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Controllability near Takens–Bogdanov points

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1 Introduction

Control systems modelling real life processes depend on some parameters. We consider control systems of the form

$$\frac{dx}{dt} = f(x, \lambda, u) \tag{1.1}$$

under the following assumptions:

(A_1). $f \in C^r(\mathbb{R}^n \times \Lambda \times U, \mathbb{R}^n)$ where Λ and U are bounded open subsets in $\mathbb{R}^p$ and $\mathbb{R}^k$ respectively, and r is sufficiently large .

(A_2). (γ_0, λ_0) is a bifurcation element of the uncontrolled system

$$\frac{dx}{dt} = f_0(x, \lambda) := f(x, \lambda, 0) \tag{1.2}$$

that is, to given any small neighborhood $\mathcal{N}_{\gamma_0}$ of the trajectory γ_0 of (1.2) for $\lambda = \lambda_0$ and to given any small neighborhood $\mathcal{N}_{\lambda_0}$ of λ_0 the topological structure of (1.2) in $\mathcal{N}_{\gamma_0}$ is not the same for all λ in $\mathcal{N}_{\lambda_0}$.

Under some additional conditions hypotheses (A_2) implies that there is a trajectory γ_λ of (1.2) bifurcating from γ_0 when λ crosses λ_0. Well–known examples are Hopf–bifurcation and homoclinic bifurcation. With respect to the supposed bifurcation we may ask the following questions:

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(Q_1). Is it possible to control the stability of the bifurcating family of trajectories γ_λ?

(Q_2). How does a bifurcation in the uncontrolled system (1.2) influence the controllability of (1.1)?

Problem (Q_1) has been treated by H. Abed and J.–H. Fu in the codimension one cases of Hopf bifurcation [1] and of stationary bifurcation [2]. They proved that under some conditions and by applying an affine feedback control the bifurcating trajectory can be made stable.

An answer to question (Q_2) has been given by F. Colonius et al. In [9] they studied the influence of a Hopf bifurcation in the uncontrolled system on the control sets (the regions of complete controllability) of an n–dimensional affine control system. By means of an additional parameter characterizing the control range it could be shown that in case of a sufficiently small control range the occurence of Hopf bifurcation in the uncontrolled system implies a branching of control sets.

In this paper we shall study an affine control system whose uncontrolled system coincides with a universal unfolding of a Takens–Bogdanov singularity [6, 24, 25]. It is well-known that such a singularity represents the simplest case of a codimension two bifurcation, and that the corresponding unfolding shows the following bifurcations: 1. Stationary bifurcation 2. Hopf bifurcation 3. Homoclinic bifurcation. The corresponding bifurcation diagram can be found in section 2.

Our main interest is devoted to the dependence of the control sets on the unfolding parameters and on an additional parameter characterizing the control range. The obtained results can be summarized as follows: Each limit set of the uncontrolled system is contained in a control set. Stable limit sets correspond to invariant control sets and unstable limit sets correspond to variant control sets. If there is no limit set for constant control functions, we get no control set at all. For sufficiently small control range, there are bifurcation curves in the unfolding parameter plane, which are connected with a change of the number of the control sets or of their topological structure. The bifurcation curves for the control sets approach the bifurcation curves of the uncontrolled system as the control range tends to zero. Moreover, the qualitative behavior of the control systems can be different to the behavior of the system with constant control function. Especially we find parameter regions and control ranges, such that for constant control there is no homoclinic orbit, whereas there exists a "controlled homoclinic orbit" belonging to the interior of a control set.

2 The bifurcation diagram of the uncontrolled Takens – Bogdanov unfolding system

Let G be a neighborhood of the origin in $\mathbb{R}^2$. In G we consider the two–dimensional autonomous differential system

$$\frac{dz}{dt} = \varphi(z) \tag{2.1}$$

under the conditions

(A_1). $\varphi \in C^r(G, \mathbb{R}^2)$, r sufficiently large.

(A_2). $\varphi(0) = 0$.

(A_3). The Jacobian A of φ at $z = 0$ has zero as algebraically double and geometrically simple eigenvalue.

Under these assumptions, the equilibrium $z = 0$ is called a Takens–Bogdanov singularity [24, 6]. Thus (2.1) can be represented as

$$\frac{dz}{dt} = Az + \psi(z) \tag{2.2}$$

where the matrix A has the form

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.3}$$

ψ belongs to $C^r(G, \mathbb{R}^2)$ and satisfies $\psi(0) = 0$, $\psi'(0) = 0$.

The l–jet normal form of (2.2) under the condition (2.3) can be written in either of two ways [14]:

$$\begin{aligned} \frac{dx}{dt} &= y + \sum_{j=2}^{l} a_j x^j + O(\|(x,y)\|^{l+1}) \\ \frac{dy}{dt} &= \sum_{j=2}^{l} b_j x^j + O(\|(x,y)\|^{l+1}), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \frac{dx}{dt} &= y + O(\|(x,y)\|^{l+1}) \\ \frac{dy}{dt} &= \sum_{j=2}^{l} (a_j x^j + b_j x^{j-1}) + O(\|(x,y)\|^{l+1}). \end{aligned} \tag{2.5}$$

Bogdanov proved in 1971 that the two–parameter differential system

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \lambda_1 + \lambda_2 x + x^2 \pm xy\end{aligned} \tag{2.6$^\pm$}$$

is an universal unfolding of any smooth two–dimensional autonomous vector field near a Takens–Bogdanov singularity, that means, system (2.6$^\pm$) shows all possible topological structures of the trajectories of any smooth vector field near a Takens–Bogdanov singularity. This result was reported first in Arnold's paper [3] in 1972. Bogdanov published his results 1975 [4] without proofs, and 1976 [5, 6] with proofs. F. Takens studied the same type of singularity by using the normal form (2.4), he published his results in 1974 [25, 26].

A Takens–Bogdanov point is the simplest example of a codimension–two singularity. A qualitative study of (2.6$^\pm$) [14, 8, 22, 24] shows that in a two–parameter family of vector fields a Takens–Bogdanov point arises naturally as the common endpoint (or start point) of a Hopf–bifurcation curve and a homoclinic bifurcation curve (separatrix loop). Hence, there are a lot of processes in nature and technology whose modelling leads to dynamical systems with Takens–Bogdanov points: motion of a thin panel in a flow [16, 17, 18] shock waves, [21, 20], population dynamics [7], solar gravity [23].

In what follows we describe the phase picture of (2.6$^+$) near the origin in dependence on λ for small λ.

Let K_r be the disk in the phase plane defined by $K_r := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < r^2\}$, analogously let Σ_s be the open disk with radius s in the parameter plane centered at the origin. We denote a curve k in Σ_s as *bifurcation curve* with respect to the qualitative behavior of system (2.6$^+$) in K_r if it consists of *bifurcation points*. A point $p \in \Sigma_s$ is called a *bifurcation point* if in each neighborhood $\mathcal{N}_\lambda^p$ of p in Σ_s there are λ_1 and λ_2 such that the corresponding systems (2.6$^+$) have different topological structures of their trajectories in K_r. The following theorem describes the set of bifurcation curves of system (2.6$^+$) in Σ_s (see Fig. 2).

Theorem 2.1 *[5] There are sufficiently small positive constants $\overline{r}$ and $\overline{s}$ such that in $\Sigma_{\overline{s}}$ there exist exactly three bifurcation curves k_E, k_H, k_S of system (2.6$^+$) with respect to $K_{\overline{r}}$. All bifurcation curves contain the origin as limit point:*

(i) The bifurcation curve $k_E := \{\lambda \in \Sigma_{\overline{s}}\colon \lambda_1 = \frac{1}{4}\lambda_2^2\}$ separates regions in $\Sigma_{\overline{s}}$ with different numbers of equilibria in $K_{\overline{r}}$. We denote by S_0 the region in $\Sigma_{\overline{s}}$ bounded by k_E and containing the positive λ_1–axis. We decompose k_E into the curves k_E^+ $(\lambda_2 > 0)$ and k_E^- $(\lambda_2 < 0)$ by dropping the origin.

(ii) The bifurcation curve $k_H := \{\lambda \in \Sigma_{\overline{s}} : \lambda_1 = 0, \lambda_2 < 0\}$ is connected with the generation of a limit cycle from an equilibrium (Hopf bifurcation). We denote by B the region in $\Sigma_{\overline{s}}$ bounded by k_H and k_E^-.

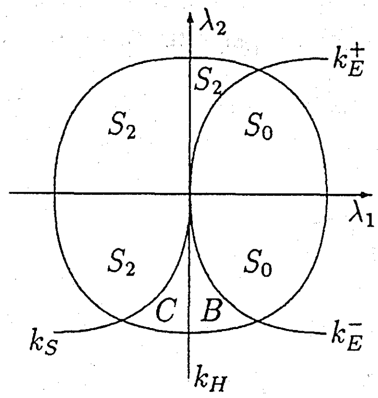

Figure 1: Bifurcation diagram to system (2.6^+) in the λ–plane

(iii) *The bifurcation curve k_S is connected with the bifurcation of a limit cycle from a separatrix loop (homoclinic bifurcation). Near the origin in the parameter plane, k_S can be described as $k_S := \{\lambda \in \Sigma_{\bar{s}} : \lambda_1 = -\frac{6}{25}\lambda_2^2 + o(\lambda_2^2),\ \lambda_2 < 0\ \}$. The region in $\Sigma_{\bar{s}}$ bounded by k_S and k_H is denoted by C, the region bounded by k_S and k_E^+ is denoted, by S_2.*

The next theorem describes the qualitative behavior of system (2.6^+) in $K_{\bar{r}}$ for $\lambda \in \Sigma_{\bar{s}}$.

Theorem 2.2 *[5] Let $\bar{r}$ and $\bar{s}$ as in Theorem 2.1 Then we have:*

(i) *For $\lambda \in S_0$, system (2.6^+) has no equilibrium in $K_{\bar{r}}$, the corresponding flow is parallelizable (see fig. 2(i)).*

(ii) *For $\lambda \in k_E^-$, system (2.6^+) has exactly one equilibrium E in $K_{\bar{r}}$. E has three separatrices, one of them tends to E as t tends to $+\infty$ (stable separatrix), two are unstable separatrices (tend to E as $t \to -\infty$) (see fig. 2(ii)).*

(iii) *For $\lambda \in B$, system (2.6^+) has exactly two equilibria, one saddle point E_S and one unstable antisaddle point E_A, it has no periodic solution (see fig. 2(iii)).*

(iv) *For $\lambda \in k_H$, system (2.6^+) has exactly two equilibria, one saddle point E_S and one unstable week focus E_F, that is, the corresponding characteristic roots of E_F are purely imaginary (see fig. 2(iv)).*

(v) *For $\lambda \in C$, system (2.6^+) has exactly two equilibria, one saddle point E_S and one stable focus E_F surrounded by exactly one limit cycle which is unstable (see fig. 2(v)).*

(vi) *For $\lambda \in k_S$, system (2.6^+) has exactly two equilibria, one saddle point E_S and one stable focus E_F. Two separatrices of the saddle point form a closed separatrix loop surrounding the stable focus. There is no periodic solution (see fig. 2(vi)).*

(vii) *For* $\lambda \in S_2$, *system* (2.6$^+$) *has exactly two equilibria, one saddle point* E_S *and one stable focus* E_F *but no periodic solution (see fig. 2(vii)).*

(viii) *For* $\lambda \in k_E^+$, *system* (2.6$^+$) *has exactly one equilibrium* E *in* $K_{\bar{r}}$. E *has three separatrices, one unstable and two stable (see fig. 2(viii)).*

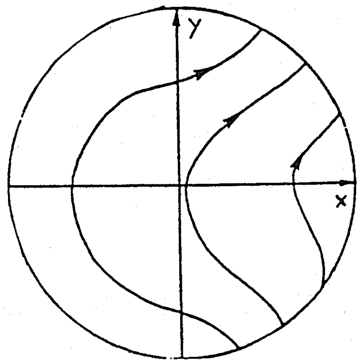

Fig. 2 (i)

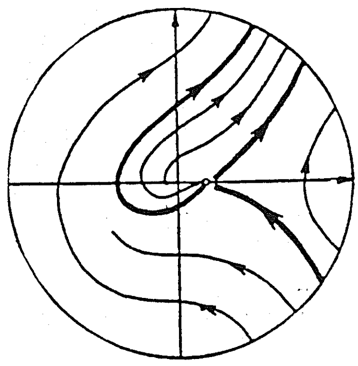

Fig. 2 (ii)

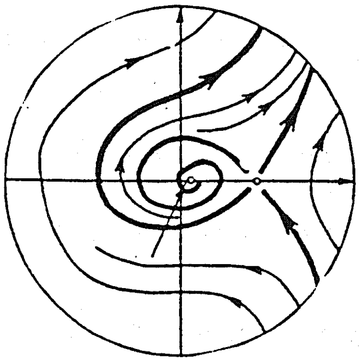

Fig. 2 (iii)

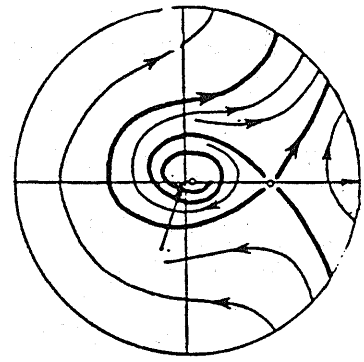

Fig. 2 (iv)

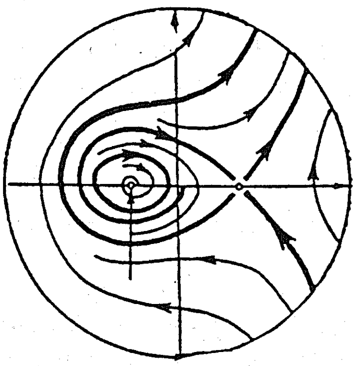

Fig. 2 (v)

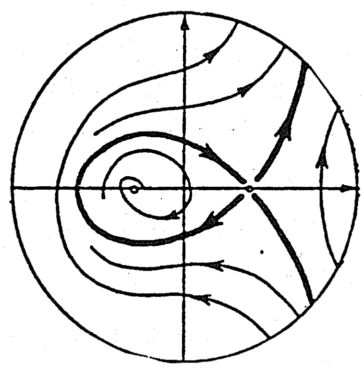

Fig. 2 (vi)

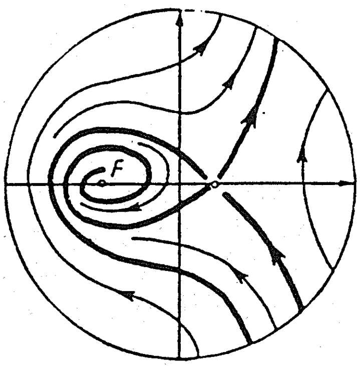

Fig. 2 (vii)

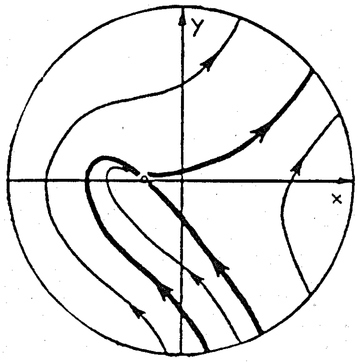

Fig. 2 (viii)

3 Control flows and Control sets

In this section we recall some basic definitions and results from control theory and from the theory of dynamical systems.

Let us consider the affine control system

$$\frac{dx}{dt} = f(x) + \sum_{i=1}^{m} u_i g_i(x) \tag{3.1$^\varrho$}$$

under the hypotheses

(H_1). $f, g_1, \ldots, g_m \in C^r(\mathbb{R}^n, \mathbb{R}^n)$, r sufficiently large.

(H_2). $u = (u_1, ..., u_m) \in \mathcal{U}^\varrho$, $\mathcal{U}^\varrho := \{\tilde{u} \in L^\infty(\mathbb{R}, \mathbb{R}^m) :\ \text{ess sup}|\tilde{u}| \leq \varrho,\ \varrho > 0\}$.

(H_3). For all $u \in \mathcal{U}^\varrho$ and for all $x \in \mathbb{R}^n$ there exists a unique solution $\phi(t, x, u)$ of (3.1^ϱ) defined for all $t \in \mathbb{R}$ and satisfying $\phi(0, x, u) = x$.

We introduce the positive ($t \geq 0$) reachable sets of (3.1^ϱ) as follows:

$$\begin{aligned} O^{+,\varrho}(x,t) &:= \{y \in \mathbb{R}^n \mid \text{there is a } u \in \mathcal{U}^\varrho \text{ such that } y = \phi(t,x,u)\} \\ O^{+,\varrho}_{\leq T}(x) &:= \bigcup_{0 \leq t \leq T} O^{+,\varrho}(x,t) \\ O^{+,\varrho}(x) &:= \bigcup_{0 \leq t} O^{+,\varrho}(x,t). \end{aligned}$$

Similarly, we define the negative reachable sets:

$$O^{-,\varrho}(x,t) := \{y \in \mathbb{R}^n \mid \text{there is a } u \in \mathcal{U}^\varrho \text{ such that } \phi(t,y,u) = x\}$$

$$
\begin{aligned}
O^{-,\varrho}_{\leq T}(x) &:= \bigcup_{0\leq t\leq T} O^{-,\varrho}(x,t) \\
O^{-,\varrho}(x) &:= \bigcup_{0\leq t} O^{-,\varrho}(x,t).
\end{aligned}
$$

In the sequel we need the property that the reachables sets $O^{\pm,\varrho}_{\leq T}(x)$ have a nonempty interior for each $T > 0$. In order to formulate a condition for the control system (3.1$^\varrho$) ensuring this property we denote by $\mathcal{LA}$ the Lie-Algebra generated by the vector fields $f, g_1, \ldots, g_m$. Let $\Delta_{\mathcal{LA}}(x)$ be the corresponding distribution. By a general result ([19], pp.56–74), the validity of the assumption

(H_4). $\dim \Delta_{\mathcal{LA}}(x) = n$ for all $x \in \mathbb{R}^n$

implies that the reachable sets $O^{\pm,\varrho}_{\leq T}(x)$ of the control system (3.1$^\varrho$) have a nonempty interior.

The following definition is basic for our investigations.

Definition 3.1 *A set $D^\varrho \subset \mathbb{R}^n$ is called a* control set *of* (3.1$^\varrho$) *with respect to $\mathcal{U}^\varrho$ if it has the following properties:*

> *(a): $D^\varrho \subset \overline{O^{+,\varrho}(x)}$ for all $x \in D^\varrho$.*
>
> *(b): For all $x \in D^\varrho$ there exists $u \in \mathcal{U}^\varrho$ with $\phi(t,x,u) \in D^\varrho \; \forall\, t \geq 0$.*
>
> *(c): D^ϱ is maximal (w.r.t. set inclusion) with these properties.*

A control set D^ϱ is called invariant *if additionally:*

$$\overline{D^\varrho} = \overline{O^{+,\varrho}(x)} \text{ for all } x \in D^\varrho.$$

All other control sets are called variant.

Note that control sets are always connected and pairwise disjoint.

In the set of control sets an order relation $\prec$ can be introduced as follows: Let D_1^ϱ and D_2^ϱ be control sets. We say the relation $D_1^\varrho \prec D_2^\varrho$ is valid if there exists an $x \in D_1^\varrho$ such that $O^{+,\varrho}(x) \cap D_2^\varrho \neq \emptyset$, that is D_2^ϱ is reachable from a point $x \in D_1^\varrho$. Concerning this order relation, invariant control sets are maximal elements and open control sets are minimal.

In what follows our main interest is devoted to control sets with a nonempty interior. For such control sets, condition (b) in Definition 3.1 is redundant and due to a result in [11] we have

$$intD^\varrho \subset O^{+,\varrho}(x) \text{ for all } x \in D^\varrho$$

that is we have exact controllability in the interior of a control set.

Let $\varphi(t,x)$ be the solution of the uncontrolled system

$$\frac{dx}{dt} = f(x) \tag{3.2}$$

satisfying $\varphi(0,x) = x$. We recall the following definition from the theory of dynamical systems which gives a weak idea of recurrence.

Definition 3.2 *The point $x \in \mathbb{R}^n$ is called* chain recurrent *if for every $\varepsilon > 0$ and for every $T > 0$ there are points $x = x_0, x_1, ..., x_n = x$ and times $t_0, ..., t_{n-1} > T$ such that $|\varphi(t_{i-1}, x_{i-1}) - x_i| < \varepsilon$ for $i = 1, ..., n$. The set of all chain recurrent points of* (3.2) *is called the* chain recurrent set $\mathcal{CR}$ *of* (3.2)*. We call a closed connected maximal subset of* $\mathcal{CR}$ *a* component *of* $\mathcal{CR}$.

All limit points of bounded trajectories, e.g. equilibria, periodic and homoclinic orbits, are contained in the set $\mathcal{CR}$.

In the sequel the correspondence between the components of the chain recurrent set $\mathcal{CR}$ and the control sets D^ϱ plays an important role. Concerning this correspondence we introduce an order relation $\prec$ between the components of the chain recurrent set $\mathcal{CR}$. The relation $C_1 \prec C_2$ means that there are points $x_0, \ldots, x_n$ where $x_0 \in C_1$ and $x_n \in C_2$ and orbits $\gamma_1, \ldots, \gamma_n$ connecting the points x_{i-1} and x_i such that $x_{i-1} \in \alpha(\gamma_i)$ and $x_i \in \omega(\gamma_i)$ for $i = 1, \ldots, n$ (see [13]).

The following definition is useful to formulate a result on the existence of a control set containing a component of the set $\mathcal{CR}$.

Definition 3.3 *(Inner–Pair–Condition). A pair $(u,x) \in \mathcal{U}^\varrho \times \mathbb{R}^n$ is called an* inner pair *to the control system* (3.1^ϱ) *if there exist $T > 0$ and $S > 0$ such that*

$$\phi(T,x,u) \in int\, O^{+,\varrho}_{\leq T+S}(x). \tag{3.3}$$

In order to formulate a suitable sufficient condition for a pair (u,x) to satisfy the inner–pair–condition we introduce the notation

$$ad^0_h g_i(x) := g_i(x) \text{ and } ad^k_h g_i(y) \; := (ad^{k-1}_h g_i)_x(y)\, h(y) - h_x(y)\, ad^{k-1}_h g_i(y)$$

where $h, g \in C^r(\mathbb{R}^n, \mathbb{R}^n)$ and r is sufficiently large. According to Corollary 4.6 in [10] we have

Lemma 3.4 *Let $u^0 \in \mathcal{U}^\varrho$ be a constant control with $|u^0| < \varrho$ and let $x \in \mathbb{R}^n$ such that $\phi(t,x,u^0)$ is bounded for $t \leq 0$. Instead of* (3.3) *we assume the following stronger condition: With $h(x) := f(x) + \sum_{i=1}^m u^0_i\, g_i(x)$ we have*

$$span\{ad^k_h g_i(z), i = 1, ..., m, k = 0, 1, 2, ...\} = \mathbb{R}^n$$

for each $z \in \omega(u^0, x)$.

Then each element (u^0, y) *where* y *is an element of the* ω*–limit set of* $\phi(t, x, u^0)$ *is an inner pair.*

The next results providing first relationships between the components of the chain recurrent set $\mathcal{CR}$ and the set of control sets are immediate consequences of Corollary 5.3 in [10].

Theorem 3.5 *Assume hypotheses* $(H_1) - (H_3)$ *to be valid. Additionally we suppose that* $(0, x)$ *is an inner pair to* (3.1^ϱ) *for all* $x \in \mathcal{CR}$ *of* (3.2) *and* $0 < \varrho < \rho_0$*. Then to any bounded isolated component* M *of the chain recurrent set* $\mathcal{CR}$ *of* (3.1^ϱ) *there is a decreasing sequence of control sets* D^ϱ *such that* $M \subset int(D^\varrho)$ *for each* $\varrho > 0$ *and* $M = \bigcap_{0<\varrho<\rho_0} D^\varrho$.

Vice versa we have

Theorem 3.6 *Assume hypotheses* $(H_1) - (H_3)$ *hold true. Further suppose the existence of a sequence of control sets* D^{ρ_k} *of* (3.1^{ρ_k}) *such that*

a) $\rho_k \to 0$ *as* $k \to \infty$.

b) The set $L := \{y \in \mathbb{R}^n :$ *there is a sequence* $x^k \in D^{\rho_k}$ *with* $x^k \to y$ *as* $k \to \infty\}$ *is nonempty.*

Then L *is a component of the chain recurrent set of* (3.2).

Let M and $\tilde{M}$ be different components of the chain recurrent set $\mathcal{CR}$ of (3.2), let D^ϱ and $\tilde{D}^\varrho$ be the associated families of control sets, that is, $M = \bigcap_{\varrho>0} D^\varrho, \tilde{M} = \bigcap_{\varrho>0} \tilde{D}^\varrho$. Colonius and Kliemann showed in [12] that the order of the chain recurrent components of $\mathcal{CR}$ is preserved by the associated family of control sets:

Theorem 3.7 *Suppose the assumptions of Theorem 3.5 hold. Then* $M \prec \tilde{M}$ *implies* $D^\varrho \prec \tilde{D}^\varrho$ *for all* $\varrho > 0$.

Theorem 3.8 *Assume the hypotheses of Theorem 3.6 to be valid. Further suppose that there is a* $\rho_0 > 0$ *such that* $D^\varrho \prec \tilde{D}^\varrho$ *for* $0 < \varrho < \rho_0$*. Then we have* $M \prec \tilde{M}$.

Finally, we need a continuity property of control sets in parameter dependent control systems (see e.g. [27])

$$\frac{dx}{dt} = f(x, \lambda) + \sum_{i=1}^{m} u_i g_i(x, \lambda), \tag{3.4}$$

where λ belongs to the open set $\Lambda \subset \mathbb{R}^k$.

We replace hypotheses (H_1) and (H_4) by

$(\tilde{H}_1)$. $f, g_1, ..., g_m \in C^r(\mathbb{R}^n \times \Lambda, \mathbb{R}^n)$ where r is sufficiently large.

$(\tilde{H}_4)$. $\dim \Delta_{\mathcal{LA}}(x, \lambda) = n$ for all $(x, \lambda) \in \mathbb{R}^n \times \Lambda$.

Lemma 3.9 *Suppose hypotheses $(\tilde{H}_1), (H_2), (H_3), (\tilde{H}_4)$ to be valid. Let $D^{\rho_0}_{\lambda_0}$ be a control set of (3.4) with $\lambda = \lambda_0$ with nonempty interior. Let K be a compact subset of int $D^{\rho_0}_{\lambda_0}$. Then there is a small number $\varepsilon_0 > 0$ such that for $\lambda \in \Lambda_0 := \{\lambda \in \Lambda : |\lambda - \lambda_0| < \varepsilon_0\}$ $D^{\rho_0}_{\lambda}$ has a nonempty interior and $K \subset$ int $D^{\rho_0}_{\lambda}$.*

4 Bifurcation of Control Sets near a Takens–Bogdanov–singularity

Now we consider the control affine system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \lambda_1 + \lambda_2 x + x^2 + xy + u(t) \\ u &\in \mathcal{U}^\varrho = \{u \in \mathcal{L}^\infty(\mathbb{R}, \mathbb{R}) \mid \text{ ess sup } |u| < \varrho, \varrho > 0\}. \end{aligned} \tag{4.1$^\varrho$}$$

Obviously, hypotheses (H_1) and (H_2) are satisfied. Since we are interested in the behavior of the control sets of (4.1$^\varrho$) near the origin we may modify the uncontrolled system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \lambda_1 + \lambda_2 x + x^2 + xy \end{aligned} \tag{4.2}$$

outside some neighborhood of the origin such that also hypothesis (H_3) is valid. Hypothesis (H_4) can be verified by a straightforward calculation.

In case of the two-dimensional system (4.2) the set of chain recurrent points coincides with the set $\mathcal{L}(\lambda)$ of limit points. According to Theorem 2.2 $\mathcal{L}(\lambda)$ has at most three components in $K_{\bar{r}}$ for $\lambda \in \Sigma_{\bar{s}}$. The number of these components changes when λ crosses a bifurcation curve in $\Sigma_{\bar{s}}$, so we can apply the results of the previous section.

In system (4.1$^\varrho$) the inner–pair–condition is satisfied for all $(0, x) \in \mathcal{U}^\varrho \times \mathcal{L}(\lambda), \lambda \in \Sigma_{\bar{s}}$.

In what follows we consider the control sets $D^{\lambda,\varrho}$ of (4.1$^\varrho$) in $K_{\bar{r}}$ when λ varies in $\Sigma_{\bar{s}}$ and study its dependence on the control range characterized by ϱ. The approach is in the same spirit as in [9].

Theorem 4.1 *Consider the control system* (4.1^{ϱ}) *in* $K_{\bar{r}}$.

1. *To any given* $\varrho^* > 0$ *there is a positive number* l^* *such that for all* $\lambda \in \Sigma_{l^*}$ *there exists a control set* D^{λ,ϱ^*} *with nonempty interior such that* $\mathcal{L}(\lambda) \subset int\ D^{\lambda,\varrho^*}$.

2. *To given* $\lambda \in \Sigma_{\bar{s}}$ *there is a* $\varrho(\lambda)$ *such that for* $\varrho \in (0, \varrho(\lambda))$ *to each component* $\mathcal{L}_i(\lambda)$ *of* $\mathcal{L}(\lambda)$ *there exists a control set* $D_i^{\lambda,\varrho}$ *with nonempty interior containing* $\mathcal{L}_i(\lambda)$ *where* $D_i^{\lambda,\varrho} \cap D_j^{\lambda,\varrho} = \emptyset$ *for* $i \neq j$.

Remark 4.2 *As an example in [10] shows, we cannot exclude the existence of further control sets with nonempty interior which do not contain any component of* $\mathcal{L}(\lambda)$.

Proof.
1. Let $\varrho^* > 0$ be any fixed number. By Theorem 2.2, $\mathcal{L}(\lambda)$ has at most three components which depend continuously on λ and converge to the equilibrium point $0 \in \mathbb{R}^2$ as $\lambda \to 0$. Thus, Theorem 3.5 yields a sequence of control sets $D^{0,\varrho}$ which increase with ϱ such that

$$\{0\} \subset int D^{0,\varrho} \text{ and } \{0\} = \bigcap_{\varrho>0} D^{0,\varrho}.$$

Since $0 = \mathcal{L}(0) \subset int\ D^{0,\varrho}$ for each ϱ, we can set $\varrho = \varrho^*$. Then to ϱ^* there is a $l_0 > 0$ such that $\mathcal{L}(\lambda) \subset int D^{0,\varrho^*}$ for $\lambda \in \Sigma_{l_0}$. Hence we have

$$\overline{\bigcup_{\lambda \in \Sigma_{l_0}} \mathcal{L}(\lambda)} \subset int\ D^{0,\varrho^*}.$$

Now Lemma 3.9 guarantees the existence of $l^* \in (0, l_0)$, such that for each $\lambda \in \Sigma_{l^*}$

$$\overline{\bigcup_{\lambda \in \Sigma_{l^*}} \mathcal{L}(\lambda)} \subset int\ D^{\lambda,\varrho^*}.$$

Therefore, $\mathcal{L}(\lambda) \subset int(D^{\lambda,\varrho^*})$ for all $\lambda \in \Sigma_{l^*}$.
2. Let $\lambda \in \Sigma_{\bar{s}}$ be given. By Theorem 3.5, to each component $\mathcal{L}_k(\lambda)$ of $\mathcal{L}(\lambda)$ there is a sequence of control sets $D_k^{\lambda,\varrho}$ with

$$\begin{aligned} \mathcal{L}_k(\lambda) &\subset int\ D_k^{\lambda,\varrho} \\ D_k^{\lambda,\bar{\varrho}} &\subset D_k^{\lambda,\varrho} \qquad \text{for } \bar{\varrho} < \varrho \\ \bigcap_{\varrho>0} D_k^{\lambda,\varrho} &= \mathcal{L}_k(\lambda) \end{aligned}$$

Hence there is a sufficiently small $\varrho(\lambda)$ such that for $0 < \varrho \leq \varrho(\lambda)$ $D_k^{\lambda,\varrho}$ is a control set with nonempty interior containing exactly one component of $\mathcal{L}(\lambda)$, namely $\mathcal{L}_k(\lambda)$ such that $D_k^{\lambda,\varrho} \cap D_l^{\lambda,\varrho} = \emptyset$ for $k \neq l$. □

By using the notation introduced in section 2 (see also fig. 1) we get immediately from Theorem 4.1:

Corollary 4.3 *Consider the control system* (4.1^ϱ) *in* $K_{\bar{r}}$.

1. *For* $\lambda \in k_E$ *there is a* $\varrho(\lambda)$ *such that for* $0 < \varrho < \varrho(\lambda)$ *system* (4.1^ϱ) *has a control set* Ψ_λ^ϱ *with nonempty interior where* Ψ_λ^ϱ *contains the multiple equilibrium point* $E(\lambda)$.

2. *For* $\lambda \in S_2$, $\lambda \in B$, *and* $\lambda \in k_H$ *there is a* $\varrho(\lambda)$ *such that for* $0 < \varrho < \varrho(\lambda)$ *system* (4.1^ϱ) *has two control sets* Π_λ^ϱ *and* Φ_λ^ϱ *with nonempty interior where* Π_λ^ϱ *contains the saddle point* $E_S(\lambda)$, Φ_λ^ϱ *contains the antisaddle point* $E_A(\lambda)$ *and is invariant for* $\lambda \in S_2$ *(see fig. 12).*

3. *For* $\lambda \in C$ *there is a* $\varrho(\lambda)$ *such that for* $0 < \varrho < \varrho(\lambda)$ *system* (4.1^ϱ) *has three control sets* Γ_λ^ϱ, Π_λ^ϱ *and* Φ_λ^ϱ *with nonempty interior and such that* Γ_λ^ϱ *contains the limit cycle* γ_λ, Π_λ^ϱ *the saddle point* $E_S(\lambda)$, *and* Φ_λ^ϱ *the focus* $E_F(\lambda)$. Φ_λ^ϱ *is invariant (see fig. 4). Since* $E_F(\lambda)$ *is located in the interior of the region bounded by* γ_λ *the control set* Γ_λ^ϱ *is at least doubly connected.*

4. *For* $\lambda \in k_S$, *there is a* $\varrho(\lambda)$ *such that for* $0 < \varrho < \varrho(\lambda)$ *system* (4.1^ϱ) *has two control sets* Γ_λ^ϱ *and* Φ_λ^ϱ *with nonempty interior where* Γ_λ^ϱ *contains the homoclinic curve* γ_{λ_S} *and the saddle point* $E_S(\lambda)$, Φ_λ^ϱ *contains the antisaddle point* $E_A(\lambda)$ *where* Φ_λ^ϱ *is invariant (see fig. 12).*

From the special structure of the second equation in (4.1^ϱ) it follows that each result on the existence and the structure of a control set remains true if we replace λ_1 by $\bar{\lambda}_1$ and $u(t)$ by $u(t) + \lambda_1 - \bar{\lambda}_1$. To indicate the special relation between the parameter λ_1 and the set of control functions we introduce the following notation: ($4.1_{\bar{\lambda}_1}^{[\alpha,\beta]}$) means that the set of control functions $U^{[\alpha,\beta]}$ is defined by $U^{[\alpha,\beta]} := \{u \in L^\infty(R,R) : \alpha \leq \text{ess sup } u \leq \beta\}$ and that λ_1 takes the value $\bar{\lambda}_1$. It is easy to verify that the control systems ($4.1_{\lambda_1}^{[\alpha,\beta]}$) and ($4.1_0^{[\lambda_1+\alpha,\lambda_1+\beta]}$) have identical control sets, the same is valid for ($4.1_{\bar{\lambda}_1}^{[\alpha,\beta]}$) and ($4.1^\varrho$) with $\varrho = \frac{\beta-\alpha}{2}$ and $\lambda_1 = \bar{\lambda}_1 + \frac{\alpha+\beta}{2}$.

Using this property we may formulate conditions about the set of control functions such that the controlled system has the same control sets as system (4.1^ϱ). The following theorem serves as prototyp.

Lemma 4.4 *Let* $(\lambda_1^s, \lambda_2^s) \in k_S$, *let* $0 < \varrho < \varrho(\lambda_s)$ *such that* (4.1^ϱ) *has the control sets* $\Gamma_{\lambda_s}^\varrho$ *and* $\Phi_{\lambda_s}^\varrho$ *as described in Corollary* 4.3(4) . *Then all control system* ($4.1_{\lambda_1+\delta}^{[-\varrho-\delta,\varrho-\delta]}$) *where* δ *is any real number have the same control sets.*

Now we study the behavior of control sets of (4.1^ϱ) when λ is close to a bifurcation curve of system (4.2). First we consider the case of the bifurcation curve k_E which is connected with a bifurcation of two equilibria from a multiple equilibrium E .

Theorem 4.5 *Consider the control system* (4.1^{ϱ}) *in* $K_{\bar{r}}$. *To each* $\lambda_E \in k_E$ *and* $\varrho > 0$ *there is a* $\kappa(\lambda_E, \varrho)$ *such that for all* λ *satisfying* $|\lambda - \lambda_E| < \kappa(\lambda_E, \varrho)$ *there is a control set* D_λ^ϱ *with nonempty interior containing the point* E.

Proof. By Lemma 4.4, for $\lambda \in k_E$ system (4.2) has a unique multiple equilibrium E in $K_{\bar{r}}$. Then, according to Theorem 3.5, to each $\varrho > 0$ there is a control set $\Psi_{\lambda_E}^\varrho$ with nonempty interior such that $E \in \Psi_{\lambda_E}^\varrho$. Let K be a compact subset of $int\ \Psi_{\lambda_E}^\varrho$ with $E \in K$. Then, by Lemma 3.9 there is a small number $\kappa(\lambda_E, \varrho)$ such that $K \subset int\ \Psi_\lambda^\varrho$ for $|\lambda - \lambda_E| < \kappa(\lambda_E, \varrho)$. □

If λ belongs to S_0 the following theorem provides a condition that no control set exists at all. To formulate this result we note that if λ belongs to S_0 then we can write $\lambda_1 = \frac{1}{4}\lambda_2^2 + \delta$ with $\delta > 0$.

Theorem 4.6 *If* $\lambda \in S_0$ *and if* $\lambda_1 - \frac{1}{4}\lambda_2^2 = \delta > \varrho$ *then there is no control set of* (4.1^{ϱ}) *at all. If* $\varrho = \lambda_1 - \frac{1}{4}\lambda_2^2$ *then* $D_\lambda^\rho = E$.

Proof. Let $V : R^2 \to R$ be the functional defined by $V(x,y) = y - \frac{x^2}{2}$. $V(x,y) = c$ is a family of curves covering $K_{\bar{r}}$. The derivative of V along the trajectories of (4.1^{ϱ}) reads

$$\left.\frac{dV(x,y)}{dt}\right|_{(4.1^{\varrho})} = \lambda_1 + \lambda_2 x + x^2 + xy + u(t) - xy.$$

Let $\lambda_1 = \frac{1}{4}\lambda_2^2 + \varrho + \varepsilon$, $\varepsilon \geq 0$. Then we have

$$\begin{aligned}\left.\frac{dV(x,y)}{dt}\right|_{(4.1^{\varrho})} &= \frac{1}{4}\lambda_2^2 + \varrho + \epsilon + \lambda_2 x + x^2 + u(t)\\ &= (\frac{1}{2}\lambda_2 + x)^2 + \varrho + \epsilon + u(t).\end{aligned}$$

For $u(t) = u_0$ with $|u_0| \leq \varrho$ we get

$$\left.\frac{dV(x,y)}{dt}\right|_{(4.1^{\varrho})} \geq (\frac{1}{2}\lambda_2 + x)^2 + \varrho + u_0 \geq 0.$$

Hence, for $\varepsilon > 0$ we get that there is no control set at all in $K_{\bar{r}}$.
In case $\varepsilon = 0$ and $u = u_0$ the straightline $x = -\frac{\lambda_2}{2}$ is no trajectorie of (4.1^{ϱ}) except the equilibrium point $x = -\frac{\lambda_2}{2}, y = 0$. Thus, the functional V is increasing along each solution except the equilibrium point. Hence there is a unique control set consisting of the equilibrium point E.

□

Next we consider the behavior of control sets of (4.1^{ϱ}) when λ is close to the bifurcation curve k_S which is connected with the existence of a homoclinic curve γ_S.

Theorem 4.7 *Let $\lambda_s = (\lambda_1^s, \lambda_2^s)$ be a point of the bifurcation curve k_S. Then to each sufficiently small control range $\tilde{\varrho}$ there is $\nu(\tilde{\varrho}) > 0$ such that for $|\lambda_1 - \lambda_1^s| < \nu(\tilde{\varrho})$ the control system (4.1^{ϱ}) has an at least doubly connected control set $\Gamma_\lambda^{\tilde{\varrho}}$ containing the separatrix loop γ_{λ_s} in its interior.*

Proof. According to Corollary 4.3,(4) to $\lambda_s \in k_S$ belongs a $\varrho_s > 0$ such that for $0 < \varrho < 2\varrho_s$ the control system (4.1^{ϱ}) has a control set $\Gamma_{\lambda_s}^{\varrho}$ which is at least doubly connected and contains the homoclinic orbit γ_{λ_s}. Thus $(4.1_{\lambda_1^s}^{[-\varrho_s,\varrho_s]})$ has a control set $\Gamma_{\lambda_s}^{\varrho_s}$ with the same property. Now we replace λ_1^s by $\lambda_1^s - \frac{\varrho_s}{4}$. Then, by Lemma 4.4, $(4.1_{\lambda_1^s - \frac{\varrho_s}{4}}^{[-\frac{3}{4}\varrho_s,\frac{5}{4}\varrho_s]})$ has $\Gamma_{\lambda_s}^{\varrho_s}$ as control set. If we enlarge the control range the corresponding control set becomes also larger. Hence, $(4.1_{\lambda_1^s - \frac{\varrho_4}{4}}^{[-\frac{5}{4}\varrho_s,\frac{5}{4}\varrho_s]})$ has a control set $\tilde{\Gamma}_{\lambda_s}^{\varrho_s}$ which contains $\Gamma_{\lambda_s}^{\varrho_s}$ In order to prove that $\tilde{\Gamma}_{\lambda_s}^{\varrho_s}$ is at least doubly connected we replace $\lambda_1^s - \frac{\varrho_s}{4}$ by λ_1^s. Therefore, $(4.1_{\lambda_1^s}^{[-\frac{3}{2}\varrho_s,\varrho_s]})$ has $\tilde{\Gamma}_{\lambda_s}^{\varrho_s}$ as control set. By enlarging the set of control functions we get that $(4.1_{\lambda_1^s}^{[-\frac{3}{2}\varrho_s,\frac{3}{2}\varrho_s]})$ has the control set $\Gamma_{\lambda_s}^{\frac{3}{2}\varrho_s}$ which contains $\tilde{\Gamma}_{\lambda_s}^{\varrho_s}$. It follows from our assumptions above that $\Gamma_{\lambda_s}^{\frac{3}{2}\varrho_s}$ is at least doubly connected and contains the homoclinic orbit γ_{λ_s}. Thus, $\tilde{\Gamma}_{\lambda_s}^{\varrho_s}$ is also at least doubly connected.

□

In what follows we sharpen the previous result in the following way: We prove that if the uncontrolled system is represented by a point $\lambda_1 \in S_2$ sufficiently near to the bifurcation curve k_S then there exists an at least doubly connected control set where all constant controlled systems belong to S_2. That is the constant controlled systems have no homoclinic curve. Our approach to establish this result is based on the intersection of unstable and stable separatrices for different constant control functions.

First we introduce some notation. For $\lambda \in S_2$ system (4.2) has two equilibria, we denote by $E_S(\lambda)$ the saddle point and by $E_A(\lambda)$ the antisaddle point. For $\lambda^* \in k_S$ two separatrices of $E_S(\lambda^*)$ form a loop. We denote these separatrices by $s^-(\lambda^*)$ and $s^+(\lambda^*)$ which have $E_S(\lambda^*)$ as ω–limit set and α–limit set respectively (stable and unstable separatrices). Let $P(\lambda^*)$ be their common intersection point with the x–axis. Thus, there is a (small) $\kappa > 0$ such that $\tilde{s}^-(\delta) := s^-(\lambda_1^* - \delta, \lambda_2^*)$ and $\tilde{s}^+(\delta) := s^+(\lambda_1^* - \delta, \lambda_2^*)$ intersect the x–axis near $P(\lambda^*)$ for $0 < \delta < \kappa$. We denote their first intersection point by $P^-(\delta)$ and $P^+(\delta)$ respectively. It is obvious that $P^-(\delta)$ and $P^+(\delta)$ depend continuously on δ. It follows from (4.2) that the segment $\sigma^-(\delta)$ of $\tilde{s}^-(\delta)$ bounded by $P^-(\delta)$ and $\tilde{E}_S(\delta) := E_S(\lambda_1^* - \delta, \lambda_2^*)$ is located in the upper half–plane while the corresponding segment $\sigma^+(\delta)$ of $\tilde{s}^+(\delta)$ lies in the lower half–plane. By Theorem 2.2, the antisaddle point $\tilde{E}_A(\delta) := E_A(\lambda_1^* - \delta, \lambda_2^*)$ is the ω–limit set of $\tilde{s}^+(\delta)$ for $\delta > 0$. Therefore, we have $P^-(\delta) < P^+(\delta)$.

The following lemma is basic in establishing our result.

Lemma 4.8 *Let $\lambda^* = (\lambda_1^*, \lambda_2^*)$ be an arbitrary point of the bifurcation curve k_S. For any $\delta > 0$ small enough there is an $\varepsilon > 0$ such that $P^-(\delta) = P^+(\delta - \varepsilon)$. That is, the stable manifold $s^-(\delta)$ and the unstable manifold $s^+(\delta - \varepsilon)$ have a nonempty intersection.*

Proof.

Let $\Theta(x, y, \lambda_1, \lambda_2)$ be the angle between the vector $v(x, y, \lambda_1, \lambda_2)$ defined by (4.2) and the positive x–axis. If we consider Θ as a function of λ_1 we get from (4.2)

$$\Theta_{\lambda_1}(x, y, \lambda_1, \lambda_2) = \frac{y}{y^2 + (\lambda_1 + \lambda_2 x + x^2 + xy)^2} \tag{4.3}$$

That means, the vector field v rotates clockwise in the half–plane $y > 0$ and anti-clockwise in the half–plane $y < 0$ for increasing λ_1.

An immediate consequence of this fact is, that $P^-(\delta)$ and $P^+(\delta)$ are increasing for increasing δ.

From the qualitative results in Theorem 2.2 we get $P^-(\delta) < P^+(\delta)$ for $\delta > 0$.

Now fix $\delta > 0$. We have

$$P^+(0) = P^-(0) < P^-(\delta) < P^+(\delta).$$

Since P^+ is continuous on $[0, \kappa]$, we have $P^+(0) < P^-(\delta)$ and $P^+(\delta) > P^-(\delta)$. The intermediate value theorem yields a $\xi \in (0, \delta)$, such that $P^+(\xi) = P^-(\delta)$. If we set $\varepsilon := \delta - \xi$ then we get the required settings. □

Let $\lambda = (\lambda_1, \lambda_2)$ be such that (4.2) has a saddle point E_S, let D_λ^ϱ be a control set of (4.1$^\varrho$) containing E_S. Then there are control functions $u(t) \equiv c_1$ and $u(t) \equiv c_2$ where c_1 and c_2 are sufficiently small $|c_1| \leq \varrho, |c_2| \leq \varrho$, $c_1 \neq c_2$ and such that for $\lambda_1 + c_1$ and $\lambda_2 + c_2$ system (4.2) has a saddle point E_S^1 and E_S^2 respectively which are located in $intD_\lambda^\varrho$.

Lemma 4.9 *Assume that the unstable separatrix $M^u_{E_S^2}$ of E_S^2 intersects the stable separatrix $M^s_{E_S^1}$ of E_S^1 in some point M. Then the segments of the separatices $M^s_{E_S^1}$ and $M^u_{E_S^2}$ bounded by E_S^1 and M, and E_S^2 and M respectively belong to $intD_\lambda^\varrho$.*

Proof. To $E_S^1, E_S^2 \in D_\lambda^\varrho$ there are neighborhoods N_1 and N_2 with $N_1 \subset intD_\lambda^\varrho$ and $N_2 \subset intD_\lambda^\varrho$. Since $M \in M^n_{E_S^2} \cap M^s_{E_S^1}$ we have

$$\begin{aligned} \lim_{t\to\infty} \phi(t, M, c_1) &= E_S^1. \\ \lim_{t\to\infty} \phi(-t, M.c_2) &= E_S^2 \end{aligned}$$

Hence we get positive numbers T^+ and T^- such that

$$M_1 := \phi(T^+, M, c_1) \in N_1 \ , \ M_2 := \phi(-T^-, M.c_2) \in N_2.$$

Since M_1 and M_2 belong to $intD_\lambda^\varrho$ we found a trajectory of (4.1^ϱ) connecting M_1 and M_2. Thus this trajectory belongs to $intD_\lambda^\varrho$. □

The next theorem gives the most interesting statement of this section. It states the existence of an at least doubly connected control set in a case, where the limit points for constant control functions are only fixed points. Hence we have a situation where the structure of the limit sets for constant control functions is different to the structure of the control sets.

Theorem 4.10 *For system $(4.1_{\lambda_1}^{[\alpha,\beta]})$ there is a $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2)$ and a control range $[\tilde{\alpha}, \tilde{\beta}]$ such that the systems corresponding to constant control functions $u(t) \equiv \tilde{u} \in [\tilde{\alpha}, \tilde{\beta}]$ have only equilibrium points as limit sets, but there exists an at least doubly connected control set $\Gamma_{\tilde{\lambda}}^\varrho$.*

Proof. For $\lambda \in k_S$ Corollary 4.3(4) yields a control range $[-\varrho, \varrho]$ such that the system $(4.1_0^{[-\varrho,\varrho]})$ has two control sets Γ_λ^ρ and Φ_λ^ρ with nonempty interior and $\gamma_\lambda \subset int\Gamma_\lambda^\varrho$ and $E_A(\lambda) \in \Phi_\lambda^\rho$.

Since the equilibrium point $E_A(\lambda)$ is located in the simply connected region bounded by the homoclinic orbit γ_λ the control set $\Gamma_{\lambda_S}^\rho$ is at least doubly connected (see fig. 8).

From Lemma 4.8 we get a $\delta \in (0, \varrho)$ and an $\varepsilon \in (0, \delta)$ with $P^-(\delta) = P^+(\delta - \varepsilon)$. So we get an intersection of an unstable and a stable manifold as we need in Lemma 4.9.

Now we restrict the control range to $[-\varrho, -\frac{\delta-\varepsilon}{2}]$. Obviously $E_S(\delta)$ and $E_S(\delta-\varepsilon)$ are contained in the interior of some control set. Since there is an intersection point of $s^+(\delta)$ and $s^-(\delta-\varepsilon)$ Lemma 4.9 yields a closed Jordan curve containing $E_A(\lambda)$ in its interior.

Since we have restricted the control range, the control sets must be smaller and so the control set Φ_λ^ϱ is not contained in Γ_λ^ρ.

Hence, to some control range we have found a control set Γ_λ^ρ which is at least doubly connected such that for constant control functions we have no doubly connected limit set. □

5 Numerical Results

In [15] an algorithm computing control sets has been introduced which is based on a solution method to solve ordinary differential equations using piecewise constant

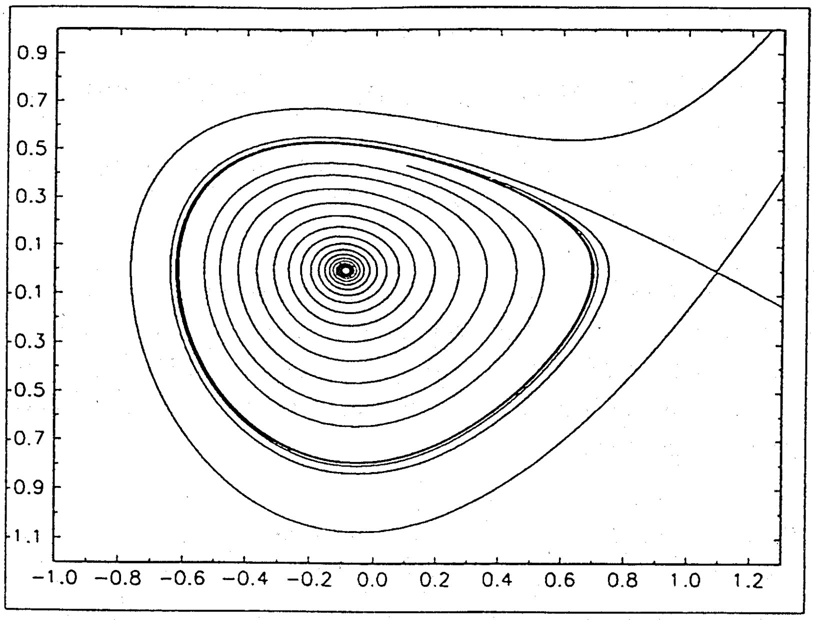

Figure 3: Phase portrait for $u \equiv 0$ and $\lambda \in C$

We present some numerical results on the control sets of system (4.1) for three points in the parameter plane and for different control ranges. We have chosen the parameter values so that the corresponding uncontrolled systems (4.2) have different qualitative behavior.

The point ($\lambda_1 = -0.1, \lambda_2 = -1$) lies in region C of the bifurcation diagram in figure 1. According to Theorem 2.2, the corresponding uncontrolled system (4.2) has as limit sets a saddle point E_S, a stable focus E_F and an unstable limit cycle (see fig. 3) As stated in Corollary 4.3 we have three control sets if the control range is small enough (see fig. 4). With increasing control range the control sets merge (see fig. 5 and 6).

The point ($\lambda_1 = -0.213605, \lambda_2 = -1$) corresponds approximately to a point on the bifurcation curve k_S. For $\lambda \in k_S$ system (4.2) has a stable focus and a homoclinic curve to a saddle point as limit sets (see fig. 7). By Corollary 4.3 and by Theorem 4.1 for sufficiently small ϱ there are two control sets containing these limit sets (see fig. 8). For increasing ϱ the control sets merge (see fig. 9 and 10).

The point ($\lambda_1 = -0.3, \lambda_2 = -1$) is located in S_2 where the system (4.2) has a saddle point and a stable antisaddle point as limit sets (see fig. 11). As long as the control range is small enough we obtain one control set to each limit set. (see fig. 12) If the control range exceeds a certain value ρ_0 we get a global bifurcation of the control set containg the saddle point. As stated in Theorem 4.10 we get a global control set around the saddle point although there is no homoclinic orbit for system (4.1) with constant control value (see fig. 13). With increasing control range the control sets merge (see fig. 13–15).

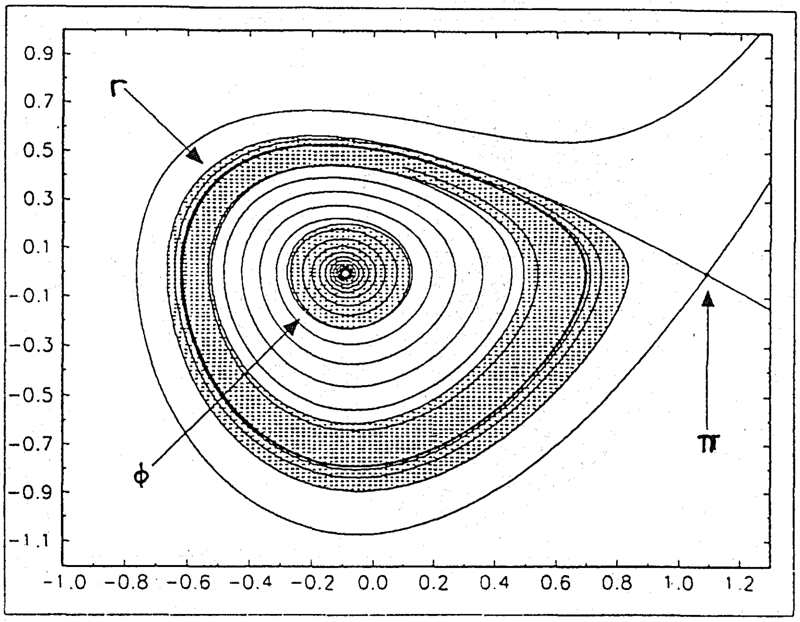

Figure 4: Control sets for $\lambda \in C$ and $u(t) \in [-0.014, 0.014]$

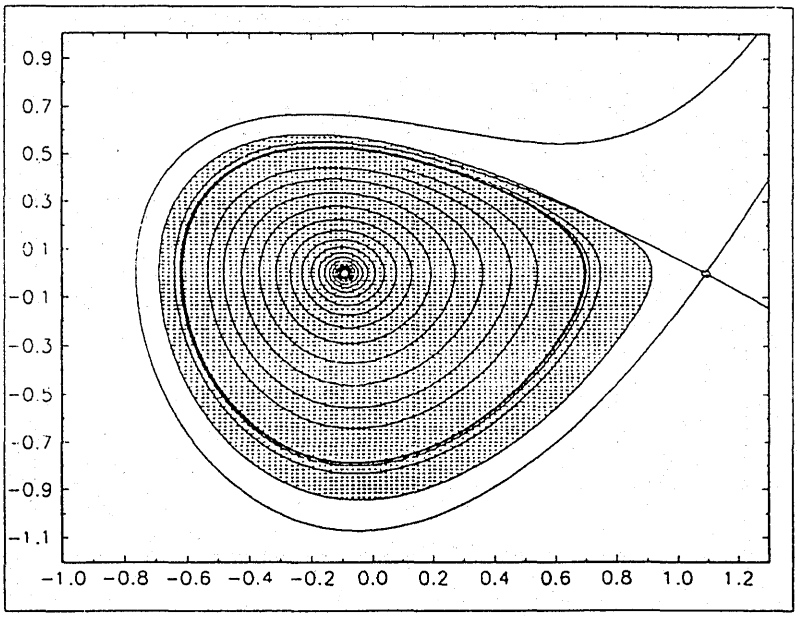

Figure 5: Control sets for $\lambda \in C$ and $u(t) \in [-0.022, 0.022]$

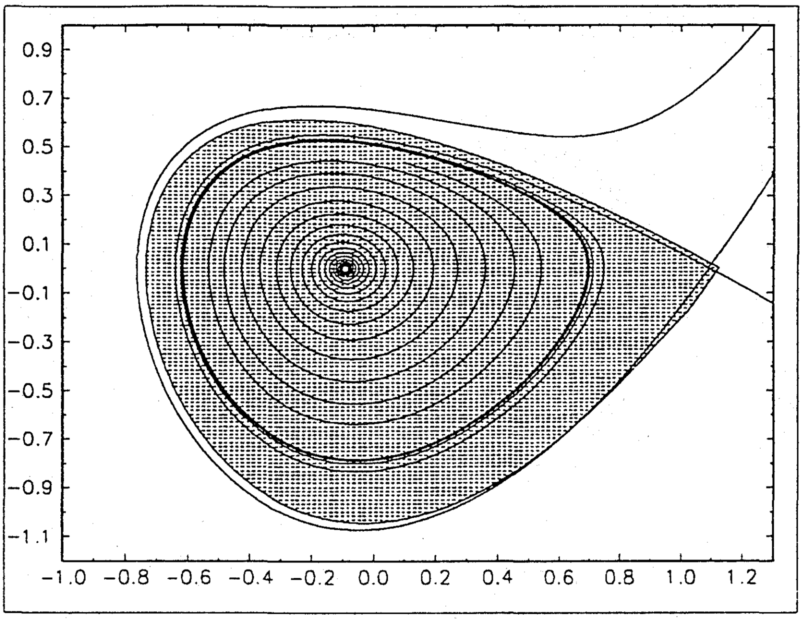

Figure 6: Control sets for $\lambda \in C$ and $u(t) \in [-0.04, 0.04]$

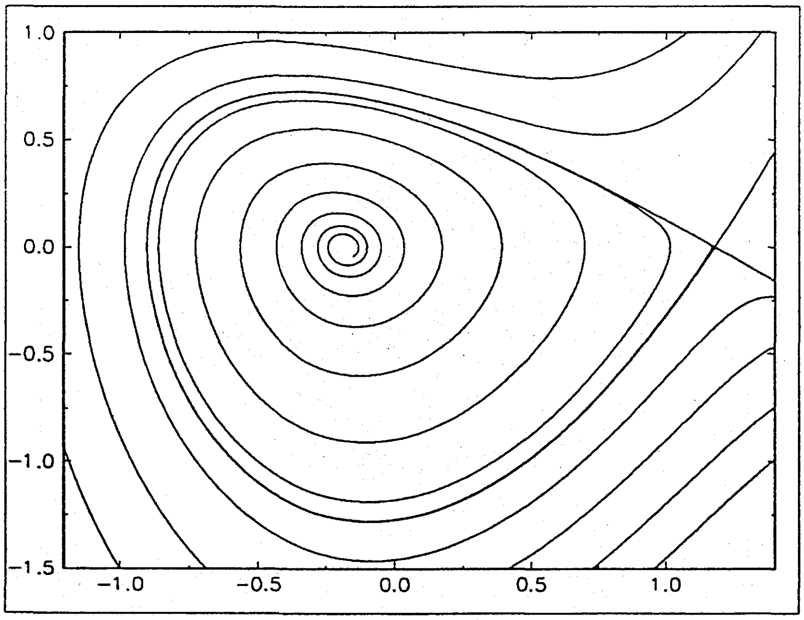

Figure 7: Phase portrait $u \equiv 0$ and $\lambda \in k_S$

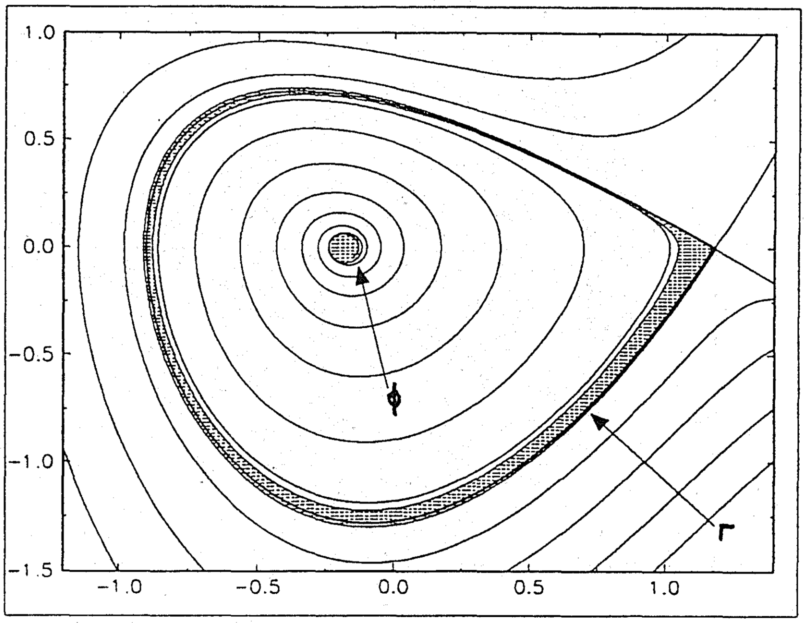

Figure 8: Control sets for $\lambda \in k_S$ and $u(t) \in [-0.01, 0.01]$

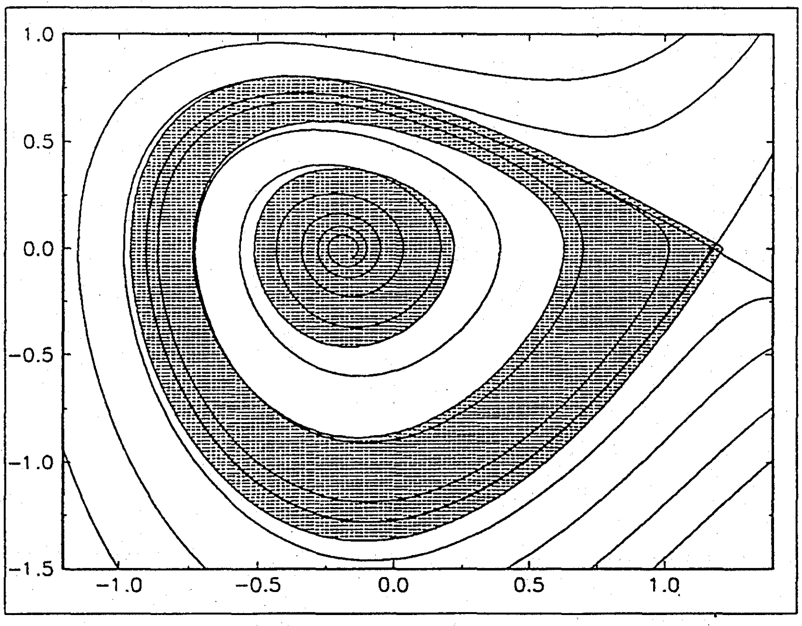

Figure 9: Control sets for $\lambda \in k_S$ and $u(t) \in [-0.05, 0.05]$

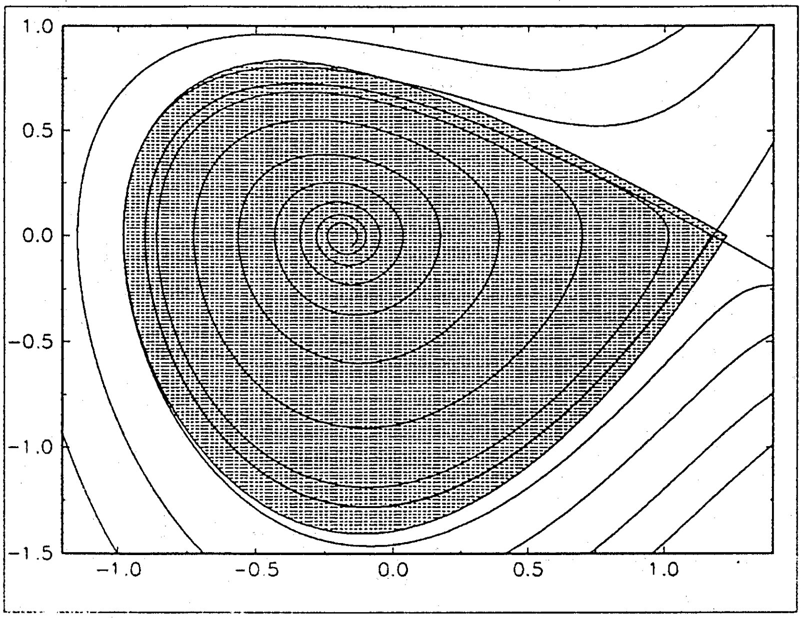

Figure 10: Control sets for $\lambda \in k_S$ and $u(t) \in [-0.07, 0.07]$

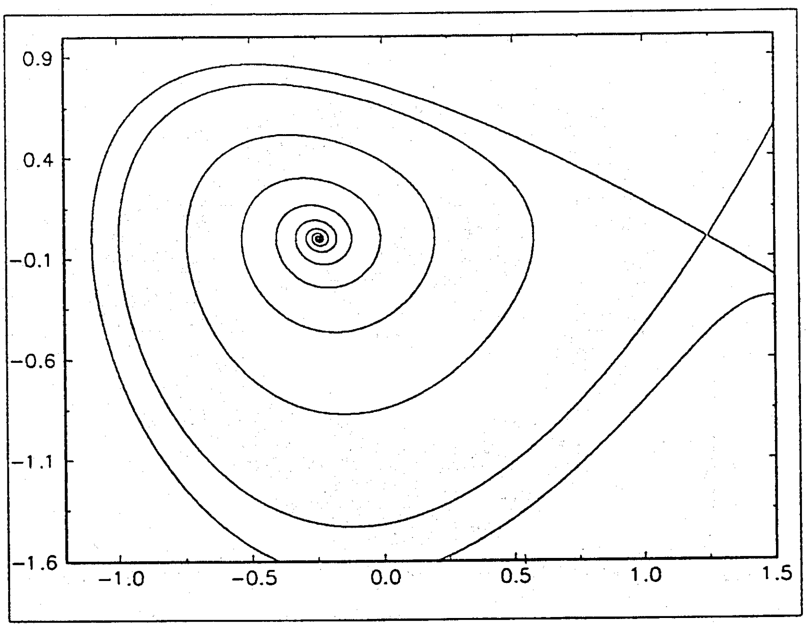

Figure 11: Phase portrait for $u \equiv 0$ and $\lambda \in S_2$

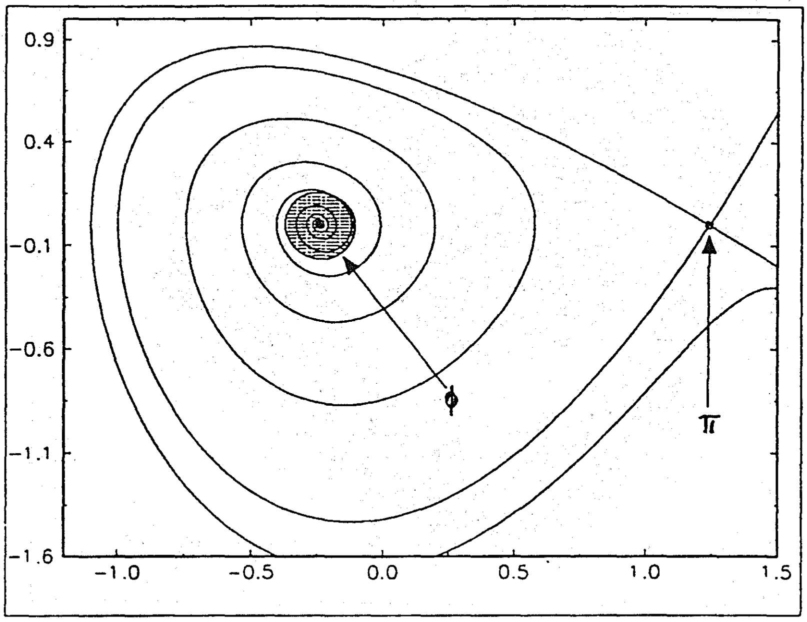

Figure 12: Control sets for $\lambda \in S_2$ and $u(t) \in [-0.03, 0.03]$

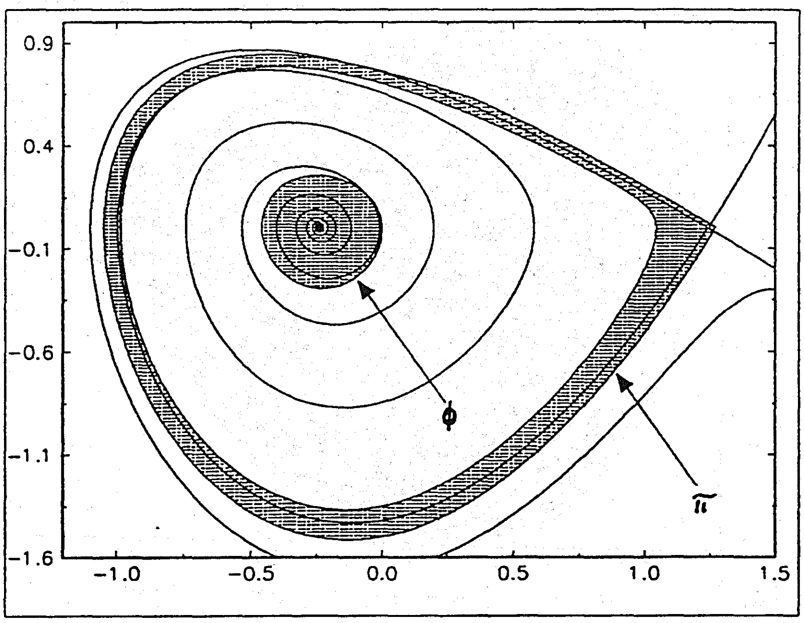

Figure 13: Control sets for $\lambda \in S_2$ and $u(t) \in [-0.05, 0.05]$

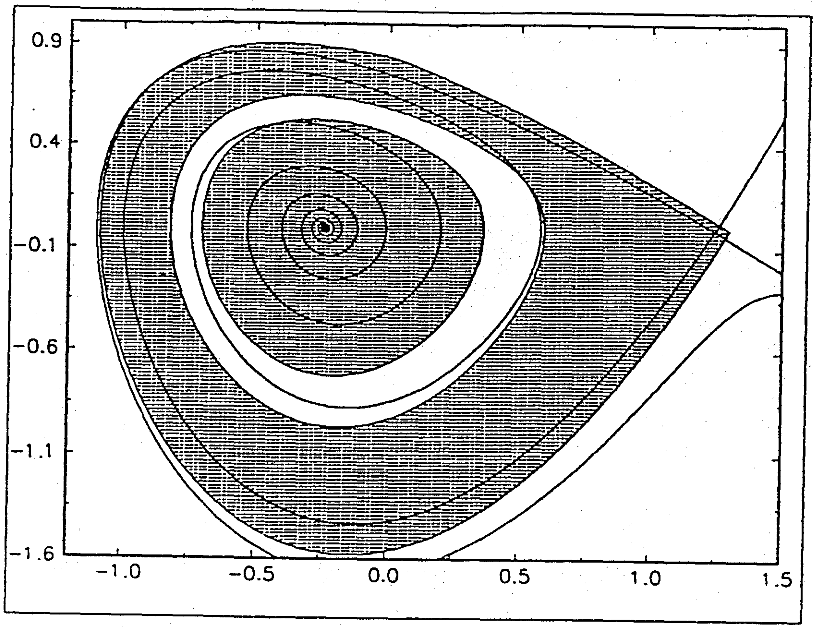

Figure 14: Control sets for $\lambda \in S_2$ and $u(t) \in [-0.09, 0.09]$

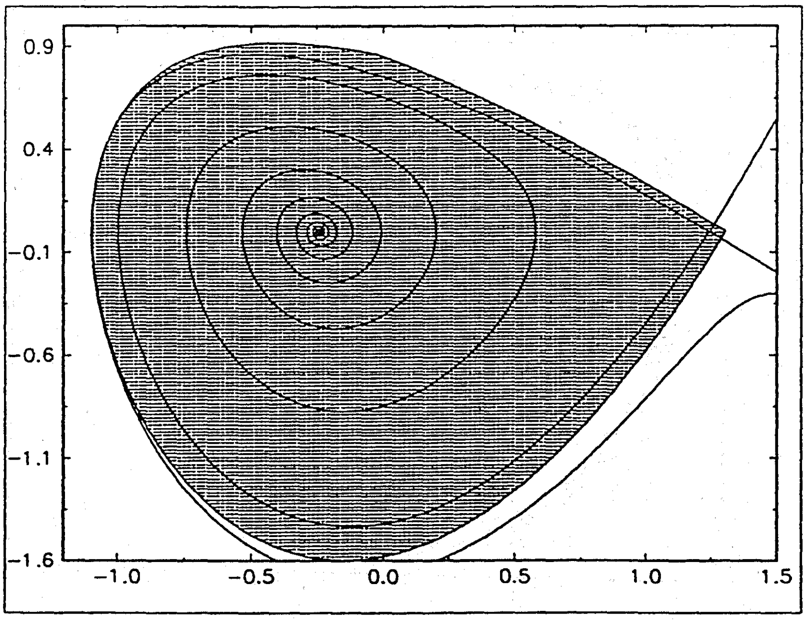

Figure 15: Control sets for $\lambda \in S_2$ and $u(t) \in [-0.1, 0.1]$

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