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On the large scale behavior of super–Brownian motion in three dimensions with a single point source

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ABSTRACT. In a recent work, Fleischmann and Mueller (2004) showed the existence of a super-Brownian motion in \mathbb{R}^d , $d = 2, 3$, with extra birth at the origin. Their construction made use of an analytical approach based on the fundamental solution of the heat equation with a one point potential worked out by Albeverio et al. (1995). The present note addresses two properties of this measure-valued process in the three-dimensional case, namely the scaling of the process and the large scale behavior of its mean.

1. INTRODUCTION

A super-Brownian motion in \mathbb{R} with a single point source δ_0 was constructed in Engländer & Fleischmann [EF00]. It was shown that its expected mass grows exponentially in time, and is in the mass-rescaled limit distributed in space as $x \mapsto e^{-|x|}$. In Engländer & Turaev [ET02] it is even proved that the random measures themselves grow in law exponentially as time increases, and are otherwise in the mass-rescaled limit spatially situated with the same shape except an overall random factor. The probabilistic effect behind the non-trivial existence of the model is the fact that a Brownian particle in \mathbb{R} hits the origin with certainty and that it has there a non-degenerate local time, serving as an additional birth rate for the random creation of mass.

In higher dimensions, a Brownian particle fails to hit the origin, and a local time would degenerate. Nevertheless, Fleischmann & Mueller [FM04] succeeded in constructing a super-Brownian motion in \mathbb{R}^d , $d = 2, 3$, with a single point source. They heavily used well-known analytical facts from mathematical physics concerning Laplace operators with one-point-potentials. Heuristically, some additional rescaling enters the regularization of the delta function (serving as single point source). Properties of this new super-Brownian motion are not known so far. The purpose of the present note is to get some progress by studying its scaling and the large scale behavior of its expectation in the three-dimensional case.

1.1. The heat equation with one-point-potential. The Schrödinger equation with a one-point-potential is studied in quantum theory to describe singular electromagnetic effects on quantum particles, see e.g. the monograph Albeverio et al. [AGHKKH88, Part I]. By analytic continuation, solutions to the Schrödinger equation can be (at least formally) obtained via solutions of the heat equation.

Formally, the heat equation with a one-point-potential is given by

$$(1) \quad \partial_t u = \Delta u + \delta_0^{(\alpha)} u =: \Delta^{(\alpha)} u,$$

where ∂_t denotes the derivative with respect to time, Δ is the d -dimensional Laplacian, and $u : (0, \infty) \times \mathring{\mathbb{R}}^d \rightarrow \mathbb{R}_+$ is a time-space field, where $\mathring{\mathbb{R}}^d := \mathbb{R}^d \setminus \{0\}$ with the Euclidean metric is locally compact. If we denote by $B_\varepsilon(y)$ an open ball around $y \in \mathbb{R}^d$ of radius $\varepsilon > 0$, then having in mind that $\varepsilon^{-d} \mathbf{1}_{B_\varepsilon(0)} \approx \delta_0$, the operator $\Delta^{(\alpha)} := \Delta + \delta_0^{(\alpha)}$ is heuristically the limit as $\varepsilon \downarrow 0$ of the operator

$$(2) \quad \Delta_\varepsilon^{(\alpha)} := \Delta + h(d, \alpha, \varepsilon) \varepsilon^{-d} \mathbf{1}_{B_\varepsilon(0)},$$

where $h(d, \alpha, \varepsilon)$ is some additional rescaling factor which depends on a parameter α at least. Restricting to $d = 3$, the function h can be chosen as

$$(3) \quad h(3, \alpha, \varepsilon) := \frac{\pi^2}{4} \varepsilon - 8\pi^2 \alpha \varepsilon^2, \quad \alpha \in \mathbb{R}, \quad \varepsilon > 0,$$

(cf. [AGHKKH88, (H.74)]).

Physically, α in the case $\alpha < 0$ is related to the *scattering length* $\text{sl}_\alpha := -(4\pi\alpha)^{-1}$ of the free Laplace operator Δ with respect to the interaction Laplacian $\Delta^{(\alpha)}$. Roughly speaking, the scattering length describes the average distance a free particle manages to go before any interaction takes place. So, if $\alpha \downarrow -\infty$ the scattering length $\text{sl}_\alpha \downarrow 0$ becomes smaller and we expect more interaction. For $\alpha \geq 0$ there is *no* proper physical interpretation of sl_α as the point spectrum of $\Delta^{(\alpha)}$ is empty (see [AGHKKH88, Theorem I.1.4]).

The *fundamental solution* p^α to the equation

$$(4) \quad \partial_t u = \Delta^{(\alpha)} u \quad \text{on } (0, \infty) \times \mathring{\mathbb{R}}^d, \quad d = 2, 3,$$

which provides the basis for the analytical construction of the superprocess in [FM04], have been computed in Albeverio et al. [ABD95]. In $d = 3$ (the two-dimensional case is analytically more delicate, which is the reason we restrict to $d = 3$), the *one-point-interaction heat kernel* p^α is given by

$$(5) \quad p_t^\alpha(x, y) = p_t(x, y) + \frac{2t}{|x||y|} p_t(|x| + |y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty du e^{-4\pi\alpha u} p_t(u + |x| + |y|),$$

$t > 0$, $x, y \neq 0$, where p is the usual *free* heat kernel defined by,

$$(6) \quad p_t(x, y) := (4\pi t)^{-d/2} \exp(-|y-x|^2/4t),$$

and with a slight abuse of notation,

$$(7) \quad p_t(r) := (4\pi t)^{-d/2} \exp(-r^2/4t), \quad t > 0, \quad r \geq 0.$$

Also recall the scaling of the free heat kernel, i.e. for all $k, t > 0$ and $x, y \in \mathbb{R}^d$,

$$(8) \quad p_t(x, y) = k^{d/2} p_{kt}(k^{1/2}x, k^{1/2}y).$$

Note, that the last term in (5) is always finite and disappears for $\alpha = 0$. Moreover, $\alpha \mapsto p^\alpha$ is pointwise continuous and decreasing, and we have the (pointwise) convergences $p^\alpha \uparrow +\infty$ as $\alpha \downarrow -\infty$ (i.e. the fundamental solution explodes which can be interpreted as *immediate interaction*), whereas $p^\alpha \downarrow p$ as $\alpha \uparrow +\infty$ leads the free case (i.e. the interaction disappears).

Rigorously, the family $\{\Delta^{(\alpha)} : \alpha \in \mathbb{R}\}$ of operators are defined as *all* self-adjoint extensions on the Hilbert space $\mathcal{L}^2(\mathring{\mathbb{R}}^d, dx)$ of the Laplacian Δ acting on $C_{\text{com}}^\infty(\mathring{\mathbb{R}}^d)$, the space of unboundedly differentiable functions on $\mathring{\mathbb{R}}^d = \mathbb{R}^d \setminus \{0\}$ with compact support (see e.g. [AGHKH88, Chapters I.1 and I.5]). Hence, although the p^α differ from the free heat kernel p , they solve the heat equation

$$(9) \quad \partial_t p_t^\alpha(x, y) = \Delta p_t^\alpha(x, y) \quad \text{on } (0, \infty) \times \mathring{\mathbb{R}}^d,$$

with the Laplacian Δ acting either on the variable x or y . In particular, $(t, x, y) \mapsto p_t^\alpha(x, y)$ is jointly continuous on $(0, \infty) \times \mathring{\mathbb{R}}^d \times \mathring{\mathbb{R}}^d$. Let us denote by S^α the semigroup associated with the kernel p^α , i.e.

$$(10) \quad S_t^\alpha \varphi(x) := \int_{\mathring{\mathbb{R}}^d} dy \varphi(y) p_t^\alpha(x, y).$$

Note that S^α is *not* a contraction semigroup and so there is *no* stochastic process generated by this flow. The following Lemma shows that the kernel p^α has a similar scaling behavior as the free heat kernel p .

Lemma 1 (Scaling of the p^α). *We have, for all $k, t > 0$, $x, y \in \mathring{\mathbb{R}}^3$, and $\alpha \in \mathbb{R}$,*

$$(11) \quad p_t^\alpha(x, y) = k^{3/2} p_{kt}^{k^{-1/2}\alpha}(k^{1/2}x, k^{1/2}y).$$

Proof. That follows immediately from the definition (5) of the p^α and the scaling (8) of the free heat kernel p . \square

1.2. The flow associated with the one-point-interaction heat kernel. This section is devoted to introduce a space of functions Φ on which the flow S^α acts as a strongly continuous linear semigroup (see [FM04, Section 2] for details). Let ϕ denote the *weight and reference function*

$$(12) \quad \phi(x) := |x|^{-1}, \quad x \in \mathring{\mathbb{R}}^3 = \mathbb{R}^3 \setminus \{0\}.$$

For fixed $\varrho \in (1, 2)$, let $\mathcal{H} = \mathcal{H}^\varrho$ denote the space of measurable functions φ on $\mathring{\mathbb{R}}^3$ for which

$$(13) \quad \|\varphi\|_{\mathcal{H}} := \left(\int_{\mathring{\mathbb{R}}^3} dx \phi(x) |\varphi(x)|^\varrho \right)^{1/\varrho} < \infty.$$

Then $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$ is a Banach space, where as usual we do not distinguish between equivalence classes and their representatives. Now, let $\Phi = \Phi^\varrho$ denote the set of all *continuous* functions $\varphi : \mathring{\mathbb{R}}^3 \rightarrow \mathbb{R}$ such that $\varphi \in \mathcal{H}$ and

$$(14) \quad 0 \leq \varphi \leq C \phi \quad \text{for some constant } C = C_\varphi > 0.$$

We endow Φ with the topology inherited from \mathcal{H} . Note that the set $C_{\text{com}}^+ = C_{\text{com}}^+(\mathring{\mathbb{R}}^3)$ of all non-negative, continuous functions on $\mathring{\mathbb{R}}^3$ with compact support is contained in Φ . We remark that $\varphi \in \Phi$ might have a singularity at $x = 0$ of order $|x|^{-\xi}$ with $0 < \xi < 1$. The linear semigroup S^α introduced in (10) is strongly continuous on the cone $\Phi = \Phi^\varrho$, cf. Corollary 2.12 in [FM04].

1.3. Super-Brownian motion with a single point source. Denote by $\mathcal{M} = \mathcal{M}(\mathbb{R}^3)$ the set of all (Radon) measures μ on \mathbb{R}^3 such that $\langle \mu, \varphi \rangle < \infty$ for all $\varphi \in \Phi$. Recalling that $\mathcal{C}_{\text{com}}^+ \subset \Phi$, endow \mathcal{M} with the vague topology.

Fix a constant $\eta > 0$ (branching rate). Suppose $0 < \beta < 1$ (the finite variance branching case $\beta = 1$ has been excluded in [FM04] for $d = 3$ for technical reasons). Then for each $\alpha \in \mathbb{R}$, there is a non-degenerate \mathcal{M} -valued (time-homogeneous) Markov process X^α such that for (deterministic) starting measures $\mu \in \mathcal{M}$ and for $\varphi \in \Phi$,

$$(15) \quad -\log \mathbb{E}_\mu \exp \langle X_t^\alpha, -\varphi \rangle = \langle \mu, v(t, \cdot) \rangle, \quad t > 0,$$

where $\{v(t, x) : t \geq 0, x \in \mathbb{R}^3\}$ is the unique non-negative solution of the integral equation related to the Φ -valued evolution equation

$$(16) \quad \begin{cases} \partial_t v = \Delta^{(\alpha)} v - \eta v^{1+\beta} & \text{on } (0, \infty), \\ v(0+, \cdot) = \varphi \end{cases}$$

(see [FM04, Theorem 4.4]). That is,

$$(17) \quad v(t, x) = \int_{\mathbb{R}^3} dy p_t^\alpha(x, y) \varphi(y) - \eta \int_0^t ds \int_{\mathbb{R}^3} dy p_{t-s}^\alpha(x, y) v^{1+\beta}(s, y),$$

$t > 0, x \in \mathbb{R}^3$. Clearly, the first moments of X^α are determined by the S^α flow to be

$$(18) \quad \mathbb{E}_\mu \langle X_t^\alpha, \varphi \rangle = \langle \mu, S_t^\alpha \varphi \rangle,$$

for all starting measures $\mu \in \mathcal{M}$, $t \geq 0$, and $\varphi \in \Phi$.

2. LARGE SCALE BEHAVIOR

2.1. A scaling property.

Proposition 2 (A scaling property). *Let $t, k > 0$, $\mu \in \mathcal{M}$, and $\alpha, \lambda_k \in \mathbb{R}$. Then*

$$(19) \quad \left\{ k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} \cdot) \mid X_0^{\lambda_k \alpha} = k^{1/\beta} \mu(k^{-1/2} \cdot) \right\} \stackrel{\mathcal{L}}{=} \left\{ X_t^{k^{1/2} \lambda_k \alpha} \mid X_0^{k^{1/2} \lambda_k \alpha} = \mu \right\}.$$

Of course, the cases $\lambda_k = k^{-1/2}$ or even $\alpha = 0$ are particularly nice.

Proof. For $\varphi \in \Phi$ fixed,

$$(20) \quad \langle k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} dy), \varphi \rangle = \langle X_{kt}^{\lambda_k \alpha}, k^{-1/\beta} \varphi(k^{-1/2} \cdot) \rangle,$$

hence, by (15) and (17),

$$(21) \quad \begin{aligned} -\log \mathbb{E}_{k^{1/\beta} \mu(k^{-1/2} \cdot)} \exp \langle k^{-1/\beta} X_{kt}^{\lambda_k \alpha}(k^{1/2} dy), \varphi \rangle &= -\log \mathbb{E}_{k^{1/\beta} \mu(k^{-1/2} \cdot)} \exp \langle X_{kt}^{\lambda_k \alpha}, k^{-1/\beta} \varphi(k^{-1/2} \cdot) \rangle \\ &= \langle k^{1/\beta} \mu(k^{-1/2} \cdot), v(kt, \cdot) \rangle = \langle \mu, k^{1/\beta} v(kt, k^{1/2} \cdot) \rangle \end{aligned}$$

where $\{v(t', x') : t' \geq 0, x' \in \mathbb{R}^3\}$ is the non-negative solution of the integral equation related to the function-valued evolution equation

$$(22) \quad \begin{cases} \partial_t v = \Delta^{(\lambda_k \alpha)} v - \eta v^{1+\beta} & \text{on } (0, \infty), \\ v(0+, \cdot) = k^{-1/\beta} \varphi(k^{-1/2} \cdot). \end{cases}$$

More precisely,

$$(23) \quad \begin{aligned} k^{1/\beta} v(kt, k^{1/2} x) &= k^{1/\beta} \int_{\mathbb{R}^3} dy p_{kt}^{\lambda_k \alpha}(k^{1/2} x, y) k^{-1/\beta} \varphi(k^{-1/2} y) \\ &\quad - k^{1/\beta} \eta \int_0^{kt} ds \int_{\mathbb{R}^3} dy p_{kt-s}^{\lambda_k \alpha}(k^{1/2} x, y) v^{1+\beta}(s, y). \end{aligned}$$

By a change of variable,

$$(24) \quad \begin{aligned} k^{1/\beta} v(kt, k^{1/2} x) &= \int_{\mathbb{R}^3} dy k^{3/2} p_{kt}^{\lambda_k \alpha}(k^{1/2} x, k^{1/2} y) \varphi(y) \\ &\quad - k^{1/\beta} \eta \int_0^t ds k \int_{\mathbb{R}^3} dy k^{3/2} p_{kt-ks}^{\lambda_k \alpha}(k^{1/2} x, k^{1/2} y) v^{1+\beta}(ks, k^{1/2} y). \end{aligned}$$

Hence, by Lemma 1,

$$k^{1/\beta} v(kt, k^{1/2}x) = \int_{\mathbb{R}^3} dy p_t^{k^{1/2}\lambda_k\alpha}(x, y) \varphi(y) - k^{1/\beta} \eta \int_0^t ds k \int_{\mathbb{R}^3} dy p_{t-s}^{k^{1/2}\lambda_k\alpha}(x, y) v^{1+\beta}(ks, k^{1/2}y).$$

Since $1/\beta + 1 - (1/\beta)(1 + \beta) = 0$ we see that $k^{1/\beta} v(kt, k^{1/2}x) =: w_k(t, x)$ satisfies the equation

$$(25) \quad w_k(t', x') = \int_{\mathbb{R}^3} dy p_{t'}^{k^{1/2}\lambda_k\alpha}(x', y) \varphi(y) - \eta \int_0^{t'} ds \int_{\mathbb{R}^3} dy p_{t'-s}^{k^{1/2}\lambda_k\alpha}(x', y) w_k^{1+\beta}(s, y),$$

$t' > 0$, $x' \in \mathbb{R}^3$. By uniqueness of solutions of the log-Laplace equation (17) and by (15), claim (19) follows. \square

2.2. Expectation of the scaled X^α . Before we can state the result, we have to introduce some notation. The limiting measure will be expressed by means of the kernel

$$(26) \quad \vartheta_t^\alpha(x, y) := \frac{2t}{|x||y|} p_t(|y|) - \frac{8\pi\alpha t}{|x||y|} \int_0^\infty du e^{-4\pi\alpha u} p_t(u + |y|),$$

for $\alpha \in \mathbb{R}$, $t > 0$, and $x, y \in \mathbb{R}^3$. Note that the integral is always finite, hence for $\alpha = 0$ the second term disappears. Moreover, the kernel ϑ^α is always non-negative. This holds trivially whenever $\alpha < 0$, and to see this for $\alpha > 0$, use the estimate

$$(27) \quad p_t(u + |y|) \leq p_t(|y|).$$

We extend the definition of ϑ^α by setting

$$(28) \quad \vartheta_t^\alpha(x, y) := \begin{cases} 0, & \text{if } \alpha = +\infty, \\ +\infty, & \text{if } \alpha = -\infty. \end{cases}$$

The so defined kernels ϑ^α turn out to be pointwise continuous in $\alpha \in [-\infty, +\infty]$ (which follows from the arguments of the proof of Theorem 3 below).

Theorem 3 (Large scale behavior of the mean). *For $t > 0$, $\alpha, \lambda_k \in \mathbb{R}$, and all starting measures $X_0^\alpha = \mu \in \mathcal{M}$ satisfying $\langle \mu, \phi \rangle < \infty$, we have the convergence in \mathcal{M} ,*

$$(29) \quad \lim_{k \uparrow \infty} k^{-1/2} \mathbb{E}_\mu [X_{kt}^{\lambda_k\alpha}(k^{1/2} dy)] = \langle \mu, \vartheta_t^{\alpha^*}(\cdot, y) \rangle dy,$$

provided that $\alpha^* := \lim_{k \uparrow \infty} k^{1/2}\lambda_k\alpha \in [-\infty, +\infty]$.

Proof. Fix $\varphi \in \mathcal{C}_{\text{com}}^+(\mathbb{R}^3)$. Using formula (18) for the first moment of X^α and substitution, we obtain

$$(30) \quad \begin{aligned} k^{-1/2} \mathbb{E}_\mu \langle X_{kt}^{\lambda_k\alpha}, \varphi(k^{-1/2}\cdot) \rangle &= k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy p_{kt}^{\lambda_k\alpha}(x, y) \varphi(k^{-1/2}y) \\ &= k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy k^{3/2} p_{kt}^{\lambda_k\alpha}(x, k^{1/2}y) \varphi(y). \end{aligned}$$

By Lemma 1 this is equal to

$$(31) \quad k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy p_t^{k^{1/2}\lambda_k\alpha}(k^{-1/2}x, y) \varphi(y).$$

Inserting according to definition (5) of p^α , we get three terms, we will deal with separately.

1° (*First term*). The first term equals,

$$(32) \quad k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy p_t(k^{-1/2}x, y) \varphi(y).$$

This double integral is finite and vanishes as $k \uparrow \infty$. To see this, let us restrict the outer integral first to $|x| > K$ where we specify $K \geq 1$ later. We call this restricted integral I_K . We use $\varphi \leq C\phi$ (since $\varphi \in \Phi$) and, with S denoting the free heat flow,

$$(33) \quad S_t\phi \leq C\phi, \quad t \geq 0,$$

with changed constant C (see [FM04, Lemma 2.1]) to arrive at

$$(34) \quad I_K \leq Ck^{-1/2} \int_{|k^{-1/2}x| > K} \mu(dx) \phi(k^{-1/2}x) = C \int_{|x| > k^{1/2}K} \mu(dx) \phi(x)$$

which can be made arbitrarily small uniformly in k by choosing K sufficiently large (by our assumption on μ). It remains to deal with the case $|x| \leq K$ for fixed K . We split the internal integral in (32) as follows. First, if $|k^{-1/2}x - y| \geq |y|/2$, then

$$(35) \quad p_t(k^{-1/2}x, y) \leq p_t(|y|/2),$$

which leads to the bound

$$(36) \quad k^{-1/2} \int_{|x| \leq K} \mu(dx) \int_{\mathbb{R}^3} dy p_t(|y|/2) \varphi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty,$$

the $\mu(dx)$ -integral is finite as $\langle \mu, \phi \rangle < \infty$. On the other hand, if $|k^{-1/2}x - y| < |y|/2$, then $-k^{-1/2}|x| + |y| < |y|/2$ which implies $|y| < 2k^{-1/2}|x| \leq 2K$. Hence as $p_t(k^{-1/2}x, y) \leq Ct^{-3/2}$ and $\varphi \leq C\phi$, we get the upper estimate

$$(37) \quad Ct k^{-1/2} \int_{|x| \leq K} \mu(dx) \int_{|y| < 2K} dy \phi(y) \longrightarrow 0 \quad \text{as } k \uparrow \infty$$

(the dy -integral is finite, since we are in dimension three).

2° (*Second term*). The second term reads

$$(38) \quad \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{2t}{|x||y|} p_t(k^{-1/2}|x| + |y|) \varphi(y) =: II_k.$$

Observe that,

$$(39) \quad p_t(k^{-1/2}|x| + |y|) \uparrow p_t(|y|) \quad \text{as } k \uparrow \infty.$$

We can apply the monotone convergence theorem to obtain the limit,

$$(40) \quad \lim_{k \uparrow \infty} II_k = \int_{\mathbb{R}^3} \mu(dx) \frac{2t}{|x|} \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) \varphi(y),$$

where finiteness follows from $\langle \mu, \phi \rangle < \infty$ and $\varphi \leq C\phi$. Hence, this summand gives the first part of the kernel ϑ^{α^*} .

3° (*Third term*). It remains to insert the scaled third term from (5) into (31) which reads as

$$(41) \quad k^{-1/2} \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{-8\pi t k^{1/2} \lambda_k \alpha}{k^{-1/2}|x||y|} \int_0^\infty du e^{-4\pi k^{1/2} \lambda_k \alpha u} p_t(u + k^{-1/2}|x| + |y|) \varphi(y).$$

We distinguish several cases: If $\lambda_k \alpha = 0$ for all sufficiently large k , then the third term disappears and we are done. From now on assume $\lambda_k \alpha \neq 0$ for all k . Substituting $u \mapsto 4\pi k^{1/2} |\lambda_k \alpha| u$ into (41) yields,

$$(42) \quad \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{-2t \operatorname{sign}(\lambda_k \alpha)}{|x||y|} \int_0^\infty du e^{-\operatorname{sign}(\lambda_k \alpha) u} p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \varphi(y).$$

Now let $k^{1/2} |\lambda_k \alpha| \rightarrow \infty$. We may consider a monotone subsequence of $k^{1/2} \lambda_k \alpha$. Clearly,

$$(43) \quad p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \uparrow p_t(|y|) \quad \text{as } k \uparrow \infty,$$

and by monotone convergence the expression (42) converges along the subsequence to

$$(44) \quad - \int_{\mathbb{R}^3} \mu(dx) \frac{2t \operatorname{sign}(\alpha^*)}{|x|} \int_0^\infty du e^{-\operatorname{sign}(\alpha^*) u} \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) \varphi(y),$$

which is independent of the choice of the subsequence. Note that

$$(45) \quad \operatorname{sign}(\alpha^*) \int_0^\infty du e^{-\operatorname{sign}(\alpha^*) u} = \begin{cases} 1 & \text{if } \operatorname{sign}(\alpha^*) = 1, \\ +\infty & \text{if } \operatorname{sign}(\alpha^*) = -1. \end{cases}$$

In the first case the second and the third limiting terms cancel.

Next, let $k^{1/2} \lambda_k \alpha \rightarrow 0$. Note, that

$$(46) \quad p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|} + k^{-1/2}|x| + |y|\right) \leq p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|}\right) e^{-|y|^2/4t}.$$

In this case the double integral (42) is in absolute value bounded by

$$(47) \quad \int_{\mathbb{R}^3} \mu(dx) \int_{\mathbb{R}^3} dy \frac{2t}{|x||y|} e^{-|y|^2/4t} \varphi(y) \int_0^\infty du e^u p_t\left(\frac{u}{4\pi k^{1/2} |\lambda_k \alpha|}\right),$$

which tends to 0 as $k \uparrow \infty$ as the $\mu(dx)$ and dy -integrals are finite and in the du -integral the p_t -term compensates the e^u .

It remains to deal with the case $k^{1/2}\lambda_k\alpha \rightarrow \alpha^* \in \dot{\mathbb{R}}^1$. Note, that we only have to justify to change the limit and integration in (42), as substituting $u \mapsto (4\pi|\alpha^*|)^{-1}u$ leads to the desired expression. To justify the interchange, we estimate as in (46). The resulting $\mu(dx)$ and dy -integrals are independent of k and finite, whereas to dominate in the second integral we use $k^{1/2}|\lambda_k\alpha| \leq |\alpha^*| + 1$ for all sufficiently large k . \square

Remark 4 (Large scale total mass). Taking $\lambda_k \equiv 1$ and choosing formally $\varphi = 1$ as test function in (29) yields

$$(48) \quad \lim_{k \uparrow \infty} k^{-1/2} \mathbb{E}_\mu \langle X_{kt}^\alpha, 1 \rangle = \begin{cases} 0 & \text{if } \alpha > 0, \\ 2t \langle \mu, \phi \rangle \int_{\mathbb{R}^3} dy \frac{1}{|y|} p_t(y) & \text{if } \alpha = 0, \\ \infty & \text{if } \alpha < 0. \end{cases}$$

A rigorous argument can be given along the lines of the previous proof. \diamond

2.3. Discussion and open problems. Let us comment on the *three cases* $\alpha^* = +\infty$, $\alpha^* \in \mathbb{R}$, and $\alpha^* = -\infty$ in Theorem 3. In the first case, the limiting mass disappears, more precisely, the scaled expression $\mathbb{E}_\mu [X_{kt}^{\lambda_k\alpha}(k^{1/2} dy)]$ is of order $o(k^{1/2})$. Roughly speaking, if $\alpha^* = +\infty$, then there are no interactions in the scaling limit (free case). In the second case, $\alpha^* \in \mathbb{R}$, the former expectation is about $k^{1/2} \langle \mu, \vartheta_t^{\alpha^*}(\cdot, y) \rangle dy$. Note that these measures are decreasing in α^* . Finally, if $\alpha^* = -\infty$, we have immediate interaction in the large scale limit leading to the explosion of the expected mass.

Clearly, to describe only the large scale behavior of the *expected processes* is unsatisfactory. It is desirable to get something similar for the processes themselves. Recall that in the one-dimensional case the large time behavior of the process itself is known from [ET02]. However, we stress the fact, that the process in three dimensions is expected to have quite different features. For instance, if $\alpha = 0$, then according to Remark 4 the total mass grows with a power order, whereas in one dimension the growth is exponential. Moreover, in the three-dimensional case one needs additionally to contract the normalized measures to get a limit. For the measures themselves, scaled as in Theorem 3, *there might be extinction in law* despite convergence of their expectations.

Another *open problem* is the large scale behavior of $\mathbb{E}X^\alpha$ in the two-dimensional case, in which the fundamental solutions p^α from [ABD95] are analytically more delicate, see e.g. [FM04, formula (2.30)]. In particular, a scaling property as in Lemma 1 is not available.

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