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Survival and complete convergence for a spatial branching system with local regulation

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Abstract

We study a discrete time spatial branching system on $\mathbb{Z}^d$ with logistic-type local regulation at each deme depending on a weighted average of the population in neighbouring demes. We show that the system survives for all time with positive probability if the competition term is small enough. For a restricted set of parameter values, we also obtain uniqueness of the non-trivial equilibrium and complete convergence, as well as long-term coexistence in a related two-type model.

1 Introduction and main results

An interesting problem from the field of mathematical ecological modelling is to find plausible stochastic models on the level of individuals for the time evolution of a 'population', say of animals or plants, which live, move – in the case of plants, we think rather of the dispersal of seeds – and reproduce in a 2-dimensional space, subject to individual random fluctuations. The mathematically simplest class of stochastic models one might come up with, namely branching random walk and its relatives in which individuals do not interact, are not adequate because in dimension 2, they virtually never exhibit stable long-time behaviour: it is well known that they will die out locally if the branching is (sub-)critical, and grow locally beyond all bounds if it is supercritical.

To describe an 'old' population, which corresponds mathematically to a non-trivial equilibrium situation, one has to introduce some interaction among individuals, which is of course also natural from the modelling perspective. A very drastic solution, that is frequently used in the context of population genetics models, is to force the population size, or the population size per deme in a spatially extended scenario, to be constant, i.e. each birth is exactly matched by a death in the population. More natural ecological models allow variable population sizes or densities, and introduce a self-regulation mechanism which, for example, makes individual reproduction super-critical in presently sparsely populated regions and subcritical in crowded areas – accounting for stress or competition for resources. Such models with explicit space have been studied in the ecological literature, see e.g. (Bolker & Pacala 1999, Law & Dieckmann 2002), mostly using computer simulations and heuristic arguments. Recently, some variants of models of locally regulated populations have been studied in the mathematics literature (Etheridge 2004, Blath, Etheridge & Meredith 2005, Fournier & Méléard 2004), and the possibility of long-time survival in certain parts of the parameter space has been rigorously proved for a continuous mass model.

We add to this literature a variant where particles live in discrete demes (arranged on $\mathbb{Z}^d$) in non-overlapping generations, which looks as follows: In the absence of competition, an individual has on average $m > 1$ offspring. Due to competition, e.g. for local resources, the average reproductive success of an individual at position x is reduced by an amount of

$\lambda_{xy} \geq 0$ by each individual at position y. Here λ_{xy} is a finite range kernel on $\mathbb{Z}^d$. Thus, an individual at x in generation n will have a random number of offspring with mean given by

$$\Big(m - \sum_{y \in \mathbb{Z}^d} \lambda_{xy}\xi_n(y)\Big)^+, \tag{1}$$

where $\xi_n(y)$ denotes the number of individuals at spatial position y in generation n. In particular, if the occupancy of neighbouring sites is so high that the term in brackets is negative, no offspring are generated at site x in this generation. For definiteness and simplicity, we assume that the actual number of offspring, given the present configuration, is Poisson-distributed with the above mean, and independent for different individuals. Once created, offspring take an independent random walk step from the location of their mother. In this way, our model incorporates individual-based random fluctuations in the number and spatial dispersal of offspring.

A formal specification of the model is given as follows: We assume that the motion/dispersal kernel $p = (p_{xy})_{x,y \in \mathbb{Z}^d}$ and the competition kernel $\lambda = (\lambda_{xy})_{x,y \in \mathbb{Z}^d}$ satisfy the following conditions.

(A1) The kernel $(p_{xy})_{x,y\in\mathbb{Z}^d} = (p_{y-x})_{x,y\in\mathbb{Z}^d}$ is a zero mean aperiodic stochastic kernel with finite range $R_p \geq 1$, i.e. for all $x, y \in \mathbb{Z}^d$: $p_{xy} = 0$ for $\|x-y\|_\infty > R_p$.

(A2) $0 \leq \lambda_{xy} = \lambda_{0,y-x}$, $\lambda_0 := \lambda_{00} > 0$ and $\lambda_{xy} = 0$ for $\|y-x\|_\infty > R_\lambda$, where $1 \leq R_\lambda < \infty$.

For a configuration $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$ define

$$f(x;\eta) := \eta(x)\big(m - \lambda_0\eta(x) - \sum_{z \neq x} \lambda_{xz}\eta(z)\big)^+ \tag{2}$$

and

$$F(x;\eta) := \sum_{y \in \mathbb{Z}^d} f(y;\eta)p_{yx}, \tag{3}$$

i.e. $F(x;\eta)$ is the expected number of individuals at x in the daughter generation if the present configuration is η. Let $N^{(x,n)}$, $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+$ be independent standard Poisson processes on $\mathbb{R}_+$. Given ξ_n, the configuration of the n-th generation, ξ_{n+1} arises as

$$\xi_{n+1}(x) = N^{(x,n)}\big(F(x;\xi_n)\big), \quad x \in \mathbb{Z}^d. \tag{4}$$

By well known properties of the Poisson distribution this definition is consistent with the intuitive description given above. Note that technically, this model is a 'probabilistic cellular automaton' with countably infinitely many possible states at each site.

As for all $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$ we have $f_\kappa(x;\eta) \leq m\eta(x)$, for $m \leq 1$ one can easily construct a coupling of (ξ_n) with a subcritical branching random walk. In that case (ξ_n) becomes extinct in finite time with probability 1 starting from any finite initial condition. Our first result roughly states in the case $m \in (1,4)$ that if the competition kernel is small enough, the population, starting from any non-trivial initial condition, will survive for all time with positive probability.

Theorem 1. *For each $m \in (1,4)$ and p satisfying (A1) there are choices of positive numbers $\lambda_0^* = \lambda_0^*(m,p)$ and $\kappa^* = \kappa^*(m,p)$ such that if $\lambda_0 \le \lambda_0^*$ and $\sum_{x \neq 0} \lambda_{0x} \le \kappa^* \lambda_0$ then the population survives with positive probability, i.e.*

$$\mathbb{P}_{\xi_0}\big[\forall n \in \mathbb{N}, \exists x \in \mathbb{Z}^d : \xi_n(x) > 0\big] > 0$$

for all ξ_0 with $f(x;\xi_0) > 0$ for some $x \in \mathbb{Z}^d$. Furthermore, conditioned on non-extinction

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbb{1}_{\{\xi_n(0)>0\}} > 0 \quad a.s.,$$

in particular the origin (and in fact any site $x \in \mathbb{Z}^d$) will be occupied at arbitrarily large times.

Note that this result as well as Theorem 3 and Corollary 4 below work in any dimension $d \ge 1$ (with threshold values λ_0^*, κ^* depending on d), in particular it establishes the possibility of long-term survival in $d = 2$.

The small competition coefficients mean that the system will typically be able to maintain a high number of particles per site. In this sense, our result concerns a 'high density regime'. Technically, we follow the natural path of comparison with oriented percolation, that might be paraphrased as 'life plus good randomness leads to more life, so show that bad randomness has small probability'. We call a space-time point *occupied* if there are enough particles there and not too many in the neighbourhood (see Definition 6 for details). The definition is such that in the corresponding deterministic model (which is a 'coupled map lattice' in dynamical systems jargon)

$$\zeta_{n+1}(x) = F(x;\zeta_n), \quad x \in \mathbb{Z}^d,\ n = 0,1,\dots \tag{5}$$

in which the Poisson variables are replaced by their means, an occupied site would after finitely many steps 'colonise' its neighbours, i.e. make them occupied as well. Then we control the probability that this remains the case under stochastic perturbation. Choosing small competition coefficients we increase the 'typical number of particles' per site in the deterministic model. Then we use the fact that the relative deviation of a Poisson random variable from its mean is typically small if the parameter is large. Finally, the finite range of competition and motion kernels allows to compare the set of occupied space-time sites with finite-range dependent oriented percolation on a suitable sub-grid of the space-time lattice.

The method can be adapted to a situation of two competing species to show that if in addition to the conditions of Theorem 1 the interspecific competition is weak enough then long term coexistence is possible (see Proposition 8).

The logistic map $\phi(x) = x(m - \lambda x)^+$ and especially the one dimensional deterministic dynamical system

$$x_{n+1} = \phi(x_n) \tag{6}$$

play an important role throughout the paper. For example, in Theorem 1 the restriction to $m < 4$ comes from the fact that otherwise the function ϕ would not map the set

$\{x \in \mathbb{R} : \phi(x) > 0\}$ into itself. The function ϕ has two fixed points, namely 0 and $(m-1)/\lambda$. For $m \in (1,3)$ it is well known that 0 is repelling and $(m-1)/\lambda$ is attracting, i.e. if $x_1 \neq 0$ then the sequence (x_n) converges to $(m-1)/\lambda$, whereas for $m \geq 3$, there are no stable fixed points. The former fact can be generalised to the coupled map lattice (5) which is a spatially extended version of (6). It can be easily seen that $\eta \equiv (m-1)/\sum_x \lambda_{0x}$ is a fixed point of F. Of course, one cannot expect that ζ_n converges uniformly to η for any initial condition, but, as the following proposition shows it converges at least locally.

Proposition 2. *Let $m \in (1,3)$, p, λ satisfying (A1) and (A2) be given. Then there exists a positive number $\kappa^* = \kappa^*(m,p)$ such that if $\sum_{x\neq 0} \lambda_{0x} \leq \kappa^* \lambda_0$ and $f(x;\zeta_0) > 0$ for some $x \in \mathbb{Z}^d$, then (ζ_n) converges locally (i.e. pointwise w.r.t. $z \in \mathbb{Z}^d$) to $(m-1)/\sum_x \lambda_{0x}$.*

Note that under the assumptions of Proposition 2, we obtain a complete classification of the equilibria of (5) and their domains of attraction: if (ζ_n) does not hit the all zero configuration $\mathbf{0} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ after the first step, it is attracted by $\eta \equiv (m-1)/\sum_x \lambda_{0x}$.

Obviously $\mathbf{0} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ is an absorbing state for (ξ_n), so the Dirac measure in this state is an invariant distribution for (ξ_n). In view of Theorem 1 it is natural to ask if there exist non-trivial stationary distributions, and one might expect that if the process does not go extinct, its distribution converges to some unique invariant distribution. A powerful method to address this problem is coupling. Let $(\xi_n^{(1)})$ and $(\xi_n^{(2)})$ be versions of the process (ξ_n) introduced in (4). Let $N_0^{(x,n)}$, $N_+^{(x,n)}$ and $N_-^{(x,n)}$, $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+$ be independent standard Poisson processes. We define the coupling of $(\xi_n^{(1)})$ and $(\xi_n^{(2)})$ as follows:

$$\begin{aligned}
\xi_{n+1}^{(1)}(x) &= N_0^{(x,n+1)}\big(F(x;\xi_n^{(1)}) \wedge F(x;\xi_n^{(2)})\big) + N_+^{(x,n+1)}\big(F(x;\xi_n^{(1)}) - F(x;\xi_n^{(1)}) \wedge F(x;\xi_n^{(2)})\big) \\
\xi_{n+1}^{(2)}(x) &= N_0^{(x,n+1)}\big(F(x;\xi_n^{(1)}) \wedge F(x;\xi_n^{(2)})\big) + N_-^{(x,n+1)}\big(F(x;\xi_n^{(2)}) - F(x;\xi_n^{(1)}) \wedge F(x;\xi_n^{(2)})\big).
\end{aligned} \tag{7}$$

Theorem 3. *Let $m \in (1,3)$, and p, λ as in (A1), (A2) be given. There are $\lambda_0^{**} = \lambda_0^{**}(m,p) > 0$ and $\kappa^{**} = \kappa^{**}(m,p) > 0$ such that if $\lambda_0 \leq \lambda_0^{**}$ and $\sum_{x\neq 0} \lambda_{0x} \leq \kappa^{**}\lambda_0$, then, conditioned on non-extinction of both populations, the coupling of $(\xi_n^{(1)})$ and $(\xi_n^{(2)})$ is successful in the sense that for each finite $\Lambda \subset \mathbb{Z}^d$ there is a random time T, such that*

$$\xi_n^{(1)}(x) = \xi_n^{(2)}(x) \quad \text{for all } x \in \Lambda \text{ and } n \geq T.$$

Obviously we have $\lambda_0^{**} \leq \lambda_0^*$, $\kappa^{**} \leq \kappa^*$. We do not know if in the case $m \in (1,3)$ the inequalities are strict (but certainly the bounds obtained in in the proof of Thm. 3 are much smaller than those obtained in the proof of Thm. 1).

Corollary 4. *Under the conditions of Theorem 3 the process (ξ_n) has two extremal invariant distributions. These distributions are translation invariant. Conditioned on non-extinction, (ξ_n) converges in distribution in the vague topology to a random measure distributed according to the non-trivial extremal invariant distribution, i.e. we have complete convergence.*

Remark 5. 1. To our knowledge, we present here the first rigorous result showing the possibility of long-time survival in a locally regulated population in $d = 2$ for a particle-based model allowing multiple occupancy (but for particular cases in a continuous-time version cf. Fournier & Méléard (2004), Proposition 6.4, where the competition acts strictly within-deme, and Proposition 7.9, where competition and dispersal kernel must be identical). As the mathematically rigorous investigation of spatial stochastic systems with local regulation terms is still in its infancy, we think it is justified to study the phenomenon in several mathematical guises. Furthermore, many species do live in discrete generations, and it is well known that discrete time dynamics can have a much richer behaviour than their continuous time analogues. This shows up in our model as well, see point 4 below.

Being honest one has to admit that the results of this paper, as well as those in Fournier & Méléard (2004), Blath et al. (2005), are still too weak to capture many ecologically interesting phenomena. Up to now, all the rigorous results are more of a conceptual nature, showing that survival resp. coexistence of several types is possible if the interaction terms are weak enough, but giving little clues about what realistic sizes of threshold values enabling/excluding survival or coexistence might be. This stems from the fact that in order to apply comparison with finite-range dependent directed percolation, one usually has to keep far away from the true critical values. For example, we have little rigorous information about properties of the non-trivial equilibrium guaranteed by Corollary 4 apart from the fact that its mean is close to the deterministic prediction $(m-1)/\sum_x \lambda_{0x}$ when the competition terms are small. One would suspect that correlations decay exponentially, but we have no rigorous proof.

Thus, the contribution of these mathematical investigations to the question how a population or several populations arrange themselves in space in order to survive in a (ecologically very interesting) situation of scarce resources and hence appreciable competition is at present rather limited. It appears that more powerful mathematical tools need to be invented in order to make rigorous progress in this direction.

2. The Poisson offspring distribution in our model is a somewhat artificial choice, which helps to streamline calculations, but is not essential for the result. To formulate a more general form of the model, one would need a one-parameter family of probability distributions (say, indexed by their mean) which includes sub- and supercritical distributions. A natural way would be to start with a fixed supercritical offspring distribution and then superimpose a 'thinning' according to the local weighted density. A nice feature of the Poisson distribution is that we can in fact think of it in this way. Another feature of the Poisson distribution is that the variance of the total number of offspring produced at some site x (given the present configuration) and its mean are the same. While it is natural for a 'branching model' to assume that conditional variance and mean of the size of the new generation are of the same order, a general class of offspring distributions would allow for different proportionality factors.

3. Our results require that λ_0, the on-site competition coefficient, is (substantially) larger than the total competition with neighbouring sites. Thus they apply to a situation where most of the competition is felt by individuals within the same 'colony'. One can think e.g. of colonies arranged on $\mathbb{Z}^d$ and λ_0 governing a rather strong population regulation inside each colony, whereas the competition λ_{0x}, $x \neq 0$, with surrounding colonies is of a lower order.

   This is certainly a technical condition which is not necessary for survival, but which intuitively helps quite a bit because it prevents the occupancy of a site from becoming so big that it would 'eradicate' its neighbourhood in the next step. It is in part owed to the discreteness of time in our model: no such condition is necessary for the continuous-time continuous-mass result in Thm. 1.5, 2 b) in Etheridge (2004) (on the other hand, unlike Etheridge (2004), we do not need the requirement that the range of λ must not exceed that of p).

Simulations suggest that the system may survive also when λ_0 and λ_{0x}, $0 < ||x|| \le R_\lambda$ are the same or similar (but sufficiently small), but occupancy numbers will fluctuate much more wildly than in the scenario treated in Theorem 1. On the other hand, with a highly asymmetric competition kernel one observes in simulations the appearance of 'fronts' of occupied sites moving in in the direction of smaller λ. This might indicate local extinction despite global survival when starting from a finite initial population in such a case.

4. As the model is in some sense a stochastic version of a spatial system of coupled logistic maps, the restrictions on m in our results are inherited from the behaviour of (6): When $m > 4$, (6) would 'live' only on a Cantor-like set, and the technique employed in the proof of Theorem 1 would fail. On the other hand, simulations suggest that even in the case $m > 4$, the random fluctuations can 'smooth out' the trajectories so that (4) might survive from initial conditions which would drive (5) to extinction in finitely many steps. The restriction to $m \in (1, 3)$ in Theorem 3 stems of course from the fact that this guarantees a unique stable fixed point of the logistic map. It is unclear if Corollary 4 would hold in a situation where (6) has periodic orbits: Then, one can see in simulations large regions of space which are 'oscillating out of phase', it might be the case that there are several non-trivial equilibria.

5. We note that the 'stepping stone version of the Bolker-Pacala model' introduced in Definition 1.3 of Etheridge (2004) can be obtained as a scaling limit of a sequence of models considered above: Assume that the parameters of the N-th model are given by

$$m^{(N)} = 1 + \frac{\alpha M}{N},\ p^{(N)}_{xy} = \frac{1}{N} m_{xy} + \Big(1 - \frac{1}{N}\sum_x m_{0x}\Big)\delta_{xy},\ \lambda^{(N)}_{xy} = \frac{\alpha\kappa\lambda_{xy}}{N^2},$$

   where $\alpha, M, m_{xy}, \lambda_{xy}$ are as in (Etheridge 2004, p.191). Let $\xi^{(N)}_0(x) = [N\mu(x)]$, where μ is some finite measure on $\mathbb{Z}^d$, and define $X^{(N)}_t(x) := \frac{1}{N}\xi^{(N)}_{[Nt]}(x)$. Then $X^{(N)}$ converges in distribution on $D_{[0,\infty)}(\mathcal{M}_f(\mathbb{Z}^d))$ to X, the solution of (5) on page 191 of Etheridge (2004), i.e. the stepping stone version of the Bolker-Pacala model, with $\gamma = 1$.

6. Hutzenthaler & Wakolbinger (2005) have shown that (at least in the case of within-site competition only) the stepping stone version of the Bolker-Pacala model from Etheridge (2004) dies out in any dimension if the carrying capacity, which would correspond to $(m-1)/\sum_x \lambda_{0x}$ in our model, is too small. Similarly, one would expect that our model, even when $m \in (1, 3)$, will die out when λ_{xy} are too large. Simulations suggest that this is indeed the case, but we have no rigorous proof.

The rest of this paper is organised as follows: in Section 2, we provide a basic lemma showing how 'occupancy' spreads through space and prove Theorem 1, in Section 3, we briefly discuss how the results can be transferred to a two-species scenario with (weak) interspecific competition. Section 4 provides results about the deterministic system (5) and proves Proposition 2. These results will be required in Section 5, where we prove Theorem 3 and Corollary 4.

To simplify the notation in the proofs we will use in the sequel a transformed version of the kernel λ,

$$\lambda_{xy} = \kappa\gamma_{xy}, \quad x \neq y \tag{8}$$

where we assume that $\sum_{y\neq x}\gamma_{xy} = 1$. That is, we separate the non-diagonal part of λ into $\kappa := \sum_{x\neq 0}\lambda_{0x}$, the total 'non-diagonal' competition and the normalised kernel $\gamma_{0x} = \lambda_{0x}/\kappa$ ($\gamma_{xx} := 0$). For $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$, $x \in \mathbb{Z}^d$ and $\kappa \geq 0$ we write

$$f_\kappa(x;\eta) := \eta(x)\big(m - \lambda_0\eta(x) - \kappa\sum_{z\neq x}\gamma_{xz}\eta(z)\big)^+ \tag{9}$$

and

$$F(x;\eta) := \sum_{y\in\mathbb{Z}^d} f_\kappa(y;\eta)p_{yx}. \tag{10}$$

Note that this is just (2) and (3) in the new parametrisation.

2 Survival

The value $f_\kappa(x;\eta)$ is the mean number of offspring at site x if the present configuration is η. The maximal (mean) number of offspring at one site in one generation will be denoted by $m^*_{\lambda_0} := \max_{\eta\in\mathbb{R}_+^{\mathbb{Z}^d}} f_\kappa(0;\eta) = m^2/(4\lambda_0)$. If the number of particles at some site x exceeds $M_{\lambda_0} := m/\lambda_0$ then, as the term in the parenthesis in (2) resp. (9) is negative, no offspring is produced at this site. Furthermore let us introduce

$$\bar{m}(\lambda_0,\kappa) := \frac{m-1}{\lambda_0+\kappa} \quad \text{and} \quad \bar{m}_{\lambda_0} := \bar{m}(\lambda_0, 0), \tag{11}$$

the deterministic equilibrium values when the non-diagonal regulation term is κ resp. 0. Note that for $\eta \equiv \bar{m}(\lambda_0,\kappa)$ we have $f_\kappa(x;\eta) = \bar{m}(\lambda_0,\kappa)$ and therefore $\eta(x) = F(x;\eta)$ for all $x \in \mathbb{Z}^d$.

Definition 6. Let $\eta \in \mathbb{R}_+^{\mathbb{Z}^d}$. For a pair of positive numbers $(\varepsilon_1,\varepsilon_2)$ we will say that a site x is $(\varepsilon_1,\varepsilon_2)$-*occupied with respect to* η if

$$\eta(x) \in [\varepsilon_1\bar{m}_{\lambda_0}, (1-\varepsilon_2)M_{\lambda_0}], \text{ and } \eta(y) \leq (1-\varepsilon_2)M_{\lambda_0}, \|x-y\|_\infty \leq R_\lambda.$$

We will often say that $\eta(x)$ is $(\varepsilon_1,\varepsilon_2)$-occupied, or just occupied if there is no risk of confusion, meaning that x is $(\varepsilon_1,\varepsilon_2)$-occupied with respect to η.

To prove Theorem 1 we compare the process (ξ_n) with oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$. The main thing in doing that is to show that if a site is $(\varepsilon_1,\varepsilon_2)$-occupied with respect to some ξ_n then in a while its neighbours will be also $(\varepsilon_1, \varepsilon_2)$-occupied with high probability. To this end we consider a perturbed coupled map lattice

$$\zeta_{n+1}(x) = F(x; \zeta_n) + \delta_n(x), \tag{12}$$

where the perturbation δ_n is assumed to satisfy $\delta_n(x) \geq -F(x; \zeta_n)$, such that (ζ_n) is nonnegative. We will show that under certain additional conditions on the perturbation term the system (ζ_n) has the desired property. Then the original process (ξ_n) can be seen as a perturbed dynamical system and we will see that the conditions mentioned above are satisfied with high probability if the competition is weak enough.

Let us now introduce and explain some notation which will be used in the sequel. We denote by p^n_{xy} the n-step transition probability of a random walk with kernel p. As mentioned above, our goal is to show that an occupied site colonises its neighbours in a couple of steps and remains itself occupied. In the first step the offspring are distributed according to the kernel p. Thus, there is in general no reason why an occupied site should remain occupied after one step. Let us fix some $\tilde{m} \in (1, m)$. By the Local Central Limit Theorem the number

$$n^* = \min\bigl\{j \in \mathbb{N} : p^j_{0x} \tilde{m}^j \geq 1 \text{ for all } x \text{ with } \|x\|_\infty \leq 1\bigr\} \tag{13}$$

is finite. We set

$$\mathcal{I} = \bigl\{(y, j) \in \mathbb{Z}^d \times \mathbb{Z}_+ : \ p^j_{0y} > 0, \ 0 \leq j \leq n^*\bigr\} \subset \bigl\{(y, j) : \ \|y\|_\infty \leq j R_p, \ 0 \leq j \leq n^*\bigr\}.$$

Suppose that the site 0 is $(\varepsilon_1, \varepsilon_2)$-occupied with respect to ζ_0 and that there is no mass at the other sites. Let us also assume for the moment that the perturbation term vanishes and that the competition between individuals at different sites is zero, i.e. $\kappa = 0$. We set $\tilde{f}(z) = z(m - \lambda_0 z)^+$. If for some positive a we have $z \in [a\varepsilon_1 \bar{m}_{\lambda_0}, (1 - \varepsilon_2) M_{\lambda_0}]$ then

$$\tilde{f}(z) \geq \begin{cases} \tilde{f}(\varepsilon_2 M_{\lambda_0}) = \varepsilon_2 M_{\lambda_0} m (1 - \varepsilon_2) & : a\varepsilon_1 \bar{m}_{\lambda_0} \geq \varepsilon_2 M_{\lambda_0} \\ \tilde{f}(a\varepsilon_1 \bar{m}_{\lambda_0}) = a\varepsilon_1 \bar{m}_{\lambda_0} m (1 - a\varepsilon_1 + \frac{1}{m}) & : a\varepsilon_1 \bar{m}_{\lambda_0} < \varepsilon_2 M_{\lambda_0} \end{cases}. \tag{14}$$

This means that the number of offspring at site 0 is at least $\tilde{m}\varepsilon_1 \bar{m}_{\lambda_0}$ if ε_1 is sufficiently small. Then the offspring are distributed in the neighbourhood according to the kernel p. In this neighbourhood the mass is again multiplied by at least $\tilde{m}$ and then distributed according to p. Hence after k steps the mass at a site y is larger than or equal to $p^k_{0y} \tilde{m}^k \varepsilon_1 \bar{m}_{\lambda_0}$. The living space of the whole population at this time is the k-th timeslice of $\mathcal{I}$ which is contained in the ball of radius kR_p. By the definition of n^*, after n^* steps the mass in 0 and in points with norm one reaches or maybe exceeds the level $\varepsilon_1 \bar{m}_{\lambda_0}$. Thus these sites are occupied at that time if the mass there and in the R_λ-neighbourhood does not exceed $(1 - \varepsilon_2) M_{\lambda_0}$.

We need some additional conditions on the perturbation term. Let $\mathcal{X} = \{(y, n) \in \mathbb{Z}^d \times \mathbb{Z}_+ : n < n^*, \ \|y\|_\infty \leq n(R_p + R_\lambda)\}$.

$(\mathrm{B1})_{\varepsilon_2}$ For all $(y, n) \in \mathcal{X}$: $F(y; \zeta_n) + \delta_n(y) \leq (1 - \varepsilon_2) M_{\lambda_0}$.

$(\mathrm{B2})_{\delta,\mathrm{K}}$ For all $(y, n) \in \mathcal{X}$: $F(y; \zeta_n) \geq K$ implies $|\delta_n(y)| \leq \delta F(y; \zeta_n)$.

Lemma 7. *Assume that m and p are as in Theorem 1. For each $K > 0$ and δ satisfying $m(1-\delta) > \tilde{m} > 1$ there are choices of positive numbers ε_1, ε_2, λ_0^* and κ^* such that whenever*

$$\lambda_0 \leq \lambda_0^* \quad \text{and} \quad \kappa \leq \kappa^* \lambda_0 \tag{15}$$

the following holds:

$$\begin{aligned} &\zeta_0(0) \text{ is } (\varepsilon_1, \varepsilon_2)\text{-occupied, } (\mathrm{B1})_{\varepsilon_2},\ (\mathrm{B2})_{\delta,\mathrm{K}} \text{ are satisfied} \\ &\quad \Longrightarrow \zeta_{n^*}(x) \text{ are } (\varepsilon_1, \varepsilon_2)\text{-occupied for all } x \text{ with } \|x\|_\infty \leq 1. \end{aligned}$$

Proof. Let $K > 0$ be given. We choose $\varepsilon_2 > 0$ such that

$$m(1-\delta)(1-\varepsilon_2 \frac{m}{m-1}) \geq \tilde{m} \quad \text{and} \quad m^*_{\lambda_0} \leq (1-2\varepsilon_2) M_{\lambda_0}. \tag{16}$$

For the second inequality we need $m < 4$. Then we choose $\varepsilon_1 > 0$ satisfying

$$p_{0y}^n \tilde{m}^n \varepsilon_1 \leq \varepsilon_2 \frac{m+1}{m} \leq \varepsilon_2 \frac{m}{m-1} \quad \text{for all } (n,y) \in \mathcal{I}. \tag{17}$$

Note that this choice guarantees

$$p_{0y}^n \tilde{m}^n \varepsilon_1 \bar{m}_{\lambda_0} \leq \varepsilon_2 \frac{m+1}{m} \bar{m}_{\lambda_0} \leq \varepsilon_2 M_{\lambda_0} \quad \text{for all } (n,y) \in \mathcal{I}. \tag{18}$$

By construction of $\mathcal{I}$, the number $\mathcal{I}_{\min} = \min\{\tilde{m}^n p_{0y}^n : (n,y) \in \mathcal{I}\}$ is positive. Therefore we may choose λ_0^* such that for $\lambda_0 \leq \lambda_0^*$

$$\varepsilon_1 \bar{m}_{\lambda_0} \mathcal{I}_{\min} \geq K.$$

Finally, we choose κ^* such that for some α satisfying $\tilde{m} < m - \alpha$

$$(1-\varepsilon_2) M_{\lambda_0} \kappa^* \lambda_0^* \leq \alpha.$$

Let us first consider the case $\kappa = 0$. We have to show that

$$\zeta_{n^*}(x) \in [\varepsilon_1 \bar{m}_{\lambda_0}, (1-\varepsilon_2) M_{\lambda_0}],\ \|x\|_\infty \leq 1.$$

By $(\mathrm{B1})_{\varepsilon_2}$ we have

$$\zeta_{n+1}(x) = F(x; \zeta_n) + \delta_n(x) \leq (1-\varepsilon_2) M_{\lambda_0} \quad \text{for all } (x,n) \in \mathcal{X}. \tag{19}$$

This means in particular $\zeta_{n^*}(x) \leq (1-\varepsilon_2) M_{\lambda_0}$ for $\|x\|_\infty \leq 1$. To complete the proof for that case we show by induction on n that

$$\zeta_n(y) \in [p_{0y}^n \tilde{m}^n \varepsilon_1 \bar{m}_{\lambda_0}, (1-\varepsilon_2) M_{\lambda_0}],\ 0 \leq n \leq n^* \text{ and } (y,n) \in \mathcal{I}.$$

By definition of n^*, the assertion of the lemma then follows. For $n = 0$ the claim holds by assumption. If it holds for some $n < n^*$ then, first using (18) and (14), then (17) and the first part of (16), we obtain

$$\begin{aligned} (1-\delta) f(y; \zeta_n) &\geq (1-\delta) \tilde{f}\big(p_{0y}^n \tilde{m}^n \varepsilon_1 \bar{m}_{\lambda_0}\big) \\ &\geq (1-\delta) p_{0y}^n \tilde{m}^n \varepsilon_1 \bar{m}_{\lambda_0} \cdot m\big(1 - p_{0y}^n \tilde{m}^n \varepsilon_1 + \frac{1}{m}\big) \\ &\geq p_{0y}^n \tilde{m}^{n+1} \varepsilon_1 \bar{m}_{\lambda_0}. \end{aligned}$$

Hence

$$\begin{aligned}(1-\delta)F(y;\zeta_n) &= \sum_{z\in\mathbb{Z}^d}(1-\delta)f(z;\zeta_n)p_{zy}\\ &\geq \sum_{z\in\mathbb{Z}^d} p_{0z}^n\tilde{m}^{n+1}\varepsilon_1\bar{m}_{\lambda_0}p_{zy} = \varepsilon_1\bar{m}_{\lambda_0}\tilde{m}^{n+1}p_{0y}^{n+1},\ (y,n)\in\mathcal{I}.\end{aligned}$$

In particular we have $F(y;\zeta_n) \geq \varepsilon_1\bar{m}_{\lambda_0}\tilde{m}^{n+1}p_{0y}^{n+1} \geq K$ for $\lambda_0 \leq \lambda_0^*$. Therefore $(\mathrm{B2})_{\delta,K}$ applies and from the last display we obtain

$$\zeta_{n+1}(y) \geq (1-\delta)F(y;\zeta_n) \geq \varepsilon_1\bar{m}_{\lambda_0}\tilde{m}^{n+1}p_{0y}^{n+1}.$$

This concludes the proof of the induction and proves the Lemma in the special case $\kappa = 0$.

Now let us turn to the case $\kappa > 0$. Assumption $(\mathrm{B1})_{\varepsilon_2}$, (15) and the choice of κ^* imply that

$$0 \leq \kappa\sum_{y\neq x}\gamma_{xy}\zeta_n(y) \leq \kappa^*\lambda_0(1-\varepsilon_2)M_{\lambda_0} \leq \alpha,\ \|x\|_\infty \leq n(R_\lambda+R_p)-R_\lambda,\ n<n^*,$$

where $\alpha > 0$ satisfies $m-\alpha > \tilde{m}$. We obtain

$$\begin{aligned}f_l(x;\zeta_n) := \zeta_n(x)(m-\alpha-\lambda_0\zeta_n(x))^+ &\leq f_\kappa(x;\zeta_n(x))\\ &\leq \zeta_n(x)(m-\lambda_0\zeta_n(x))^+ =: f_u(x;\zeta_n).\end{aligned}$$

So we can use the same induction as in the diagonal case. For the lower bound estimates we use f_l and for the upper bound estimates we use f_u. □

We set $\zeta_0 = \xi_0$ and assume that (ζ_n) is the solution of (12) with the perturbation term

$$\delta_n(x) = N^{(n,x)}\big(F(x;\xi_n)\big) - F(x;\xi_n).$$

Thus, (ξ_n) with $\xi_n = \zeta_n$ can be considered as a perturbed coupled map lattice.

Proof. **(Theorem 1)** For $(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+$ we define

$$X(x,n) = \big\{N^{(y,j)} : (y,j)\in(x,n)+\mathcal{X}\big\}.$$

Consider the events

$$A(x,n) = \{N^{(y,j)}(m_{\lambda_0}^*) \leq (1-\varepsilon_2)M_{\lambda_0}, (y,j)\in(x,n)+\mathcal{X}\}$$

and

$$B(x,n) = \left\{\sup_{(y,j)\in(x,n)+\mathcal{X}}\ \sup_{t\geq K}\left|\frac{N^{(y,j)}(t)}{t}-1\right| \leq \delta\right\}$$

We say that $X(x,n)$ is *good* if $A(x,n)\cap B(x,n)$ holds. First we want to show that the probability of a good realization can be made arbitrarily large by choosing small λ_0. It is of course enough to consider the corresponding problem in the space-time point $(0,0)$. As

$A(0,0)$ implies $(\mathrm{B1})_{\varepsilon_2}$ and $B(0,0)$ implies $(\mathrm{B2})_{\delta,K}$ on the event $A(0,0)\cap B(0,0)$ Lemma 7 yields

$$\{\xi_0(0)\ (\varepsilon_1,\varepsilon_2)\text{-occupied}\}\cap (A(0,0)\cap B(0,0))\subset \{\xi_{n^*}(y),\ \|y\|_\infty\le 1\ (\varepsilon_1,\varepsilon_2)\text{-occupied}\}.$$

By translation invariance the corresponding statement is also true for all $(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+$. Furthermore we point out that $X(x,n)$ and $X(x',n')$ are independent if $\|x-x'\|_\infty\ge 2n(R_\lambda+R_p)$ or $|n-n'|>n^*$.

Let Δ be the number of points in $\mathcal{X}$ and let $\big(N(t)\big)_{t\ge 0}$ be a standard Poisson process. Then we have

$$\mathbb{P}[A(0,0)]=(1-a(\lambda_0))^\Delta \quad \text{where} \quad a(\lambda_0)=\mathbb{P}\big[N(m^*_{\lambda_0})>(1-\varepsilon_2)M_{\lambda_0}\big].$$

According to (16) we have $m^*_{\lambda_0}\le(1-2\varepsilon_2)M_{\lambda_0}$. Thus, for some $\tilde c_1>0$ we have

$$a(\lambda_0)=\mathbb{P}\Big[\frac{N(m^*_{\lambda_0})}{m^*_{\lambda_0}}-1>\frac{(1-\varepsilon_2)M_{\lambda_0}}{m^*_{\lambda_0}}-1\Big]\le\mathbb{P}\Big[\frac{N(m^*_{\lambda_0})}{m^*_{\lambda_0}}-1>\varepsilon_2\Big]\le\exp\Big(-\frac{\tilde c_1\varepsilon_2^2}{\lambda_0}\Big).$$

Furthermore, by standard large deviation results for Poisson processes, we have for some $\tilde c_2>0$ and sufficiently large K we have W

$$\begin{aligned}\mathbb{P}\big[B(0,0)\big]&=\mathbb{P}\Big[\sup_{t\ge K}\Big|\frac{N(t)}{t}-1\Big|\le\delta\Big]^\Delta=\Big(1-\mathbb{P}\Big[\sup_{t\ge K}\Big|\frac{N(t)}{t}-1\Big|>\delta\Big]\Big)^\Delta\\&\ge\big(1-\exp(-\tilde c_2\delta^2K)\big)^\Delta.\end{aligned}$$

From the proof of Lemma 7 one can see that making K large corresponds to making λ_0 small. Hence

$$\mathbb{P}[(A(0,0)\cap B(0,0))^c]\le\mathbb{P}[A(0,0)^c]+\mathbb{P}[B(0,0)^c]\le\theta(\lambda_0), \tag{20}$$

where $\theta(\lambda_0)\le\exp(-c/\lambda_0)$ for some suitable positive constant $c=c(p,m,R_\lambda)$. This implies

$$\mathbb{P}\big[X(0,0)\text{ is good}\big]\ge 1-\theta(\lambda_0)=1-\big(1-\sqrt{p(\lambda_0)}\big)^\Delta,$$

where $p(\lambda_0)=\big(1-\theta(\lambda_0)^{1/\Delta}\big)^2$. Since $p(\lambda_0)$ converges to one as λ_0 goes to 0 we may apply a result by Liggett, Schonmann and Stacey (see (Liggett 1999, Theorem 26)) to show that for fixed n, the distribution of the random field $\mathbb{1}_{\{X(x,n)\text{ is good}\}}$ dominates the product measure $\nu_{p(\lambda_0)}=\otimes_{\mathbb{Z}^d}\mathrm{Ber}\big(p(\lambda_0)\big)$ on $\{0,1\}^{\mathbb{Z}^d\times\mathbb{Z}_+}$ Comparison of the process $\big(\mathbb{1}_{\{X(x,n)\text{ is good}\}}\big)_{x\in\mathbb{Z}^d\times n^*\mathbb{Z}_+}$ with independent oriented percolation concludes the proof. □

3 A competing species model

In this section we consider two processes $(\xi^{(1)}_n)$ and $(\xi^{(2)}_n)$, modeling for example two different species or genetic types living in the same habitat and competing for similar (or the same) resources. In the absence of the other type each of them is a version of the basic process described in the introduction, possibly with different parameters.

Let $(\lambda^{(ij)}_{xy})_{x,y\in\mathbb{Z}^d}$, $i,j\in\{1,2\}$ be translation invariant nonegative kernels on $\mathbb{Z}^d$ with finite range R_λ. These kernels will determine the intra- resp. interspecific competition: The average reproductive success of an i-individual at x is reduced by each j-individual at y by $\lambda^{(ij)}_{xy}$. The evolution of $(\xi^{(1)}_n,\xi^{(2)}_n)$ may then be described as follows. Similar to the single species model we define

$$\begin{aligned}
f_1\big(x;\xi^{(1)}_n,\xi^{(2)}_n\big) &= \xi^{(1)}_n(x)\big(m_1-\sum_y \lambda^{(11)}_{xy}\xi^{(1)}_n(y)-\sum_y \lambda^{(12)}_{xy}\xi^{(2)}_n(y)\big)^+,\\
f_2\big(x;\xi^{(1)}_n,\xi^{(2)}_n\big) &= \xi^{(2)}_n(x)\big(m_2-\sum_y \lambda^{(22)}_{xy}\xi^{(2)}_n(y)-\sum_y \lambda^{(21)}_{xy}\xi^{(1)}_n(y)\big)^+,\\
F_1\big(x;\xi^{(1)}_n,\xi^{(2)}_n\big) &= \sum_y f_1(y;\xi^{(1)}_n,\xi^{(2)}_n)p^{(1)}_{yx},\\
F_2\big(x;\xi^{(1)}_n,\xi^{(2)}_n\big) &= \sum_y f_2(y;\xi^{(1)}_n,\xi^{(2)}_n)p^{(2)}_{yx},
\end{aligned}$$

where m_i is the mean number of offspring of a type i individual in the absence of competition. If $N^{(x,n)}_1$, $N^{(x,n)}_2$, $(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+$ are independent standard Poisson processes on $\mathbb{R}_+$ then, given $(\xi^{(1)}_n,\xi^{(2)}_n)$, the configuration of the next generation is given by

$$(\xi^{(1)}_{n+1},\xi^{(2)}_{n+1})=\Big(N^{(x,n)}_1\big(F_1(x;\xi^{(1)}_n,\xi^{(2)}_n)\big),N^{(x,n)}_2\big(F_2(x;\xi^{(1)}_n,\xi^{(2)}_n)\big)\Big).$$

We obtain the following about long-term coexistence if the competition terms are weak enough:

Proposition 8. *For given $m_1,m_2\in(1,4)$, $p^{(1)}$ and $p^{(2)}$ satisfying (A1) there are positive numbers λ^*_1, λ^*_2, κ^*_1, κ^*_2 and γ^* such that if the conditions*

- *(i) $\lambda^{(ii)}_0\le\lambda^*_0$, $\sum_{y\neq x}\lambda^{(ii)}_{xy}\le\lambda^{(ii)}_0\kappa^*_i$, $i\in\{1,2\}$;*
- *(ii) $\sum_y\lambda^{(12)}_{xy},\sum_y\lambda^{(21)}_{xy}\le\gamma^*\min\{\lambda^{(11)}_0,\lambda^{(22)}_0\}$;*

are satisfied then both populations survive with positive probability provided that for some $x,y\in\mathbb{Z}$ we have $f_1(x;\xi^{(1)}_0,\xi^{(2)}(0))>0$ and $f_2(y;\xi^{(11)}_0,\xi^{(22)}_0)>0$. Furthermore, conditioned on survival of both populations

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{1}_{\{\xi^{(1)}_n(0)\,\xi^{(2)}_n(0)>0\}}>0\quad a.s.,$$

i.e. we have local coexistence.

To prove this proposition one can essentially use the same argument as we have used in the proof of Lemma 7 to reduce the case $\kappa>0$ to the case $\kappa=0$.

4 Results for the deterministic system

In this section we will prove Proposition 2. For clarity of exposition, we start with the 'diagonal case' $\kappa=0$. Let us consider more generally a coupled map lattice (ζ_n) on $\mathbb{Z}^d$, defined via

$$\zeta_{n+1}(x) = \sum_{y \in \mathbb{Z}^d} g\big(\zeta_n(y)\big) p_{yx}, \quad x \in \mathbb{Z}^d, \tag{21}$$

where $(p_{yx})_{x,y\in\mathbb{Z}^d}$ is a translation invariant stochastic kernel with finite range satisfying (A1) and $g : [0, G] \to [0, G]$ is a continuously differentiable function. We think of the single site function g as having 0 as a repelling fixed point and and another stable fixed point $\bar{a} \in (0, G]$ which attracts $(0, G]$, i.e. for any $x_0 \in (0, G]$, the sequence (x_n) defined through $x_{n+1} = g(x_n)$ converges to $\bar{a}$. (Thus in particular $g'(0) > 1$, $g(G) > 0$.) Then obviously $\zeta \equiv 0$ and $\zeta \equiv \bar{a}$ are fixed points of (21), and one is strongly inclined to believe that in this well-behaved scenario there are no others. We will say that a dynamical system (η_n) on $\mathbb{Z}^d$ *converges locally* to $a \in \mathbb{R}$ if for each finite $\Lambda \subset \mathbb{Z}^d$ and each $\varepsilon > 0$ there exists N_0 such that

$$|\eta_n(x) - a| \leq \varepsilon \quad \text{for all } x \in \Lambda \text{ and } n \geq N_0.$$

Having been unable to find the result we need in the literature, we provide Lemma 9 below. Assume

(DS1) For each $a > 0$ there exist sequences (α_n) and (β_n) such that $0 < \alpha_0 \leq a$, $\beta_0 = G$, $\alpha_n \uparrow \bar{a}$, $\beta_n \downarrow \bar{a}$ and $g([\alpha_n, \beta_n]) \subset [\alpha_{n+1}, \beta_{n+1}]$.

Note that this implies

(DS2) There exists $a \in (0, \bar{a})$ with the following property:
If $\zeta_0(0) \in [a, G]$ then there is $N_0 \in \mathbb{N}$ such that $\zeta_{N_0}(x) \in [a, G]$, $\|x\|_\infty \leq 1$.

A proof that (DS1) $\Rightarrow$ (DS2) is basically a reformulation of the proof of Lemma 7. Note that (DS1) holds true e.g. if we assume additionally that g is concave (see e.g. the construction given in Lemma 12). We refrain from pursuing the most general conditions for (DS1), but observe that this together with Lemma 9 already yields a proof of Proposition 2 in the diagonal case $\kappa = 0$.

Lemma 9. *If $\zeta_0(x) \in (0, G]$ for some $x \in \mathbb{Z}^d$ and* (DS1) *holds then (ζ_n) converges locally to $\bar{a}$.*

In the following we will call the set $\mathcal{N}_k(A) := \{x \in \mathbb{Z}^d : \inf_{y\in A} \|x - y\|_\infty \leq k\}$ the k-neighbourhood of A. If $A = \{x\}$, then we write $\mathcal{N}_k(x)$ for the k-neighbourhood of x.

Proof. Let Λ be a finite subset of $\mathbb{Z}^d$. We may assume that Λ is a ball with respect to the sup norm. Let (α_n) and (β_n) be sequences from (DS1). Given $\varepsilon > 0$, we choose n_0 such that $\beta_n - \alpha_n < \varepsilon$ holds for all $n \geq n_0$. According to (DS2) there exist $a \in (0, \bar{a})$ and $n_1 \in \mathbb{N}$ such that

$$\zeta_n(x) \geq a \Rightarrow \zeta_{n+n_1}(y) \geq a \quad \text{for all } y \text{ with } \|x - y\|_\infty \leq 1.$$

Since 0 and $\bar{a}$ are the only fixed points of g, $g'(0) > 1$ and $a \in (0, \bar{a}]$ we have $g(a) \geq a$. It follows that if for all y in the R_p-neighbourhood of some point x we have $\zeta_0(y) \geq a$, then

$$\zeta_1(x) = \sum_y g\big(\zeta_0(y)\big) p_{yx} \geq a.$$

We set

$$\Lambda' := \mathcal{N}_{R_p(n_0+n_1)}(\Lambda) \quad \text{and} \quad \Lambda_i = \mathcal{N}_{R_p(n_0-i)}(\Lambda), \quad i \in \{0, \dots, n_0\}.$$

Note that $\Lambda_i = \mathcal{N}_{R_p}(\Lambda_{i+1})$ and $\Lambda_{n_0} = \Lambda$. By (DS2) there is some time point $n_2 \in \mathbb{N}$ such that $\zeta_{n_2}(x) \geq a$ for all $x \in \Lambda'$. We claim that $\zeta_{n_2+n}(x) \geq a$ for all $x \in \Lambda_0$ and all $n \geq 0$. Indeed, during the next $n_1 - 1$ steps from time n_2 on, the mass in the points of the R_p-neighbourhood of Λ_0 remain bounded from below by a. According to (DS2) by the time $n_2 + n_1$ each point in the 1-neighbourhood of Λ' is bounded below by a. Hence we are in particular again in the above situation.

For simplicity of notation we assume that $\zeta_0(x) \geq a$ for all $x \in \Lambda'$. We need to show that $\zeta_n(x) \in [\alpha_{n_0}, \beta_{n_0}]$ for all $x \in \Lambda$ and $n \geq n_0$. First, we check inductively that for $n = 0, 1, \dots, n_0$ we have

$$\begin{array}{rl} i) & \zeta_n(x) \in [\alpha_n, \beta_n] \quad \text{for } x \in \Lambda_n, \\ ii) & \zeta_k(x) \in [\alpha_k, \beta_k] \quad \text{for } x \in \Lambda_k \setminus \Lambda_{k+1},\ k = 0, 1, \dots, n-1. \end{array} \tag{22}$$

For $n = 0$, *i)* is true by assumption, and *ii)* is void. Assume that *i)* and *ii)* hold true for some $n < n_0$, let $k \in \{0, 1 \dots, n+1\}$ and $x \in \Lambda_k \setminus \Lambda_{k-1}$, resp. $x \in \Lambda_{n+1}$ if $k = n+1$. As $\Lambda_{k-1} = \mathcal{N}_{R_p}(\Lambda_k)$ we have

$$\zeta_{n+1}(x) = \sum_y g\big(\zeta_n(y)\big) p_{yx} = \sum_{y \in \Lambda_{k-1}} g\big(\zeta_n(y)\big) p_{yx} \in [\alpha_k, \beta_k]$$

by (DS1), proving *i)* and *ii)* for $n + 1$.

To conclude the proof note that by the argument above, the set of configurations ζ such that

$$\begin{array}{l} \zeta(x) \geq a \ \text{ for } x \in \Lambda', \quad \zeta(x) \in [\alpha_k, \beta_k] \quad \text{for } x \in \Lambda_k \setminus \Lambda_{k+1},\ k = 0, 1, \dots, n_0 - 1, \\ \zeta(x) \in [\alpha_{n_0}, \beta_{n_0}] \quad \text{for } x \in \Lambda_{n_0} \end{array}$$

is invariant under the dynamics (21), hence we have in particular for $n \geq n_0$

$$\zeta_n(x) \in [\alpha_{n_0}, \beta_{n_0}] \quad \text{for } x \in \Lambda\ \big(= \Lambda_{n_0}\big).$$

□

For the 'non-diagonal' case $\kappa > 0$ we need three more lemmas. Note that we only need to consider the case $\lambda_0 = 1$. Otherwise consider $\tilde{\zeta}$ defined by $\tilde{\zeta}_n(x) = \lambda_0 \zeta_n(x)$, which solves the iteration given by (9) and (10) with λ_0 replaced by 1 and κ by κ/λ_0. Until the end of this section we write $\bar{m}_{1,\kappa} = \bar{m}(1, \kappa)$, $m^* = m_1^* = m^2/4$ and $\bar{m} = \bar{m}_1 = m - 1$ (see (11)).

Lemma 10. *There exist positive κ^* and δ such that for $\kappa \leq \kappa^*$ exist sequences (α_n), (β_n) in $[\bar{m}_{1,0} - \delta, \bar{m}_{1,0} + \delta]$ satisfying*

1. $a_n \uparrow \bar{m}_{1,\kappa}$, $\beta_n \downarrow \bar{m}_{1,\kappa}$;
2. *If $\zeta(y) \in [\alpha_n, \beta_n]$ for all $y \in \mathcal{N}_{R_\lambda}(x)$, then $f_\kappa(x; \zeta) \in [\alpha_{n+1}, \beta_{n+1}]$.*

Proof. For fixed $x \in \mathbb{Z}^d$ we may consider the mapping $\zeta \mapsto f_\kappa(x;\zeta)$ as a function of the restriction of ζ to the R_λ-neighbourhood of x (viewed as an element of $\mathbb{R}^k$ where k is the number of points in $\mathcal{N}_{R_\lambda}(x)$). We denote by $\vec{m}_{1,\kappa}$ the vector of length k with all entries equal to $\bar{m}_{1,\kappa}$ and by $B_\delta(\vec{m}_{1,\kappa})$ the δ-neighbourhood of $\vec{m}_{1,\kappa}$ with respect to sup norm.

The gradient of $\zeta \mapsto f_\kappa(x;\zeta)$ is given by (we assume that the positive part appearing in (9) is not 0)

$$\begin{aligned}\partial_{\zeta(x)} f_\kappa(x;\zeta) &= m - 2\zeta(x) - \kappa \sum_{y \neq x} \gamma_{xy}\zeta(y)\\ \partial_{\zeta(y)} f_\kappa(x;\zeta) &= -\kappa\gamma_{xy}\zeta(x) \quad \text{for } y \neq x.\end{aligned}$$

Choose positive ε, δ and κ^* satisfying

$$(|m-2| + 2\delta + \kappa^*(\delta + m - 1))^2 < 1 - \varepsilon \quad \text{and} \quad \kappa^* < \min\Big\{\frac{\delta}{m-1}, \frac{\sqrt{\varepsilon}}{\sqrt{2}(2+\delta)}\Big\}. \tag{23}$$

For $\zeta \in B_\delta(\vec{m}_{1,0})$ we have

$$\partial_{\zeta(x)} f_\kappa(x;\zeta) \leq m - 2(\bar{m}_{1,0} - \delta) - \kappa \sum_{y \neq x} \gamma_{xy}(\bar{m}_{1,0} - \delta) = 2 - m + 2\delta - \kappa(m-1) + \kappa\delta$$

and

$$\partial_{\zeta(x)} f_\kappa(x;\zeta) \geq m - 2(\bar{m}_{1,0} + \delta) - \kappa \sum_{y \neq x} \gamma_{xy}(\bar{m}_{1,0} + \delta) = 2 - m - 2\delta - \kappa(m-1) - \kappa\delta,$$

hence

$$|\partial_{\zeta(x)} f_\kappa(x;\zeta)| \leq |m-2| + (m-1)\kappa + 2\delta + \kappa\delta$$

and due to (23) we obtain for $\kappa \leq \kappa^*$

$$\big(\partial_{\zeta(x)} f_\kappa(x;\zeta)\big)^2 < 1 - \varepsilon. \tag{24}$$

For $y \neq x$ we have

$$\begin{aligned}|\partial_{\zeta(y)} f_\kappa(y;\zeta)| &= \kappa\gamma_{xy}\zeta(x) \leq \kappa\gamma_{xy}(\bar{m}_0 + \delta)\\ &\leq \kappa\gamma_{xy}(m-1) + \delta\gamma_{xy}\kappa < \kappa\gamma_{xy}(2+\delta).\end{aligned}$$

Consequently

$$\sum_{y \neq x} \big(\partial_{\zeta(y)} f_\kappa(y;\zeta)\big)^2 < (2+\delta)^2\kappa^2 \sum_{x \neq y} \gamma_{xy}^2 \leq (2+\delta)^2\kappa^2 < \frac{\varepsilon}{2}, \tag{25}$$

where the last inequality holds if (23) is satisfied.

Altogether, the above implies that for all $\zeta \in B_\delta(\vec{m}_{1,\kappa})$ and $\kappa \leq \kappa^*$ we have

$$\|\nabla f_\kappa(x;\zeta)\|_2^2 < 1 - \frac{\varepsilon}{2}. \tag{26}$$

Due to the mean value theorem for all $\zeta, \zeta' \in B_\delta(\vec{m}_{1,0})$ exists $\tilde{\zeta} \in B_\delta(\vec{m}_{1,0})$ such that

$$\begin{aligned}|f_\kappa(x;\zeta) - f_\kappa(x;\zeta')| &= |\nabla f_\kappa(x;\tilde{\zeta})(\zeta - \zeta')| \\ &\le \|\nabla f_\kappa(x;\tilde{\zeta})\|_2 \cdot \|\zeta - \zeta'\|_2 \le c\|\zeta - \zeta'\|_2,\end{aligned}$$

where $c = \sqrt{1 - \varepsilon/2} < 1$. Thus, the claim of the lemma follows. We only need to note that $f_\kappa\big(x; \vec{m}_{1,\kappa}\big) = m_{1,\kappa}$ and that $|\bar{m}_{1,0} - \bar{m}_{1,\kappa}| < \delta$ if $\kappa < \delta/(m-1)$ which holds by (23). ☐

Lemma 11. *For each $\delta > 0$ exists $\kappa^* > 0$ such that whenever $\kappa \le \kappa^*$ and $f_\kappa(x;\zeta_0) > 0$ for some $x \in \mathbb{Z}^d$, the following holds: For each finite $\Lambda \subset \mathbb{Z}^d$ there exists $N \in \mathbb{N}$ such that $\zeta_n(x) \in [\bar{m}_{1,0} - \delta, \bar{m}_{1,0} + \delta]$ for all $x \in \Lambda$ and all $n \ge N$.*

Proof. Recall our assumption $\lambda_0 = 1$, which implies $M_{\lambda_0} = m$. For all $x \in \mathbb{Z}^d$, $\zeta \in [0,m]^{\mathbb{Z}^d}$ and $\tilde{\delta} > 0$ we have

$$\kappa \sum_{z \ne x} \gamma_{xz} \zeta(z) \le m\kappa.$$

That implies

$$f_{\kappa,l}(\zeta(x)) := \zeta(x)(m - m\kappa - \zeta(x)) \le f_\kappa(x;\zeta) \le \zeta(x)(m - \zeta(x)) =: f_u(\zeta(x)).$$

The non-zero fixed points of $f_{\kappa,l}$ and f_u are respectively $\bar{m}_l = m - m\kappa - 1$ and $\bar{m}_{1,0}$. Furthermore if $m\kappa < \delta$ then $\bar{m}_{1,0} - \bar{m}_l < \delta$.

According to Lemma 7 there is $n_1 \in \mathbb{N}$ and $a > 0$ with the property

$$\zeta_n(x) \ge a \Rightarrow \zeta_{n+n_1}(y) \ge a, \quad \|x - y\|_\infty \le 1. \tag{27}$$

Thus, for each finite $\Lambda' \subset \mathbb{Z}^d$ there is $n_2 \in \mathbb{N}$ such that $\zeta_{n_2}(x) \ge a$ for all $x \in \Lambda'$.

According to Lemma 12 for each $\delta > 0$ one can choose κ^* and sequences (a_n) and (b_n) such that for all $\kappa \le \kappa^*$ the following holds

$$\begin{gathered} a_0 \le a \\ f_{\kappa,l}([a_n, b_n]), f_u([a_n, b_n]) \subset [a_{n+1}, b_{n+1}] \\ \text{for some } n_0 \in \mathbb{N}: \quad a_n, b_n \in [\bar{m}_0 - \delta, \bar{m}_0 + \delta] \text{ for all } n \ge n_0. \end{gathered}$$

A construction analogous to the proof of Lemma 9 concludes the proof. ☐

The following lemma is a deterministic ingredient in our construction (see (DS1)), providing a shrinking sequence of intervals which the one-point iteration maps into themselves. Having been unable to find a proof in the literature, we provide one here. The property in question will hold for a concave f with 0 a repelling and another attracting fixed point and does not depend on the particular functional form of f. On the other hand, as we also need to consider a slightly perturbed version f_δ (where in our case the perturbation is of a particular functional form), we refrain from generality and stick to $f_\delta, f : [0,m] \to [0, m^*]$,

$$f_\delta(x) = x(m - \delta - x)^+, \quad f(x) = x(m - x)^+, \tag{28}$$

where $m^* = m^2/4 = \max f = f(m/2)$. Recall that $\bar{m} = m - 1$, $\bar{m}_\delta = m - \delta - 1$ are the (unique) attracting fixed points of f resp. f_δ (we think of small δ).

Lemma 12. *Let $m \in (1,3)$, consider f, f_δ as defined in (28). For each $\varepsilon > 0$ one can choose positive γ and $\tilde{\varepsilon}$, a strictly increasing sequence (α_n), and a strictly decreasing sequence (β_n) with the following properties:*

- *(A) There exists $N_0 \in \mathbb{N}$ s.t. $\alpha_n, \beta_n \in [\bar{m} - \varepsilon, \bar{m} + \varepsilon]$ for all $n \geq N_0$.*
- *(B) For all $n \leq N_0$ and $0 \leq \delta \leq \gamma$: $f_\delta([\alpha_n, \beta_n]), f([\alpha_n, \beta_n]) \subset \left[\frac{\alpha_{n+1}}{1-\tilde{\varepsilon}}, \frac{\beta_{n+1}}{1+\tilde{\varepsilon}}\right]$.*

Furthermore $\alpha_0 > 0$ can be chosen arbitrarily small and $\beta_0 < m$ can be chosen arbitrarily close to m.

Proof. We wish to construct the sequences (α_n) and (β_n) in such a way that

$$\alpha_n < \alpha_{n+1} < \bar{m}_\gamma \leq \bar{m} < \beta_{n+1} < \beta_n \quad \text{and} \tag{29}$$

$$f_\gamma([\alpha_n, \beta_n]), f([\alpha_n, \beta_n]) \subset (\alpha_{n+1}, \beta_{n+1}) \tag{30}$$

for all n. This together with

$$\bar{m} - \varepsilon < \lim_{n\to\infty} \alpha_n \leq \lim_{n\to\infty} \beta_n < \bar{m} + \varepsilon \tag{31}$$

will suffice to conclude, as $f_\gamma(x) \leq f_\delta(x) \leq f(x)$ for $0 \leq \delta \leq \gamma$ and (30) implies (B) for each finite N_0 and sufficiently small $\tilde{\varepsilon}$. The construction is slightly different depending on whether the slope of f at its attractive fixed point $\bar{m}$ is $\in (0,1)$, $= 0$ or $\in (-1, 0)$, thus we consider the cases $m \in (1,2)$, $m = 2$ and $m \in (2,3)$ separately.

Let $m \in (1,2)$, choose $\gamma \in (0, \varepsilon)$ s.t. $m - \gamma \in (1,2)$. Take arbitrary $\alpha_0 \in (0, \bar{m} - \gamma)$ and $\beta_0 > m/2$ s.t. $f_\gamma(\beta_0) \geq f_\gamma(\alpha_0)$. This guarantees $f([\alpha_0, \beta_0]), f_\gamma([\alpha_0, \beta_0]) \subset [f_\gamma(\alpha_0), m^*]$. Define

$$\alpha_{n+1} = \frac{\alpha_n + f_\gamma(\alpha_n)}{2}, \quad n \geq 0,$$

$$\beta_1 = \frac{m^* + \frac{m}{2}}{2} \quad \text{and} \quad \beta_{n+1} = \frac{f(\beta_n) + \beta_n}{2}, \quad n \geq 1.$$

Note that $m^* < m/2$ in the case considered, so the choice of β_1 ensures (30) for $n = 0$ and that f, f_γ are increasing on $[\alpha_0, \beta_1]$. As $f_\gamma(x) > x$ on $(0, \bar{m}_\gamma)$ and $f'_\gamma(\bar{m}_\gamma) \geq 0$, we have $\alpha_n < \alpha_{n+1} < f_\gamma(\alpha_n)$ for $n \geq 1$. Thus $\alpha_n \nearrow \bar{m}_\gamma$. Similarly, observing that $x > f(x) \geq \bar{m}$ for $x \in (\bar{m}, m/2)$, we have $\beta_n > \beta_{n+1} > f_\gamma(\beta_n)$ for $n \geq 1$, hence $\beta_n \searrow \bar{m}$. This proves (29), (30) and (31) in this case.

Let $m = 2$. In this case $f(m/2) = m^*$, so the values of $f(\beta_n)$ cannot be decreasing, and we modify the construction as follows: Choose $0 < \gamma < \varepsilon$. Pick $\alpha_0 \in (0, \gamma)$, define

$$\alpha_{n+1} = \frac{f_\delta(\alpha_n) + \alpha_n}{2}, \quad \beta_n = \frac{2-\gamma}{2} + \sqrt{(2-\gamma)^2/4 - f_\gamma(\alpha_n)}, \ n = 0, 1, \ldots$$

As above, we have $\alpha_n \nearrow \bar{m}_\gamma = 2 - \gamma$. Note that β_n is the larger root of $f_\gamma(x) = f_\gamma(\alpha_n)$, and that the solutions of $f_\gamma(x) = \bar{m}_\gamma$ are $\bar{m}_\gamma = 1 - \gamma$ and 1 in the case $m = 2$, so that so that $f(\beta_n) \geq f_\gamma(\beta_n) > \alpha_{n+1}$ and $\beta_n \searrow 1$. Hence (29), (30) and (31) are satisfied.

Finally, let $m \in (2,3)$. Here, as $\bar{m} > m/2$, we need to observe that $f([\alpha_n, \beta_n])$ will contain m^* as long as $\alpha_n \leq m/2$, so β_n must not decrease too quickly. Furthermore, as $f'(\bar{m}) < 0$, once we come close to $\bar{m}$, the roles of the lower and upper boundary are interchanged in each step.

Choose $\gamma > 0$ s.t. $m - \gamma \in (2,3)$ and $\bar{m}_\gamma > f_\gamma(m^*) > m/2$. Pick $\alpha_0 \in (0,(m-\gamma)/2)$. While $(\alpha_n + f_\gamma(\alpha_n))/2 \le m/2$ we set

$$\alpha_{n+1} = \frac{\alpha_n + f_\gamma(\alpha_n)}{2}.$$

Let n_0 be the smallest integer satisfying $(\alpha_{n_0} + f_\gamma(\alpha_{n_0}))/2 > m/2$. We set

$$\alpha_{n_0+1} = \frac{\alpha_{n_0} + f_\gamma(\alpha_{n_0})}{2} \wedge \frac{1}{2}\Big(\frac{m}{2} + f_\gamma(m^*)\Big).$$

Now we choose $\beta_0, \dots, \beta_{n_0}$ s.t. $m^* < \beta_i < \beta_{i-1} < m$ and $f_\gamma(\beta_i) > \alpha_{i+1}$, $i = 1, \dots, n_0$. Note that this is possible because $f_\gamma(m^*) > m/2$. Put $\beta_{n_0+1} = (\beta_{n_0} + m^*)/2$.

Let us check (29) and (30) for $n \le n_0$: as $f_\gamma(x) > x$ for $x \in (0, \bar{m}_\gamma)$ and $f_\gamma(m^*) < \bar{m}_\gamma$, the sequence $(\alpha_n)_{n\in\{0,\dots,n_0+1\}}$ is strictly increasing. $(\beta_n)_{n\in\{0,\dots,n_0+1\}}$ is strictly decreasing by construction. By definition we have

$$f_\gamma(\alpha_n) \ge 2\alpha_{n+1} - \alpha_n > \alpha_{n+1}.$$

Note that while $\alpha_n \le m/2$, i.e. $n \le n_0$ we always have

$$f_\gamma([\alpha_n, \beta_n]), f([\alpha_n, \beta_n]) \subset (\alpha_{n+1}, m^*] \subset (\alpha_{n+1}, \beta_{n+1}).$$

For $n \ge n_0 + 1$ define

$$\alpha_{n+1} = \frac{1}{2}(f_\gamma(\beta_n) + \alpha_n), \quad \beta_{n+1} = \frac{1}{2}(\beta_n + f(\alpha_n)). \tag{32}$$

In order to verify (29) and (30) for $n \ge n_0 + 1$, consider

$$a \in (\frac{m}{2}, \bar{m}_\gamma),\ b \in (\bar{m}, m) \quad \text{satisfying} \quad f(a) < b,\ f_\gamma(b) > a. \tag{33}$$

Note that then

$$a' = \frac{1}{2}(a + f_\gamma(b)) \quad \text{and} \quad b' = \frac{1}{2}(b + f(a))$$

fulfil

$$a' \in (a, \bar{m}_\gamma),\ b' \in (\bar{m}, b) \quad \text{and} \quad f(a') < b',\ f_\gamma(b') > a'.$$

Indeed, by assumption we have $f_\gamma(b) > a$, so $a' > a$. On the other hand $f_\gamma(b) < \bar{m}_\gamma$ because f_γ is decreasing in $[\bar{m}_\gamma, m]$ and $b > \bar{m}_\gamma = f_\gamma(\bar{m}_\gamma)$. As f is decreasing in the considered region, we have

$$f(a') < f(a) < \frac{1}{2}(b + f(a)) = b'.$$

Similarly, $b' \in (\bar{m}, b)$ and $f_\gamma(b') > a'$.

Obviously $a = \alpha_{n_0+1}$ and $b = \beta_{n_0+1}$ satisfy the condition (33), hence (29) and (30) hold true for $n > n_0$ as well.

By the above construction, $\alpha_n \nearrow \alpha \in (m/2, \bar{m}_\gamma]$, $\beta_n \searrow \beta \in [\bar{m}, m^*)$, where (α, β) solves $f(\alpha) = \beta$, $f_\gamma(\beta) = \alpha$. For $\gamma = 0$, the unique solution would be $\alpha = \beta = \bar{m}$, for γ sufficiently small, we have (31). $\square$

Proof. **(Proposition 2)** Let Λ be a finite ball in $\mathbb{Z}^d$ and $\varepsilon > 0$. Let $n_1 \in \mathbb{N}$ be such that (27) is fulfilled. For the sequences (α_n), (β_n) from Lemma 10 choose n_0 s.t. $\beta_n - \alpha_n \le \varepsilon$ for all $n \ge n_0$. Define Λ' and $\Lambda_0, \dots, \Lambda_{n_0}$ through

$$\Lambda' = \mathcal{N}_{(n_0+n_1)(R_\lambda+R_p)}(\Lambda) \quad \text{and} \quad \Lambda_i = \mathcal{N}_{(n_0-i)(R_\lambda+R_p)}(\Lambda),\ i \in \{0, \dots, n_0\}.$$

According to Lemma 11 there exists $n_2 \in \mathbb{N}$ such that $\zeta_n(x) \in [\bar{m}_{1,0} - \delta, \bar{m}_{1,0} + \delta]$ for all $x \in \Lambda_0$ and $n \ge n_2$. Then, for simplicity of notation we may assume $n_2 = 0$. Now the rest of the proof is a reproduction of the arguments from the proof of Lemma 9. □

5 Coupling

In this section we prove Proposition 3 and Corollary 4. Let us first describe the idea behind the successful coupling. Recall in (7) the definition of the processes $\xi^{(1)}$ and $\xi^{(2)}$. Consider three large (but finite) boxes $B_1 \subset B_2 \subset B_3 \subset \mathbb{Z}^d$ and assume that $\xi^{(1)}$ and $\xi^{(2)}$ agree on B_1 with values close to $\bar{m}_{\lambda_0}$, that they are close to $\bar{m}_{\lambda_0}$ but do not necessarily agree on B_2, and that on B_3 all sites are occupied in both systems. In view of Lemma 7 we expect that the region of sites which are occupied in both systems grows. If the competition is not too strong the random system 'follows closely' the deterministic one. Thus, in view of Proposition 2 we can hope that the region where both systems are close to the deterministic equilibrium $\bar{m}_{\lambda_0}$ is growing as well. Finally there is a chance that Poisson variables whose means are close to each other produce the same realization. Therefore there is also hope that the region where both systems are the same grows too.

Thus, for suitably tuned parameters we expect that with high probability the above situation will reproduce itself after some time on larger Boxes $B_1' \subset B_2' \subset B_3'$. As before this observation lends itself to a comparison with finite range dependent percolation on a coarse grained space-time grid. A certain subtlety stems from the problem that the coarse graining must be chosen depending on λ_0 in such a way that the dependence range of the percolation does not diverge when taking λ_0 small.

For $k, l \in \mathbb{N}$ we set $A_k = \mathcal{N}_{k(R_\lambda+R_p+1)}(0)$ and $A_{k,l} = \mathcal{N}_{k(R_\lambda+R_p+1)+l}(0)$. Let $\mathcal{X}(y,n)$, $(y,n) \in \mathbb{Z}^d \times \mathbb{Z}_+$ be the event that for some $N \in \mathbb{N}$, to be chosen later, the following holds

$$\begin{aligned} \xi_n^{(1)}(x) = \xi_n^{(2)}(x) \in \Big[\frac{m-1-\delta}{\lambda_0}, \frac{m-1+\delta}{\lambda_0}\Big] &=: I(m,\delta,\lambda_0) \quad \text{for all } x \in y + A_N \\ \xi_n^{(1)}(x), \xi_n^{(2)}(x) \in I(m,\delta,\lambda_0) &\quad \text{for all } x \in y + A_{4N} \setminus A_N \\ \xi_n^{(1)}(x), \xi_n^{(2)}(x) \in \big[\varepsilon_1 \bar{m}_{\lambda_0}, (1-\varepsilon_2) M_{\lambda_0}\big] &=: J(m,\lambda_0) \quad \text{for all } x \in y + A_{7N} \setminus A_{4N}. \end{aligned} \tag{34}$$

Our goal is to show that the process $\mathbb{1}_{\mathcal{X}(y,n)}$ dominates oriented independent percolation on a suitable sub-grid of $\mathbb{Z}^d \times \mathbb{Z}_+$. The main part of the proof is carried out in Lemma 13 below. With this lemma one can use the Liggett, Schonmann and Stacey argument as we have done in the proof of Theorem 1.

Let n^* be as defined in (13) and note that this number only depends on m and on the kernel p. As we will later choose N large, we will be able to choose it as a multiple of n^*. In the sequel we will assume that N/n^* is an integer.

Lemma 13. *For $m \in (1,3)$, p as in assumption (A1) and $\tilde{\varepsilon} > 0$ there exist $\lambda_0^*, \kappa^* > 0$ such that for each $\lambda_0 \leq \lambda_0^*$, $\kappa \leq \kappa^* \lambda_0$ one can choose N such that*

$$\mathbb{P}\big[\mathcal{X}(y, n+N) \text{ for all } y \text{ with } \|x-y\|_\infty \leq N/n^* \,\big|\mathcal{X}(x,n)\big] \geq 1 - \tilde{\varepsilon} \tag{35}$$

holds for all $x \in \mathbb{Z}^d$.

Proof. Let $m \in (1,3)$ and $\tilde{\varepsilon} > 0$ be given. Due to translation invariance and the Markov property the left hand side in (35) does not depend on (x,n). Thus, it is enough to prove

$$\mathbb{P}\big[\mathcal{X}(y, N) \text{ for all } y \text{ with } \|y\|_\infty \leq N/n^* \,\big|\mathcal{X}(0,0)\big] \geq 1 - \tilde{\varepsilon}. \tag{36}$$

Choose positive ε, δ and κ^* satisfying

$$|m-2| + 2\delta + \kappa^*(\delta + m - 1) < 1 - \varepsilon \quad \text{and} \quad \kappa^* < \min\Big\{\frac{\delta}{m-1}, \frac{\varepsilon}{2(2+\delta)}\Big\}. \tag{37}$$

These constants also satisfy (23). Thus the properties of f_κ (see (9)), proven in Lemma 10, are preserved. Note that unlike the situation in Lemma 10 we do not set $\lambda_0 = 1$ here. Furthermore, similar to (24), (25) and (26) we obtain

$$\|\nabla f_\kappa(x;\zeta)\|_1 \leq 1 - \frac{\varepsilon}{2}, \quad \text{if } \zeta(y) \in \Big[\frac{m-1-\delta}{\lambda_0}, \frac{m-1+\delta}{\lambda_0}\Big] \text{ for all } y \in \mathcal{N}_{R_\lambda}(x). \tag{38}$$

We choose k_0 such that for all $k \geq k_0$ we have

$$\frac{|A_{k+1}|}{|A_k|}\Big(1 - \frac{\varepsilon}{2}\Big) \leq \frac{|A_{k_0+1}|}{|A_{k_0}|}\Big(1 - \frac{\varepsilon}{2}\Big) =: c(\varepsilon) < 1. \tag{39}$$

We will assume that $N \geq k_0$. We set $\mathcal{X}_0 = \mathcal{X}(0,0)$ and $\mathcal{X}_N = \mathcal{X}_{N,1} \cap \mathcal{X}_{N,2} \cap \mathcal{X}_{N,3}$, where

$$\begin{aligned}
\mathcal{X}_{N,1} &= \big\{\xi_N^{(1)}(x) = \xi_N^{(2)}(x) \in I(m,\delta,\lambda_0) \quad \text{for all } x \in A_{3N}\big\} \\
\mathcal{X}_{N,2} &= \big\{\xi_N^{(1)}(x), \xi_N^{(2)}(x) \in I(m,\delta,\lambda_0) \quad \text{for all } x \in A_{6N} \setminus A_{3N}\big\} \\
\mathcal{X}_{N,3} &= \big\{\xi_N^{(1)}(x), \xi_N^{(2)}(x) \in J(m,\lambda_0) \quad \text{for all } x \in A_{7N,N/n^*} \setminus A_{6N}\big\}.
\end{aligned}$$

Furthermore we define for each $n \leq N$ the event Ψ_n by

$$\Psi_n = \Big\{\forall (x,k) \in \bigcup_{j=1}^{n} A_{4(N-j)} \times \{j\} : \xi_k^{(1)}(x), \xi_k^{(2)}(x) \in I(m,\delta,\lambda_0)\Big\}.$$

As $\mathcal{X}_N$ implies that $\mathcal{X}(y,N)$ holds for all y with $\|y\|_\infty \leq N/n^*$, $\mathbb{P}\big[\mathcal{X}_N \big| \mathcal{X}_0\big]$ is a lower bound for the left hand side of (36). Therefore it suffices to show $\mathbb{P}\big[\mathcal{X}_N^c \big| \mathcal{X}_0\big] \leq \tilde{\varepsilon}$. Because

$$\mathbb{P}\big[\mathcal{X}_N^c \big| \mathcal{X}_0\big] \leq \mathbb{P}\big[\mathcal{X}_{N,1}^c \cap \Psi_N \big| \mathcal{X}_0\big] + \mathbb{P}\big[\mathcal{X}_{N,2}^c \cap \Psi_N \big| \mathcal{X}_0\big] + \mathbb{P}\big[\mathcal{X}_{N,3}^c \big| \mathcal{X}_0\big] + \mathbb{P}\big[\Psi_N^c \big| \mathcal{X}_0\big], \tag{40}$$

it suffices to estimate each of the summands. To do this we will repeatedly use large deviation estimates for Poisson random variables. There are constants c_1 and δ_1 such that

$$\mathbb{P}\big[\Psi_N^c \big| \mathcal{X}_0\big] \leq N|A_{4N}| \exp\Big(-\frac{c_1 \delta_1^2}{\lambda_0}\Big). \tag{41}$$

Now let us consider the first term on the right hand side of (40). We have

$$\begin{aligned}
&\frac{1}{|A_{3N}|}\sum_{x\in A_{3N}}\mathbb{E}[|\xi_N^{(1)}(x)-\xi_N^{(2)}(x)|\mathbb{1}_{\Psi_N}|\mathcal{F}_{N-1}]\\
&\qquad\le \mathbb{1}_{\Psi_{N-1}}\frac{1}{|A_{3N}|}\sum_{x\in A_{3N}}\mathbb{E}[|\xi_N^{(1)}(x)-\xi_N^{(2)}(x)||\mathcal{F}_{N-1}]\\
&\qquad\le \mathbb{1}_{\Psi_{N-1}}\frac{1}{|A_{3N}|}\sum_{x\in A_{3N}}\sum_{y\in\mathcal{N}_{R_p}(x)}p_{yx}\sum_{z\in\mathcal{N}_{R_\lambda}(y)}|\nabla_z f_\kappa(y;\tilde{\xi})||\xi_{N-1}^{(1)}(z)-\xi_{N-1}^{(2)}(z)|\\
&\qquad\le \mathbb{1}_{\Psi_{N-1}}\sum_{z\in A_{3N+1}}|\xi_{N-1}^{(1)}(z)-\xi_{N-1}^{(2)}(z)|\frac{1}{|A_{3N}|}\sum_{y\in\mathcal{N}_{R_\lambda}(z)}|\nabla_z f_\kappa(y;\tilde{\xi})|\sum_{x\in A_{3N}}p_{xy}\\
&\qquad\le \mathbb{1}_{\Psi_{N-1}}\frac{|A_{3N+1}|}{|A_{3N}|}\Big(1-\frac{\varepsilon}{2}\Big)\frac{1}{|A_{3N+1}|}\sum_{z\in A_{3N+1}}|\xi_{N-1}^{(1)}(z)-\xi_{N-1}^{(2)}(z)|\\
&\qquad\le \mathbb{1}_{\Psi_{N-1}}c(\varepsilon)\frac{1}{|A_{3N+1}|}\sum_{z\in A_{3N+1}}|\xi_{N-1}^{(1)}(z)-\xi_{N-1}^{(2)}(z)|.
\end{aligned}$$

We can iterate the above argument to obtain on $\mathcal{X}_0$

$$\begin{aligned}
&\frac{1}{|A_{3N}|}\sum_{x\in A_{3N}}\mathbb{E}[|\xi_N^{(1)}(x)-\xi_N^{(2)}(x)|\mathbb{1}_{\Psi_N}|\mathcal{F}_0]\le c(\varepsilon)^N\frac{1}{|A_{4N}|}\sum_{z\in A_{4N}}|\xi_0^{(1)}(z)-\xi_0^{(2)}(z)|\\
&\qquad\le c(\varepsilon)^N\frac{1}{|A_{4N}|}\sum_{z\in A_{4N}\setminus A_N}2\delta\bar{m}_{\lambda_0}=c(\varepsilon)^N\frac{|A_{4N}\setminus A_N|}{|A_{4N}|}2\delta\bar{m}_{\lambda_0}\le c(\varepsilon)^N 2\delta\bar{m}_{\lambda_0}
\end{aligned}\tag{42}$$

From this we obtain

$$\mathbb{P}\big[\mathcal{X}_{1,N}^c\cap\Psi_N\big|\mathcal{X}_0\big]\le\sum_{x\in A_{3N}}\mathbb{E}\big[|\xi_N^{(1)}(x)-\xi_N^{(2)}(x)|\mathbb{1}_{\Psi_N}\big]\le c(\varepsilon)^N 2\bar{m}_{\lambda_0}\delta|A_{3N}|.\tag{43}$$

Note that on $\mathcal{X}_0$ for all $|x|\le R_\lambda+R_p$ we have $\xi_n^{(1)}(x)=\xi_n^{(2)}(x)$ for all $n\le N-1$.

To estimate the second term of the right hand side of (40) let (α_n) and (β_n) be sequences from Lemma 12 satisfying $\alpha_0\le\varepsilon_1(m-1)$ and $\beta_0\ge(1-\varepsilon_2)m$. Let κ^* be small enough for Theorem 1 and Proposition 2 to apply. Let N_0 be the number from Lemma 12 such that for all $n\ge N_0$ we have $\alpha_n/((1-\tilde{\delta})\lambda_0),\beta_n/((1+\tilde{\delta})\lambda_0)\in I(m,\lambda_0,\delta)$. Recall that in the formulation of Lemma 12 we have chosen $\lambda_0=1$ but it holds for general λ_0. We assume $N_0\le N$. If for all $x\in\mathcal{N}_{R_\lambda+R_p}(0)$ we have $\xi_0(x)\in[\alpha_n/\lambda_0,\beta_n/\lambda_0]$, where ξ is a version of the processes considered, then there exist positive constants c_2 and δ_2 such that for all $n\le N_0$ we have

$$\mathbb{P}\Big[\xi_1(0)\notin\big[\frac{\alpha_{n+1}}{\lambda_0},\frac{\beta_{n+1}}{\lambda_0}\big]\Big]=\mathbb{P}\big[N^{(0,0)}\big(F(0;\xi_0)\big)\notin\big[\frac{\alpha_{n+1}}{\lambda_0},\frac{\beta_{n+1}}{\lambda_0}\big]\big]\le\exp\Big(-\frac{c_2\delta_2^2}{\lambda_0}\Big),$$

because $F(0;\xi_0)\in[\alpha_{n+1}/((1-\tilde{\delta})\lambda_0),\beta_{n+1}/((1+\tilde{\delta})\lambda_0)]$. It follows that

$$\mathbb{P}\big[\mathcal{X}_{N,2}^c\cap\Psi_N\big|\mathcal{X}_0\big]\le N|A_{7N}\setminus A_{4N}|\exp\Big(-\frac{c_2\delta_2^2}{\lambda_0}\Big).\tag{44}$$

The upper bound for the third term on the right hand side of (40) is obtained as follows

$$\begin{aligned}\mathbb{P}\big[\mathcal{X}_{N,3}^c\big|\mathcal{X}_0\big] &= \mathbb{P}\big[\exists x \in A_{7N,N/n^*} : \xi_N^{(1)}(x), \xi_N^{(2)}(x) \notin J(m,\lambda_0)\big|\mathcal{X}_0\big] \\ &\le \mathbb{P}\big[\exists k \in \{1,\dots,\frac{N}{n^*}-1\}\ \exists x \in A_{7N,k}\setminus A_{6N} : \xi_{kn^*}^{(1)}(x) \text{ or } \xi_{kn^*}^{(2)}(x) \notin J(m,\lambda_0)\big|\mathcal{X}_0\big] \\ &\le \frac{2N|A_{7N,N/n^*-1}\setminus A_{6N}|}{n^*}\theta(\lambda_0), \end{aligned} \tag{45}$$

where $\theta(\lambda_0) \le \exp(-c_3/\lambda_0)$ for some positive c_3 is defined in (20).

Let N be the smallest multiple of n^* larger than $1/\lambda_0$. Using the above estimates one can choose some positive c and $r \in \mathbb{N}$ such that

$$\mathbb{P}\big[\mathcal{X}_N^c\big|\mathcal{X}_0\big] \le \exp\Big(-\frac{c}{\lambda_0}\Big)N^r.$$

The right hand side goes to zero as λ_0 goes to zero. Thus, (36) follows. □

Before we turn to the proof of Proposition 3 we need a result about oriented percolation. Let $\theta \in (0,1)$ be given and let $A(x,n)$, $(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+$ be i.i.d. Bernoulli random variables with parameter θ. For $k < n$ we say that (x,k) *is connected to* (y,n), this will be denoted by $(x,k) \to (y,n)$, if there is a sequence $x = x_0,\dots,x_{n-k} = y$ such that $\|x_i - x_{i-1}\|_\infty \le 1$ and $A(x_i, k+i) = 1$ for $i = 1,\dots,n-k$. Let $C_0 = \{(x,n) : (0,0) \to (x,n)\}$ be the cluster of the origin. We call a space time-point (y,n) C_0*-exposed* if there exists a sequence $y_n,\dots,y_0$ such that $y_n = y$, $\|y_k - y_{k-1}\|_\infty \le 1$, and $(y_k,k) \notin C_0$, $k = 1,\dots,n$.

The next lemma follows from Durrett (1992). The idea behind the proof is as follows: With the 'usual' percolation interpretation in mind, let us call a site (x,n) *wet* if there is a backwards path $(x,n) = (x_0,n),(x_1,n-1),\dots,(x_n,0)$ with $\|x_i - x_{i-1}\| \le 1$ consisting only of open sites, i.e. $A(x_i,n-i) = 1$, $i = 0,1,\dots,n-1$. Otherwise, the site will be called *dry*. Lemma 7 in Durrett (1992) shows, using a contour-counting argument, that if θ is sufficiently close to 1, the dry sites do not percolate. In fact, this lemma even obtains an exponential bound on the tail of the size of the cluster of dry sites containing a given site. The next ingredient is complete convergence for oriented percolation (Durrett 1992, Lemma 8): When θ is close enough to 1, there is a fixed $c > 0$ and a random N_0 such that on $\{|C_0| = \infty\}$, $\{(x,n) : (x,n) \text{ wet and } \|x\| \le cn\} \subset C_0$ for all $n \ge N_0$. In words, any wet site inside the 'cone' $\{(x,n) : \|x\| \le cn, n \ge N_0\}$ is also connected to $(0,0)$ by an open path. Fix $c' \in (0,c)$. Assume that $\{|C_0| = \infty\}$, consider (y,n) with $\|y\| \le c'n$ and $n \ge 2N_0$, say. If (y,n) is C_0-exposed, there must be a backwards path $(y,n) = (y_0,n),(y_1,n-1),\dots,(y_n,0)$ with $(y_i,n-i) \notin C_0$. By the above, at least the initial $n(c-c')/2$ of these sites must be dry (for otherwise, they would be in C_0, as they must satisfy $\|y_i\| \le c(n-i)$. Hence, there must be a cluster of dry sites containing a point in $\{(x,n) : \|x\| \le c'n\}$ of size at least $n(c-c')/2$. By the exponential bound on the cluster size distribution and the Borel-Cantelli Lemma, this does not occur for n sufficiently large.

Lemma 14. *If θ is sufficiently close to 1 then there is a positive constant c such that for large enough times n conditional on $\{|C_0| = \infty\}$ there are no C_0-exposed sites in $\{x \in \mathbb{Z}^d : \|x\|_\infty \le cn\}$.*

Proof. **(Proposition 3)** Recall the definition of the event $\mathcal{X}(y,n)$ from (34). Theorem 1 implies that, conditioned on non-extinction of $\big(\xi_n^{(1)}\big)$ and $\big(\xi_n^{(2)}\big)$, with probability one there exist some finite time N_0 such that the event $\mathcal{X}(0, N_0)$ holds. Therefore we may assume a priori that $\mathcal{X}(0,0)$ holds.

We set $\widetilde{N} = \big[N/(2n^*)\big]$, $B = \{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+ \big| \|x\|_\infty \leq N, n \leq N\}$, $L = \widetilde{N}\mathbb{Z}^d$ and $K = N\mathbb{Z}_+$. Then we have

$$\mathbb{Z}^d \times \mathbb{Z}_+ = \bigcup_{(\alpha,\nu)\in L\times K} \Big((\alpha,\nu) + B\Big).$$

Let $\|\cdot\|_L$ be the norm on L defined by $\|\alpha\|_L = \|\alpha\|_\infty/\widetilde{N}$. To prove the theorem it is enough to show that for each $x^* \in \mathbb{Z}^d$ there is time T, such that $\xi_n^{(1)}(x^*) = \xi_n^{(2)}(x^*)$ holds for all $n \geq T$. Let us fix an arbitrary $x^* \in \mathbb{Z}^d$ and let $\alpha^* \in L$ be such that $\|\alpha^* - x^*\|_\infty \leq \widetilde{N}$. We define a process (η_ν) on the coarse-grained lattice $L \times K$ by

$$\eta_0(\alpha) = \mathbb{1}_{\mathcal{X}(\alpha,0)} \quad \text{and} \quad \eta_\nu(x) = \mathbb{1}_{\mathcal{X}(\alpha,\nu-N)}, \; \nu > 0.$$

Note that $\mathbb{1}_{\mathcal{X}(\alpha,\nu-N)} = 1$ for $\nu > 0$ ensures that $\xi_k^{(1)}(y) = \xi_k^{(2)}(y)$ holds for all $(y,k) \in (\alpha, \nu - N) + B$, because any backwards in time path starting in (y,k) will at time $\nu - N$ be inside $\alpha + A_N$, where $\xi^{(1)}$ and $\xi^{(2)}$ are the same on the event $\mathcal{X}(\alpha, \nu - N)$. In particular $\eta_\nu(\alpha^*) = 1$ implies $\xi_k^{(1)}(x^*) = \xi_k^{(2)}(x^*)$ for all $k \in \{\nu - N, \ldots, \nu\}$. We aim at showing that for suitable choice of parameters the process (η_ν) dominates oriented percolation on $L \times K$. To this end we need to estimate

$$\mathbb{P}\big[\eta_{\nu+N}(\beta) = 1,\ \|\alpha - \beta\|_L \leq 1 \;\big|\; \eta_\nu(\alpha) = 1\big],$$

whereas, due to translation invariance, it is enough to consider the corresponding probability for $(\alpha,\nu) = (0,0)$. By the construction of (η_ν) and Lemma 13 for each positive $\tilde{\varepsilon}$ one can choose λ_0, κ and N such that

$$\mathbb{P}\big[\eta_N(\beta) = 1,\ \|\beta\|_L \leq 1 \;\big|\; \eta_0(0) = 1\big] \geq \mathbb{P}\Big[\mathcal{X}(z,N),\ \|z\|_\infty \leq \frac{N}{n^*} \;\Big|\; \mathcal{X}(0,0)\Big] \geq 1 - \tilde{\varepsilon}.$$

From the proofs of Theorem 1 and Lemma 13 it can be seen that for x with $\|x\|_\infty \leq 1$ the event $\mathcal{X}(\tilde{N}x, N)$ is independent of the Poisson processes (which generate $\big(\xi_n^{(1)}\big)$ and $\big(\xi_n^{(2)}\big)$) outside the box $\big\{(y,k) \in \mathbb{Z}^d \times \mathbb{Z}_+ : k \leq N, \|y\|_\infty \leq (8N+2)(R_\lambda + R_p)\big\}$. Therefore, (η_ν) can be considered as M-dependent oriented percolation on $L \times K$, where $M = 20n^*(R_\lambda + R_p) \geq (8N+2)(R_\lambda + R_p)/\widetilde{N}$. Note that M does not depend on N and λ_0. Thus, the fact that we need to make λ_0 small does not affect the comparison.

Let θ be close enough to 1 such that Lemma 14 holds. For $\tilde{\varepsilon} \in \big(0, (1 - \sqrt{\theta})^\Delta\big)$, where $\Delta = |\{(\alpha,\nu) \in L \times K : \nu \in \{0, N\}, \|\alpha\|_L \leq M\}|$, we have

$$\mathbb{P}\big[\eta_N(\beta) = 1,\ \|\beta\|_L \leq 1 \;\big|\; \eta_0(0) = 1\big] \geq 1 - (1 - \sqrt{\theta})^\Delta.$$

As in the proof of Theorem 1, according to Theorem B26 in Liggett (1999), (η_ν) dominates nearest neighbour oriented percolation build from the product measure ν_θ on $\{0,1\}^{L\times K}$. Thus we obtain $\mathbb{P}\big[|C_\eta| = \infty\big] > 0$, where $C_\eta \subset L \times K$ is the cluster of the origin generated by (η_ν). By Lemma 14, conditioned on $\{|C_\eta| = \infty\}$, there is a time T such that the points $(\alpha^*, \nu) \in L \times K$ with $\nu \geq T$ are not C_η-exposed.

We claim that for each $n \geq T$, conditioned on $\{|C_\eta| = \infty\}$, we have $\xi_n^{(1)}(x^*) = \xi_n^{(2)}(x^*)$. If we assume the contrary then there must be a path $(x^*, n) = (x_n, n), (x_{n-1}, n-1), \ldots, (x_0, 0)$

in $\mathbb{Z}^d \times \mathbb{Z}_+$ such that $\|x_{i+1}-x_i\|_\infty \le R_\lambda + R_p$ and $\xi_i^{(1)}(x_i) \neq \xi_i^{(2)}(x_i)$ for all $i \in \{0, \dots, n-1\}$. From this path we discard the points (x_i, i) for which i is not a multiple of N, thus obtaining for some integer k the path $(x_{kN}, kN), (x_{(k-1)N}, (k-1)N), \dots, (x_0, 0)$. To this path belongs a path $(\alpha^*, (k+1)N), (\alpha_{kN}, kN), \dots, (\alpha_0, 0)$ in $L \times K$ where for $j \in \{1, \dots k\}$ we choose α_{jN} such that $(x_{(j-1)N}, (j-1)N) \in (\alpha_{jN}, jN) + B$ and α_0 such that $\|\alpha_0 - x_0\|_\infty \le N$. The assumption means that $\eta_{iN}(\alpha_{iN}) = 0$ for all $i \in \{0, \dots, k\}$. This contradicts the fact that $(\alpha^*, (k+1)N)$ is not C_η-exposed. ☐

Proof. **(Corollary 4)** The sequence (ξ_n), seen as a sequence of random measures on $\mathbb{Z}^d$, is relatively compact with respect to convergence in distribution in the vague topology because the expectation of $\xi_n(x)$ is bounded uniformly by $m^*_{\lambda_0}$.

It is clear that Dirac measure in $\mathbf{0} \in \mathbb{Z}_+^{\mathbb{Z}^d}$ is invariant. If there were two invariant distributions assigning probability 0 to the configuration $\mathbf{0}$, then Proposition 3 would imply that they coincide on finite subsets of $\mathbb{Z}^d$ and therefore they must be equal.

It remains to prove the existence of a limiting invariant distribution μ satisfying $\mu(\mathbf{0}) = 0$. Let the initial distribution μ_0 be the product measure on $\mathbb{Z}^d$ such that $\xi_0(x) = N^{(0,x)}\big(\bar{m}(\lambda_0, \kappa)\big)$ for all $x \in \mathbb{Z}^d$. Let μ_n be the distribution of ξ_n. Then the Cesaro average $1/N \sum_{n=0}^N \mu_n$ converges along some subsequence $\{N_k\}$ to some measure $\bar{\mu}$. This measure is invariant for (ξ_n) (see e.g. Liggett (1985, Proposition I.1.8)).

To show $\bar{\mu}(\mathbf{0}) = 0$ it is enough to prove that the restriction of (ξ_n) to $\mathbb{Z}$ survives with probability 1. At time 0 each site is occupied in the sense of Definition 6 with probability

$$\mathbb{P}\big[N^{(0,0)}(\bar{m}(\lambda_0, \kappa)) \in [\varepsilon_1 \bar{m}_{\lambda_0}, (1-\varepsilon_2) M_{\lambda_0}]\big],$$

where ε_1 and ε_2 are as in the proof of Lemma 7. In particular at time 0 there are infinitely many occupied sites. Again by comparison with oriented percolation we have $\mathbb{P}_{\xi_0}[\xi_n = \mathbf{0} \text{ for some } n] = 0$ because supercritical percolation starting from infinitely many wet sites does not die out (see e.g. Theorem B24 in Liggett (1999)).

☐

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