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SPECTRAL MAPPING THEOREM FOR LINEAR HYPERBOLIC SYSTEMS

MARK LICHTNER

Abstract. We show high frequency resolvent and spectral estimates and prove spectral mapping theorem for linear hyperbolic systems in one space dimension.

1. Introduction

As is well known stability and dichotomy of a linear system of ODEs

$$\dot{z} = Az \tag{1.1}$$

are determined by the spectrum $\sigma(A)$ of the matrix A. For infinite dimensional evolution equations the issue is more complex. Let A be an infinitesimal generator of a C_0 semigroup e^{At}. We say that A has the Spectral Mapping Property (SMP) if

$$\text{(SMP)} \quad \sigma(e^{At}) \setminus \{0\} = \overline{e^{\sigma(A)t}} \setminus \{0\} \quad \text{for} \quad t \geq 0.$$

It is well known that (SMP) does not hold, in general, if A is a generator of a C_0 semigroup. Several counterexamples are known for abstract C_0 semigroups [8, 19]. Renardy gave a more applied counterexample [18]: He constructed a lower order derivative perturbation of a two dimensional wave equation, where the growth bound $\omega(A) := \inf\{\omega \in \mathbb{R} \mid \exists M = M(\omega) \forall t \geq 0 : \|e^{At}\| \leq Me^{\omega t}\}$ is strictly larger than the spectral bound $s(A) := \sup\{\mathfrak{Re}\,\lambda \mid \lambda \in \sigma(A)\}$. One has the following formula for ω [8]: For each $t_0 > 0$ $\omega(A) = t_0^{-1} \log r\left(e^{At_0}\right) = \lim_{t\to\infty} t^{-1} \log \|e^{At}\|$, where $r\left(e^{At_0}\right)$ denotes the spectral radius of e^{At_0}. Hence (SMP) contains spectrum determined growth $\omega(A) = s(A)$. Renardys counterexample not only illustrates that (SMP) is violated, in general, for linear hyperbolic PDEs in higher dimensions (see also [7]), moreover it destroys hopes that "applied" problems are somehow well behaved. This is annoying since computing spectra of linearizations is the most widely used practical method for assessing stability and bifurcations of physical systems. It is known that for C_0 semigroups in Hilbert space the set of counterexamples, where $\omega(A) > s(A)$, is small in the sense of Baire category [19].

Moreover, (SMP) implies existence of dichotomic projections for (1.1) under the presence of a spectral gap for A. Exponential dichotomies are required for proof of center manifold theorems for nonlinear evolution equations, which are often applied to analyse the local dynamics and bifurcations of (infinite dimensional) dynamical

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systems. The theory is, to our knowledge, restricted to classes of evolution equations where (SMP) is known, which includes ODEs and large classes of DDEs and semilinear parabolic PDEs. The unavailability/failure of (SMP) has been a main obstacle for extending the theory [1, 2, 3, 20] to hyperbolic PDEs. Quite recently (SMP) has been proven for Schrödinger equations [6].

Our main result is that (SMP) holds true for a general class of linear hyperbolic systems in one space dimension. We prove (SMP) by establishing high frequency estimates for spectra and resolvents in terms of reduced (block)diagonal systems and applying the Gearhart-Prüss spectral mapping theorem [5, 15] (see Theorem 4.2). Related results can be found in [14, 17], but precise resolvent and spectral estimates as well as the spectral mapping property (SMP) are missing. Our estimates can be used even in cases where the Gearhart-Prüss theorem does not apply. This has important applications for nonlinear hyperbolic systems, see [9, 10, 11, 16], which often appear in applied problems.

2. Results

We consider linear hyperbolic systems in the following normal form

$$\text{(H)} \quad \begin{cases} \frac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\frac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ \frac{d}{dt}\left[v(t,l) - Du(t,l)\right] = Fu(t,\cdot) + Gv(t,\cdot), \\ u(t,0) = Ev(t,0). \end{cases}$$

The assumptions are:
(HI) K is a blockdiagonal $n \times n$ matrix of the form

$$K = \text{blockdiag}\,(k_i I_{d_i})_{1 \le i \le \alpha+\beta},$$

where $d_i \in \mathbb{N}$, $d_i > 0$, $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}$, $\sum_{i=1}^{\alpha} d_i = n_1$, $\sum_{i=1}^{\beta} d_{\alpha+i} = n_2$, I_{d_i} denotes the identity matrix in $\mathbb{C}^{d_i \times d_i}$ and $k_i \in C^1([0,l],\mathbb{R})$ satisfy for $x \in [0,l]$

$$k_i(x) > 0,\ 1 \le i \le \alpha, \quad k_j(x) < 0,\ \alpha+1 \le j \le \alpha+\beta.$$

(HII) $C(x) = (C_{ij}(x))_{1 \le i,j \le \alpha+\beta} \in \mathbb{C}^{n \times n}$ with $C_{ij}(x) \in \mathbb{C}^{d_i \times d_j}$ and

$$C_{ii} \in L^\infty\left(]0,l[, \mathbb{C}^{d_i \times d_i}\right),\ C_{ij} \in \mathrm{BV}\left([0,l], \mathbb{C}^{d_i \times d_j}\right),\ 1 \le i,j \le \alpha+\beta,\ i \ne j.$$

(HIII) If $i \ne j$ and $k_i(x) = k_j(x)$ for some $x \in [0,l]$, then C_{ij} vanishes on $[0,l]$.
(HIV) $u(t,x) = (u_i(t,x))_{1 \le i \le n_1} \in \mathbb{C}^{n_1}$, $v(t,x) = (v_i(t,x))_{1 \le i \le n_2} \in \mathbb{C}^{n_2}$.
(HV) $D \in \mathbb{C}^{n_2 \times n_1}$, $E \in \mathbb{C}^{n_1 \times n_2}$ and

$$F : C([0,l],\mathbb{C}^{n_1}) \to \mathbb{C}^{n_2}, \quad G : C([0,l],\mathbb{C}^{n_2}) \to \mathbb{C}^{n_2}$$

are linear continuous operators.

Let

$$A\begin{pmatrix} u \\ v \\ d \end{pmatrix} := \begin{pmatrix} -K\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - C\begin{pmatrix} u \\ v \end{pmatrix} \\ Fu + Gv \end{pmatrix}$$

denote the infinitesimal generator for (H) with domain

$$\mathcal{D}(A) := \left\{(u,v,d) \in W^{1,2}\left(]0,l[, \mathbb{C}^n\right) \times \mathbb{C}^{n_2} \mid u(0) = Ev(0),\ d = v(l) - Du(l)\right\}.$$

Then A generates a C_0 semigroup e^{At} on the Hilbert space $L^2\left(]0,l[, \mathbb{C}^n\right) \times \mathbb{C}^{n_2}$ (see [12, 13, 14]). Let $R(\lambda, A) = (\lambda I - A)^{-1}$ denote resolvent, $\sigma(A)$ spectrum, $\sigma_p(A)$ pointspectrum and $\rho(A) = \mathbb{C} \setminus \sigma(A)$ the resolvent set of A. Our main result is

Theorem 2.1 (Spectral Mapping Theorem)**.**

$$\text{(SMP)} \quad \sigma(e^{At}) \setminus \{0\} = \overline{e^{\sigma(A)t}} \setminus \{0\} \quad \textit{for} \quad t \geq 0.$$

In Theorem 2.1 $\overline{e^{\sigma(A)t}}$ denotes the closure of the set $e^{\sigma(A)t}$. We prove (SMP) by establishing high frequency estimates for spectra and resolvents of A in terms of reduced (block)diagonal systems: Let C_{b0} be the block diagonal matrix containing the square matrices C_{ii}

$$C_{b0} := \text{blockdiag } (C_{ii})_{1 \leq i \leq \alpha+\beta}\,. \tag{2.1}$$

Then the reduced system is per definitionem given by

$$\text{(H}_0\text{)} \quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + K(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} + C_{b0}(x)\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = 0, \\ u(t,0) = Ev(t,0), \quad v(t,l) = Du(t,l). \end{cases}$$

We denote the infinitesimal generator to (H$_0$) by

$$\begin{aligned} A_0\begin{pmatrix} u \\ v \end{pmatrix} &:= -K\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - C_{b0}\begin{pmatrix} u \\ v \end{pmatrix}, \\ \mathcal{D}(A_0) &:= \left\{(u,v) \in W^{1,2}\,(]0,l[\,,\mathbb{C}^n) \mid u(0) = Ev(0),\ v(l) = Du(l)\right\}. \end{aligned}$$

A_0 generates a C_0 semigroup on $L^2(]0,l[\,,\mathbb{C}^n)$.

Lemma 2.2. *The resolvent $R(\lambda, A)$ ($R(\lambda, A_0)$) is compact, $\sigma(A) = \sigma_p(A)$ ($\sigma(A_0) = \sigma_p(A_0)$).*

Denote $\mathbb{C}_{r,d} := \{\lambda \in \mathbb{C} \mid -r < \mathfrak{Re}\,\lambda < r,\ |\mathfrak{Im}\,\lambda| > d\}, \quad \mathbb{C}_r := \mathbb{C}_{r,-1}$.

Lemma 2.3 (Estimates for spectrum)**.** *There exists an exponential polynomial h_0 and an entire function h with*[1]*:*

- $\sigma(A) = \{\lambda \in \mathbb{C} \mid h(\lambda) = 0\}$, $\sigma(A_0) = \{\lambda \in \mathbb{C} \mid h_0(\lambda) = 0\}$.
- *Put* $\tilde{h}(\lambda) := \lambda^{-n} h(\lambda)$.

$$\forall r > 0 \exists c, d > 0 \forall \lambda \in \mathbb{C}_{r,d}: \quad \left|\tilde{h}(\lambda) - h_0(\lambda)\right| \leq c\frac{1}{|\lambda|}. \tag{2.2}$$

If A_0 has nonempty spectrum define $\gamma_- := \inf\{\mathfrak{Re}\lambda \mid h_0(\lambda) = 0\}$ and $\gamma_+ := \sup\{\mathfrak{Re}\lambda \mid h_0(\lambda) = 0\}$. Since h_0 is an exponential polynomial γ_- and γ_+ are finite. If A_0 has empty spectrum put $\gamma_+ := -\infty$ and $\gamma_- := \infty$. For $\lambda \in \mathbb{C}$ and $\epsilon > 0$ let $B_\epsilon(\lambda) := \{z \in \mathbb{C} \mid |z - \lambda| < \epsilon\}$ denote the ball around λ with radius ϵ.

Lemma 2.4. *For each $\gamma > \gamma_+$ there exist only finitely many eigenvalues λ of A that satisfy $\mathfrak{Re}\,\lambda \geq \gamma$.*

Lemma 2.5. *Suppose A_0 has nonempty spectrum. Then the following hold:*

- *For each $\delta > 0$ there are only finitely many eigenvalues λ of A which satisfy $\mathfrak{Re}\lambda \leq \gamma_- - \delta$ or $\mathfrak{Re}\lambda \geq \gamma_+ + \delta$.*
- *For $\epsilon > 0$ there exists $d > 0$ such that $\sigma(A) \cap \{\lambda \in \mathbb{C} \mid |\mathfrak{Im}\,\lambda| \geq d\} \subset \bigcup_{h_0(\lambda)=0} B_\epsilon(\lambda)$.*

[1] h_0 and h are defined in formulas (3.4) and (3.6)

- *Suppose* $\rho = \inf\{|\lambda_1 - \lambda_2| \mid \lambda_1 \neq \lambda_2,\ h_0(\lambda_1) = h_0(\lambda_2) = 0\} > 0$. *Then for* $\eta < \frac{\rho}{2}$ *there exists* $d > 0$ *such that for each* $\lambda_0 \in \mathbb{C}$ *with* $h_0(\lambda_0) = 0$ *and* $|\mathfrak{Im}\,\lambda_0| \geq d$ *there exists* $\lambda \in B_\eta(\lambda_0)$ *with* $h(\lambda) = 0$. *Both* h *and* h_0 *have the same number of zeros in each* $B_\eta(\lambda_0)$ *counted with multiplicities. If* A_0 *only possesses algebraically simple eigenvalues, then the eigenvalues* $\lambda \in B_\eta(\lambda_0)$ *of* A *are unique and algebraically simple.*

Lemma 2.6 (Estimates for resolvent)**.** *Let* $U \subset \rho(A)$ *so that* $\sup_{\lambda\in U} |\mathfrak{Re}\,\lambda| < \infty$ *and* $\inf_{\lambda\in U} |h_0(\lambda)| > 0$. *Then there exists* $d > 0$ *so that for* $\lambda \in U$ *and* $|\mathfrak{Im}\,\lambda| \geq d$

- $R(\lambda, A)\begin{pmatrix} f \\ g \\ b \end{pmatrix} = \begin{pmatrix} u \\ v \\ v(l) - Du(l) \end{pmatrix}$, $\begin{pmatrix} u \\ v \end{pmatrix} = R(\lambda, A_0)\begin{pmatrix} f \\ g \end{pmatrix} + \frac{1}{\lambda}\mathcal{E}(\lambda, A)\begin{pmatrix} f \\ g \\ b \end{pmatrix}$.
- $R(\lambda, A_0)$ *and* $\mathcal{E}(\lambda, A)$ *are bounded. In particular the resolvent* $R(\lambda, A)$ *is bounded on* U.
- $\mathcal{E}(\lambda, A)$ *can be represented as a series* $\sum_{k=0}^{\infty} \lambda^{-k} R_k(\lambda, A)$, *where for each* R_k *there exists a closed formula in terms of integrals of elementary functions of* λ.

Remark 2.7. For all applications we know of conditions (HI)-(HV) are fulfilled. Our conditions are slightly more general than [14], we include equal speed systems which are required for applications to semiconductor laser dynamics [9, 16].

3. Estimates for spectra and resolvents

Consider (H) and assume (HI) – (HV). We prove Lemmas 2.2-2.6.

Let T denote the fundamental matrix satisfying

$$\begin{aligned} \frac{d}{dx}T(x,y,\lambda) &= -K(x)^{-1}\left(\lambda I + C(x)\right)T(x,y,\lambda) \quad \text{for } x,y \in [0,l], \\ T(y,y,\lambda) &= I \quad \text{for } y \in [0,l]. \end{aligned} \tag{3.1}$$

and T_0 the fundamental system for

$$\begin{aligned} \frac{d}{dx}T_0(x,y,\lambda) &= -K^{-1}(x)\left(\lambda I + C_{b0}(x)\right)T_0(x,y,\lambda) \quad \text{for } x,y \in [0,l], \\ T_0(y,y,\lambda) &= I \quad \text{for } y \in [0,l]. \end{aligned} \tag{3.2}$$

For $1 \leq i \leq \alpha+\beta$ there exist F_i depending only on C_{b0} and K, $F_i : [0,l]^2 \to \mathbb{C}^{d_i \times d_i}$, $F_i(\cdot, y) \in W^{1,\infty}([0,l], \mathbb{C}^{d_i\times d_i})$ for $y \in [0,l]$, so that for $F := (\operatorname{blockdiag} F_i)_{1\leq i\leq \alpha+\beta}$ we have

$$T_0(x,y,\lambda) = \exp\left(-\lambda \int_y^x K^{-1}(z)\,dz\right) F(x,y). \tag{3.3}$$

Moreover for $x \geq y \geq z$ we have $F_i(x,z) = F_i(x,y)F_i(y,z)$. Indeed: Define F_i to be the solution to

$$\frac{d}{dx}F_i(x,y) = -k_i^{-1}(x)C_{ii}(x)F_i(x,y), \quad F_i(y,y) = I_{d_i}.$$

From (2.1) and (HI) it follows that K, C_{b0} and $\exp\left(-\lambda \int_y^x K^{-1}(z)\,dz\right)$ commute. Hence the right hand side of (3.3) solves (3.2).

We have $\lambda \in \sigma_p(A)$ iff there exists $v_0 \in \mathbb{C}^{n_2}$, $v_0 \neq 0$, such that

$$\begin{pmatrix} u \\ v \end{pmatrix}(x) = T(x,0,\lambda)\begin{pmatrix} E \\ I \end{pmatrix} v_0 \quad \text{and} \quad (-\lambda D\delta_l - F, \lambda I\delta_l - G)\begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

Hence we define

$$H(\lambda) := (-\lambda D\delta_l - F, \lambda I\delta_l - G)T(\cdot,0,\lambda)\begin{pmatrix}E\\I\end{pmatrix} \quad (\in \mathbb{C}^{n_2\times n_2}),$$

$$h(\lambda) := \det H(\lambda). \tag{3.4}$$

Using this notation we have

$$\sigma_p(A) = \{\lambda\in\mathbb{C} \mid h(\lambda) = 0\}.$$

We call $h(\lambda)$ the characteristic function and $H(\lambda)$ the characteristic matrix to (H). For $\lambda\in\sigma(A_p)$ the eigenspace can be represented as

$$\operatorname{Eig}(A,\lambda) = \left\{ \begin{pmatrix} u\\ v\\ v(l) - Du(l)\end{pmatrix} \mid \begin{pmatrix}u(\cdot)\\ v(\cdot)\end{pmatrix} = T(\cdot,0,\lambda)\begin{pmatrix}E\\I\end{pmatrix} v_0, \quad v_0\in\operatorname{Ker} H(\lambda)\right\}.$$

The geometric multiplicity of each eigenvalue is less than or equal to n_2.

The resolvent equation $R(\lambda,A)(f,g,b) = (u,v,d)$ for $(u,v,d)\in\mathcal{D}(A)$ and $(f,g,b)\in L^2(]0,l[,\mathbb{C}^n)\times\mathbb{C}^{n_2}$ reads

$$K\frac{\partial}{\partial x}\begin{pmatrix}u\\v\end{pmatrix} + (\lambda I + C)\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}f\\g\end{pmatrix}, \quad (-\lambda D\delta_l - F, \lambda I\delta_l - G)\begin{pmatrix}u\\v\end{pmatrix} = b.$$

This is equivalent to

$$\begin{pmatrix}u\\v\end{pmatrix}(x) = T(x,0,\lambda)\begin{pmatrix}E\\I\end{pmatrix}v(0) + \int_0^x T(x,y,\lambda)K(y)^{-1}\begin{pmatrix}f\\g\end{pmatrix}(y)\,dy,$$

$$b = (-\lambda D\delta_l - F,\ \lambda I\delta_l - G)\begin{pmatrix}u\\v\end{pmatrix}.$$

If $\lambda\notin\sigma_p(A)$, by inserting the first equation into the second one, we get that the resolvent equation has a unique solution (hence $\sigma_p(A) = \sigma(A)$) and $v(0) = H(\lambda)^{-1}\beta(\lambda)(f,g,b)$, where

$$\beta(\lambda)(f,g,b) := b + (\lambda D\delta_l + F, G - \lambda I\delta_l)\int_0^{\cdot} T(\cdot,y,\lambda)K(y)^{-1}\begin{pmatrix}f(y)\\g(y)\end{pmatrix}dy.$$

Thus we have:

Proposition 3.1. For any λ such that $h(\lambda)\neq 0$ the resolvent $R(\lambda,A) = (\lambda I - A)^{-1}$ is given by

$$\left[R(\lambda,A)\begin{pmatrix}f\\g\\b\end{pmatrix}\right](x) = \begin{pmatrix}u(x)\\v(x)\\v(l) - Du(l)\end{pmatrix}, \tag{3.5}$$

where

$$\begin{pmatrix}u\\v\end{pmatrix} = T(\cdot,0,\lambda)\begin{pmatrix}E\\I\end{pmatrix}H(\lambda)^{-1}\beta(\lambda)(f,g,b) + \int_0^{\cdot} T(\cdot,y,\lambda)K(y)^{-1}\begin{pmatrix}f(y)\\g(y)\end{pmatrix}dy$$

From Proposition 3.1 we get Lemma 2.2.

Remark 3.2. $\lambda I - A$ is a Fredholm operator of index zero.

Denote

$$h_0(\lambda) := \det H_0(\lambda), \quad H_0(\lambda) := (-D, I)T_0(l,0,\lambda)\begin{pmatrix}E\\I\end{pmatrix} \quad (\in\mathbb{C}^{n_2\times n_2}). \tag{3.6}$$

h_0 is called characteristic function and H_0 characteristic matrix to the reduced system (H_0).

For any λ such that $h_0(\lambda) \neq 0$ we have the following explicit formula for the resolvent $R(\lambda, A_0) = (\lambda I - A_0)^{-1}$

$$R(\lambda, A_0)\begin{pmatrix} f \\ g \end{pmatrix} = T_0(\cdot, 0, \lambda)\begin{pmatrix} E \\ I \end{pmatrix} H_0(\lambda)^{-1}\beta_0(\lambda)(f,g) + \int_0^{\cdot} T_0(\cdot, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy,$$

where

$$\beta_0(\lambda)(f,g) := (D, -I)\int_0^l T_0(l, y, \lambda)K(y)^{-1}\begin{pmatrix} f(y) \\ g(y) \end{pmatrix} dy.$$

Define the coupling matrix $C_1(x) := C(x) - C_{b0}(x)$ and for $k \geq 1$

$$T_k(x, y, \lambda) := -\lambda T_0(x, y, \lambda)\int_y^x T_0(y, z, \lambda)K^{-1}(z)C_1(z)T_{k-1}(z, y, \lambda)\, dz. \tag{3.7}$$

Each T_k, $k \geq 1$, satisfies the initial value problem

$$\frac{d}{dx}T_k(x, y, \lambda) = -K^{-1}(x)\left(\lambda I + C_{b0}(x)\right)T_k(x, y, \lambda) - \lambda K^{-1}(x)C_1(x)T_{k-1}(x, y, \lambda)$$

with $T_k(x, x, \lambda) = 0$, and can be calculated recursively in terms of integrals of elementary functions and F. The only unknown F does not depend on λ. We will see that the series $\sum_{k=0}^{\infty}\lambda^{-k}T_k(x, y, \lambda)$ converges in $W^{1,\infty}$ for sufficiently large $|\Im\mathrm{m}\lambda|$ to the fundamental matrix T of (3.1). However, T_k, $k \geq 2$, are not bounded for λ chosen from any stripe $\mathbb{C}_r$. Indeed we will see in the following that the expressions T_k, $k \geq 2$, contain some power terms λ^i with i up to the lower integer part of $k/2$ which will be due to successive failures in partial integration in the formula (3.7). After reordering terms for any finite $\kappa \in \mathbb{N}$ we will obtain an explicit representation of the form

$$T(x, y, \lambda) = \sum_{k=0}^{\kappa}\lambda^{-k}F_k(x, y, \lambda) + O(\lambda^{-(\kappa+1)}),$$

for λ in a stripe $\mathbb{C}_r$ and sufficiently large $|\Im\mathrm{m}\lambda|$, where each F_k is of order 1 with respect to λ on $\mathbb{C}_r$. To see this we calculate the first two steps T_1 and T_2. Put

$$f_0(x, y, \lambda) := T_0(x, y, \lambda)(y_0^{(1)}, \ldots, y_0^{(\alpha+\beta)})^t$$

with the arbitrary fixed initial data $y_0^{(i)} \in \mathbb{C}^{d_i}$, $1 \leq i \leq \alpha + \beta$. Define

$$f_k := -\lambda\int_y^x T_0(x, z, \lambda)K^{-1}(z)C_1(z)f_{k-1}(z, y, \lambda)\, dz \text{ for } k \geq 1.$$

Then according to (3.3) the i-th component, $1 \leq i \leq \alpha + \beta$ of f_k is

$$\begin{aligned} f_0^{(i)}(x, y, \lambda) &= \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) F_i(x, y)y_0^{(i)}, \\ f_k^{(i)}(x, y, \lambda) &= -\lambda\exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\, du\right) F_i(x, y) \\ &\quad \sum_{\substack{1 \leq l \leq \alpha+\beta \\ l \neq i}}\int_y^x \exp\left(\lambda\textstyle\int_y^z k_i^{-1}(u)\, du\right) F_i(y, z)\frac{C_{il}(z)}{k_i(z)}f_{k-1}^{(l)}(z, y, \lambda)\, dz. \end{aligned}$$

By (HII) and (HIII) we can perform partial integration and get rid of the λ factor appearing in the recursion formula for $f_k^{(i)}$: Denote $\kappa_{il}(x) := k_i^{-1}(x) - k_l^{-1}(x)$, then

$$\begin{aligned} f_1^{(i)}(x,y,\lambda) = & -\exp\left(-\lambda\textstyle\int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \\ & \sum_{\substack{1\le l\le\alpha+\beta\\ l\neq i}} \int_y^x \lambda\,\kappa_{il}(z)\exp\left(\textstyle\int_y^z \lambda\kappa_{il}(u)\right) F_i(y,z)\frac{C_{il}(z)}{k_i(z)}\frac{F_l(z,y)}{\kappa_{il}(z)} y_0^{(l)}\,dz \\ = & \sum_{\substack{1\le l\le\alpha+\beta\\ l\neq i}} \Bigg\{ -\exp\left(-\lambda\textstyle\int_y^x k_l^{-1}(u)\,du\right)\frac{C_{il}(x)}{k_i(x)}\frac{F_l(x,y)}{\kappa_{il}(x)} + \exp\left(-\textstyle\int_y^x \lambda k_i^{-1}(u)\,du\right) F_i(x,y) \\ & \left[\frac{C_{il}(y)}{k_i(y)}\frac{1}{\kappa_{il}(y)} + \int_y^x \exp\left(\textstyle\int_y^z \lambda\,\kappa_{il}(u)\,du\right)\frac{d}{dz}\left(F_i(y,z)\frac{C_{il}(z)}{k_i(z)}\frac{F_l(z,y)}{\kappa_{il}(z)}\right) dz\right]\Bigg\} y_0^{(l)}. \end{aligned}$$

Note that for partial integration we used that in the sum for $l \neq i$ in the formula for $f_1^{(i)}$ the leading λ-exponential of $f_0^{(l)}$ is $e^{-\int_y^x \lambda k_l^{-1}(u)\,du}$. However, now $f_1^{(i)}$ not only contains $2(\alpha+\beta-1)$ terms with λ-exponential $e^{-\int_y^x \lambda k_i^{-1}(u)\,du}$ but also $(\alpha+\beta-1)$ terms of the form $e^{-\int_y^x \lambda k_l^{-1}(u)\,du}$, $1 \le l \le \alpha+\beta$, $l \neq i$. Therefore, in the next step for f_2 we are not able to get rid of all λ terms by partial integration as in the first step:

$$\begin{aligned} f_2^{(i)}(x,y,\lambda) = & -\exp\left(-\lambda\textstyle\int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \sum_{\substack{1\le l_2,l_1\le\alpha+\beta\\ l_2\neq i, l_1\neq l_2}} \lambda \\ & \int_y^x \Bigg\{ -\exp\left(\textstyle\int_y^{z_2}\lambda\kappa_{il_1}(u)\,du\right) F_i(y,z)\frac{C_{il_2}(z_2)C_{l_2l_1}(z_2)}{k_i(z_2)k_{l_2}(z_2)}\frac{F_{l_1}(z_2,y)}{\kappa_{l_2l_1}(z_2)} \\ & + \exp\left(\lambda\textstyle\int_y^{z_2}\kappa_{il_2}(u)\,du\right) F_i(y,z_2)\frac{C_{il_2}(z_2)}{k_i(z_2)} F_{l_2}(z_2,y)\left[\frac{C_{l_2l_1}(y)}{k_{l_2}(y)}\frac{1}{\kappa_{l_2l_1}(y)}\right. \\ & \left. + \int_y^{z_2}\exp\left(\lambda\textstyle\int_y^{z_1}\kappa_{l_2l_1}(u)\,du\right)\frac{d}{dz_1}\left(F_{l_2}(y,z_1)\frac{C_{l_2l_1}(z_1)}{k_{l_2}(z_1)}\frac{F_{l_1}(z_1,y)}{\kappa_{l_2l_1}(z_1)}\right) dz_1\right]\Bigg\} y_0^{(l_1)}\,dz_2. \end{aligned}$$

Partial integration is not possible for the terms in the sum corresponding to $l_1 = i$. Therefore we are forced to keep $(\alpha+\beta-1)$ terms containing λ factors:

$$\begin{aligned} f_2^{(i)}(x,y,\lambda) = & -\lambda\exp\left(-\lambda\textstyle\int_y^x k_i^{-1}(u)\,du\right) F_i(x,y) \sum_{\substack{1\le l_2\le\alpha+\beta\\ l_2\neq i}} \int_y^x \frac{C_{il_2}(z_2)C_{l_2i}(z_2)}{k_i(z_2)k_{l_2}(z_2)}\frac{F_i(z_2,y)}{\kappa_{l_2i}(z_2)}\,dz_2\cdot y_0^{(i)} \\ & + \text{terms of order } 1. \end{aligned}$$

However, in the next third step for these $(\alpha+\beta-1)$ terms containing a λ factor partial integration can be done, so that in the third step there will be no λ^2 factors, only λ or 1 factors. Factors with λ^2 in the multisums will first appear in the fourth step. Thus, generally for $m \in \mathbb{N}$, terms containing λ^m factors appear for the first time in the $(2m)$-th recursion step. Besides these λ^m terms there only appear terms, which are bounded for $\lambda \in \mathbb{C}_r$. After reordering terms we have proven the following

Lemma 3.3. *There exists a sequence $F_k(x,y,\lambda)$ of matrices with*

- $F_0 = T_0$,
- $\forall r>0\,\exists c>0\,\forall\lambda\in\mathbb{C}_r, (x,y)\in[0,l]^2, k\in\mathbb{N}:\ \|F_k(x,y,\lambda)\| \le c^k$.
- *For $r>0$ there exists $d>0$ such that for $\lambda\in\mathbb{C}_{r,d}$ the series $\sum_{k=0}^\infty \lambda^{-k}F_k(x,y,\lambda)$ converges absolutely (in $L^\infty([0,l]\times[0,l],\mathbb{C}^{n\times n})$) to $T(x,y,\lambda)$,*
- $\forall r>0\,\exists c,d>0\,\forall\lambda\in\mathbb{C}_{r,d}: \|T(x,y,\lambda) - T_0(x,y,\lambda)\| \le c\frac{1}{|\lambda|}$.

Remark 3.4. Each F_k can be calculated from T_n for $n = 1,\dots,2k$.

Consequently we get:

Lemma 3.5. *For $r > 0$ there exist $c, d > 0$ such that for $\lambda \in \mathbb{C}_{r,d}$ we have*

- $\left\| \tilde{H}(\lambda) - H_0(\lambda) \right\| \leq c\frac{1}{|\lambda|}$, *where* $\tilde{H}(\lambda) := \lambda^{-1} H(\lambda)$,
- *Put* $\tilde{h}(\lambda) := \det \tilde{H}(\lambda)$. *Then*

$$\left| \tilde{h}(\lambda) - h_0(\lambda) \right| \leq c\frac{1}{|\lambda|}, \tag{3.8}$$

- $\left\| \tilde{H}(\lambda)^{-1} - H_0(\lambda)^{-1} \right\| \leq c\frac{1}{|\lambda|}$.

We need the following Lemma

Lemma 3.6. *Let f be an exponential polynomial of the form $f(\lambda) = \sum_{j=1}^{r} a_j e^{b_j \lambda}$ $(\lambda, a_j \in \mathbb{C},\ b_j \in \mathbb{R})$. Let $Z = \{\lambda \in \mathbb{C} \mid f(\lambda) = 0\}$ denote the zero set of f. For all $\delta > 0$, $\alpha, \beta \in \mathbb{R}$ with $\alpha \leq \beta$ there exists a constant $m = m(\delta, \alpha, \beta) > 0$ such that for all $\lambda \in \mathbb{C}$, which satisfy $\operatorname{dist}(\lambda, Z) \geq \delta$, $\alpha \leq \mathfrak{Re}\lambda \leq \beta$, we have $|f(\lambda)| > m(\delta, \alpha, \beta)$.*

Proof. We give a correct proof of [4, Lemma 2.2]. Suppose the assertion was false. Then there exists a sequence $(\lambda_m)_{m\in\mathbb{N}}$, $\alpha \leq \mathfrak{Re}\,\lambda_m \leq \beta$, so that $f(\lambda_m) \to_{m\to\infty} 0$, $\operatorname{dist}(\lambda_m, Z) \geq \delta > 0$. Put $g_m(\lambda) := f(\lambda+\lambda_m) = \sum_{j=1}^{r} a_j e^{b_j \lambda} e^{b_j x_m} e^{ib_j y_m}$, where $\lambda_m =: x_m + iy_m$. Then there exists a subsequence so that g_m converges uniformly to some function g on each stripe $\mathbb{C}_r$ for any given $r > 0$. Indeed, by passing to a subsequence we can assume that $x_m \to x \in [\alpha, \beta]$ and $e^{ib_j y_m} \to s_j \in S^1$. Then g_m converges uniformly to $g(\lambda) = \sum_{j=1}^{r} a_j e^{b_j \lambda} e^{b_j x} s_j$ on each $\mathbb{C}_r$. Let $U := \{\lambda \in \mathbb{C} \mid |\lambda| < \frac{\delta}{2}\}$. By assumption $g_m(\lambda) \neq 0$ for $\lambda \in U$ and $g(0) = \lim g_m(0) = \lim f(\lambda_m) = 0$. By Hurwitz theorem g must be identical to zero on U, and therefore on $\mathbb{C}$. Hence for all $\lambda \in \mathbb{C}$ we get that $f(\lambda) = g_m(\lambda - \lambda_m) \to_{m\to\infty} 0$. Which is a contradiction to the assumption that $\operatorname{dist}(\lambda_m, Z) \geq \delta > 0$. □

We will use the following Proposition [4, Proposition 2.1] on the distribution of zeros of an exponential polynomial

Proposition 3.7. Let f be an exponential polynomial. Then its set Z of zeros (counted with multiplicities) is a finite union of separable sets: there exist $m < \infty$ sets Z_i with $Z = \cup_{i=1}^{m} Z_i$ and $\inf_{\lambda,\mu\in Z_i,\, \lambda\neq\mu} |\lambda - \mu| > 0$.

Denote $S_{\gamma,\omega} := \{z \in \mathbb{C} \mid \gamma \leq \mathfrak{Re}\, z \leq \omega\}$.

Proof of Lemma 2.4. Let $\gamma > \gamma_+$. Denote $\sigma_+ := \{\lambda \in \sigma(A) \mid \mathfrak{Re}\lambda \geq \gamma\}$. Suppose σ_+ was infinite. Since A generates a C_0 semigroup there exists an $\omega > \gamma$ so that $\sigma_+ \subset S_{\gamma,\omega}$. For the characteristic function $h(\lambda)$ of (H) is analytic the infinitely many eigenvalues of σ_+ must accumulate at infinity within the closed stripe $S_{\gamma,\omega}$. Since $\gamma > \gamma_+$ Lemma 3.6 yields that $\inf_{\lambda\in S_{\gamma,\omega}} |h_0(\lambda)| > 0$. Therefore for sufficiently large $d > 0$ (3.8) implies that $\inf\{|\tilde{h}(\lambda)| \mid \lambda \in S_{\gamma,\omega},\ |\mathfrak{Im}\,\lambda| \geq d\} > 0$. Hence, by choosing $d > 0$ sufficiently large, we get a contradiction to the assumption that there existed infinitely many $\lambda \in S_{\gamma,\omega}$ with $|\mathfrak{Im}\,\lambda| \geq d$ and $\tilde{h}(\lambda) = \lambda^{-n} h(\lambda) = 0$. Hence σ_+ is finite. □

Proof of Lemma 2.5. Because h_0 is an exponential polynomial A_0 has infinitely many eigenvalues. Let $\delta > 0$ be arbitrary and fixed. For $\theta \in [0, 1]$ consider the family of operators corresponding to a perturbation from the diagonal operator with $C = C_0$ to the nondiagonal one with $C = C_0 + C_1$

$$A_\theta(u, v, d) := \left(-K(x)\frac{\partial}{\partial x} \begin{pmatrix} u \\ v \end{pmatrix} - (C_0(x) + \theta C_1(x)) \begin{pmatrix} u \\ v \end{pmatrix};\quad \theta F u(\cdot) + \theta G v(\cdot) \right).$$

Let $h^\theta(\lambda)$ denote the corresponding characteristic function. Put $\tilde{h}^\theta(\lambda) := \frac{1}{\lambda^n} h^\theta(\lambda)$. Note that $\tilde{h}^0 = h_0$. According to Lemma 3.6 outside of balls of radius $\frac{\delta}{2}$ around each of the zeros of h_0 in the stripe $S_{\gamma_- - \delta, \gamma_+ + \delta}$ the function $|h_0|$ has an infimum

$m > 0$. From (3.8) in Lemma 3.5 there exist $c, d > 0$ such that for $\lambda \in S_{\gamma_- - \delta, \gamma_+ + \delta}$ with $|\mathfrak{Im}(\lambda)| > d$

$$|\tilde{h}^\theta(\lambda) - h_0(\lambda)| \leq c\frac{1}{\lambda}. \tag{3.9}$$

Therefore for $\lambda \in S_{\gamma_- - \delta, \gamma_+ + \delta} \setminus \cup_{h_0(\lambda_0)=0} B(\lambda_0, \delta/2)$ with $|\mathfrak{Im}(\lambda)|$ sufficiently large it follows from (3.9) that $|\tilde{h}^\theta(\lambda)| \geq m/2 > 0$. And this holds true uniformly in $\theta \in [0,1]$. Starting from $\theta = 0$ this shows that, as long as we increase θ up to 1 all but finitely many zeros of $h^\theta(\lambda)$ must stay in a $\delta/2$-ball of an zero of $h_0(\lambda)$ (apply Proposition 3.7 and choose $\delta > 0$ sufficiently small). By the continuity of a finite system of zeros with respect to the perturbation parameter θ it follows that $\{\lambda \in \mathbb{C} \mid \mathfrak{Re}\lambda \leq \gamma_- - \delta\} \cup \{\lambda \in \mathbb{C} \mid \mathfrak{Re}\lambda \geq \gamma_+ + \delta\}$ contains only finitely many eigenvalues. The remaining assertions follow by applying Lemma 3.6, Rouchés Theorem and [12]. □

From formula (3.5) and Lemmas 3.3 and 3.5 we obtain Lemma 2.6.

4. Spectral mapping theorem

We prove Theorem 2.1. For this we note some observations.

Theorem 4.1 (Spectral inclusion [8, Thm. 2.6, p.25]). $\forall t \geq 0 : e^{t\sigma(A)} \subset \sigma(e^{At})$.

Theorem 4.2 (Gearhart-Prüss spectral mapping theorem [5, 15]). *Let $(e^{At})_{t\geq 0}$ be a C_0 semigroup generated by A in a Hilbert space. Then $e^{\lambda t} \in \rho(e^{At})$ iff*

$$\{\lambda + i2\pi t^{-1} z \mid z \in \mathbb{Z}\} \subset \rho(A) \quad \textit{and} \quad \sup_{z\in\mathbb{Z}} \left\| \left(\lambda + i2\pi t^{-1} z - A\right)^{-1} \right\| < \infty.$$

Proof of Theorem 2.1: Theorem 4.1 yields $\sigma(e^{At}) \setminus \{0\} \supset \overline{e^{\sigma(A)t}} \setminus \{0\}$. Hence it remains to show $\sigma(e^{At}) \setminus \{0\} \subset \overline{e^{\sigma(A)t}} \setminus \{0\}$. Suppose $e^{\lambda t} \notin \overline{e^{\sigma(A)t}}$. Then Theorem 4.1 implies that there exists a $\delta > 0$ so that

$$U_\delta := \bigcup_{z\in\mathbb{Z}} B_\delta\left(\lambda + i2\pi t^{-1} z\right) \subset \rho(A). \tag{4.1}$$

There exists $d > 0$ so that for all $\lambda \in U_{\frac{\delta}{2}}$ with $|\mathfrak{Im}\,\lambda| \geq d$ we have $h_0(\lambda) \neq 0$. If that was false then we get a contradiction to (4.1) that h had a zero on U_δ: Indeed, suppose the contrary. Then there exists a sequence $(\lambda_m)_{m\in\mathbb{N}}$ in $U_{\frac{\delta}{2}}$ with $|\mathfrak{Im}\,\lambda_m| \to_{m\to\infty} \infty$ and $h_0(\lambda_m) = 0$. Define $h_{0m}(\lambda) := h_0(\lambda + \lambda_m)$, $\tilde{h}_m(\lambda) := \tilde{h}(\lambda + \lambda_m)$. Since h_0 is an exponential polynomial, the real parts of λ_m are bounded and the circle S^1 is compact we have that h_{0m} converges uniformly to an exponential polynomial $h_*(\lambda)$ on each stripe $\mathbb{C}_r$ for any given $r > 0$ (compare with the proof of Lemma 3.6). Moreover, estimate (2.2) yields that $\tilde{h}_m$ converges locally uniformly to h_*. Since $h_*(0) = 0$ Hurwitz' theorem implies that for any $\epsilon > 0$ there exists $\lambda_\epsilon \in \mathbb{C}$, $|\lambda_\epsilon| < \epsilon$, and $m \in \mathbb{N}$ so that $\tilde{h}(\lambda_\epsilon + \lambda_m) = \tilde{h}_m(\lambda_\epsilon) = 0$. By choosing ϵ sufficiently small we get $\lambda_\epsilon + \lambda_m \in U_\delta$ with $h(\lambda_\epsilon + \lambda_m) = \tilde{h}(\lambda_\epsilon + \lambda_m) = 0$. Hence, using Lemma 3.6 it follows that there exists $d > 0$ so that $\inf_{\lambda \in U_{\frac{\delta}{4}}, |\mathfrak{Im}\,\lambda| > 2d} |h_0(\lambda)| > 0$. By applying Lemma 2.6 the resolvent is bounded on $U_{\frac{\delta}{4}}$. Theorem 4.2 yields that $e^{\lambda t} \in \rho(e^{At})$. □

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References

1. P. W. Bates and C. K. R. T. Jones, *Invariant manifolds for semilinear partial differential equations*, Dynamics Reported, vol. 2 (old series), John Wiley & Sons, 1989, ed. U. Kirchgraber and H.-O. Walther, p. 1-38.
2. P. W. Bates, K. Lu, and C. Zeng, *Existence and persistence of invariant manifolds for semiflows in Banach spaces*, Mem. Amer. Math. Soc. **645** (1998), 129.
3. ———, *Persistence of overflowing manifolds for semiflow*, Comm. Pure Appl. Math. **LII** (1999), 983–1046.
4. B.Z.Guo and G.Q.Xu, *On basis property of a hyperbolic system with dynamic boundary condition*, Differential Integral Equations **18** (2005), no. 1, 35–60.
5. L. Gearhart, *Spectral theory for contraction semigroups in hilbert space*, Trans. Amer. Math. Soc. **236** (1978), 385–394.
6. F. Gesztesy, C. K. R. T. Jones, Y. Latushkin, and M. Stanislavova, *A spectral mapping theorem and invariant manifolds for nonlinear Schrödinger equations*, Indiana Univ. Math. J. **49** (2000), 221–243.
7. H. Koch and D. Tataru, *On the spectrum of hyperbolic semigroups*, Comm. Partial Differential Equations **20** (1994), 901–937.
8. Y. Latushkin and C. Chicone, *Evolution semigroups in dynamical systems and differential equations*, vol. 70, AMS, Providence, 1999, Math. Surv. Monogr.
9. M. Lichtner, *Exponential dichotomy and smooth invariant center manifolds for semilinear hyperbolic systems*, PhD thesis (2006), 167, http://dochost.rz.hu-berlin.de/browsing/dissertationen/.
10. ———, *Principle of linearized stability and center manifold theorem for semilinear hyperbolic systems*, WIAS preprint No. 1155 (2006), 28.
11. ———, *Variation of constants formula for hyperbolic systems*, WIAS preprint No. 1212 (2007), 19.
12. X-B Lin and A. F. Neves, *A Multiplicity Theorem for Hyperbolic Systems*, J. Differential Equations **76** (1988), 339–352.
13. Z. H. Luo, B. Z. Guo, and O. Morgul, *Stability and stabilization of infinite dimensional systems with applications*, Springer, 1999.
14. A. F. Neves, H. Ribeiro, and O. Lopes, *On the spectrum of evolution operators generated by hyperbolic systems*, J. Funct. Anal. **67** (1986), 320–344.
15. J. Prüss, *On the spectrum of c_0 semigroups*, Trans. Amer. Math. Soc. **284** (1984), no. 2, 847–857.
16. M. Radziunas, *Numerical bifurcation analysis of traveling wave model of multisection semiconductor lasers*, Physica D **213(1)** (2006), 98–112.
17. M. Renardy, *On the type of certain c_0-semigroups*, Comm. Partial Differential Equations **18** (1993), 1299–1307.
18. ———, *On the linear stability of hyperbolic pdes and viscoelastic flows*, Z. Angew. Math. Phys. (ZAMP) **45** (1994), 854–865.
19. ———, *Spectrally determined growth is generic*, Proc. Amer. Math. Soc. **124** (1996), 2451–2453.
20. A. Vanderbauwhede and G. Iooss, *Center manifold theory in infinite dimensions*, Dynamics Reported, vol. 1 (new series), Springer, Heidelberg, 1992, ed. C. K. R. T. Jones and U. Kirchgraber and H.-O. Walther, p. 123-163.

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