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The von Mises model for one-dimensional elastoplastic beams and Prandtl-Ishlinskii hysteresis operators

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Abstract

In this paper, the one-dimensional equation for the transversal vibrations of an elastoplastic beam is derived from a general three-dimensional system. The plastic behavior is modeled using the classical three-dimensional von Mises plasticity model. It turns out that this single-yield model without hardening leads after a dimensional reduction to a multi-yield one-dimensional hysteresis model with kinematic hardening, given by a hysteresis operator of Prandtl-Ishlinskii type whose density function can be determined explicitly. This result indicates that the use of Prandtl-Ishlinskii hysteresis operators in the modeling of elastoplasticity is not just a questionable phenomenological approach, but in fact quite natural. In addition to the derivation of the model, it is shown that the resulting partial differential equation with hysteresis can be transformed into an equivalent system for which the existence and uniqueness of a strong solution is proved. The proof employs techniques from the mathematical theory of hysteresis operators.

1 Introduction

The use of hysteresis operators in the modeling of the hysteretic stress-strain relations that are commonplace in nonlinear elastoplasticity, dates back to at least the early 20th century. Back in 1928, Prandtl in his pioneering work [8] introduced the input-output relation that was independently studied by Ishlinskii in [2] in the 1940's and nowadays is called the *Prandtl-Ishlinskii operator*. It describes the time-evolution of the relation between strain ε (input) and stress σ (output) in one-dimensional elastoplasticity in the form

$$\sigma(t) = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[\varepsilon](t)\, dq\,. \tag{1.1}$$

Here, t denotes the time variable, φ is some nonnegative weight function that satisfies the growth condition

$$\int_0^\infty (1+q)\,\varphi(q)\, dq < +\infty\,, \tag{1.2}$$

and $\mathfrak{s}_q$ denotes the one-dimensional *stop operator* or *Prandtl's elastic-perfectly plastic element* with thresholds $\pm q$, which is a basic hysteresis operator whose dynamic input-output behavior is described in Fig. 1.

Between the thresholds $\pm q$, the behavior is linear elastic (with elasticity modulus 1), while along the upper (lower) threshold $+q$ $(-q)$ we have irreversible plastic yielding and can only move to the right (left). The operator $\mathfrak{s}_q$ is a special one-dimensional case of the abstract stop operator $\mathfrak{S}_Z$ in a separable Hilbert space X associated with a closed and convex set $Z \subset X$ containing 0. This operator is defined in the following way: for a given input function $v \in W^{1,1}(0,T;X)$, consider the variational inequality

$$\begin{aligned} &\chi(t) \in Z \quad \forall\, t \in [0,T]\,, \quad \chi(0) = \chi_0\,, \\ &(\dot\chi(t) - \dot v(t), z - \chi(t)) \ge 0 \quad \forall\, z \in Z\,, \quad \text{for a.e. } t \in (0,T)\,. \end{aligned} \tag{1.3}$$

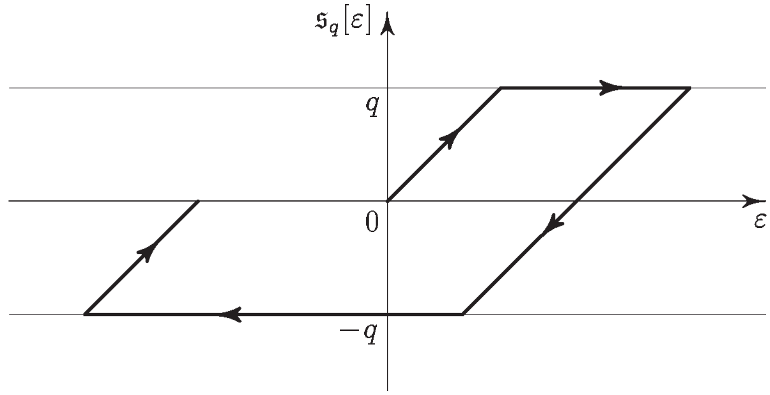

Figure 1: Hysteretic input-output behavior of $\mathfrak{s}_q$.

Here, and throughout the paper, the superimposed dot stands for differentiation with respect to time, and $(\cdot,\cdot)$ is a scalar product in X. The investigation of such problems goes back to [7], and the existence and uniqueness of a solution $\chi \in W^{1,1}(0,T;X)$ for any given initial value $\chi_0 \in Z$ is obtained as a special case of the general theory. This allows us to define the corresponding solution operator $\mathfrak{S}_Z$ as

$$\mathfrak{S}_Z : Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)\,, \quad [\chi_0, v] \mapsto \chi\,. \tag{1.4}$$

It is proved in [4, Section I.3] that this operator is continuous and, if Z has non-empty interior, admits a continuous extension to

$$\mathfrak{S}_Z : Z \times C([0,T];X) \to C([0,T];X)\,.$$

In the case $X = \mathbb{R}^1$, we set $\mathfrak{s}_q = \mathfrak{S}_{[-q,q]}$. Notice that since $z = 0 \in Z$, we obtain from (1.3) the fundamental *energy dissipation inequality*

$$\frac{1}{2}\frac{d}{dt}\left|\mathfrak{S}_Z[\chi_0,v](t)\right|^2 \le \left(\mathfrak{S}_Z[\chi_0,v](t), v_t(t)\right), \quad \text{a. e. in } (0,T)\,. \tag{1.5}$$

In this paper, we restrict ourselves to the canonical choice of initial conditions

$$\chi_0 \;=\; \mathrm{Proj}_Z(v(0))\,, \tag{1.6}$$

where $\mathrm{Proj}_Z : X \to Z$ is the orthogonal projection onto Z. We then simply write $\chi = \mathfrak{S}_Z[v]$ instead of $\chi = \mathfrak{S}_Z[\chi_0, v]$. The operator

$$\mathfrak{P}_Z = I - \mathfrak{S}_Z\,, \tag{1.7}$$

where I denotes the identity mapping, is called the *vector play operator* associated with Z. We similarly denote $\mathfrak{p}_q = \mathfrak{P}_{[-q,q]}$. The stop and play operators form the corner stones of the mathematical theory of hysteresis operators. In the 1D case in particular, every hysteresis relation with the so-called "return point memory" (which is a common property of hysteresis relations in plasticity, ferromagnetism, piezoelectricity, etc.) can be represented by some functional on the one-parametric play system $\{\mathfrak{p}_q\,;\, q > 0\}$, see

[1, Theorem 2.7.7]. The Prandtl-Ishlinskii operators (1.1) correspond in this respect to *linear functionals*. For a thorough treatment of their analytical and geometrical properties, we refer the reader to the monographs [1, 3, 4, 9]. Some important facts concerning $\mathfrak{s}_q$, which will be needed in the analysis below, are collected in Propositions 3.4, 3.5 in Section 3.

Although the Prandtl-Ishlinskii operator is easily understood and rather intuitive, its use in the physical and engineering literature is still nonstandard. The main reasons are the following: on the one hand, the operator appears to be entirely phenomenological, and its weight function φ is a priori unknown and must be identified; on the other hand, other well-established three-dimensional plasticity models like those by von Mises or Tresca are available.

The aim of this paper is twofold: first, we demonstrate that in the modeling of the (one-dimensional) transversal vibrations of an elastoplastic beam the use of the three-dimensional von Mises model leads (after normalizing all physical constants to unity) to the following beam equation for the transversal displacement:

$$w_{tt} - w_{xxtt} + \mathcal{P}\,[w_{xx}]_{xx} + w_{xxxx} = g\,. \tag{1.8}$$

Here, $\mathcal{P}$ is a Prandtl-Ishlinskii operator *whose weight function φ can be determined explicitly*, and g is given. Observe that the Prandtl-Ishlinski operator $\mathcal{P}$ is (as most nontrivial hysteresis operators) non-differentiable, so that (1.8) has to be given a proper meaning.

The existence and uniqueness analysis of the problem is carried out by transforming (1.8) into a system, in which no differentiation of the hysteresis operator occurs. The strong solution of this system is then interpreted as a weak solution to (1.8). The proof employs techniques from the mathematical theory of hysteresis operators; in particular, the properties of the stop operators $\mathfrak{s}_q$ will play a crucial role in the analysis.

The paper is organized as follows: in Section 2, will derive our model equation from a three-dimensional model using dimensional reduction. In Section 3, we will state the main existence and uniqueness result, which will be proved in the last two sections.

2 Derivation of the model

In this section, we derive our model from a general three-dimensional system. We restrict ourselves to *rectangular beams*, that is, to sets $\Omega \subset \mathbb{R}^3$ of the form $\Omega = (0,L) \times \omega$, where $L > 0$ is the *length* of the beam, and where, with some $h > 0$ and $b > 0$, the set $\omega = (-b,b) \times (-h,h)$ represents its (rectangular) *cross section*. We denote by $x \in (0,L)$ the longitudinal coordinate, by $(y,z) \in \omega$ the transversal coordinates, and by $t \in [0,T]$ the time, where $T > 0$ is given.

Having in mind applications to plasticity, we use an approach to dimensional reduction that slightly differs from the classical one in [6]. In order to compare the resulting equations, we start with the linear elastic isotropic case (Subsection 2.1), and then pass to the elastoplastic model under further simplifying assumptions (Subsection 2.2). We proceed first as in [6] and consider smooth deformations $\mathbf{F} : \Omega \times (0,T) \to \mathbb{R}^3$ under the following hypotheses (the precise assumptions on the data will be specified later).

(A1) The deformation of the central surface $\mathcal{C} = \{(x,y) \in \mathbb{R}^2;\, (x,y,0) \in \Omega\}$ is independent of y, that is,

$$\mathbf{F}(x,y,0,t) = \begin{pmatrix} x + v(x,t) \\ y \\ w(x,t) \end{pmatrix} \quad \forall\,(x,y) \in \mathcal{C}\,, \quad \forall\, t \in (0,T)\,, \tag{2.1}$$

with given functions $v, w : (0,L) \times (0,T) \to \mathbb{R}$.

(A2) The thickness $2h$ and the derivatives v_x, w_x, w_{xx} of v, w with respect to x are "sufficiently small".

(A3) The cross sections $\{x\} \times \omega$ remain perpendicular to the central surface, and their deformation is proportional to their distance to it; that is,

$$\mathbf{F}(x,y,z,t) = \mathbf{F}(x,y,0,t) + z\,\mathbf{n}(x,y,t) \quad \forall\,(x,y,z,t) \in \Omega \times (0,T)\,, \tag{2.2}$$

where $\mathbf{n}(x,y,t)$ is the unit "upward" normal to the deformed central surface $\mathcal{C}(t) = \mathbf{F}(\mathcal{C},0,t)$ at time t.

Under the hypothesis (A2), we can linearize the problem by replacing

$$\mathbf{n}(x,y,t) = \frac{1}{\sqrt{(1+v_x(x,t))^2 + w_x^2(x,t)}} \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1 + v_x(x,t) \end{pmatrix}$$

with

$$\tilde{\mathbf{n}}(x,y,t) := \begin{pmatrix} -w_x(x,t) \\ 0 \\ 1 \end{pmatrix}. \tag{2.3}$$

This is justified, since an elementary computation yields that

$$|\,\tilde{\mathbf{n}}(x,y,t) - \mathbf{n}(x,y,t)| \,<\, \left(|v_x(x,t)| + |w_x(x,t)|\right)^2$$

whenever $|v_x(x,t)| < 1$, $|w_x(x,t)| < 1$. This enables us to define for every $(x,y,z,t) \in \Omega \times (0,T)$ the linearized deformation $\tilde{\mathbf{F}}(x,y,z,t)$ by

$$\tilde{\mathbf{F}}(x,y,z,t) = \begin{pmatrix} x + v(x,t) - z\,w_x(x,t) \\ y \\ z + w(x,t) \end{pmatrix}, \tag{2.4}$$

and the displacement vector $\mathbf{u}(x,y,z,t)$ by

$$\mathbf{u}(x,y,z,t) = \begin{pmatrix} v(x,t) - z\,w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}. \tag{2.5}$$

The meaning of Hypothesis **(A2)** is to ensure that the Jacobian $D\tilde{\mathbf{F}}(x,y,z,t)$ satisfy, with some $\bar{c} > 0$,

$$\det D\tilde{\mathbf{F}}(x,y,z,t) = 1 + v_x(x,t) - z\, w_{xx}(x,t) + w_x^2(x,t) \geq \bar{c}\,.$$

Thus, $\tilde{\mathbf{F}}$ defines a local isomorphism. Moreover,

$$\nabla \mathbf{u}(x,y,z,t) = \begin{pmatrix} v_x(x,t) - z\, w_{xx}(x,t) & 0 & -w_x(x,t) \\ 0 & 0 & 0 \\ w_x(x,t) & 0 & 0 \end{pmatrix}, \tag{2.6}$$

and the linearized strain tensor $\boldsymbol{\varepsilon} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ becomes

$$\boldsymbol{\varepsilon}(x,y,z,t) = \begin{pmatrix} v_x(x,t) - z\, w_{xx}(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

2.1 Small elastic deformations

We denote by " $:$ " the canonical scalar product in the space of (3×3)-tensors, i.e.,

$$\boldsymbol{\xi} : \boldsymbol{\eta} = \sum_{i,j=1}^{3} \xi_{ij}\, \eta_{ij}\,, \quad \forall\, \boldsymbol{\xi} = (\xi_{ij})\,,\ \boldsymbol{\eta} = (\eta_{ij})\,, \quad i,j = 1,2,3\,. \tag{2.8}$$

Moreover, we define for any given (3×3)-tensor $\boldsymbol{\xi}$ its (trace-free) *deviator* $\mathbf{d}(\boldsymbol{\xi})$ by

$$\mathbf{d}(\boldsymbol{\xi}) = \boldsymbol{\xi} - \frac{1}{3}(\boldsymbol{\xi} : \boldsymbol{\delta})\, \boldsymbol{\delta}\,, \tag{2.9}$$

where $\boldsymbol{\delta} = (\delta_{ij})$ denotes the Kronecker tensor.

To motivate the elastoplastic case treated below, we first study the case of linear isotropic elasticity, in which the strain tensor $\boldsymbol{\varepsilon}$ and the stress tensor $\boldsymbol{\sigma}$ are related to each other through the formula

$$\boldsymbol{\sigma} \;=\; 2\mu\, \boldsymbol{\varepsilon} + \lambda\, (\boldsymbol{\varepsilon} : \boldsymbol{\delta})\, \boldsymbol{\delta}\,, \tag{2.10}$$

where μ, λ are the Lamé constants. Then, with $\varepsilon_{11} = v_x - z\, w_{xx}$, we infer from (2.7) that

$$\boldsymbol{\sigma} \;=\; \begin{pmatrix} (2\,\mu + \lambda)\, \varepsilon_{11} & 0 & 0 \\ 0 & \lambda\, \varepsilon_{11} & 0 \\ 0 & 0 & \lambda\, \varepsilon_{11} \end{pmatrix}. \tag{2.11}$$

Owing to the choice (2.1) of $\mathbf{F}$, we cannot prescribe boundary conditions on the lateral surface $(0,L) \times \omega$. On the left surface, where $x = 0$, $(y,z) \in \omega$, we restrict ourselves to the case of *vanishing normal stress* $\boldsymbol{\sigma}(0,y,z,t) \cdot \boldsymbol{\nu}_0 = 0$, where $\boldsymbol{\nu}_0 = (-1,0,0)^T$; that is, we have

$$w_{xx}(0,t) = v_x(0,t) = 0\,, \quad w(0,t) = 0\,, \tag{2.12}$$

where the latter boundary condition is added in order to eliminate possible rigid body displacements. An analogous choice of the boundary conditions is made at the right surface $\{L\} \times \omega$. In accordance with these boundary conditions, we consider the Sobolev space

$$V = \left\{(v,w) \in H^1(0,L) \times H^2(0,L);\; w(0) = w(L) = 0\right\}. \tag{2.13}$$

Now suppose that a constant mass density $\rho > 0$, an external force density of the form $\mathbf{f}(x,y,z,t) = (f_1(x,z,t), 0, f_3(x,z,t))^T$, and the initial conditions

$$v(x,0) = v^0(x)\,,\;\; v_t(x,0) = v^1(x)\,,\;\; w(x,0) = w^0(x)\,,\;\; w_t(x,0) = w^1(x)\,, \tag{2.14}$$

are given. We then want to solve the *momentum balance equation*

$$\int_\Omega \rho\, \mathbf{u}_{tt} \cdot \hat{\mathbf{u}}\; dx\, dy\, dz \;+\; \int_\Omega \boldsymbol{\sigma} : \hat{\varepsilon}\, dx\, dy\, dz = \int_\Omega \mathbf{f} \cdot \hat{\mathbf{u}}\; dx\, dy\, dz\,, \tag{2.15}$$

for the unknown vector function $\mathbf{u}$ given as in (2.5), where the admissible displacements $\hat{\mathbf{u}}$ and strains $\hat{\varepsilon}$ are also of the form (2.5) and (2.7), respectively; i. e., we have

$$\hat{\mathbf{u}}(x,y,z) = \begin{pmatrix} \hat{v}(x) - z\,\hat{w}_x(x) \\ 0 \\ \hat{w}(x) \end{pmatrix}, \quad \hat{\varepsilon}(x,y,z) = \begin{pmatrix} \hat{v}_x(x) - z\,\hat{w}_{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.16}$$

where $(\hat{v}, \hat{w})$ varies over the space V. The test functions $\hat{v}, \hat{w}$ are independent of each other, and a straightforward calculation shows that (2.15) decouples into the system

$$\rho \int_0^L v_{tt}(x,t)\,\hat{v}(x)\,dx + (2\mu+\lambda)\int_0^L v_x(x,t)\,\hat{v}_x(x)\,dx \;=\; \int_0^L g_1(x,t)\,\hat{v}(x)\,dx\,, \tag{2.17}$$

$$\rho \int_0^L \left(w_{tt}(x,t)\,\hat{w}(x) \;+\; \frac{h^2}{3}\, w_{xtt}(x,t)\,\hat{w}_x(x)\right) dx \;+\; \frac{(2\mu+\lambda)\,h^2}{3} \int_0^L w_{xx}(x,t)\,\hat{w}_{xx}(x)\,dx$$

$$= \int_0^L g_2(x,t)\,\hat{w}(x)\,dx\,, \tag{2.18}$$

where we have set

$$g_1(x,t) = \frac{1}{2h}\int_{-h}^{h} f_1(x,z,t)\,dz\,, \quad g_2(x,t) = \frac{1}{2h}\int_{-h}^{h} \left(f_3(x,z,t) + z\,(f_1)_x(x,z,t)\right)\,dz\,. \tag{2.19}$$

The variational system (2.17), (2.18) leads formally to the partial differential equations

$$\rho\, v_{tt} \;-\; (2\mu+\lambda)\, v_{xx} \;=\; g_1\,, \tag{2.20}$$

$$\rho\, w_{tt} \;-\; \frac{\rho\, h^2}{3}\, w_{xxtt} \;+\; \frac{(2\mu+\lambda)\,h^2}{3}\, w_{xxxx} \;=\; g_2\,, \tag{2.21}$$

which describe the longitudinal (Eq. (2.20)) and transversal (Eq. (2.21)) vibrations of a straight elastic beam. Note that the coefficient multiplying w_{xxxx} differs from the one in [6, p. 13]. This is due to the fact that we do not assume $\sigma_{33} = 0$ as in [6, p. 8].

2.2 Transversal elastoplastic oscillations

We now turn our interest to elastoplasticity. As further simplifications, we assume:

(A4) The motion is only transversal, that is, the admissible displacements and strains have the form

$$\mathbf{u}(x,y,z,t) = \begin{pmatrix} -z\, w_x(x,t) \\ 0 \\ w(x,t) \end{pmatrix}, \quad \boldsymbol{\varepsilon}(x,y,z,t) = \begin{pmatrix} -z\, w_{xx}(x,t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.22}$$

(A5) The strain tensor is decomposed in elastic and plastic components $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p$, and it holds

$$\boldsymbol{\sigma} = 2\mu\, \boldsymbol{\varepsilon}^e \,+\, \lambda\,(\boldsymbol{\varepsilon}^e : \boldsymbol{\delta})\,\boldsymbol{\delta}\,. \tag{2.23}$$

(A6) The plastic deformations are *volume preserving* in the sense that

$$\boldsymbol{\varepsilon}^p : \boldsymbol{\delta} \;=\; 0\,. \tag{2.24}$$

(A7) The stress deviators are bounded in norm by a *yield limit* $R > 0$, that is,

$$\mathbf{d}(\boldsymbol{\sigma}(x,y,z,t)) : \mathbf{d}(\boldsymbol{\sigma}(x,y,z,t)) \le R^2 \quad \forall\,(x,y,z,t)\,, \tag{2.25}$$

and the plastic strain rate obeys the *normality flow rule.*

$$\boldsymbol{\varepsilon}^p_t : (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}) \;\ge\; 0 \quad \forall\, \tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{(3\times 3)} : \quad \mathbf{d}(\tilde{\boldsymbol{\sigma}}) : \mathbf{d}(\tilde{\boldsymbol{\sigma}}) \le R^2\,. \tag{2.26}$$

The choice (2.25) of the domain

$$Z := \left\{ \tilde{\boldsymbol{\sigma}} \in \mathbb{R}^{(3\times 3)};\ \mathbf{d}(\tilde{\boldsymbol{\sigma}}) : \mathbf{d}(\tilde{\boldsymbol{\sigma}}) \le R^2 \right\}$$

of admissible stresses corresponds to the *von Mises model of elastoplasticity.* In this connection, $\operatorname{int}(Z)$ is called the *elasticity domain*, while ∂Z is the *yield surface.* Observe that

$$\mathbf{d}(\boldsymbol{\sigma}) = 2\mu\, \mathbf{d}(\boldsymbol{\varepsilon}^e)\,, \tag{2.27}$$

which automatically imposes a restriction on $\boldsymbol{\varepsilon}^e(x,y,z,t)$, namely (omitting the arguments)

$$\mathbf{d}(\boldsymbol{\varepsilon}^e) : \mathbf{d}(\boldsymbol{\varepsilon}^e) \le \frac{R^2}{4\mu^2}\,. \tag{2.28}$$

In view of (2.24) and (2.27), the normality flow rule (2.26) can be rewritten as

$$\boldsymbol{\varepsilon}^e \;=\; \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p \in \frac{1}{2\mu}\, Z\,, \qquad \boldsymbol{\varepsilon}^p_t : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p - \tilde{\boldsymbol{\varepsilon}}) \;\ge\; 0 \quad \forall\, \tilde{\boldsymbol{\varepsilon}} \in \frac{1}{2\mu}\, Z\,. \tag{2.29}$$

At this point, the notion of hysteresis operators comes into play: for every initial condition $\boldsymbol{\varepsilon}^p(0)$ and every fixed $(x,y,z) \in \Omega$, we may rewrite (2.29) as

$$\boldsymbol{\varepsilon}^e \;=\; \mathfrak{S}_{Z/(2\mu)}[\boldsymbol{\varepsilon}(0) - \boldsymbol{\varepsilon}^p(0), \boldsymbol{\varepsilon}]\,, \quad \boldsymbol{\varepsilon}^p \;=\; \mathfrak{P}_{Z/(2\mu)}[\boldsymbol{\varepsilon}(0) - \boldsymbol{\varepsilon}^p(0), \boldsymbol{\varepsilon}]$$

in agreement with (1.4), (1.7) for $X = \mathbb{R}^{(3\times 3)}$. We fix a tensor

$$\boldsymbol{\eta} = \begin{pmatrix} -\frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} . \tag{2.30}$$

Then it is easily seen that

$$\mathbf{d}(\boldsymbol{\varepsilon}(x,y,z,t)) = z\, w_{xx}(x,t)\, \boldsymbol{\eta} . \tag{2.31}$$

Now suppose that the initial condition $\boldsymbol{\varepsilon}_0^p(x,y,z)$ for $\boldsymbol{\varepsilon}^p$ points in the direction of $\boldsymbol{\eta}$, i. e., that there exists a scalar-valued function p_0 such that

$$\boldsymbol{\varepsilon}_0^p(x,y,z) = p_0(x,y,z)\, \boldsymbol{\eta} . \tag{2.32}$$

We claim that the solution $\boldsymbol{\varepsilon}^p$ of Eq. (2.29) has the form

$$\boldsymbol{\varepsilon}^p(x,y,z,t) = p(x,y,z,t)\, \boldsymbol{\eta} \tag{2.33}$$

with some scalar-valued function p, more precisely,

$$p \;=\; \frac{1}{\sqrt{\boldsymbol{\eta} : \boldsymbol{\eta}}}\, \boldsymbol{\varepsilon}^p : \boldsymbol{\eta} . \tag{2.34}$$

Indeed, putting $\boldsymbol{\varepsilon}^{p*} = \boldsymbol{\varepsilon}^p - p\, \boldsymbol{\eta}$, we have

$$\boldsymbol{\varepsilon}^{p*} : \boldsymbol{\eta} \;=\; 0 , \quad \boldsymbol{\varepsilon}^{p*} : \boldsymbol{\delta} \;=\; 0 , \quad \boldsymbol{\varepsilon}^{p*}(0) \;=\; 0 . \tag{2.35}$$

Moreover,

$$\mathbf{d}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) : \mathbf{d}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \;=\; \mathbf{d}(\boldsymbol{\varepsilon} - p\, \boldsymbol{\eta}) : \mathbf{d}(\boldsymbol{\varepsilon} - p\, \boldsymbol{\eta}) + \boldsymbol{\varepsilon}^{p*} : \boldsymbol{\varepsilon}^{p*} .$$

In particular, we may choose $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} - p\, \boldsymbol{\eta}$ in (2.29) and obtain

$$\boldsymbol{\varepsilon}_t^{p*} : \boldsymbol{\varepsilon}^{p*} \;\le\; 0 ,$$

hence $\boldsymbol{\varepsilon}^{p*} = 0$, and (2.33)–(2.34) hold. It thus follows from (2.29) that $p(x,y,z,\cdot)$ is the (unique) solution to the scalar variational inequality

$$\begin{aligned} |z\, w_{xx}(x,t) - p(x,y,z,t)| &\le \frac{R}{2\mu \sqrt{\boldsymbol{\eta} : \boldsymbol{\eta}}} = \frac{\sqrt{3}\, R}{\sqrt{8}\, \mu} =: r , \\ p_t(x,y,z,t)\, \left(z\, w_{xx}(x,t) - p(x,y,z,t) - q\right) &\ge 0 \quad \forall\, |q| \le r , \quad t \in (0,T) , \end{aligned} \tag{2.36}$$

with the initial condition $p_0(x,y,z)$. Restricting ourselves to the canonical choice of the initial condition,

$$p_0(x,y,z) = P_r(z\, w_{xx}(x,0)) , \quad \text{where } P_r(s) = \min\{s + r, \max\{0, s - r\}\} \quad \forall\, s \in \mathbb{R} , \tag{2.37}$$

we then arrive at the conclusion that

$$p(x,y,z,t) = \mathfrak{p}_r\left[z\, w_{xx}(x,\cdot)\right](t)\,, \tag{2.38}$$

where $\mathfrak{p}_r = I - \mathfrak{s}_r$ is the scalar play operator with threshold r. In the following, we will abbreviate this identity by simply writing

$$p = \mathfrak{p}_r\left[z\, w_{xx}\right]\,, \tag{2.39}$$

with obvious meaning. Identities involving the one-dimensional stop operator $\mathfrak{s}_r$ will be abbreviated accordingly. In particular, we have

$$\mathbf{d}\left(\boldsymbol{\varepsilon}^e\right) \;=\; \mathbf{d}(\boldsymbol{\varepsilon}) - \boldsymbol{\varepsilon}^p \;=\; \mathfrak{s}_r\left[z\, w_{xx}\right] \boldsymbol{\eta}\,. \tag{2.40}$$

By virtue of (2.23)–(2.24), it follows that

$$\begin{aligned} \boldsymbol{\sigma} \;&=\; 2\mu\; \mathfrak{s}_r\left[z\, w_{xx}\right] \boldsymbol{\eta} + \left(\frac{2\mu}{3} + \lambda\right)(\boldsymbol{\varepsilon} : \boldsymbol{\delta})\,\boldsymbol{\delta} \\ &=\; 2\mu\, z\, \mathfrak{s}_{r/|z|}\left[w_{xx}\right] \boldsymbol{\eta} - \frac{2\mu + 3\lambda}{3}\; z\, w_{xx}\, \boldsymbol{\delta}\,. \end{aligned} \tag{2.41}$$

Here, we have used the simple identity

$$\mathfrak{s}_r[\alpha\, u] = \alpha\; \mathfrak{s}_{r/|\alpha|}[u]\,,$$

which, with the convention $\mathfrak{s}_\infty[u] := u$, holds for all $\alpha \in \mathbb{R}$ and every input function u.

We now aim to derive the momentum balance in the same way as in (2.15) to (2.19). To this end, we again make the test functions independent of $\hat{v}$, so that

$$\boldsymbol{\sigma} : \hat{\boldsymbol{\varepsilon}} \;=\; \frac{4\mu}{3}\, z^2\, \mathfrak{s}_{r/|z|}\left[w_{xx}\right]\, \hat{w}_{xx} \;+\; \frac{2\mu + 3\lambda}{3}\, z^2\, w_{xx}\, \hat{w}_{xx}\,.$$

We then obtain for the hysteresis term

$$\begin{aligned} \int_\omega z^2\; \mathfrak{s}_{r/|z|}\left[w_{xx}\right]\, dy\, dz \;&=\; 2b \int_{-h}^{h} z^2\, \mathfrak{s}_{r/|z|}\left[w_{xx}\right]\, dz \\ =\; 4b \int_0^h z^2\, \mathfrak{s}_{r/z}\left[w_{xx}\right]\, dz \;&=\; 4b\, r^3 \int_{r/h}^{\infty} q^{-4}\, \mathfrak{s}_q\left[w_{xx}\right]\, dq\,, \end{aligned}$$

where

$$\mathcal{P}[u] := \int_{r/h}^{\infty} q^{-4}\; \mathfrak{s}_q[u]\, dq \tag{2.42}$$

is a Prandtl-Ishlinskii operator with the weight function

$$\varphi(q) = \begin{cases} 0\,, & \text{if} \quad 0 \le q \le \frac{r}{h}\,, \\ q^{-4}\,, & \text{if} \quad q > \frac{r}{h}\,. \end{cases} \tag{2.43}$$

The counterpart of (2.21) then reads formally

$$\rho\, w_{tt} \;-\; \frac{\rho\, h^2}{3}\, w_{xxtt} \;+\; \frac{4\,\mu\, r^3}{3\,h}\, \mathcal{P}\left[w_{xx}\right]_{xx} \;+\; \frac{(2\mu+3\lambda)\, h^2}{9}\, w_{xxxx} \;=\; g_2\,. \tag{2.44}$$

Here, we have used the abbreviation

$$\mathcal{P}\left[w_{xx}\right]_{xx}(x,t) \;=\; \frac{\partial^2}{\partial x^2}\, \mathcal{P}\left[w_{xx}(x,\cdot)\right](t)\,. \tag{2.45}$$

Remark 2.1. The term $((2\mu+3\lambda)h^2/9)\, w_{xxxx}$ in (2.44) plays the role of *kinematic hardening* in the Prandtl-Ishlinskii theory. It ensures that the elliptic part of the momentum balance equation is coercive, so that the problem will turn out to be well posed despite the fact that the Prandtl-Ishlinskii *initial loading curve* is bounded and saturation occurs. Indeed, the initial loading curve $\sigma = \Phi(\varepsilon)$ is given by the formula (see [1], [4])

$$\Phi(\varepsilon) \;=\; \frac{4\,\mu\, r^3}{3\,h} \int_{r/h}^{\infty} q^{-4}\, \min\{q,\varepsilon\}\, dq\,, \quad \text{for } \varepsilon \geq 0\,, \tag{2.46}$$

so that

$$\Phi(\varepsilon) \;=\; \begin{cases} \frac{4\,\mu\, h^2}{9}\, \varepsilon\,, & \textit{if} \quad \varepsilon \leq \frac{r}{h}\,, \\ \frac{2\,\mu\, r^3}{3\,h} \left(\frac{h^2}{r^2} - \frac{1}{3\,\varepsilon^2}\right)\,, & \textit{if} \quad \varepsilon > \frac{r}{h}\,. \end{cases} \tag{2.47}$$

Remark 2.2. Note that (2.44) reduces to (2.21) if we replace $\mathfrak{s}_q[u]$ by u in the expression (2.42) for $\mathcal{P}[u]$ (no plasticity). Also, if we pass to the elastic limit as $r \to \infty$ in (2.44), we recover (2.21) in agreement with natural expectations.

3 Statement of the mathematical results

In what follows, we use the usual notations for the spaces of continuous functions and for the standard Lebesgue and Sobolev spaces. The L^2-norm is always denoted by $\|\cdot\|$.

We now formulate the main mathematical results of this paper. To this end, we normalize all physical constants in (2.44) to unity, which has no bearing on the mathematical analysis. We thus study the following initial-boundary value problem in Q_T, where $Q_t := (0,1) \times (0,t)$ for any $t > 0$:

$$w_{tt} \;-\; w_{xxtt} \;+\; \mathcal{P}\left[w_{xx}\right]_{xx} \;+\; w_{xxxx} \;=\; g \qquad \text{in } Q_T\,, \tag{3.1}$$

$$w(0,t) = w_{xx}(0,t) = w(1,t) = w_{xx}(1,t) \;=\; 0\,, \quad 0 \leq t \leq T\,, \tag{3.2}$$

$$w(x,0) = z_0(x)\,, \quad w_t(x,0) = z_1(x)\,, \qquad 0 \leq x \leq 1\,. \tag{3.3}$$

We make the following general assumptions on the data of the system:

(H1) $g \in L^2(Q_T)$.

(H2) $z_0 \in H^3(0,1)$, $z_1 \in H^2(0,1)$, and the following compatibility conditions are satisfied:

$$z_0(0) = z_{0,xx}(0) = z_0(1) = z_{0,xx}(1) = 0\,, \; z_1(0) = z_1(1) = 0\,. \tag{3.4}$$

(H3) The weight function $\varphi : (0,\infty) \to [0,\infty)$ of the Prandtl-Ishlinskii operator

$$\mathcal{P}[u] = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[u]\, dq$$

is measurable and satisfies the growth condition

$$\int_0^\infty \left(1+q^2\right)\, \varphi(q)\, dq < +\infty\,. \tag{3.5}$$

Remark 3.1. Under condition (3.5) the so-called *clockwise admissible potential* of $\mathcal{P}$, given by the hysteresis operator

$$\mathcal{Q}[u] = \frac{1}{2}\int_0^\infty \varphi(q)\, \mathfrak{s}_q^2[u]\, dq\,, \tag{3.6}$$

is well defined. It then follows from the dissipation inequality (1.5) for the stop operator that for any input function $u \in W^{1,1}(0,T)$ it holds

$$(\mathcal{Q}[u])_t\,(t) \;=\; \int_0^\infty \varphi(q)\, \mathfrak{s}_q[u](t)\; (\mathfrak{s}_q[u])_t\,(t)\, dq \;\le\; \mathcal{P}[u](t)\, u_t(t)\,, \quad \textit{for a. e. } \; t \in (0,T). \tag{3.7}$$

We now associate with problem (3.1)–(3.3) the following system of initial-boundary value problems

$$u_t \;=\; \mathcal{P}\left[w_{xx}\right] + w_{xx} \qquad \text{in } Q_T\,, \tag{3.8}$$

$$w_t - w_{xxt} \;=\; -u_{xx} + f(x,t) \qquad \text{in } Q_T\,, \tag{3.9}$$

$$u(0,t) = u(1,t) \;=\; 0\,, \qquad 0 \le t \le T\,, \tag{3.10}$$

$$w(0,t) = w(1,t) \;=\; 0\,, \qquad 0 \le t \le T\,, \tag{3.11}$$

$$u(x,0) \;=\; z_1(x)\,, \qquad 0 \le x \le 1\,, \tag{3.12}$$

$$w(x,0) \;=\; z_0(x)\,, \qquad 0 \le x \le 1\,, \tag{3.13}$$

which arises from (3.1)–(3.3) if we put

$$u(x,t) \;=\; z_1(x) + \int_0^t \left(\mathcal{P}\left[w_{xx}\right] + w_{xx}\right)(x,s)\, ds\,, \quad f(x,t) \;=\; z_1(x) + \int_0^t g(x,s)\, ds\,. \tag{3.14}$$

Conversely, one should expect that a sufficiently smooth solution (u,w) to the system (3.8)–(3.13) induces a solution to (3.1)–(3.3). We will therefore in the following examine the solvability of (3.8)–(3.13). It will turn out, however, that we will not be able to extract enough regularity from the system (3.8)–(3.13) so that the existence of a strong solution to (3.1)–(3.3) can be guaranteed. Instead, we will show the following weaker result.

Theorem 3.2. *Suppose that the conditions (H1)–(H3) are satisfied. Then the system (3.8)–(3.13) has a unique solution pair (u,w) having the following properties:*

(i) $u \in W^{2,\infty}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^2(0,1)) \cap H^1(0,T;H^1(0,1))$.

(ii) $w \in W^{1,\infty}(0,T;H^2(0,1)) \cap L^\infty(0,T;H^3(0,1)) \cap H^2(0,T;H^1(0,1))$.

(iii) *Eq. (3.8) is fulfilled pointwise in $\overline{Q_T}$, and Eq. (3.9) holds almost everywhere in Q_T.*

(iv) *The initial and boundary conditions (3.10)–(3.13) are satisfied pointwise, and it holds*

$$w_{xx}(0,t) = w_{xx}(1,t) = 0 \quad \forall\, t \in [0,T]\,.$$

Remark 3.3. We call (u,w) a strong solution to (3.8)–(3.13), and w a weak solution to (3.1)–(3.3). The meaning of conditions (i), (ii) in Theorem 3.2 is that

$$\left.\begin{array}{rcl} u_{tt}, u_{xx}, w_{xxt}, w_{xxx} & \in & L^\infty(0,T;L^2(0,1))\,, \\ u_{xt}, w_{xtt} & \in & L^2(Q_T)\,. \end{array}\right\} \tag{3.15}$$

By virtue of the boundary conditions and embedding theorems, we then have

$$u, u_x, u_t, w, w_x, w_t, w_{xx}, w_{xt} \;\in\; C(\overline{Q_T})\,. \tag{3.16}$$

Before proving Theorem 3.2 in the next sections, we now collect some well-known properties of the one-dimensional stop operator that can be found in a more general form in the monographs [1] or [4], and in the paper [5]. For the reader's convenience, we give a brief outline of the proofs.

Proposition 3.4. *Let $v_1, v_2 \in W^{1,1}(0,T)$ be given, $\chi_i = \mathfrak{s}_q[v_i]$, $p_i = v_i - \chi_i = \mathfrak{p}_q[v_i]$, $i = 1,2$. Then*

(i) $(\chi_1(t) - \chi_2(t))(\dot v_1(t) - \dot v_2(t)) \;\geq\; \dfrac{1}{2}\dfrac{d}{dt}(\chi_1(t) - \chi_2(t))^2$ *a. e.;*

(ii) $|\dot p_1(t) - \dot p_2(t)| + \dfrac{d}{dt}|\chi_1(t) - \chi_2(t)| \;\leq\; |\dot v_1(t) - \dot v_2(t)|$ *a. e.;*

(iii) $|\chi_1(t) - \chi_2(t)| \;\leq\; 2 \max\limits_{0\leq\tau\leq t} |v_1(\tau) - v_2(\tau)| \quad \forall t \in [0,T]\,;$

(iv) $|\dot\chi_i(t)| \;\leq\; |\dot v_i(t)|$ *a. e.*

Sketch of the proof. We have by (1.3) that $\dot p_1(\chi_1 - \chi_2) \geq 0$, $\dot p_2(\chi_2 - \chi_1) \geq 0$ a. e., hence

$$(\dot p_1(t) - \dot p_2(t))(\chi_1(t) - \chi_2(t)) \;\geq\; 0 \quad \text{a. e.}, \tag{3.17}$$

which is nothing but (i). We obtain (ii) from (3.17) whenever $\chi_1(t) \neq \chi_2(t)$. If $\chi_1(t) = \chi_2(t) \in (-q,q)$, then $\dot p_1(t) = \dot p_2(t) = 0$, while on the set of all t such that $\chi_1(t) = \chi_2(t) = \pm q$, we have

$$\dot\chi_1(t) = \dot\chi_2(t) = \frac{d}{dt}|\chi_1(t) - \chi_2(t)| = 0 \quad \text{a. e.}\,,$$

and (ii) follows. To prove (iii), we fix any $t \in (0,T]$, assume for instance that $\chi_1(t) > \chi_2(t)$, and find a smallest $t_0 < t$ such that $\chi_1(\tau) > \chi_2(\tau)$ for all $\tau \in (t_0, t]$. Then, by (3.17), $\dot p_1(\tau) \ge \dot p_2(\tau)$ for a. e. $\tau \in (t_0, t)$, hence

$$p_1(t_0) - p_2(t_0) \;\le\; p_1(t) - p_2(t) \;\le\; v_1(t) - v_2(t)$$

(note that $p_i + \chi_i = v_i$). Then either $t_0 > 0$ with $\chi_1(t_0) = \chi_2(t_0)$, or $t_0 = 0$ with $|p_1(t_0) - p_2(t_0)| \le |v_1(t_0) - v_2(t_0)|$. In both cases we have

$$|p_1(t) - p_2(t)| \;\le\; \max\{|v_1(t_0) - v_2(t_0)|, |v_1(t) - v_2(t)|\},$$

hence (iii). Part (iv) follows from the obvious identity $\dot p_i(t)\,\dot\chi_i(t) = 0$ a. e. □

Proposition 3.5. *Let $v \in C(\overline{Q_T})$ be such that $v_{xt} \in L^1(Q_T)$. For $(x,t) \in Q_T$ set $\chi(x,t) = \mathfrak{s}[v(x,\cdot)](t)$. Then $\chi_{xt} \in L^1(Q_T)$, and*

$$\left.\begin{aligned} |\chi_x(x,t)| \;&\le\; 2\max\nolimits_{0\le\tau\le t}|v_x(x,\tau)| \\ &\qquad \textit{for a. e. } x \in (0,1) \textit{ and all } t \in [0,T], \\ |\chi_{xt}(x,t)| + \tfrac{\partial}{\partial t}|\chi_x(x,t)| \;&\le\; 2\,|v_{xt}(x,t)| \quad \textit{a. e. in } Q_T. \end{aligned}\right\} \tag{3.18}$$

If moreover $v_{xt} \in L^2(Q_T)$, then for all $t \in [0,T]$ we have

$$\int_0^t\int_0^1 \chi_x(x,\tau)\,v_{xt}(x,\tau)\,dx\,d\tau \;\ge\; \frac{1}{2}\int_0^1 (\chi_x^2(x,t) - \chi_x^2(x,0))\,dx. \tag{3.19}$$

Sketch of the proof. By Proposition 3.4 (ii), (iii), we have for all $0 < x_1 < x_2 < 1$ and $t > 0$ that

$$\begin{aligned} |\chi(x_1,t) - \chi(x_2,t)| \;&\le\; 2\max_{0\le\tau\le t}|v(x_1,\tau) - v(x_2,\tau)| \;\le\; 2\int_{x_1}^{x_2}\max_{0\le\tau\le t}|v_x(x,\tau)|\,dx, \\ |\chi_t(x_1,t) - \chi_t(x_2,t)| + \frac{\partial}{\partial t}|\chi(x_1,t) - \chi(x_2,t)| \;&\le\; 2\,|v_t(x_1,t) - v_t(x_2,t)|, \end{aligned}$$

hence (3.18) holds. To prove (3.19), we first notice that by Proposition 3.4 (i), we have for each $h \in (0,1)$ and $t \in (0,T]$ that

$$\begin{aligned} &\int_0^t\int_h^1 \frac{\chi(x,\tau) - \chi(x-h,\tau)}{h}\cdot\frac{v_t(x,\tau) - v_t(x-h,\tau)}{h}\,dx\,d\tau \\ &\ge\; \frac{1}{2}\int_h^1\left(\left(\frac{\chi(x,t) - \chi(x-h,t)}{h}\right)^2 - \left(\frac{\chi(x,0) - \chi(x-h,0)}{h}\right)^2\right)dx. \end{aligned}$$

Using e. g. the Mean Continuity Theorem, we pass to the limit as $h \searrow 0+$ and obtain the assertion. □

4 Proof of existence

In this section, we will prove the existence result of Theorem 3.2. To this end, we use Faedo-Galerkin approximations. Let $\{\psi_k\}_{k\in\mathbb{N}}$ denote the system of eigenfunctions to the eigenvalue problem

$$-\psi_k'' = \lambda_k\,\psi_k\,, \quad \text{in } [0,1]\,, \quad \psi_k(0)=\psi_k(1)=0\,, \quad k\in\mathbb{N},$$

normalized with respect to the standard scalar product $\langle\cdot,\cdot\rangle$ in $L^2(0,1)$. Clearly, $\lambda_k = k^2\pi^2$ and $\psi_k(x)=\sqrt{2}\sin(k\pi x)$, for $k\in\mathbb{N}$. We set $V_m = \text{span}\,\{\psi_1,\ldots,\psi_m\}$. Then $V_m\subset V_{m+1}$, $m\in\mathbb{N}$, and $\bigcup_{m\in\mathbb{N}} V_m$ is dense in any of the spaces $L^2(0,1)$, $H_0^1(0,1)$, and $\tilde{H}_0^3(0,1) := \{v\in H^3(0,1);\, v(0)=v''(0)=v(1)=v''(1)=0\}$.

For given $m\in\mathbb{N}$, we consider approximations for u,w of the form

$$u^m(x,t)=\sum_{j=1}^m \mu_j(t)\,\psi_j(x)\,, \quad w^m(x,t)=\sum_{j=1}^m \eta_j(t)\,\psi_j(x)\,. \tag{4.1}$$

Denoting by Q_m the $L^2(0,1)$-orthogonal projection onto V_m, and using the standard notation $u(t)(x)=u(x,t)$ for functions of space and time, we consider the system of Faedo-Galerkin equations

$$\langle u_t^m(t),\psi\rangle \;=\; \langle \mathcal{P}\,[w_{xx}^m]\,(t)+w_{xx}^m(t),\psi\rangle \quad \forall\,\psi\in V_m\,, \quad 0\le t\le T\,, \tag{4.2}$$

$$\langle w_t^m(t)-w_{xxt}^m(t),\psi\rangle \;=\; \langle -u_{xx}^m(t)+f(t),\psi\rangle \quad \forall\,\psi\in V_m\,, \quad 0\le t\le T\,, \tag{4.3}$$

$$u^m(0) \;=\; Q_m(z_1)\,, \quad w^m(0) \;=\; Q_m(z_0)\,, \tag{4.4}$$

which is equivalent to the system

$$\dot\mu_k(t) \;=\; \langle \mathcal{P}\,[w_{xx}^m]\,(t),\psi_k\rangle \;-\; k^2\pi^2\,\eta_k(t)\,, \quad 0\le t\le T\,, \tag{4.5}$$

$$\dot\eta_k(t) \;=\; \frac{k^2\pi^2}{1+k^2\pi^2}\,\mu_k(t) \;+\; \frac{1}{1+k^2\pi^2}\,\langle f(t),\psi_k\rangle\,, \quad 0\le t\le T\,, \tag{4.6}$$

$$\mu_k(0) \;=\; \langle z_1,\psi_k\rangle\,, \quad \eta_k(0) \;=\; \langle z_0,\psi_k\rangle\,, \tag{4.7}$$

for $k=1,\ldots,m$. Here, we have used the abbreviation

$$\mathcal{P}\,[w_{xx}^m]\,(x,t) \;=\; \mathcal{P}\,[w_{xx}^m(x,\cdot)]\,(t) \;=\; \mathcal{P}\Big[-\sum_{j=1}^m k^2\pi^2\,\eta_j\,\psi_j(x)\Big](t)\,. \tag{4.8}$$

Obviously, (4.5)–(4.7) is an initial value problem for a system of $2m$ ordinary differential equations whose right-hand side is globally Lipschitz continuous on $C\,([0,T];\mathbb{R}^{2m})$. Indeed, owing to Proposition 3.4 (iii) and Eq. (3.5), we have for any $u_1,u_2\in C[0,T]$ the estimate

$$\begin{aligned}|\mathcal{P}\,[u_1]\,(t)-\mathcal{P}\,[u_2]\,(t)| \;&\le\; \int_0^\infty \varphi(q)\;|\mathfrak{s}_q\,[u_1]\,(t)-\mathfrak{s}_q\,[u_2]\,(t)|\;dq\\ &\le\; 2\max_{0\le s\le t}\,|u_1(s)-u_2(s)|\int_0^\infty \varphi(q)\,dq\,, \quad \forall\,t\in[0,T]\,,\end{aligned}$$

from which the claim follows. Consequently, the system (4.5)–(4.7) has a unique (global) solution $(\mu_1, \ldots, \mu_m, \eta_1, \ldots, \eta_m) \in C^1([0,T]; \mathbb{R}^{2m})$ that defines the solution (u^m, w^m) of (4.2)–(4.4) through Eq. (4.1). We have in fact $(\mu_1, \ldots, \mu_m, \eta_1, \ldots, \eta_m) \in H^2(0,T; \mathbb{R}^{2m})$ as a consequence of Proposition 3.4 (iv). In the following, we derive a series of a priori estimates to pave the way for the passage to the limit as $m \to \infty$. To this end, we differentiate Eq. (4.3) with respect to t to obtain

$$\langle w^m_{tt}(t) - w^m_{xxtt}(t), \psi \rangle \;=\; -\langle u^m_{xxt}, \psi \rangle + \langle g(t), \psi \rangle \quad \forall \psi \in V_m \quad \text{for a. e. } t \in (0,T)\,. \tag{4.9}$$

Inserting $\psi = w^m_t(t) \in V_m$ in (4.9), integrating by parts, and employing Young's inequality, we find for a. e. $t \in (0,T)$ that

$$\frac{1}{2}\frac{d}{dt}\left(\|w^m_t(t)\|^2 + \|w^m_{xt}(t)\|^2\right) + \langle u^m_t(t), w^m_{xxt}(t)\rangle \;\le\; \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|w^m_t(t)\|^2\,. \tag{4.10}$$

Now observe that $w^m_{xxt}(t) \in V_m$, so that it follows from (4.2) that

$$\langle u^m_t(t), w^m_{xxt}(t)\rangle \;=\; \langle \mathcal{P}\left[w^m_{xx}\right](t), w^m_{xxt}(t)\rangle + \frac{1}{2}\frac{d}{dt}\|w^m_{xx}(t)\|^2 \qquad \text{a. e.} \tag{4.11}$$

Recalling (3.6) and (3.7), we can infer that

$$\langle \mathcal{P}\left[w^m_{xx}\right](t), w^m_{xxt}(t)\rangle \;\ge\; \frac{d}{dt}\int_0^1 \mathcal{Q}\left[w^m_{xx}\right](t)\,dx \qquad \text{a. e.}\,, \tag{4.12}$$

where $\mathcal{Q}\left[w^m_{xx}\right](t) \ge 0$. Hence, integrating (4.10) over $[0,t]$ for any $t \ge 0$, we arrive at the estimate

$$\begin{aligned} \|w^m_t(t)\|^2 + \|w^m_{xt}(t)\|^2 + \|w^m_{xx}(t)\|^2 \;\le\; & \|w^m_t(0)\|^2 + \|w^m_{xt}(0)\|^2 + \|w^m_{xx}(0)\|^2 \\ & + 2\int_0^1 \mathcal{Q}\left[w^m_{xx}\right](0)\,dx + \int_0^t \|g(s)\|^2\,ds + \int_0^t \|w^m_t(s)\|^2\,ds\,. \end{aligned} \tag{4.13}$$

In the following, we denote by C_ℓ, $\ell \in \mathbb{N}$, positive constants that may depend on the data of the system, but not on $m \in \mathbb{N}$. First notice that

$$\|w^m_{xx}(0)\| \;=\; \|(Q_m(z_0))_{xx}\| \;=\; \|Q_m(z_{0,xx})\| \;\le\; \|z_{0,xx}\| \;\le\; C_1\,. \tag{4.14}$$

Also, by (3.5), and since $|\mathfrak{s}_q\left[w^m_{xx}\right]| \le q$, $q \ge 0$,

$$\int_0^1 \mathcal{Q}\left[w^m_{xx}\right](0)\,dx \;\le\; \int_0^1\int_0^\infty \varphi(q)\,q^2\,dq\,dx \;\le\; C_2\,. \tag{4.15}$$

Next, observe that

$$\|w^m_t(0)\|^2 \;=\; \sum_{k=1}^m \dot{\eta}_k^2(0)\,. \tag{4.16}$$

Now, in view of (3.14), (4.6), and (4.7),

$$\dot{\eta}_k(0) \;=\; \langle z_1, \psi_k\rangle\,, \tag{4.17}$$

and it follows from Bessel's inequality and (H2) that

$$\|w_t^m(0)\|^2 \le \sum_{k=1}^m \langle z_1, \psi_k\rangle^2 \le \|z_1\|^2 \le C_3 . \tag{4.18}$$

Likewise,

$$\|w_{xxt}^m(0)\|^2 = \sum_{k=1}^m k^4\pi^4\, \dot\eta_k^2(0) \le \sum_{k=1}^m k^4\pi^4 \,\langle z_1, \psi_k\rangle^2 \le \|z_{1,xx}\|^2 \le C_4 . \tag{4.19}$$

Combining the above estimates with (4.13), and invoking Gronwall's lemma, we have proved the a priori estimate

$$\max_{0\le t\le T} \left(\|w_t^m(t)\|^2 + \|w_{xt}^m(t)\|^2 + \|w_{xx}^m(t)\|^2\right) \le C_5 . \tag{4.20}$$

As second step in the proof, we insert $\psi = -w_{xxt}^m \in V_m$ in (4.9). Integrating by parts, and invoking (4.2) and Young's inequality, we find that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|w_{xt}^m(t)\|^2 + \|w_{xxt}^m(t)\|^2 + \|w_{xxx}^m(t)\|^2\right) + \langle (\mathcal{P}[w_{xx}^m])_x(t), w_{xxxt}^m(t)\rangle \\ &\le \frac{1}{2}\|g(t)\|^2 + \frac{1}{2}\|w_{xxt}^m(t)\|^2 , \end{aligned} \tag{4.21}$$

where

$$\langle (\mathcal{P}[w_{xx}^m])_x(t), w_{xxxt}^m(t)\rangle = \int_0^1\int_0^\infty \varphi(q)\, \left(\mathfrak{s}_q[w_{xx}^m(x,\cdot)]\right)_x\, w_{xxxt}^m(t)\, dq\, dx . \tag{4.22}$$

Recalling Proposition 3.5, and integrating (4.21) over $[0,t]$ for any $t\in[0,T]$, we arrive at the estimate

$$\begin{aligned} &\|w_{xt}^m(t)\|^2 + \|w_{xxt}^m(t)\|^2 + \|w_{xxx}^m(t)\|^2 + \int_0^\infty \varphi(q)\left\|\left(\mathfrak{s}_q[w_{xx}^m]\right)_x(t)\right\|^2 dq \\ &\le \|w_{xt}^m(0)\|^2 + \|w_{xxt}^m(0)\|^2 + \|w_{xxx}^m(0)\|^2 + \int_0^\infty \varphi(q)\left\|\left(\mathfrak{s}_q[w_{xx}^m]\right)_x(0)\right\|^2 dq \\ &\quad + \frac{1}{2}\int_0^t \left(\|g(s)\|^2 + \|w_{xxt}^m\|^2\right) ds . \end{aligned} \tag{4.23}$$

Now recall (4.19). Likewise,

$$\|w_{xt}^m(0)\|^2 = \sum_{k=1}^m \dot\eta_k^2(0)\, k^2\pi^2 \le C_3 . \tag{4.24}$$

Moreover, since z_0 satisfies the compatibility conditions (3.4), we have

$$\langle z_0, \psi_k\rangle = \frac{1}{k^3\pi^3}\langle z_{0,xxx}, \psi_k\rangle \quad \forall\, k\in\mathbb{N},$$

and thus

$$\|w^m_{xxx}(0)\|^2 \;=\; \sum_{k=1}^{m} |\langle z_0, \psi_k\rangle|^2\, k^6\pi^6 \;\le\; \|z_{0,xxx}\|^2\,. \tag{4.25}$$

Finally, we employ the property (3.18) of the stop operator $\mathfrak{s}_q$ to deduce that

$$\left|(\mathfrak{s}_q[w^m_{xx}])_x\,(0)\right| \;\le\; 2\,|w^m_{xxx}(0)|\,, \tag{4.26}$$

whence it follows that

$$\int_0^\infty \varphi(q)\,\left\|(\mathfrak{s}_q[w^m_{xx}])_x\,(0)\right\|^2\, dq \;\le\; C_6\,. \tag{4.27}$$

In conclusion, we have shown the estimate

$$\max_{0\le t\le T}\, \left(\|w^m_{xt}(t)\|^2 \,+\, \|w^m_{xxt}(t)\|^2 \,+\, \|w^m_{xxx}(t)\|^2\right) \;\le C_7\,. \tag{4.28}$$

Now observe that Proposition 3.4 (iv) shows that

$$|(\mathcal{P}[w^m_{xx}])_t\,(x,t)| \;\le\; \int_0^\infty \varphi(q)\,\left|(\mathfrak{s}_q[w^m_{xx}])_t\,(x,t)\right|\,dq \;\le\; C_8\,|w^m_{xxt}(x,t)| \quad \text{a.e. in } Q_T\,.$$

Hence, differentiating (4.2) with respect to t, inserting $\psi = u^m_{tt}(t) \in V_m$, and invoking (4.28), we can infer that

$$\max_{0\le t\le T}\, \|u^m_{tt}\| \;\le\; C_9\,. \tag{4.29}$$

Moreover, by inserting $\psi = u^m_{xx}(t) \in V_m$ in (4.3), we directly find that

$$\max_{0\le t\le T}\, \|u^m_{xx}(t)\| \;\le C_{10}\,. \tag{4.30}$$

We now use the elementary formula

$$\int_0^T \left(\dot\mu_k^2 + \ddot\mu_k\,\mu_k\right)(t)\,dt = \dot\mu_k(T)\,\mu_k(T) - \dot\mu_k(0)\,\mu_k(0)$$

to estimate u^m_{xt} as follows.

$$\begin{aligned}
\int_0^T \|u^m_{xt}(t)\|^2 dt \;&=\; \pi^2 \sum_{k=1}^m \int_0^T k^2\dot\mu_k^2(t)\,dt \\
&\le\; \pi^2 \sum_{k=1}^m \int_0^T k^2\,|\ddot\mu_k(t)\,\mu_k(t)|\,dt + 2\pi^2 \max_{0\le t\le T} \sum_{k=1}^m k^2\,|\dot\mu_k(t)\,\mu_k(t)| \\
&\le\; \pi^2 \int_0^T \left(\sum_{k=1}^m |\ddot\mu_k(t)|^2\right)^{1/2} \left(\sum_{k=1}^m k^4|\mu_k(t)|^2\right)^{1/2} dt \\
&\quad +\, 2\pi^2 \max_{0\le t\le T} \left(\sum_{k=1}^m |\dot\mu_k(t)|^2\right)^{1/2} \left(\sum_{k=1}^m k^4|\mu_k(t)|^2\right)^{1/2} \\
&=\; \int_0^T \|u^m_{tt}(t)\|\|u^m_{xx}(t)\|\,dt + 2\,\max_{0\le t\le T} \|u^m_t(t)\|\|u^m_{xx}(t)\| \\
&\le\; C_{11}\,,
\end{aligned} \tag{4.31}$$

by virtue of (4.29)–(4.30). To estimate w^m_{xtt}, we refer to (4.6), which yields for almost every $t \in (0,T)$ and every $k = 1, \dots, m$ that

$$
\begin{aligned}
|k\pi \ddot{\eta}_k(t)| &\leq \frac{k^3\pi^3}{1+k^2\pi^2}\, |\dot{\mu}_k(t)| + \frac{k\pi}{1+k^2\pi^2}\, |\langle g(t), \psi_k\rangle| \\
&\leq k\pi\, |\dot{\mu}_k(t)| + \frac{1}{2}\, |\langle g(t), \psi_k\rangle|\,, \qquad (4.32)
\end{aligned}
$$

hence

$$
\begin{aligned}
\int_0^T \|w^m_{xtt}(t)\|^2 dt &= \pi^2 \sum_{k=1}^m \int_0^T k^2 \ddot{\eta}_k^2(t)\, dt \\
&\leq 2\pi^2 \sum_{k=1}^m \int_0^T k^2 \dot{\mu}_k^2(t)\, dt + \frac{1}{2} \sum_{k=1}^m \int_0^T \langle g(t), \psi_k\rangle^2 dt \\
&\leq 2 \int_0^T \|u^m_{xt}(t)\|^2 dt + \frac{1}{2} \int_0^T \|g(t)\|^2 dt \\
&\leq C_{12}\,. \qquad (4.33)
\end{aligned}
$$

Combining the above estimates, and possibly selecting a suitable subsequence again indexed by m, we find that there exist functions u, w in the appropriate Sobolev spaces such that the following convergences take place:

$$
\left.
\begin{array}{rl}
\left.\begin{array}{r} u^m_{tt} \to u_{tt},\ u^m_{xx} \to u_{xx}, \\ w^m_{xxt} \to w_{xxt},\ w^m_{xxx} \to w_{xxx}, \end{array}\right\} & \text{weakly-* in } L^\infty(0,T; L^2(0,1))\,, \\
u^m_{xt} \to u_{xt},\ w^m_{xtt} \to w_{xtt}, & \text{weakly in } L^2(Q_T)\,.
\end{array}
\right\} \qquad (4.34)
$$

Then, by compact embedding,

$$
\left.
\begin{array}{r}
u^m \to u,\ u^m_x \to u_x,\ u^m_t \to u_t, \\
w^m \to w,\ w^m_x \to w_x,\ w^m_t \to w_t,\ w^m_{xx} \to w_{xx},\ w^m_{xt} \to w_{xt},
\end{array}
\right\} \text{strongly in } C(\overline{Q_T})\,, \qquad (4.35)
$$

and it follows from the Lipschitz continuity in Proposition 3.4 (iii) of the operators $\mathfrak{s}_q(x,\cdot)$ on $C[0,T]$ that, for every $(x,t) \in \overline{Q_T}$,

$$
\begin{aligned}
|\mathcal{P}[w^m_{xx}](x,t) - \mathcal{P}[w_{xx}](x,t)| &\leq \int_0^\infty \varphi(q)\, |\mathfrak{s}_q[w^m_{xx}(x,\cdot)](t) - \mathfrak{s}_q[w_{xx}(x,\cdot)](t)|\, dq \\
&\leq 2 \int_0^\infty \varphi(q)\, dq\, \max_{0 \leq s \leq t} |w^m_{xx}(x,s) - w_{xx}(x,s)|\,, \qquad (4.36)
\end{aligned}
$$

that is,

$$
\mathcal{P}[w^m_{xx}] \to \mathcal{P}[w_{xx}]\,, \qquad \text{strongly in } C(\overline{Q_T})\,. \qquad (4.37)
$$

Combining the convergences (4.31) to (4.37), it is now a standard argument (which can be omitted here) that the pair (u, w) is in fact a solution to the system (4.8)-(4.13) that enjoys the properties requested in Theorem 3.1. The existence part of Theorem 3.1 is thus proved. □

5 Proof of uniqueness and concluding remarks

Let us consider two solutions u_1, w_1, u_2, w_2 to (3.8)–(3.13), with the regularity stated in Theorem 3.2, and set $u = u_1 - u_2$, $w = w_1 - w_2$. We then have

$$\begin{aligned} u_t &= \mathcal{P}[w_{1,xx}] - \mathcal{P}[w_{2,xx}] + w_{xx} && \text{in } Q_T, && (5.1)\\ w_t - w_{xxt} &= -u_{xx} && \text{in } Q_T, && (5.2)\\ u(0,t) = u(1,t) &= 0, && 0 \le t \le T, && (5.3)\\ w(0,t) = w(1,t) &= 0, && 0 \le t \le T, && (5.4)\\ u(x,0) &= 0, && 0 \le x \le 1, && (5.5)\\ w(x,0) &= 0, && 0 \le x \le 1. && (5.6) \end{aligned}$$

By Proposition 3.4 (i), we have a. e. in Q_T that

$$(\mathcal{P}[w_{1,xx}] - \mathcal{P}[w_{2,xx}])\, w_{xxt} \ \ge\ \mathcal{R}_t\,,$$

where

$$\mathcal{R}(x,t) \ =\ \frac{1}{2}\int_0^\infty \varphi(q)\,(\mathfrak{s}_q[w_{1,xx}](x,t) - \mathfrak{s}_q[w_{2,xx}](x,t))^2 dq \ \ge\ 0\,.$$

We now test Eq. (5.1) by w_{xxt}, Eq. (5.2) by $-w_{tt}$, and sum them up. The regularity (3.15)–(3.16) enables us to obtain for almost all $t \in (0,T)$ that

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \left(\mathcal{R} + \frac{1}{2}w_{xx}^2 - \frac{1}{2}w_t^2 - \frac{1}{2}w_{xt}^2\right)(x,t)\,dx\\ &\quad\le\ \int_0^1 (u_t\, w_{xxt} + u_{xx}\, w_{tt})(x,t)\,dx\\ &\quad=\ -\int_0^1 (u_{xt}\, w_{xt} + u_x\, w_{xtt})(x,t)\,dx\\ &\quad=\ -\frac{d}{dt}\int_0^1 u_x(x,t)\, w_{xt}(x,t)\,dx\\ &\quad=\ \frac{d}{dt}\int_0^1 u_{xx}(x,t)\, w_t(x,t)\,dx\\ &\quad=\ \frac{d}{dt}\int_0^1 (w_t^2 + w_{xt}^2)(x,t)\,dx\,, \end{aligned} \qquad (5.7)$$

hence

$$\frac{d}{dt}\int_0^1 \left(\mathcal{R} + \frac{1}{2}w_{xx}^2 + \frac{1}{2}w_t^2 + \frac{1}{2}w_{xt}^2\right)(x,t)\,dx \ \le\ 0 \quad \text{a. e.} \qquad (5.8)$$

The initial conditions for w_1 and w_2 coincide, hence $w_1 = w_2$ in Q_T, and consequently also $u_1 = u_2$. This completes the proof of Theorem 3.2. □

Remark 5.1. The uniqueness of the limit pair (u, w) entails that the convergences (4.34)–(4.35) hold for the entire sequence $\{(u^m, w^m)\}$ and not only for a subsequence. Hence the Faedo-Galerkin scheme (4.2)–(4.7) constitutes a convergent method to approximate the solution numerically.

Remark 5.2. The fact that the norms of w^m_{xx} (and, eventually, of w_{xx}) in $C(\overline{Q_T})$ are uniformly bounded above by a constant $C_0 > 0$ implies that $\mathfrak{s}_q[w^m_{xx}] = w^m_{xx}$ for all $q \geq C_0$. By a suitable cut-off argument, it is thus possible to obtain the result of Theorem 3.2 even if the growth condition (3.5) is relaxed to

$$\int_0^\infty \varphi(q)\, dq \;<\; \infty\,.$$

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