dynamical systems. For a comprehensive presentation of this theory see Arnold [1].

A standard model for noise is a *metric dynamical system* $\Theta = (\Omega, \mathcal{F}, \mathbb{P}, \theta)$ which consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a flow $\theta = \{\theta_t\}_{t\in\mathbb{R}}$:

$$\theta : \mathbb{R} \times \Omega \to \Omega, \qquad \theta_0 = \mathrm{id}_\Omega, \qquad \theta_{t_1} \circ \theta_{t_2} (= \theta_{t_1}\theta_{t_2}) = \theta_{t_1+t_2}.$$

The flow θ is supposed to be $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F}$-measurable. We also assume that the measure $\mathbb{P}$ is invariant (ergodic) with respect to the mappings $\{\theta_t\}_{t\in\mathbb{R}}$.
For example, the Brownian motion represents a metric dynamical system, where $\Omega = C_0(\mathbb{R}, \mathbb{R}^d)$ is the set of continuous paths on $\mathbb{R}$ with values in $\mathbb{R}^d$ that are zero at the origin. This set is equipped with the compact open topology. $\mathcal{F}$ is supposed to be the associated Borel-σ-algebra and $\mathbb{P}$ the Wiener measure with respect to a covariance operator Q. The flow θ is given by the Wiener shifts

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

This metric dynamical system is related to a two-sided Wiener process appearing as a white noise in stochastic differential systems. Note that the measure $\mathbb{P}$ is ergodic with respect to the Wiener shifts.

Let H be some separable Banach space. A *random dynamical system* with phase space H consists of a mapping ϕ:

$$\phi : \ \mathbb{R}^+ \times \Omega \times H \to H$$

that is $\mathbb{R}^+ \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H)$ -measurable and satisfies the *cocycle* property:

$$\begin{aligned} &\phi(0, \omega, x) = x \in H, \\ &\phi(t_1 + t_2, \omega, x) = \phi(t_1, \theta_{t_2}\omega, \phi(t_2, \omega, x)). \end{aligned} \tag{5}$$

A random dynamical system is called continuous if $x \to \phi(t,\omega,x)$ is continuous for $t \geq 0$, $\omega \in \Omega$. If we omit all the ω's in (5), then ϕ becomes a semigroup. Random dynamical systems are generated by systems of differential equations with random stationary coefficients or with a white noise. For our application it is sufficient to suppose that $H = \mathbb{R}^d$.
Note that the formulation of the cocycle property does not contain the term *almost surely* which often appears in the formulation of a stochastic differential equation problem. However, in the case that (5) is only satisfied on a θ-invariant set Ω' of full measure, then we can define ϕ outside of Ω' by the identical mapping.

We now introduce some objects describing the dynamics of a random dynamical system.
A random variable $\omega \to X^*(\omega)$ with values in H is called a *random fixed point* of the random dynamical system ϕ if

$$\phi(t,\omega,X^*(\omega)) = X^*(\theta_t\omega) \quad \text{for} \quad t \geq 0,\, \omega \in \Omega.$$

Since θ_t leaves the measure $\mathbb{P}$ invariant, the random variables $\omega \to X^*(\omega)$ and $\omega \to X^*(\theta_t\omega)$ have the same distribution. Hence, the process $(t,\omega) \to X^*(\theta_t\omega)$ is a stationary process and therefore a stationary solution to the differential equation generating the random dynamical system ϕ.

A family of nonempty closed sets $M = \{M(\omega)\}_{\omega\in\Omega}$ is called a *random set* if for every $y \in H$ the mapping

$$\Omega \ni \omega \to \operatorname{dist}(y, M(\omega)) := \inf_{x\in M(\omega)} \|x-y\|_H$$

is measurable, where $||.||_H$ is some norm in H. M is called (positively) invariant with respect to the random dynamical system ϕ if

$$\phi(t,\omega,M(\omega)) \subset M(\theta_t\omega) \quad \text{for} \quad t \geq 0,\, \omega \in \Omega. \tag{6}$$

We now consider random sets defined by a Lipschitz continuous graph. In the sequel we suppose $H = \mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and denote by $|.|$ the Euclidean norm, where we omit an index characterizing the dimension. Let

$$\Omega \times \mathbb{R}^{d_2} \ni (\omega, y) \to \gamma^*(\omega, y) \in \mathbb{R}^{d_1}$$

be such a function that $\gamma^*(\omega,y)$ is globally Lipschitzian in y for all $\omega \in \Omega$ and that for any $y \in \mathbb{R}^{d_2}$ the mapping $\omega \to \gamma^*(\omega,y)$ is a random variable. We define

$$M^*(\omega) := \{(\gamma^*(\omega,y), y) | y \in \mathbb{R}^{d_2}\}.$$

Lemma 2.1. *The family of sets $\{M^*(\omega)\}_{\omega\in\Omega}$ is a random set.*

Proof. Since the mapping $\gamma^*(\omega, \cdot)$ is continuous we have for $z \in \mathbb{R}^d$

$$\begin{aligned}\Omega \ni \omega \to \inf_{(x,y)\in M^*(\omega)} |z-(x,y)| &= \inf_{y\in\mathbb{R}^{d_2}} |z-(\gamma^*(\omega,y),y)| \\ &= \inf_{y\in\mathbb{Q}^{d_2}} |z-(\gamma^*(\omega,y),y)|\end{aligned}$$

is measurable for any $z \in \mathbb{R}^d$ because the set $\mathbb{Q}^{d_2}$ of rational d_2-tuples is countable. □

Since we supposed that for any $\omega \in \Omega$ the mapping $\gamma^*(\omega, .)$ is a globally Lipschitzian function, the random set $M^*(\omega)$ satisfying (6) is called a *Lipschitz random invariant manifold.* If in addition

$$\lim_{t\to\infty} \operatorname{dist}(\phi(t,\omega,z), M^*(\theta_t\omega)) = 0 \quad \text{for} \quad z \in \mathbb{R}^d$$

with exponential decay rate, then the manifold M is called a *random inertial manifold* with respect to the random dynamical system ϕ.

Later on we have to transform a random dynamical system into another random dynamical system which is simpler to treat. Let $V : \Omega\times\mathbb{R}^d \to \mathbb{R}^d$ be a mapping such that for any fixed ω the mapping $V(\omega, \cdot)$ is a homeomorphism. The corresponding inverse mapping is denoted by $V^{-1}(\omega, \cdot)$. For any $z \in \mathbb{R}^d$ we suppose that $V(\cdot, z)$ and $V^{-1}(\cdot, z)$ are measurable. Then it follows from Lemma III.14 in [5] that $V(\cdot, \cdot)$ and $V^{-1}(\cdot, \cdot)$ are $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d)$-measurable.
Let $\phi : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ be a random dynamical system. Then the mapping $\tilde{\phi} : \mathbb{R}^+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\tilde{\phi}(t,\omega,z) := V(\theta_t\omega, \phi(t,\omega,V^{-1}(\omega,z)))$$

represents also a random dynamical system. Let us set

$$M^*(\cdot) := V^{-1}(\cdot, \tilde{M}^*(\cdot)),$$

where $\tilde{M}^*$ is a random invariant set for the random dynamical system $\tilde{\phi}$. From the properties of V it follows that M^* is a random set. In addition, M^* is also invariant with respect to the random dynamical system $\tilde{\phi}$.

3 Random dynamical systems with slow and fast variables

In what follows we will study random dynamical systems generated by systems of differential equations under the influence of a noise in different time scales. At first we consider systems of differential equations with coefficients that are stationary with respect to a metric dynamical system Θ. Next, we will be concerned with

systems of differential equations driven by white noise. The associated metric dynamical system is the Brownian motion introduced in Section 2.
The random dynamical systems under consideration contain slow and fast variables and random perturbations in different time scales. For scaled perturbations we introduce a *scaled metric dynamical system*

$$\Theta_\varepsilon = (\Omega, \mathcal{F}, \mathbb{P}, \theta^\varepsilon), \qquad \theta^\varepsilon = \{\theta_{\varepsilon t}\}_{t\in\mathbb{R}}$$

for some $\varepsilon > 0$. It is straightforward that the operators $\theta_{\varepsilon t}$ leave $\mathbb{P}$ invariant.
Let Θ_1, Θ_2 be two independent metric dynamical systems. We scale the flow in Θ_1 with the factor $1/\varepsilon$. With respect to these metric dynamical systems we consider the following singularly perturbed system of random differential equations

$$\begin{aligned} \varepsilon \frac{dx}{dt} &= A(\theta_t^{1,\frac{1}{\varepsilon}}\omega_1, \theta_t^2\omega_2)x + F(\theta_t^{1,\frac{1}{\varepsilon}}\omega_1, \theta_t^2\omega_2, x, y), \\ \frac{dy}{dt} &= B(\theta_t^{1,\frac{1}{\varepsilon}}\omega_1, \theta_t^2\omega_2)y + G(\theta_t^{1,\frac{1}{\varepsilon}}\omega_1, \theta_t^2\omega_2, x, y), \end{aligned} \tag{7}$$

where $0 < \varepsilon \ll 1$. Concerning the right hand side of (7) we suppose

(A_1). The functions

$$\Omega := \Omega_1 \times \Omega_2 \ni (\omega_1, \omega_2) \to A(\omega_1, \omega_2) \in L(\mathbb{R}^{d_1}, \mathbb{R}^{d_1}),$$

$$\Omega_1 \times \Omega_2 \ni (\omega_1, \omega_2) \to B(\omega_1, \omega_2) \in L(\mathbb{R}^{d_2}, \mathbb{R}^{d_2})$$

are measurable, their norms $||.||$ are contained in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ (see Example 2.2.8 in Arnold [1]).

(A_2). There are constants $c_A > 0$ and $c_B \geq 0$ such that

$$\begin{aligned} (A(\omega_1, \omega_2)x, x) \leq -c_A|x|^2 &\qquad \forall (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2,\ \forall x \in \mathbb{R}^{d_1}, \\ ||B(\omega_1, \omega_2)|| \leq c_B &\qquad \forall (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2, \end{aligned}$$

where $||.||$ be the norm of a matrix K mapping $\mathbb{R}^k$ into $\mathbb{R}^k$ such that $|Kx| \leq ||K|||x|$ for any $x \in \mathbb{R}^k$.

With respect to the *scaled* metric dynamical system

$$\Theta^\varepsilon := \Theta_1 \times \Theta_{2,\varepsilon} = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \times \mathbb{P}_2, (\theta^1, \theta^{2,\varepsilon}))$$

we introduce the abbreviations

$$\theta_t^\varepsilon \omega := (\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2),\ \omega := (\omega_1, \omega_2) \in \Omega.$$

Let $U_A^\varepsilon(t,\omega)$ and $U_{\varepsilon B}^\varepsilon(t,\omega)$ be the fundamental solutions of the linear systems

$$\frac{dx}{dt} = A(\theta_t^\varepsilon\omega)x, \qquad \frac{dy}{dt} = \varepsilon B(\theta_t^\varepsilon\omega)y,$$

respectively. These operators generate linear random dynamical systems such that the cocycle property is satisfied. The exponent ε in the symbols for these systems indicates that the noisy input comes from Θ^ε. In the case that A, B are independent of ω these fundamental solutions U_A, $U_{\varepsilon B}$ are groups on the spaces of linear mappings $L(\mathbb{R}^{d_i}, \mathbb{R}^{d_i})$, $i = 1, 2$.

Hypothesis (A_2) implies that there are positive constants a and b such that

$$||U_A^\varepsilon(t-s,\omega)|| \le a e^{-c_A(t-s)} \quad \text{for} \quad t \ge s,\ \omega \in \Omega, \tag{8}$$

$$||U_{\varepsilon B}^\varepsilon(t-s,\omega)|| \le b e^{\varepsilon c_B|t-s|} \quad \text{for any} \quad t, s,\ \omega \in \Omega. \tag{9}$$

(A_3). For any fixed $(x,y) \in \mathbb{R}^d$, the mappings

$$\Omega_1 \times \Omega_2 \ni (\omega_1,\omega_2) \to F(\omega_1,\omega_2,x,y) \in \mathbb{R}^{d_1},$$
$$\Omega_1 \times \Omega_2 \ni (\omega_1,\omega_2) \to G(\omega_1,\omega_2,x,y) \in \mathbb{R}^{d_2}$$

are measurable. There is a positive constant L such that for any $(\omega_1,\omega_2) \in \Omega_1 \times \Omega_2$ and for any $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^d$

$$\begin{aligned} |F(\omega_1,\omega_2,x_1,y_1) - F(\omega_1,\omega_2,x_2,y_2)| &\le L(|x_1-x_2| + |y_1-y_2|),\\ |G(\omega_1,\omega_2,x_1,y_1) - G(\omega_1,\omega_2,x_2,y_2)| &\le L(|x_1-x_2| + |y_1-y_2|). \end{aligned} \tag{10}$$

Due to Castaing and Valdier (see Lemma III.14 in [5]), assumption (A_3) implies that F and G are measurable with respect to all variables. Finally, we assume

(A_4). There exist positive numbers c_F and c_G such that

$$\sup_{\substack{(x,y)\in\mathbb{R}^d,\\ (\omega_1,\omega_2)\in\Omega_1\times\Omega_2}} |F(\omega_1,\omega_2,x,y)| =: c_F,$$
$$\sup_{\substack{(x,y)\in\mathbb{R}^d,\\ (\omega_1,\omega_2)\in\Omega_1\times\Omega_2}} |G(\omega_1,\omega_2,x,y)| =: c_G.$$

We now scale the time $t \to \varepsilon t$ such that we can rewrite (7) as

$$\begin{aligned} \frac{dx}{dt} &= A(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2)x + F(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2, x, y),\\ \frac{dy}{dt} &= \varepsilon B(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2)y + \varepsilon G(\theta_t^1\omega_1, \theta_t^{2,\varepsilon}\omega_2, x, y). \end{aligned} \tag{11}$$

System (11) is a slow-fast system with random perturbations in two time scales which has a unique global solution $\phi^\varepsilon(t,\omega,(x_0,y_0))$ for *any* initial condition

$(x_0, y_0) \in \mathbb{R}^{d_1+d_2}$ and *every* $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$.

The solution operator of the initial value problem to system (11) denoted by

$$\phi^\varepsilon(t, \omega, (x_0, y_0)) = (\phi_1^\varepsilon(t, \omega, (x_0, y_0)), \phi_2^\varepsilon(t, \omega, (x_0, y_0))) \in (\mathbb{R}^{d_1}, \mathbb{R}^{d_2}) \tag{12}$$

defines a random dynamical system.

Now, we are going to show that the stochastic differential system

$$\begin{aligned} dx &= \frac{1}{\varepsilon}\, Axdt + \frac{1}{\varepsilon}\, F(x, y)dt + \frac{1}{\sqrt{\varepsilon}} dw_1, \\ dy &= Bydt + G(x, y)dt + dw_2, \end{aligned} \tag{13}$$

where A and B are constant matrices, can be transformed into the systems (7) or (11). The processes w_1, w_2 are independent standard Wiener processes with covariance $Q_1 = \mathrm{id}_{\mathbb{R}^{d_1}}$, $Q_2 = \mathrm{id}_{\mathbb{R}^{d_2}}$ related to the Brownian motion metric dynamical systems Θ_1, Θ_2 introduced in Section 2, dw_1, dw_2 are Ito differentials. For the transformation we need particular properties of the linear systems

$$dx = \frac{1}{\varepsilon} Ax\, dt + \frac{1}{\sqrt{\varepsilon}} dw_1, \qquad dy = By\, dt + dw_2. \tag{14}$$

Lemma 3.1. *Suppose assumption* (A_2) *to be satisfied. Additionally we suppose that* B *has no eigenvalue on the imaginary axis. Then there exist random variables* $\omega_1 \to x^{1,\frac{1}{\varepsilon}}(\omega_1) \in \mathbb{R}^{d_1}$, $\omega_2 \to y^2(\omega_2) \in \mathbb{R}^{d_2}$ *such that*

$$(t, \omega_1) \to x^{1,\frac{1}{\varepsilon}}(\theta_t^1 \omega_1),\ (t, \omega_2) \to y^2(\theta_t^2 \omega_2)$$

are continuous stationary solutions to the systems in (14) *defined on* (θ^1, θ^2)*-invariant sets of full measure.*

Proof. For the proof we note that by (A_2) the fundamental solution U_A is an exponentially stable semigroup. Therefore, we can refer to the proof of a corresponding result in [7]. By the hyperbolicity assumption on B we can represent $\mathbb{R}^{d_2}$ as the direct sum of the linear eigenspaces Y^+, Y^- belonging to the eigenvalues of B with positive and negative real parts, respectively. The construction in [7] applied to the projections of the second equation of (14) on Y^+ with respect to $\mathbb{R}^+$ and Y^- with respect to $\mathbb{R}^-$ yields the existence of random variables generating stationary solutions for (14). □

Next we will transform the stochastic differential system (13) into a system of type (11). In contrast to the definition of a cocycle, a stochastic differential equation is only defined almost surely, where the initial exceptional set may depend on the initial condition. To find a version which doesn't have this dependence we introduce the random transformation

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} := V_\varepsilon(\omega_1, \omega_2, x, y) =: \begin{pmatrix} x - x^{1,\frac{1}{\varepsilon}}(\omega_1) \\ y - y^2(\omega_2) \end{pmatrix}.$$

Then $(\tilde{x}(t),\tilde{y}(t))=V_\varepsilon(\theta_t^1\omega_1,\theta_t^2\omega_2,x(t),y(t))$ satisfies

$$
\begin{aligned}
\frac{d\tilde{x}}{dt}&=\frac{1}{\varepsilon}A\tilde{x}+\frac{1}{\varepsilon}F(\tilde{x}+x^{1,\frac{1}{\varepsilon}}(\theta_t^1\omega_1),\tilde{y}+y^2(\theta_t^2\omega_2)),\\
\frac{d\tilde{y}}{dt}&=B\tilde{y}+G(\tilde{x}+x^{1,\frac{1}{\varepsilon}}(\theta_t^1\omega_1),\tilde{y}+y^2(\theta_t^2\omega_2)).
\end{aligned}
\tag{15}
$$

This can be seen by a *formal* differentiation of $x-x^{1,\frac{1}{\varepsilon}}(\omega_1)$, $y-y^2(\omega_2)$. (15) can be solved for any ω contained in a θ-invariant set of full measure and for any initial condition (x_0,y_0) such that the cocycle property is satisfied. Applying the ideas from the end of Section 2 with $V:=V_\varepsilon$ to the solution of (15), then system (13) also has a version satisfying the cocycle property. Hence by the particular structure of V_ε if (15) has an inertial manifold so has (13).

In the following we will omit the tilde in (15). Preparing the time scaling transformation we need the lemma:

Lemma 3.2. *The process $(t,\omega_1)\to x^{1,\frac{1}{\varepsilon}}(\theta_{\varepsilon t}^1\omega_1)$ has the same distribution as the process $(t,\omega_1)\to x^{1,1}(\theta_t^1\omega_1)$ (setting $\varepsilon=1$), where $x^{1,\frac{1}{\varepsilon}}$ is defined in Lemma 3.1.*

Proof. Let $U_{\frac{1}{\varepsilon}A}$ be the fundamental solution of $dx/dt=\frac{1}{\varepsilon}Ax$ and let U_A the fundamental solution for $\varepsilon=1$. The processes $(t,\omega_1)\to x^{1,\frac{1}{\varepsilon}}(\theta_{\varepsilon t}^1\omega_1)$, $(t,\omega_2)\to y^2(\theta_t^2\omega_2)$ are centered Gaussian processes. Hence, the finite dimensional distributions of these processes are uniquely determined by the covariance matrix. This matrix can be calculated by

$$
\begin{aligned}
&\operatorname{cov}(x^{1,\frac{1}{\varepsilon}}(\theta_{\varepsilon t}^1\omega_1),x^{1,\frac{1}{\varepsilon}}(\theta_{\varepsilon s}^1\omega_1))\\
&\quad=\mathbb{E}\left(\int_{-\infty}^{\varepsilon t}U_{\frac{1}{\varepsilon}A}(\varepsilon t-\tau)\frac{1}{\sqrt{\varepsilon}}dw_1(\tau,\omega_1)\otimes\int_{-\infty}^{\varepsilon s}U_{\frac{1}{\varepsilon}A}(\varepsilon s-\tau)\frac{1}{\sqrt{\varepsilon}}dw_1(\tau,\omega_1)\right)\\
&\quad=\mathbb{E}\left(\int_{-\infty}^{\varepsilon t}U_A(t-\frac{\tau}{\varepsilon})\frac{1}{\sqrt{\varepsilon}}dw_1(\tau,\omega_1)\otimes\int_{-\infty}^{\varepsilon s}U_A(s-\frac{\tau}{\varepsilon})\frac{1}{\sqrt{\varepsilon}}dw_1(\tau,\omega_1)\right)\\
&\quad=\int_{-\infty}^{s}U_A(t-\tau)U_A^*(s-\tau)d\tau\quad\text{for } t\geq s,
\end{aligned}
$$

where we have used $U_{\frac{1}{\varepsilon}A}(\varepsilon t)=U_A(t)$. Hence, the expression on the right hand side of the chain of equations is independent of $\varepsilon>0$. □

The scaling $t\to\varepsilon t$ in (15) yields

$$
\begin{aligned}
\frac{dx}{dt}&=Ax+F(x+x^{1,\frac{1}{\varepsilon}}(\theta_{\varepsilon t}^1\omega_1),y+y^2(\theta_{\varepsilon t}^2\omega_2)),\\
\frac{dy}{dt}&=\varepsilon Bx+\varepsilon G(x+x^{1,\frac{1}{\varepsilon}}(\theta_{\varepsilon t}^1\omega_1),y+y^2(\theta_{\varepsilon t}^2\omega_2)).
\end{aligned}
\tag{16}
$$

The Lipschitz constant of the functions on the right hand side are not influenced by the additional terms $x^{1,\frac{1}{\varepsilon}}$, y^2. If we now replace $x^{1,\frac{1}{\varepsilon}}(\theta^1_{\varepsilon t}\omega_1)$ by $x^{1,1}(\theta^1_t\omega_1)$ that has the same distribution by Lemma 3.2, then we obtain a system of the form (11)

$$\begin{aligned}\frac{dx}{dt} &= Ax + F(x + x^{1,1}(\theta^1_t\omega_1), y + y^2(\theta^2_{\varepsilon t}\omega_2)),\\ \frac{dy}{dt} &= \varepsilon Bx + \varepsilon G(x + x^{1,1}(\theta^1_t\omega_1), y + y^2(\theta^2_{\varepsilon t}\omega_2)).\end{aligned} \tag{17}$$

Note that by the independence of $x^{1,1}$ and y^2 the distribution of the solution of (17) is the same as the distribution of the solution of (16). In the next sections we will establish for some general class of systems including (17) as special case the existence of inertial manifolds with a particular limiting behavior for $\varepsilon \to 0$. By the method mentioned at the end of Section 2 we then can show that the original system (13) also has an inertial manifold.

In the following section we will show that system (11) generates a random dynamical system that has a random inertial manifold for sufficiently small $\varepsilon > 0$. However, it makes also sense to study (11) for $\varepsilon = 0$. In that case we can prove that there exists a random invariant manifold consisting of random fixed points. We call this manifold the *slow manifold* of the slow-fast system (11). We emphasize that this situation differs from the deterministic case.

4 Inertial manifolds for random dynamical systems

In this section we will establish the existence of a random inertial manifold for system (11) introduced in Section 2. Random invariant manifolds for different types of stochastic differential equations have been studied in [2], [6], [11]. Here, we will introduce another technique that can also be used for *random* dynamical systems. A similar approach has been applied to prove the existence of invariant manifolds for stochastic partial differential equations in [7]. Note that our technique allows us to formulate optimal conditions for the existence of inertial manifolds.

Our approach to establish an inertial manifold for system (11) is based on some graph transform. It corresponds to Hadamard's method for proving the existence of invariant manifolds for ordinary differential equations [12]. To be able to explain this idea we have to introduce two basic hypotheses which we call (H_1) and (H_2). Later on we will show that these hypotheses are valid under the assumptions $(A_1) - (A_4)$ and the additional assumption

(A_5)

$$c_A > aL.$$

This condition can be interpreted as a *gap condition* for the existence of an invariant manifold of system (11) for sufficiently small ε.

We denote by $Lip(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ the set of globally Lipschitz continuous functions γ mapping $\mathbb{R}^{d_2}$ into $\mathbb{R}^{d_1}$. On this set we have the seminorm

$$L_\gamma := \|\gamma\|_{Lip} := \sup_{y_1 \neq y_2 \in \mathbb{R}^{d_2}} \frac{|\gamma(y_1) - \gamma(y_2)|}{|y_1 - y_2|} < \infty.$$

We recall that $\phi_2^\varepsilon(t, \omega, (x_0, y_0))$ is the second component of the solution operator $\phi^\varepsilon(t, \omega, (x_0, y_0))$ of system (11) introduced in (12).

We need two basic hypotheses to explain the idea of graph transform. The first hypothesis is:

(H_1) For any $\gamma \in Lip(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ the map

$$\mathbb{R}^{d_2} \ni y_0 \to \phi_2^\varepsilon(t, \theta_{-t}^\varepsilon \omega, (\gamma(y_0), y_0)) = \tilde{y} \in \mathbb{R}^{d_2}$$

is invertible for $t = T$, where T is any positive number.

We denote the corresponding inverse mapping by $\Psi^\varepsilon(T, \theta_T^\varepsilon \omega, \gamma)$ such that we have

$$y_0 = \Psi^\varepsilon(T, \theta_T^\varepsilon \omega, \gamma)(\tilde{y}).$$

Using this map we can define the *random graph transform* $\Phi^\varepsilon(T, \omega, \gamma)$ mapping some function $\gamma \in Lip(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ into a function $\hat{\gamma} : \mathbb{R}^{d_2} \to \mathbb{R}^{d_1}$ by

$$\Phi^\varepsilon(T, \omega, \gamma)(\tilde{y}) := \phi_1^\varepsilon(T, \omega, (\gamma(\Psi^\varepsilon(T, \theta_T^\varepsilon \omega, \gamma)(\tilde{y})), \Psi^\varepsilon(T, \theta_T^\varepsilon \omega, \gamma)(\tilde{y}))). \tag{18}$$

Our second hypothesis reads as follows:

(H_2) The mapping $\Phi^\varepsilon(T, \omega, \gamma)$ satisfies the cocycle property for small $\varepsilon > 0$, i.e. for any T_1, T_2 we have

$$\Phi^\varepsilon(T_1 + T_2, \omega, \gamma) = \Phi^\varepsilon(T_1, \theta_{T_2}^\varepsilon \omega, \Phi^\varepsilon(T_2, \omega, \gamma)).$$

Under the validity of these hypotheses the following fact holds.

Lemma 4.1. *For $\omega \in \Omega$ let $\gamma^{*,\varepsilon}(\omega, .) \in Lip(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$. Suppose that $\gamma^{*,\varepsilon}(\omega, \cdot)$ is a random fixed point of Φ^ε: for $t > 0$, $\omega \in \Omega$ we have*

$$\Phi^\varepsilon(t, \omega, \gamma^{*,\varepsilon}(\omega, \cdot))(\tilde{y}) = \gamma^{*,\varepsilon}(\theta_t^\varepsilon \omega, \tilde{y}).$$

Then the random Lipschitz manifold defined by

$$M^\varepsilon(\omega) := \{(\gamma^{*,\varepsilon}(\omega, \tilde{y}), \tilde{y}) | \tilde{y} \in \mathbb{R}^{d_2}\}$$

is positively invariant.

Proof. Since $\gamma^{*,\varepsilon}$ is a fixed point of Φ^ε we have

$$\begin{aligned}\phi^\varepsilon(t,\omega,&(\gamma^{*,\varepsilon}(\omega,y_0),y_0)) = (\phi_1^\varepsilon(t,\omega,(\gamma^{*,\varepsilon}(\omega,y_0),y_0)),\phi_2^\varepsilon(t,\omega,(\gamma^{*,\varepsilon}(\omega,y_0),y_0)))\\ =&(\phi_1^\varepsilon(t,\omega,(\gamma^{*,\varepsilon}(\omega,\Psi^\varepsilon(t,\theta_t^\varepsilon\omega,\gamma^{*,\varepsilon}(\omega,\cdot))(\tilde y)),\Psi^\varepsilon(t,\theta_t^\varepsilon\omega,\gamma^{*,\varepsilon}(\omega,\cdot))(\tilde y))),\tilde y)\\ =&(\Phi^\varepsilon(t,\omega,\gamma^{*,\varepsilon}(\omega,.))(\tilde y),\tilde y) = (\gamma^{*,\varepsilon}(\theta_t^\varepsilon\omega,\tilde y),\tilde y)\in M^\varepsilon(\theta_t^\varepsilon\omega).\end{aligned}$$

□

From Lemma 4.1 we can conclude that the graph of this fixed point represents an *invariant* manifold of system (11).

The following *nonstandard* boundary value problem plays a fundamental rôle in establishing the validity of the hypotheses (H_1) and (H_2) under our assumptions.

$$\begin{aligned}\frac{dx}{dt} &= A(\theta_t^\varepsilon\omega)x + F(\theta_t^\varepsilon\omega,x,y), \qquad 0<t<T,\\ \frac{dy}{dt} &= \varepsilon B(\theta_t^\varepsilon\omega)y + \varepsilon G(\theta_t^\varepsilon\omega,x,y),\end{aligned} \tag{19}$$

$$x(0)=\gamma(y(0)), \qquad y(T)=\tilde y, \tag{20}$$

where ε is a small positive parameter, $\tilde y\in\mathbb{R}^{d_2}, T>0$, and $\gamma\in Lip(\mathbb{R}^{d_2},\mathbb{R}^{d_1})$ are given.

Remark 4.2. *Suppose this boundary value problem has the unique solution*

$$z^\varepsilon(.,\omega,T,\gamma,\tilde y) = (x^\varepsilon(.,\omega,T,\gamma,\tilde y),y^\varepsilon(.,\omega,T,\gamma,\tilde y)).$$

Then, there is a one-to-one relation between $\tilde y$ and $y^\varepsilon(0,\omega,T,\gamma,\tilde y)) = y_0$, that is, hypothesis (H_1) is valid and the inverse mapping Ψ^ε and the graph transform Φ^ε are given by

$$\Psi^\varepsilon(T,\theta_T^\varepsilon\omega,\gamma)(\tilde y) := y^\varepsilon(0,\omega,T,\gamma,\tilde y), \quad \Phi^\varepsilon(T,\omega,\gamma)(\tilde y) := x^\varepsilon(T,\omega,T,\gamma,\tilde y). \tag{21}$$

Lemma 4.3. *Assume the hypotheses $(A_1)-(A_3)$ and (A_5) to be valid. Then for any $\tilde y\in\mathbb{R}^{d_2}, T>0, \omega\in\Omega, \gamma\in Lip(\mathbb{R}^{d_1},\mathbb{R}^{d_2})$ there exists a sufficiently small positive number ε_0 such that for $0<\varepsilon<\varepsilon_0$ the boundary value problem* (19),(20) *has a unique solution $(x^\varepsilon(t,\omega,T,\gamma,\tilde y),y^\varepsilon(t,\omega,T,\gamma,\tilde y))$.*

Proof. We first note that the boundary value problem (19),(20) is equivalent to the system of integral equations

$$\begin{aligned}x(t) &= U_A^\varepsilon(t,\omega)\gamma(y(0)) + \int_0^t U_A^\varepsilon(t-s,\theta_s^\varepsilon\omega)F(\theta_s^\varepsilon\omega,x(s),y(s))ds,\\ y(t) &= U_{\varepsilon B}^\varepsilon(t-T,\theta_T^\varepsilon\omega)\tilde y + \varepsilon\int_T^t U_{\varepsilon B}^\varepsilon(t-s,\theta_s^\varepsilon\omega)G(\theta_s^\varepsilon\omega,x(s),y(s))ds.\end{aligned} \tag{22}$$

To study system (22) we introduce the following spaces

$$C_1 := C([0,T],\mathbb{R}^{d_1}),\ C_2 := C([0,T],\mathbb{R}^{d_2})$$

and endow these spaces with the norms

$$\begin{aligned} ||x||_{1,\alpha} &:= \max_{0\le t\le T} e^{-\alpha(T-t)}|x(t)| \text{ for } x\in C_1,\\ ||y||_{2,\alpha} &:= \max_{0\le t\le T} e^{-\alpha(T-t)}|y(t)| \text{ for } y\in C_2, \end{aligned} \tag{23}$$

where α is a positive number satisfying

$$c_A-\alpha > aL. \tag{24}$$

Let C be the product space $C := C_1\times C_2$, $z=(x,y)\in C$. C equipped with the norm

$$||z||_\alpha := ||x||_{1,\alpha}+||y||_{2,\alpha} \tag{25}$$

is a Banach space.
For fixed $\omega, T, \gamma, \tilde y$ we introduce the operators $\mathcal{T}_1^\varepsilon : C\to C_1$ and $\mathcal{T}_2^\varepsilon : C\to C_2$ by

$$\begin{aligned} \hat y(t) = \mathcal{T}_2^\varepsilon(z(\cdot))[t] :=& U^\varepsilon_{\varepsilon B}(t-T,\theta^\varepsilon_T\omega)\ \tilde y\\ &+\varepsilon\int_T^t U^\varepsilon_{\varepsilon B}(t-s,\theta^\varepsilon_s\omega)\ G(\theta^\varepsilon_s\omega, x(s), y(s))ds,\\ \hat x(t) = \mathcal{T}_1^\varepsilon(z(\cdot))[t] :=& U^\varepsilon_A(t,\omega)\gamma(\hat y(0))\\ &+\int_0^t U^\varepsilon_A(t-s,\theta^\varepsilon_s\omega)F(\theta^\varepsilon_s\omega, x(s), y(s))ds \end{aligned} \tag{26}$$

and define the operator $\mathcal{T}^\varepsilon$ by

$$\hat z(\cdot) = \mathcal{T}^\varepsilon(z(\cdot)) := \begin{pmatrix} \mathcal{T}_1^\varepsilon(z(\cdot))\\ \mathcal{T}_2^\varepsilon(z(\cdot)) \end{pmatrix}.$$

It is obvious that a fixed point z^ε of $\mathcal{T}^\varepsilon$ represents a solution of system (22), and thus of the boundary value problem (19), (20).

Under our assumptions above, $\mathcal{T}^\varepsilon$ maps C into itself. In what follows we show that $\mathcal{T}^\varepsilon$ is also strictly contractive.
For $z_i=(x_i,y_i)\in C, i=1,2$, we set

$$\Delta x := x_1-x_2,\ \Delta y := y_2-y_1,\ |\Delta z| := |\Delta x|+|\Delta y|. \tag{27}$$

Analogously, we define $\Delta\hat x, \Delta\hat y$, and $\Delta\hat z$.

From (26), (9), (10), (27) and (25) we get

$$\begin{aligned}
|\Delta\hat{y}(t)| &\le \varepsilon b \int_t^T e^{\varepsilon c_B(s-t)} |G(\theta_s^\varepsilon\omega, x_1(s), y_1(s)) - G(\theta_s^\varepsilon\omega, x_2(s), y_2(s))| ds \\
&\le \varepsilon b L \int_t^T e^{\varepsilon c_B(s-t)} e^{\alpha(T-s)} e^{-\alpha(T-s)} |\Delta z(s)| \; ds \\
&\le \varepsilon b L \; ||\Delta z||_\alpha \; e^{\alpha T - \varepsilon c_B t} \int_t^T e^{-(\alpha-\varepsilon c_B)s} ds.
\end{aligned}$$

For the sequel we assume ε to be so small that

$$\alpha - \varepsilon c_B > 0.$$

Taking into account this relation we have

$$|\Delta\hat{y}(t)| \le \frac{\varepsilon b L}{\alpha - \varepsilon c_B} \; ||\Delta z||_\alpha \; e^{\alpha(T-t)}, \tag{28}$$

and thus it holds by (23)

$$||\Delta\hat{y}||_{2,\alpha} \le \frac{\varepsilon b L}{\alpha - \varepsilon c_B} ||\Delta z||_\alpha. \tag{29}$$

Analogously, we obtain from (26), (27) and (8)

$$\begin{aligned}
|\Delta\hat{x}(t)| &\le e^{-c_A t} a L_\gamma |\Delta\hat{y}(0)| + aL \int_0^t e^{-c_A(t-s)} |\Delta z(s)| \; ds \\
&\le e^{-c_A t} a L_\gamma |\Delta\hat{y}(0)| + aL ||\Delta z||_\alpha \; e^{-c_A t} e^{\alpha T} \int_0^t e^{(c_A-\alpha)s} ds.
\end{aligned} \tag{30}$$

We then get from (30)

$$|\Delta\hat{x}(t)| \le e^{-c_A t} a L_\gamma |\Delta\hat{y}(0)| + \frac{aL e^{\alpha(T-t)}}{c_A - \alpha} ||\Delta z||_\alpha.$$

Hence, we have by (28) and (24)

$$||\Delta\hat{x}||_{1,\alpha} \le \left(\frac{\varepsilon a b L_\gamma L}{\alpha - \varepsilon c_B} + \frac{aL}{c_A - \alpha} \right) ||\Delta z||_\alpha. \tag{31}$$

From (25), (29), and (31) we get

$$||\Delta\hat{z}||_\alpha \le \; \kappa(\varepsilon, L_\gamma) ||\Delta z||_\alpha, \tag{32}$$

where

$$\kappa(\varepsilon) := \kappa(\varepsilon, L_\gamma) := \frac{\varepsilon b L}{\alpha - \varepsilon c_B}(1 + a L_\gamma) + \frac{aL}{c_A - \alpha}. \tag{33}$$

By (33) and assumption (A_5) it holds

$$\kappa(0) = \frac{aL}{c_A - \alpha} < 1, \quad \kappa'(\varepsilon) > 0 \quad \text{for} \quad \varepsilon > 0.$$

Hence, there is a sufficiently small positive number ε_0 and a constant κ_0 satisfying $0 < \kappa_0 < 1$, such that

$$\kappa(\varepsilon) \leq \kappa_0 < 1 \quad \text{for} \quad 0 \leq \varepsilon \leq \varepsilon_0.$$

Therefore, the operator $\mathcal{T}^\varepsilon$ is strictly contractive and has a unique fixed point $z^\varepsilon \in C$, and, consequently, the boundary value problem (19), (20) has a unique solution $(x^\varepsilon(t, \omega, T, \gamma, \tilde{y}), y^\varepsilon(t, \omega, T, \gamma, \tilde{y}))$. □

Remark 4.4. *We note that κ_0 can be chosen independently of $\omega, \tilde{y}$ and T. In addition, the constant ε_0 introduced in the formulation of the last Lemma can be chosen such that it depends only on L_γ, the Lipschitz constant of γ. Standard arguments allow to conclude that the fixed point depend Lipschitz continuously on the parameter $\tilde{y}$.*

In what follows we investigate the dependence of the fixed point z^ε of the operator $\mathcal{T}^\varepsilon$ on the function γ. To this end we restrict our space $Lip(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$. We denote by $\mathcal{L}$ the subset of $Lip(\mathbb{R}^{d_2}, \mathbb{R}^{d_1})$ consisting of bounded functions and introduce in $\mathcal{L}$ the norm

$$||\gamma||_\infty := \sup_{\tilde{y} \in \mathbb{R}^{d_2}} |\gamma(\tilde{y})|.$$

Since the operator $\mathcal{T}^\varepsilon$ explicitly depends on γ we use in the following the notation $\mathcal{T}^\varepsilon_\gamma$.

Lemma 4.5. *Suppose the hypotheses of Lemma* 4.3 *to be valid. Then to given $\omega \in \Omega, T > 0, \tilde{y} \in \mathbb{R}^{d_2}, t \in [0, T]$ and for sufficiently small ε the solution $z^\varepsilon(t, \omega, T, \gamma, \tilde{y})$ of* (19), (20) *depends Lipschitz continuously on $\gamma \in \mathcal{L}$:*

$$\sup_{\tilde{y} \in \mathbb{R}^{d_2}} \|z^\varepsilon(., \omega, T, \gamma_1, \tilde{y}) - z^\varepsilon(., \omega, T, \gamma_2, \tilde{y})\|_\alpha \leq \frac{ae^{-\alpha T}}{1 - \kappa(\varepsilon, L_{\gamma_1})} ||\gamma_1 - \gamma_2||_\infty, \tag{34}$$

where $\kappa(\varepsilon, L_\gamma)$ is defined in (33).

Proof. Let γ_1 and γ_2 be any functions from $\mathcal{L}$. By Lemma 4.3, to given $(\omega, T, \gamma, \tilde{y})$ there exists for sufficiently small ε a unique fixed point z^ε of the operator $\mathcal{T}^\varepsilon_\gamma$. We set

$$\Delta_\gamma z^\varepsilon(t) := z^\varepsilon(t, \omega, T, \gamma_1, \tilde{y}) - z^\varepsilon(t, \omega, T, \gamma_2, \tilde{y}).$$

A corresponding notation is used for the components $\Delta_\gamma x^\varepsilon(t), \Delta_\gamma y^\varepsilon(t)$ of $\Delta_\gamma z^\varepsilon(t)$. By (32) we have

$$\begin{aligned} \|\Delta_\gamma z^\varepsilon\|_\alpha =& \|\mathcal{T}^\varepsilon_{\gamma_1} z^\varepsilon(.,\omega,T,\gamma_1,\tilde{y}) - \mathcal{T}^\varepsilon_{\gamma_2} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y})\|_\alpha \\ \leq& \|\mathcal{T}^\varepsilon_{\gamma_1} z^\varepsilon(.,\omega,T,\gamma_1,\tilde{y}) - \mathcal{T}^\varepsilon_{\gamma_1} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y})\|_\alpha \\ &+ \|\mathcal{T}^\varepsilon_{\gamma_1} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y}) - \mathcal{T}^\varepsilon_{\gamma_2} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y})\|_\alpha \\ \leq& \kappa(\varepsilon, L_{\gamma_1})\|\Delta_\gamma z^\varepsilon\|_\alpha \\ &+ \|\mathcal{T}^\varepsilon_{\gamma_1} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y}) - \mathcal{T}^\varepsilon_{\gamma_2} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y})\|_\alpha. \end{aligned} \tag{35}$$

For the last term we get from (26)

$$\begin{aligned} &\|\mathcal{T}^\varepsilon_{\gamma_1} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y}) - \mathcal{T}^\varepsilon_{\gamma_2} z^\varepsilon(.,\omega,T,\gamma_2,\tilde{y})\|_\alpha \\ &\leq \|U^\varepsilon_A(.,\omega)\|_\alpha \, ||\gamma_1-\gamma_2||_\infty \leq ae^{-\alpha T}||\gamma_1-\gamma_2||_\infty. \end{aligned} \tag{36}$$

Consequently, we get from (35), (36)

$$\|\Delta_\gamma z^\varepsilon\|_\alpha \leq \frac{ae^{-\alpha T}}{1-\kappa(\varepsilon, L_{\gamma_1})} \, ||\gamma_1-\gamma_2||_\infty.$$

Since the right hand side does not depend on $\tilde{y}$, we obtain (34)

For the components $\Delta_\gamma x^\varepsilon, \Delta_\gamma y^\varepsilon$ we have

$$\sup_{\tilde{y}\in\mathbb{R}^{d_2}} \|\Delta_\gamma y^\varepsilon\|_{2,\alpha} \leq \frac{\varepsilon bL}{\alpha-\varepsilon c_B}\|\Delta_\gamma z^\varepsilon\| \leq \frac{\varepsilon bL}{\alpha-\varepsilon c_B}\frac{ae^{-\alpha T}}{1-\kappa(\varepsilon, L_{\gamma_1})} \, ||\gamma_1-\gamma_2||_\infty, \tag{37}$$

$$\sup_{\tilde{y}\in\mathbb{R}^{d_2}} \|\Delta_\gamma x^\varepsilon\|_{1,\alpha} \leq \left(\frac{\varepsilon abL_\gamma L}{\alpha-\varepsilon c_B} + \frac{aL}{c_A-\alpha}\right)\frac{ae^{-\alpha T}}{1-\kappa(\varepsilon, L_{\gamma_1})} \, ||\gamma_1-\gamma_2||_\infty.$$

□

For the following we recall that by (21) the graph transform Φ^ε is given by

$$\begin{aligned} &\Phi^\varepsilon(T,\omega,\gamma)(\tilde{y}) := x^\varepsilon(T,\omega,T,\gamma,\tilde{y}) = U^\varepsilon_A(T,\omega)\gamma(y(0)) \\ &+ \int_0^T U^\varepsilon_A(T-s,\theta^\varepsilon_s\omega)F(\theta^\varepsilon_s\omega, x^\varepsilon(s,\omega,T,\gamma,\tilde{y}), y^\varepsilon(s,\omega,T,\gamma,\tilde{y}))ds, \end{aligned} \tag{38}$$

and that the set $\mathcal{L}$ consists of all bounded globally Lipschitz continuous functions mapping $\mathbb{R}^{d_2}$ into $\mathbb{R}^{d_1}$. For the following we introduce the set $\mathcal{L}_\Gamma \subset \mathcal{L}$ whose elements γ satisfy

$$L_\gamma = ||\gamma||_{Lip} \leq \Gamma.$$

We note that the set $\mathcal{L}_\Gamma$ is complete and separable with respect to the metric $||\gamma_1 - \gamma_2||_\infty$.

Lemma 4.6. *Assume that the assumptions (A_1)-(A_4) hold. Then, for sufficiently small ε and sufficiently large T, the graph transform $\Phi^\varepsilon(T,\omega,.)$ maps the set $\mathfrak{L}_\Gamma$ into itself, where Γ is any positive number satisfying*

$$\Gamma \geq \Gamma^* := \frac{b}{1-\beta-abe^{-\frac{T\alpha}{2}}}.$$

Here, β is any given number from the interval $\left(\frac{aL}{c_A-\alpha},1\right)$.

Proof. By assumptions (A_2) and (A_4) we get from (38) for $\gamma \in \mathfrak{L}$

$$||\Phi^\varepsilon(T,\omega,\gamma)||_\infty \leq ae^{-\alpha T}||\gamma||_\infty + \frac{ac_F}{c_A} < \infty.$$

Set $\Delta_{\tilde{y}} z^\varepsilon(t) := z^\varepsilon(t,\omega,T,\gamma,\tilde{y}_1) - z^\varepsilon(t,\omega,T,\gamma,\tilde{y}_2), \quad \Delta\tilde{y} := \tilde{y}_1-\tilde{y}_2$. Similar to (28) and (30) we obtain for $\gamma \in \mathfrak{L}_{\Gamma^*}$

$$|\Delta_{\tilde{y}} y^\varepsilon(t)| \leq \frac{\varepsilon bL}{\alpha-\varepsilon c_B}\, ||\Delta_{\tilde{y}} z^\varepsilon||_\alpha \; e^{\alpha(T-t)} + |\Delta\tilde{y}|be^{\varepsilon c_B(T-t)}, \tag{39}$$

$$|\Delta_{\tilde{y}} x^\varepsilon(t)| \leq e^{-c_A t}aL_\gamma|\Delta_{\tilde{y}} y^\varepsilon(0)| + \frac{aLe^{\alpha(T-t)}}{c_A-\alpha}\|\Delta_{\tilde{y}} z^\varepsilon\|_\alpha. \tag{40}$$

According to these estimates we have

$$\begin{aligned}
\|\Delta_{\tilde{y}} y^\varepsilon\|_{2,\alpha} &\leq \frac{\varepsilon bL}{\alpha-\varepsilon c_B}\|\Delta_{\tilde{y}} z^\varepsilon\|_\alpha + b|\Delta\tilde{y}|,\\
\|\Delta_{\tilde{y}} x^\varepsilon\|_{1,\alpha} &\leq \left(\frac{aL}{c_A-\alpha}+\frac{\varepsilon abLL_\gamma}{\alpha-\varepsilon c_B}\right)\|\Delta_{\tilde{y}} z^\varepsilon\|_\alpha + abL_\gamma|\Delta\tilde{y}|e^{-T(\alpha-\varepsilon c_B)}.
\end{aligned}$$

From (39), (40) we obtain

$$\|\Delta_{\tilde{y}} z^\varepsilon\|_\alpha \leq \kappa(\varepsilon,L_\gamma)\|\Delta_{\tilde{y}} z^\varepsilon\|_\alpha + (b+abL_\gamma e^{-T(\alpha-\varepsilon c_B)})|\Delta\tilde{y}|,$$

where $\kappa(\varepsilon,L_\gamma)$ is defined in (33). According to $|\Delta_{\tilde{y}} x(T)| \leq \|\Delta_{\tilde{y}} z\|_\alpha$ we have

$$\|\Phi^\varepsilon(T,\omega,\gamma)\|_{Lip} \leq \frac{b+abL_\gamma e^{-\alpha T+\varepsilon c_B T}}{1-\kappa(\varepsilon,L_\gamma)}.$$

Let $\beta \in (\kappa(0,L_\gamma) = aL/(c_A-\alpha),1)$. Then there is a sufficiently small ε_0 such that for $0 \leq \varepsilon \leq \varepsilon_0$ and any $T>0$

$$\frac{1}{1-\kappa(\varepsilon,L_\gamma)} < \frac{1}{1-\beta}, \qquad e^{-T(\alpha-\varepsilon c_B)} \leq e^{-\frac{T\alpha}{2}}.$$

Thus, we have for $0 \leq \varepsilon \leq \varepsilon_0$

$$\frac{b(1+aL_\gamma e^{-T(\alpha-\varepsilon c_B)})}{1-\kappa(\varepsilon,L_\gamma)} \leq \frac{b(1+aL_\gamma e^{-\frac{T\alpha}{2}})}{1-\beta}.$$

There is a $T_0 > 0$ such that for $T \geq T_0$

$$1 - \beta > abe^{-\frac{T_0 \alpha}{2}}.$$

Hence, for $T \geq T_0$ the inequality

$$\frac{b(1 + aL_\gamma e^{-\frac{T\alpha}{2}})}{1 - \beta} \leq L_\gamma$$

is valid for

$$L_\gamma \geq \Gamma^* := \frac{b}{1 - \beta - abe^{-\frac{T_0 \alpha}{2}}},$$

and we have

$$||\Phi^\varepsilon(T, \omega, \gamma)||_{Lip} \leq L_\gamma$$

for sufficiently large T, sufficiently small ε, and $L_\gamma \geq \Gamma^*$. $\square$

Our intention is to establish a random fixed point of the graph transform $\Phi^\varepsilon(T, \omega, .)$. Then, by Lemma 4.1, the random inertial manifold is given by the graph of that fixed point. Hence, in what follows we will show that the graph transform $\Phi^\varepsilon(T, \omega, .)$ is strictly contractive in $\mathfrak{L}_\Gamma$ with Γ defined in Lemma 4.6, for sufficiently small ε and sufficiently large T.

Lemma 4.7. *Suppose the hypotheses* $(\mathrm{A}_1) - (\mathrm{A}_5)$ *to be satisfied. Then, there is a constant* $k(\varepsilon, T)$ *satisfying for sufficiently small* ε *and sufficiently large* T

$$0 < k(\varepsilon, T) \leq k_0 < 1,$$

such for any γ_1, $\gamma_2 \in \mathfrak{L}_\Gamma$, *where* Γ *is defined in Lemma* 4.6,

$$\|\Phi^\varepsilon(T, \omega, \gamma_1) - \Phi^\varepsilon(T, \omega, \gamma_2)\|_\infty \leq k(\varepsilon, T)\|\gamma_1 - \gamma_2\|_\infty,$$

Proof. Let γ_1, γ_2 be any functions in $\mathfrak{L}_\Gamma$, let $(x^\varepsilon(t, \omega, T, \gamma_i, \tilde{y}), y^\varepsilon(t, \omega, T, \gamma_i, \tilde{y}))$ be the corresponding unique solution of the boundary value problem (19), (20) which exists according to Lemma 4.3. We set

$$\begin{aligned}
\Delta_\gamma x^\varepsilon(t, \tilde{y}) &:= x^\varepsilon(t, \omega, T, \gamma_1, \tilde{y}) - x^\varepsilon(t, \omega, T, \gamma_2, \tilde{y}), \\
\Delta_\gamma y^\varepsilon(t, \tilde{y}) &:= y^\varepsilon(t, \omega, T, \gamma_1, \tilde{y}) - y^\varepsilon(t, \omega, T, \gamma_2, \tilde{y}).
\end{aligned}$$

By (38) we have

$$\Phi^\varepsilon(T, \omega, \gamma_1) - \Phi^\varepsilon(T, \omega, \gamma_2) = \Delta_\gamma x^\varepsilon(T, \tilde{y}).$$

Furthermore, we introduce the notation

$$\Delta_\gamma \xi^\varepsilon(t) := ||\Delta_\gamma x^\varepsilon(t, .)||_\infty, \ \Delta_\gamma \eta^\varepsilon(t) := ||\Delta_\gamma y^\varepsilon(t, .)||_\infty, \ \Delta\gamma := \gamma_1 - \gamma_2.$$

Using this notation the assertion of Lemma 4.7 takes the form

$$\Delta_\gamma \xi^\varepsilon(T) \leq k(\varepsilon, T) \|\Delta\gamma\|_\infty. \tag{41}$$

Under our assumptions we get from (22)

$$\begin{aligned} \Delta_\gamma \xi^\varepsilon(t) &\leq a e^{-c_A t}(||\gamma_1(y^\varepsilon(0,\omega,T,\gamma_1,\tilde{y}) - \gamma_2(y^\varepsilon(0,\omega,T,\gamma_2,\tilde{y})||_\infty) \\ &\quad + L\int_0^t a e^{-c_A(t-s)}\big(\Delta_\gamma\xi^\varepsilon(s) + \Delta_\gamma\eta^\varepsilon(s)\big)ds, \\ \Delta_\gamma\eta^\varepsilon(t) &\leq -\varepsilon L\int_T^t b e^{-\varepsilon c_B(t-s)}\big(\Delta_\gamma\xi^\varepsilon(s) + \Delta_\gamma\eta^\varepsilon(s)\big)ds. \end{aligned} \tag{42}$$

With respect to $||\gamma_1(y^\varepsilon(0,\omega,T,\gamma_1,\tilde{y})) - \gamma_2(y^\varepsilon(0,\omega,T,\gamma_2,\tilde{y}))||_\infty$ we obtain for $\gamma_1, \gamma_2 \in \mathcal{L}_\Gamma$

$$\begin{aligned} &||\gamma_1(y^\varepsilon(0,\omega,T,\gamma_1,\tilde{y})) - \gamma_2(y^\varepsilon(0,\omega,T,\gamma_2,\tilde{y}))||_\infty \\ &\leq ||\gamma_1(y^\varepsilon(0,\omega,T,\gamma_1,\tilde{y})) - \gamma_1(y^\varepsilon(0,\omega,T,\gamma_2,\tilde{y}))||_\infty \\ &+ ||\gamma_1(y^\varepsilon(0,\omega,T,\gamma_2,\tilde{y})) - \gamma_2(y^\varepsilon(0,\omega,T,\gamma_2,\tilde{y}))||_\infty \leq \Gamma\Delta_\gamma\eta^\varepsilon(0) + ||\Delta\gamma||_\infty. \end{aligned} \tag{43}$$

Substituting (43) into the first inequality of (42) we get

$$\begin{aligned} \Delta_\gamma\xi^\varepsilon(t) &\leq a e^{-c_A t}\Big(\Gamma\Delta_\gamma\eta^\varepsilon(0) + ||\gamma_1 - \gamma_2||_\infty\Big) \\ &\quad + L\int_0^t a e^{-c_A(t-s)}\big(\Delta_\gamma\xi^\varepsilon(s) + \Delta_\gamma\eta^\varepsilon(s)\big)ds. \end{aligned} \tag{44}$$

Using the norm

$$||f||_\alpha := \max_{0\leq t\leq T} e^{-\alpha(T-t)}|f(t)| \quad \text{for} \quad f \in C([0,T],\mathbb{R}),$$

where α is the positive number defined in (24), we obtain from the second inequality in (42)

$$\begin{aligned} \Delta_\gamma\eta^\varepsilon(t) &\leq b\varepsilon L(||\Delta_\gamma\xi^\varepsilon||_\alpha + ||\Delta_\gamma\eta^\varepsilon||_\alpha)e^{\alpha T - \varepsilon c_B t}\int_t^T e^{-(\alpha-\varepsilon c_B)s}ds \\ &\leq \frac{b\varepsilon L}{\alpha - \varepsilon c_B}(||\Delta_\gamma\xi^\varepsilon||_\alpha + ||\Delta_\gamma\eta^\varepsilon||_\alpha)e^{\alpha(T-t)}. \end{aligned} \tag{45}$$

Thus, we have

$$||\Delta_\gamma\eta^\varepsilon||_\alpha \leq \frac{b\varepsilon L}{\alpha - \varepsilon c_B}(||\Delta_\gamma\xi^\varepsilon||_\alpha + ||\Delta_\gamma\eta^\varepsilon||_\alpha). \tag{46}$$

If we assume ε to be so small that

$$\alpha - \varepsilon(c_B + bL) > 0$$

we get from (46)

$$||\Delta_\gamma \eta^\varepsilon||_\alpha \leq \frac{b\varepsilon L}{\alpha - \varepsilon(c_B + bL)} ||\Delta_\gamma \xi^\varepsilon||_\alpha. \tag{47}$$

Putting $t = 0$ in (45) and substituting the corresponding relation into (44) we get

$$\begin{aligned} \Delta_\gamma \xi^\varepsilon(t) \leq & a e^{-c_A t} \Big(\frac{\varepsilon \Gamma b L e^{\alpha T}}{\alpha - \varepsilon c_B} (||\Delta_\gamma \xi^\varepsilon||_\alpha + ||\Delta_\gamma \eta^\varepsilon||_\alpha) + ||\gamma_1 - \gamma_2||_\infty \Big) \\ & + L \int_0^t a e^{-c_A(t-s)} \big(\Delta_\gamma \xi^\varepsilon(s) + \Delta_\gamma \eta^\varepsilon(s) \big) ds \\ \leq & a e^{-c_A t} \Big(\frac{\varepsilon \Gamma b L e^{\alpha T}}{\alpha - \varepsilon c_B} (||\Delta_\gamma \xi^\varepsilon||_\alpha + ||\Delta_\gamma \eta^\varepsilon||_\alpha) + ||\Delta\gamma||_\infty \Big) \\ & + a \frac{L e^{\alpha(T-t)}}{c_A - \alpha} (||\Delta_\gamma \xi^\varepsilon||_\alpha + ||\Delta_\gamma \eta^\varepsilon||_\alpha). \end{aligned} \tag{48}$$

Taking into account (47) we obtain from (48) by an elementary calculation

$$\Delta_\gamma \xi^\varepsilon(t) \leq k_1(\varepsilon) e^{\alpha(T-t)} ||\Delta_\gamma \xi^\varepsilon||_\alpha + a e^{-c_A t} ||\Delta\gamma||_\infty, \tag{49}$$

where $k_1(\varepsilon)$ depends continuously on ε. By (24) it holds

$$0 < k_1(0) = \frac{aL}{(c_A - \alpha)} < 1,$$

such that we have

$$0 < k_1(\varepsilon) < 1 \quad \text{for sufficiently small } \varepsilon.$$

Setting

$$k(\varepsilon, T) := \frac{a e^{-\alpha T}}{1 - k_1(\varepsilon)}$$

we get from (49)

$$||\Delta_\gamma \xi^\varepsilon||_\alpha \leq k(\varepsilon, T) ||\Delta\gamma||_\infty. \tag{50}$$

It is obvious that $k(\varepsilon, T)$ satisfies $0 < k(\varepsilon, T) \leq k_0 < 1$ for sufficiently small ε and sufficiently large T. Using the inequality

$$\Delta_\gamma \xi^\varepsilon(T) \leq ||\Delta_\gamma \xi^\varepsilon||_\alpha$$

we obtain from (50) the validity of inequality (41).

□

Suppose that $T > 0$ satisfies the conclusion of the last Lemma. According to Lemma 4.6 we note that there exists a $\bar{\Gamma} \in [0, \infty)$ which can be larger than Γ^* such that

$$\sup_{t \leq T, \gamma \in \mathcal{L}_{\Gamma^*}} \|\Phi^\varepsilon(t, \omega, \gamma)\|_{Lip} \leq \bar{\Gamma} \vee \Gamma^*.$$

To ensure that $\Phi^\varepsilon(t,\omega,\gamma)$, $\gamma\in\mathcal{L}_{\bar\Gamma\vee\Gamma^*}$, $t>0$ is well defined we have to choose ε_0 (see Lemma 4.3, Remark 4.4) sufficiently small. Then for the *ω-dependent* non-empty set

$$\mathcal{H}^\varepsilon(\omega):=\bigcup_{t\geq 0}\Phi^\varepsilon(t,\theta^\varepsilon_{-t}\omega,\mathcal{L}_{\Gamma^*})\subset\mathcal{L}_{\bar\Gamma\vee\Gamma^*}$$

we have

$$\Phi^\varepsilon(t,\omega,\mathcal{H}^\varepsilon(\omega))\subset\mathcal{H}^\varepsilon(\theta^\varepsilon_t\omega)$$

for $t\geq 0$. According to this relation we are able to prove the cocycle property for the graph transform Φ^ε.

Lemma 4.8. *For sufficiently small positive ε the graph transform Φ^ε satisfies the cocycle property: For t_1, $t_2\geq 0,\omega\in\Omega$ and $\gamma\in\mathcal{H}^\varepsilon(\omega)$ we have*

$$\Phi^\varepsilon(t_1+t_2,\omega,\gamma)=\Phi^\varepsilon(t_1,\theta_{t_2}\omega,\Phi^\varepsilon(t_2,\omega,\gamma)).$$

Proof. The above considerations ensure that all appearing operators are well defined. Let

$$(x_1^\varepsilon(t),y_1^\varepsilon(t)):=(x_1^\varepsilon(t,\omega,T_1,\gamma,\tilde y),y_1^\varepsilon(t,\omega,T_1\gamma,\tilde y)),\quad t\in[0,T_1]$$

be the solution of (19) satisfying the boundary conditions

$$x_1^\varepsilon(0)=\gamma(y_1^\varepsilon(0)),\ y_1^\varepsilon(T_1)=\tilde y.$$

Then $x_1^\varepsilon(T_1)$ defines $\Phi^\varepsilon(T_1,\omega,\gamma)(\tilde y)$. Similarly, let

$$(x_2^\varepsilon(t),y_2^\varepsilon(t)):=(x_2^\varepsilon(t,\theta^\varepsilon_{T_1}\omega,T_2,\mu,\tilde y),y_2^\varepsilon(t,\theta^\varepsilon_{T_1}\omega,T_2,\mu,\tilde y)),\quad t\in[0,T_2]$$

be the solution of (19), (20) such that $x_2^\varepsilon(T_2)$ defines $\Phi^\varepsilon(T_2,\theta^\varepsilon_{T_1}\omega,\mu)(\tilde y)$ with $\mu:=\Phi^\varepsilon(T_1,\omega,\gamma)$ which is equal to $x_1(T_1,\omega,T_1,\gamma,\cdot)$. We set $T=T_1+T_2$ and define the functions y^ε and x^ε by

$$y^\varepsilon(t,\omega,T,\gamma,\tilde y):=\begin{cases} y_1^\varepsilon(t,\omega,T_1,\gamma,y_2^\varepsilon(0,\theta^\varepsilon_{T_1}\omega,T_2,\mu,\tilde y)) & \text{for}\quad t\in[0,T_1],\\ y_2^\varepsilon(t-T_1,\theta^\varepsilon_{T_1}\omega,T_2,\mu,\tilde y) & \text{for}\quad t\in(T_1,T],\end{cases}\tag{51}$$

$$x^\varepsilon(t,\omega,T,\gamma,\tilde y):=\begin{cases} x_1^\varepsilon(t,\omega,T_1,\gamma,y_2^\varepsilon(0,\theta^\varepsilon_{T_1}\omega,T_2,\mu,\tilde y)) & \text{for}\quad t\in[0,T_1],\\ x_2^\varepsilon(t-T_1,\theta^\varepsilon_{T_1}\omega,T_2,\mu,\tilde y) & \text{for}\quad t\in(T_1,T].\end{cases}\tag{52}$$

(51) yields that $y^\varepsilon(\cdot,\omega,T,\gamma,\tilde y)$ is continuous on $[0,T]$ and in particular in $t=T_1$. Due to (22) the relation (51) can be written for $t\in[0,T_1]$

$$\begin{aligned}
&U^\varepsilon_{\varepsilon B}(t-T_1,\theta^\varepsilon_{T_1}\omega)U^\varepsilon_{\varepsilon B}(-T_2,\theta^\varepsilon_{T_1+T_2}\omega)\tilde y\\
&\qquad+\varepsilon\int_{T_2}^0 U^\varepsilon_{\varepsilon B}(t-T_1,\theta^\varepsilon_{T_1}\omega)U^\varepsilon_{\varepsilon B}(-s,\theta^\varepsilon_{T_1+s}\omega)G(\theta^\varepsilon_{T_1+s}\omega,x_2^\varepsilon(s),y_2^\varepsilon(s))ds\\
&\qquad+\varepsilon\int_{T_1}^t U^\varepsilon_{\varepsilon B}(t-s,\theta^\varepsilon_s\omega)G(\theta^\varepsilon_s\omega,x_1^\varepsilon(s),y_1^\varepsilon(s))ds\\
&=U^\varepsilon_{\varepsilon B}(t-T_1-T_2,\theta^\varepsilon_{T_1+T_2}\omega)\tilde y+\varepsilon\int_{T_1+T_2}^t U^\varepsilon_{\varepsilon B}(t-s,\theta^\varepsilon_s\omega)G(\theta^\varepsilon_s\omega,x^\varepsilon(s),y^\varepsilon(s))ds.
\end{aligned}\tag{53}$$

We can conclude that the left hand side of (51) satisfies the first equation of (22) on $[0, T]$. From (53) we obtain

$$\gamma(y^\varepsilon(0)) = \gamma(y_1^\varepsilon(0, \omega, T_1, \gamma, y_2^\varepsilon(0, \theta_{T_1}^\varepsilon \omega, T_2, \mu, \tilde{y}))). \tag{54}$$

Hence by (22) and the definition of $\Phi^\varepsilon(T_1, \omega, \gamma)$ we have

$$\begin{aligned} x_1^\varepsilon(T_1, \omega, T_1, &\gamma, y_2^\varepsilon(0, \theta_{T_1}^\varepsilon \omega, T_2, \mu, \tilde{y})) = \mu(y_2^\varepsilon(0, \theta_{T_1}^\varepsilon \omega, T_2, \mu, \tilde{y})) \\ &= x_2^\varepsilon(0, \theta_{T_1}^\varepsilon \omega, T_2, \mu, \tilde{y}). \end{aligned} \tag{55}$$

By the definition of x_2^ε we have for $t \in (T_1, T_1 + T_2]$ by (55)

$$\begin{aligned} x^\varepsilon(t, \omega, T, \gamma, \tilde{y}) =& x_2^\varepsilon(t - T_1, \theta_{T_1}^\varepsilon \omega, T_2, \mu, \tilde{y}) \\ =& U_A^\varepsilon(t - T_1, \theta_{T_1}^\varepsilon \omega) \mu(y_2^\varepsilon(0, \theta_{T_1}^\varepsilon \omega, T_2, \mu, \tilde{y})) \\ &+ \int_0^{t-T_1} U_A^\varepsilon(t - T_1 - s, \theta_{T_1+s}^\varepsilon \omega) F(\theta_{T_1+s}^\varepsilon \omega, x_2^\varepsilon(s), y_2^\varepsilon(s)) ds \\ =& U_A^\varepsilon(t - T_1, \theta_{T_1}^\varepsilon \omega) U_A^\varepsilon(T_1, \omega) \gamma(y_1^\varepsilon(0, \omega, \gamma, y_2^\varepsilon(0, \theta_{T_1}^\varepsilon \omega, \mu, \tilde{y}))) \\ &+ U_A^\varepsilon(t - T_1, \theta_{T_1}^\varepsilon \omega) \int_0^{T_1} U_A^\varepsilon(T_1 - s, \theta_s^\varepsilon \omega) F(\theta_s^\varepsilon \omega, x_1^\varepsilon(s), y_1^\varepsilon(s)) ds \\ &+ \int_0^{t-T_1} U_A^\varepsilon(t - T_1 - s, \theta_{T_1+s}^\varepsilon \omega) F(\theta_{T_1+s}^\varepsilon \omega, x_2^\varepsilon(s), y_2^\varepsilon(s)) ds. \end{aligned}$$

Applying the cocycle property of U_A^ε to concentrate the integrals, (54) and (52) we see that x^ε and y^ε satisfies (22).

□

In order to establish the existence of a random inertial manifold for the random dynamical system defined by (11) we introduce the metric space $\mathcal{G}_\Gamma$ of bounded measurable mappings from Ω into $\mathcal{L}_\Gamma$ equipped with the metric

$$d_\infty(\gamma_1, \gamma_2) := \sup_{\omega \in \Omega} \sup_{\tilde{y} \in \mathbb{R}^d} |\gamma_1(\omega, \tilde{y}) - \gamma_2(\omega, \tilde{y})|.$$

It is not hard to prove that the space $(\mathcal{G}_\Gamma, d_\infty)$ is complete.
By means of the graph transform Φ^ε we define an operator $\mathcal{S}^\varepsilon$ on the space $\mathcal{G}_\Gamma$ by

$$(\omega, \tilde{y}) \to \mathcal{S}^\varepsilon(\gamma)(\omega, \tilde{y}) = \Phi^\varepsilon(T, \theta_{-T}^\varepsilon \omega, \gamma(\theta_{-T}^\varepsilon \omega))(\tilde{y}).$$

The following theorem gives conditions under which $\mathcal{S}^\varepsilon$ has a unique fixed point $\gamma^{*,\varepsilon}$ in $\mathcal{G}_{\Gamma^*}$ which defines the random inertial manifold M^ε. The proof is similar to [7], [16].

Theorem 4.9. *Assume the hypotheses $(A_1) - (A_5)$ to be valid. Suppose that $\varepsilon > 0$ is sufficiently small and T sufficiently large. Then the random dynamical system defined by (11) has a random inertial manifold whose graph is defined by $\gamma^{*,\varepsilon}(\omega) \in \mathcal{G}_{\Gamma^*}$. In addition, the following estimate holds true*

$$d_\infty(\gamma^{*,\varepsilon}, 0) = \sup_{\omega \in \Omega} \|\gamma^{*,\varepsilon}(\omega)\|_\infty \le \frac{c_F}{c_A(1 - k(\varepsilon, T))}. \tag{56}$$

Proof. Let T be sufficiently large and ε sufficiently small such that the conclusions of Lemma 4.7 hold. Then the mapping $\mathcal{S}^\varepsilon$ is a contraction on $\mathcal{G}_{\Gamma^*}$. Moreover, $\mathcal{S}^\varepsilon$ maps $\mathcal{G}_{\Gamma^*}$ into itself by Lemma 4.6 and by the fact that $\Phi^\varepsilon(T,\cdot,\gamma)$ is measurable because the contraction constant for the fixed point problem of Lemma 4.3 can be chosen independently of $\tilde{y}$ and ω (see Remark 4.4). Hence, $\mathcal{S}^\varepsilon$ has a unique fixed point $\gamma^{*,\varepsilon} \in \mathcal{G}_{\Gamma^*}$. Hence, replacing ω by $\theta_T^\varepsilon\omega$

$$\Phi^\varepsilon(T,\omega,\gamma^{*,\varepsilon})(\cdot) = \gamma^{*,\varepsilon}(\theta_T^\varepsilon\omega,\cdot).$$

According to Lemma 4.8 we have for $t > 0$

$$\begin{aligned}
&\Phi^\varepsilon(t,\omega,\gamma^{*,\varepsilon}(\omega)) = \Phi^\varepsilon(t,\cdot,\mathcal{S}^\varepsilon(\gamma^{*,\varepsilon}(\cdot)))(\omega) = \Phi^\varepsilon(t,\omega,\Phi^\varepsilon(T,\theta_{-T}^\varepsilon\omega,\gamma^{*,\varepsilon}(\theta_{-T}^\varepsilon\omega)))\\
=&\Phi^\varepsilon(t+T,\theta_{-T}^\varepsilon\omega,\gamma^{*,\varepsilon}(\theta_{-T}^\varepsilon\omega)) = \Phi^\varepsilon(T,\theta_{-T+t}^\varepsilon\omega,\Phi^\varepsilon(t,\theta_{-T}^\varepsilon\omega,\gamma^{*,\varepsilon}(\theta_{-T}^\varepsilon\omega)))\\
=&\mathcal{S}^\varepsilon(\Phi^\varepsilon(t,\theta_{-t}^\varepsilon\cdot,\gamma^{*,\varepsilon}(\theta_{-t}^\varepsilon\cdot))(\theta_t^\varepsilon\omega).
\end{aligned}$$

The left hand side $\Phi^\varepsilon(t,\omega,\gamma^{*,\varepsilon}(\omega))$ of this equation can be written as

$$\Phi^\varepsilon(t,\theta_{-t}^\varepsilon\theta_t^\varepsilon\omega,\gamma^{*,\varepsilon}(\theta_{-t}^\varepsilon\theta_t^\varepsilon\omega)).$$

Since $\mathcal{S}^\varepsilon$ has for small ε and large T a *unique* fixed point also in $\mathcal{G}_{\bar{\Gamma}\vee\Gamma^*}$ we have $\Phi^\varepsilon(t,\omega,\gamma^{*,\varepsilon}(\omega)) = \gamma^{*,\varepsilon}(\theta_t^\varepsilon\omega)$ which has to be in $\mathcal{G}_{\Gamma^*}$. According to Lemma 4.1 the random dynamical system has a random invariant Lipschitz manifold.
To see that this manifold is exponentially attracting we first note that the convergence

$$\lim_{t\to\infty} d_\infty(\Phi^\varepsilon(t,\cdot,\gamma),\Phi^\varepsilon(t,\cdot,\gamma^{*,\varepsilon}(\cdot))) = \lim_{t\to\infty} d_\infty(\Phi^\varepsilon(t,\cdot,\gamma),\gamma^{*,\varepsilon}(\theta_t^\varepsilon\cdot)) = 0 \text{ for } \gamma\in\mathcal{G}_{\Gamma^*}$$

is exponentially fast. Indeed, we have

$$\begin{aligned}
\mathcal{S}^\varepsilon\circ\mathcal{S}^\varepsilon(\gamma)(\omega,\tilde{y}) &= \Phi^\varepsilon(T,\theta_{-T}^\varepsilon\omega,\Phi^\varepsilon(T,\theta_{-2T}^\varepsilon\omega,\gamma(\theta_{-2T}^\varepsilon\omega)))(\tilde{y})\\
&= \Phi^\varepsilon(2T,\theta_{-2T}^\varepsilon\omega,\gamma(\theta_{-2T}^\varepsilon\omega))(\tilde{y})
\end{aligned}$$

and similarly

$$\underbrace{S^\varepsilon\circ\cdots\circ\mathcal{S}^\varepsilon(\gamma)}_{n-\text{times}}(\omega,\tilde{y}) = \Phi^\varepsilon(nT,\theta_{-nT}^\varepsilon\omega,\gamma(\theta_{-nT}^\varepsilon\omega))(\tilde{y})$$

such that

$$\lim_{n\to\infty} d_\infty(\Phi^\varepsilon(nT,\cdot,\gamma),\gamma^{*,\varepsilon}(\theta_T^\varepsilon\cdot)) = 0 \qquad \text{for } \gamma\in\mathcal{G}_{\Gamma^*}$$

is exponentially fast by the contraction property of $\mathcal{S}^\varepsilon$. On the other hand, by the cocycle property we have

$$\Phi^\varepsilon(t,\omega,\gamma(\omega)) = \Phi^\varepsilon(t-n(t)T+T,\theta_{n(t)T-T}^\varepsilon\omega,\Phi^\varepsilon(n(t)T-T,\omega,\gamma(\omega))),$$

where $n(t)$ is the biggest integer such that $n(t)T$ is smaller than or equal to t. The above convergence now follows by the Lipschitz continuity of Φ^ε, what follows from

Lemma 4.7.
For some given $(x_0, y_0) \in \mathbb{R}^d$ we set

$$(x^\varepsilon(t), \tilde{y}_t) := \phi^\varepsilon(t, \omega, (x_0, y_0)).$$

Choose some graph $\gamma \in \mathcal{G}_{\Gamma^*}$ such that $\gamma(\omega, y_0) = x_0$. Then $\Phi^\varepsilon(t, \cdot, \gamma)$ tends to $\gamma^{*,\varepsilon}(\theta_t^\varepsilon \cdot)$ exponentially fast in $\mathcal{G}_{\Gamma^*}$. Hence

$$\begin{aligned}\inf_{z \in M^\varepsilon(\theta_t^\varepsilon \omega)} |\phi^\varepsilon(t, \omega, (x_0, y_0)) - z| &\leq |x^\varepsilon(t) - \gamma^{*,\varepsilon}(\theta_t^\varepsilon \omega, \tilde{y}_t)| \\ = |\Phi^\varepsilon(t, \omega, \gamma)(\tilde{y}_t) - \gamma^{*,\varepsilon}(\theta_t^\varepsilon \omega, \tilde{y}_t)| &\leq d_\infty(\Phi^\varepsilon(t, \cdot, \gamma), \gamma^{*,\varepsilon}(\theta_t^\varepsilon \cdot))\end{aligned}$$

gives us the asserted convergence to the invariant manifold.

To see the inequality (56) we define $\gamma_1 := \mathcal{S}^\varepsilon(\gamma_0)$, $\gamma_0 \equiv 0$. It follows by estimating the first component of (22) that

$$\|\gamma_1 - \gamma_0\|_\infty = \|\gamma_1\|_\infty \leq \frac{1}{c_A}(1 - e^{-c_A T})c_F.$$

The standard a priori estimate of the Banach fixed point theorem (see Zeidler [25] Theorem 1.A.c.) implies (56). $\square$

Remark 4.10. *We have shown that the random dynamical system generated by* (11) *has an inertial manifold with graph* $\gamma^{*,\varepsilon}$ *for small* $\varepsilon > 0$. *Thus also the system derived from* (7) *has an inertial manifold with the same graph but interpreted in the dynamics of* $(\theta^1_{\frac{t}{\varepsilon}} \omega_1, \theta_t^2 \omega_2)$.

5 Existence of a slow manifold

In this section we consider system (11) for $\varepsilon = 0$. We show that there exists a random invariant manifold as a family of random fixed points depending on some parameter $\tilde{y} \in \mathbb{R}^{d_2}$. This manifold is called the *random slow manifold* for (7), (11). The random slow manifold will be the limit of the random inertial manifold $M^\varepsilon(\omega)$ introduced in the last section. Using the inertial form for the inertial manifold we are also able to describe the limiting behavior of system (7) using the graph of the slow manifold. But this is the topic of a forthcoming paper.

In the following we denote by U_A^0 the linear random dynamical system generated by $A(\theta_t^1 \omega_1, \omega_2)$.

(A_6). Suppose that for any $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$ and $t \geq 0$

$$\begin{aligned}&\lim_{\varepsilon \to 0^+} \|A(\omega_1, \theta_t^{2,\varepsilon} \omega_2) - A(\omega_1, \omega_2)\| = 0, \qquad \text{and} \\ &\lim_{\varepsilon \to 0^+} |F(\omega_1, \theta_t^{2,\varepsilon} \omega_2, x, y) - F(\omega_1, \omega_2, x, y)| = 0\end{aligned}$$

uniformly for x, y on compact sets.

From the first property we have by (A_2) that

$$\lim_{\varepsilon\to 0^+}\|U_A^\varepsilon(t,\theta_s^1\omega_1,\theta_{\varepsilon s}^2\omega_2)-U_A^0(t,\theta_s^1\omega_1,\omega_2)\|=0.$$

If F has the structure as in (15) with the Brownian motion metric dynamical system, the second equation is always satisfied. We start to consider (11) for $\varepsilon=0$. Let $y^0(t,\tilde{y})$ be the solution to (19), (20) for $\varepsilon=0$ with end condition $\tilde{y}$. Then $\tilde{y}\to y^0(t,\tilde{y})=\tilde{y}$ for $t\in[0,T]$. Let us denote the solution operator of the first equation of (11) for $x^0(0)=x_0\in\mathbb{R}^{d_1}$ by $\phi^0_{\tilde{y},\omega_2}(t,\omega_1,x_0)$ which depends on the parameters $\tilde{y}$, ω_2. This operator generates for every $\tilde{y}$, ω_2 a random dynamical system on Θ_1. In particular, we have the cocycle property

$$\phi^0_{\tilde{y},\omega_2}(t_1+t_2,\omega_1,x_0)=\phi^0_{\tilde{y},\omega_2}(t_1,\theta^1_{t_2}\omega_1,\phi^0_{\tilde{y},\omega_2}(t_2,\omega_1,x_0)).$$

For $\gamma\in\mathcal{G}_\mathcal{L}$ we introduce the operator

$$\Phi^0(t,\omega,\gamma)(\tilde{y}):=\phi^0_{\tilde{y},\omega_2}(t,\omega_1,(\gamma(\tilde{y}),\tilde{y})).$$

This operator is defined analogously as the operator Φ^ε in (18). The corresponding inverse mapping Ψ^0 introduced in Section 3 is the identical mapping on $\mathbb{R}^{d_2}$.
Using the notation $\theta^0_t\omega=(\theta^1_t\omega_1,\omega_2)$ we define on $\mathcal{G}_\Gamma$ the operator $\mathcal{S}^0$ by

$$\mathcal{S}^0(\gamma)(\omega,\tilde{y}):=\Phi^0(T,\theta^0_{-T}\omega,\gamma(\theta^0_{-T}\omega))(\tilde{y})$$

where T, Γ are given in Lemma 4.8, Theorem 4.9.

Theorem 5.1. *Suppose the hypotheses $(A_1)-(A_5)$ hold. Then the operator $\mathcal{S}^0$ has a unique random fixed point $\gamma^{*,0}\in\mathcal{G}_{\Gamma^*}$.*

Proof. The proof is analogous to the proof of Theorem 4.9. However, the proofs for Lemma 4.6, 4.7 remains true for $\varepsilon=0$. The associated contraction constant $k_1(0)<1$. □

The random fixed point $\gamma^{*,0}$ defines an invariant manifold

$$M^0(\omega_1,\omega_2,\tilde{y})=\{(\gamma^{*,0}(\omega_1,\omega_2,\tilde{y}),\tilde{y})|\tilde{y}\in\mathbb{R}^{d_2}\}.$$

This manifold will be called the random *slow* manifold for (7) or (11). But this manifold does not have any dynamics with respect to $\tilde{y}$–direction. Hence, to describe this objective it is more appropriate to say that $\gamma^{*,0}$ is a family of random fixed points for the random dynamical system $\phi^0_{\tilde{y},\omega_2}$ parameterized by $\tilde{y},\omega_2$.

It remains to prove that the inertial manifolds M^ε for (11) tend to the slow manifold M^0.

Theorem 5.2. *Assume the hypotheses $(A_1)-(A_6)$ to valid. Then we have for $\omega_1\in\Omega_1$, $\omega_2\in\Omega_2$, $k\in\mathbb{N}$ and $\varepsilon>0$ small*

$$\lim_{\varepsilon\to 0^+}\sup_{|\tilde{y}|\le k}|\gamma^{*,\varepsilon}(\omega_1,\omega_2,\tilde{y})-\gamma^{*,0}(\omega_1,\omega_2,\tilde{y})|=0.$$

Proof. Let $x^\varepsilon(t,\omega,\tilde y)$, $y^\varepsilon(t,\omega,\tilde y)$, $t\in[0,T]$ for some $T>0$ and $\varepsilon\ge 0$ be the solution of (19), (20) for some $\gamma\in\mathcal{G}_{\Gamma^*}$. Note that the constant Γ^* introduced in Lemma 4.6 can be chosen independently of $\varepsilon\ge 0$.

It is straightforward that for $t\in[0,T]$

$$\lim_{\varepsilon\to 0^+}|\int_0^t \varepsilon U^\varepsilon_{\varepsilon B}(t-s,\theta^\varepsilon_s\omega)G(\theta^\varepsilon_s\omega,x^\varepsilon(s,\tilde y),y^\varepsilon(s,\tilde y))ds|\le \lim_{\varepsilon\to 0}T\varepsilon e^{\varepsilon c_B T}c_G=0$$

uniformly for $\tilde y\in\mathbb{R}^{d_2}$ and $\omega\in\Omega$. In addition, we have for any $t\in[0,T]$

$$\lim_{\varepsilon\to 0^+}U^\varepsilon_{\varepsilon B}(t-T,\theta^\varepsilon_T\omega)\tilde y=\tilde y$$

uniformly for $\tilde y$ in a compact set. Hence

$$\lim_{\varepsilon\to 0^+}y^\varepsilon(t,\omega,\tilde y)=\tilde y,\qquad \lim_{\varepsilon\to 0^+}\gamma(y^\varepsilon(0,\omega,\tilde y))=\gamma(\tilde y) \tag{57}$$

uniformly for $\tilde y$ contained in a compact set.
Suppose that $\gamma\in\mathcal{G}_{\Gamma^*}$. We now show that

$$\lim_{\varepsilon\to 0^+}x^\varepsilon(T,\omega,\tilde y)=x^0(T,\omega,\tilde y)$$

uniformly for $\tilde y$ in a compact set. By the definition of Φ^ε the last equality is equivalent to

$$\lim_{\varepsilon\to 0^+}\Phi^\varepsilon(T,\omega,\gamma)(\tilde y)=\Phi^0(T,\omega,\gamma)(\tilde y) \tag{58}$$

uniformly for $\tilde y$ contained in a compact set. For $\gamma\in\mathcal{G}_{\Gamma^*}$ we have that $\Phi^\varepsilon(T,\omega,\gamma)$ is well defined if $t\ge 0$ and ε is small. In order to compare $x^\varepsilon(T)$ and $x^0(T)$ we derive the estimate

$$\begin{aligned}|x^\varepsilon(T)-x^0(T)|\le{}&|\gamma(y^\varepsilon(0))-\gamma(\tilde y)|\|U^\varepsilon_A(T,\omega)\|+|\gamma(\tilde y)|\|U^\varepsilon_A(T,\omega)-U^0_A(T,\omega)\|\\&+|\int_0^T U^\varepsilon_A(T-s,\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2)(F(\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2,x^\varepsilon(s),y^\varepsilon(s))-F(\theta^1_s\omega_1,\omega_2,x^0(s),\tilde y))ds|\\&+|\int_0^T(U^\varepsilon_A(T-s,\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2)-U^0_A(T-s,\theta^1_s\omega_1,\omega_2))F(\theta^1_s\omega_1,\omega_2,x^0(s),\tilde y)ds|.\end{aligned}$$

From

$$\begin{aligned}F(\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2,x^\varepsilon,y^\varepsilon)-F(\theta^1_s\omega_1,\omega_2,x^0,\tilde y)={}&F(\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2,x^\varepsilon,y^\varepsilon)-F(\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2,x^0,\tilde y)\\&+F(\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2,x^0,\tilde y)-F(\theta^1_s\omega_1,\omega_2,x^0,\tilde y)\end{aligned}$$

we obtain

$$\begin{aligned}|x^\varepsilon(T)-x^0(T)|\le{}&|\gamma(y^\varepsilon(0))-\gamma(\tilde y)|ae^{-c_At}+|\gamma(\tilde y)|\|U^\varepsilon_A(T,\omega)-U^0_A(T,\omega)\|\\&+a\int_0^T e^{-c_A(t-s)}(L|x^\varepsilon(s)-x^0(s)|+L|y^\varepsilon(s)-\tilde y|\\&+|F(\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2,x^0(s),\tilde y)-F(\theta^1_s\omega_1,\omega_2,x^0(s),\tilde y)|)\\&+\|U^\varepsilon_A(T-s,\theta^1_s\omega_1,\theta^2_{\varepsilon s}\omega_2)-U^0_A(T-s,\theta^1_s\omega_1,\omega_2)\|c_Fds.\end{aligned}$$

The asserted convergence (58) follows from the Gronwall lemma, Lebesgue's theorem, (A_6) with (A_2) and (57). The same result holds if we replace ω by $\theta^\varepsilon_\rho\omega$ for some $\rho \in \mathbb{R}$.

We have for $n \in \mathbb{N}$:

$$\begin{aligned}
&|\gamma^{*,\varepsilon}(\omega,\tilde{y}) - \gamma^{*,0}(\omega,\tilde{y})| \\
&\qquad =|\Phi^\varepsilon(nT,\theta^\varepsilon_{-nT}\omega,\gamma^{*,\varepsilon}(\theta^\varepsilon_{-nT}\omega))(\tilde{y}) - \Phi^0(nT,\theta^0_{-nT}\omega,\gamma^{*,0}(\theta^0_{-nT}\omega))(\tilde{y})| \\
&\qquad \le\|\Phi^\varepsilon(nT,\theta^\varepsilon_{-nT}\omega,\gamma^{*,\varepsilon}(\theta^\varepsilon_{-nT}\omega)) - \Phi^\varepsilon(nT,\theta^\varepsilon_{-nT}\omega,\gamma^{*,0}(\theta^0_{-nT}\omega))\|_\infty \\
&\qquad\quad + |\Phi^\varepsilon(nT,\theta^\varepsilon_{-nT}\omega,\gamma^{*,0}(\theta^0_{-nT}\omega))(\tilde{y}) - \Phi^0(nT,\theta^0_{-nT}\omega,\gamma^{*,0}(\theta^0_{-nT}\omega))(\tilde{y})| \\
&\qquad \le k_0^n\frac{2c_F}{c_A(1-k_0)} \\
&\qquad\quad + |\Phi^\varepsilon(nT,\theta^\varepsilon_{-nT}\omega,\gamma^{*,0}(\theta^0_{-nT}\omega))(\tilde{y}) - \Phi^0(nT,\theta^0_{-nT}\omega,\gamma^{*,0}(\theta^0_{-nT}\omega))(\tilde{y})|
\end{aligned}$$

where $\theta^0_t\omega = (\theta^1_t\omega_1,\omega_2)$. According to Lemma 4.7 and Theorem 4.9 the first term on the right hand side can be made arbitrarily small if n is chosen sufficiently large independently of ε and ω. On account of the calculations in the first part of the proof also the second term becomes arbitrarily small for sufficiently small ε. □

For an interpretation of the manifold $\gamma^{*,0}$ for the fast system (7) we refer to a forthcoming paper by the authors. However in absence of ω_1 similar to the deterministic theory we can determinate $\gamma^{*,0}$ explicitly.

Corollary 5.3. *Suppose (A_5) and (8) with $a = 1$ are satisfied. Then the equation*

$$A(\omega_2)x + F(\omega_2,x,y) = 0 \tag{59}$$

has a unique solution $x^*(\omega_2,y) =: \gamma^{*,0}(\omega_2,y)$ for any ω_2, y. This solution depends Lipschitz continuously on y for any $\omega_2 \in \Omega_2$.

Proof. By (A_2) the operator $A(\omega_2)$ has an inverse $A(\omega_2)^{-1}$ with a norm $\|A(\omega_2)^{-1}\| \le \frac{1}{c_A}$. (59) can be written as

$$x = -A(\omega_2)^{-1}F(\omega_2,x,y)$$

The the conclusion follows by (A5) and the Banach fixed point theorem. □

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