Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Global attractors for semigroups of closed operators

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submitted: 6 Jun 2006

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No. 1141
Berlin 2006

2000 *Mathematics Subject Classification.* 34D45, 47H20, 47J35.

Key words and phrases. Semigroups of operators, abstract Cauchy problems, closed operators, global attractors, connected attractors.

This work was partially supported by the Weierstrass Postdoctoral Fellowship Program.

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Abstract. In this note, we establish a general result on the existence of global attractors for semigroups $S(t)$ of operators acting on a Banach space $\mathcal{X}$, where the strong continuity $S(t) \in C(\mathcal{X}, \mathcal{X})$ is replaced by the much weaker requirement that $S(t)$ be a closed map.

1. Introduction

Let $\mathbb{K}$ denote either $\mathbb{R}^+$ or $\mathbb{N}$, and let $\mathcal{X}$ be a Banach space or, more generally, a complete metric space. A *closed semigroup* on $\mathcal{X}$ is a one-parameter family of (nonlinear) operators $S(t) : \mathcal{X} \to \mathcal{X}$ $(t \in \mathbb{K})$ satisfying the conditions

- (S.1) $S(0) = \mathbb{I}_{\mathcal{X}}$;
- (S.2) $S(t + \tau) = S(t)S(\tau)$, for all $t, \tau \in \mathbb{K}$;
- (S.3) $x_n \to x$ and $S(t)x_n \to y$ imply that $y = S(t)x$.

When $\mathbb{K} = \mathbb{N}$, $S(t)$ is called a *discrete* semigroup. Assumptions (S.1)-(S.2) are the semigroup properties, while (S.3) says that $S(t)$ is a closed (nonlinear) map.

The interest in considering such an object is motivated by the study of differential equations in Banach spaces. Assuming to have, for every $x_0 \in \mathcal{X}$, a unique global solution $x(t) \in \mathcal{X}$ to the abstract Cauchy problem

$$\begin{cases} x'(t) = A(x(t)), & t > 0, \\ x(0) = x_0, \end{cases}$$

where A is a given (nonlinear) operator defined on a dense domain $\mathcal{D} \subset \mathcal{X}$, and writing $x(t) = S(t)x_0$, it is readily seen that $S(t)$ fulfills (S.1)-(S.2).

According to the standard terminology (cf. [1, 2, 3, 4, 6, 7, 8]), a semigroup $S(t)$ is *dissipative* when there exists an absorbing set. This is a bounded set $\mathcal{B} \subset \mathcal{X}$ for which, given any bounded set $\mathcal{U} \subset \mathcal{X}$, there exists $t_0 = t_0(\mathcal{U}) \in \mathbb{K}$ (the entering time) such that

$$S(t)\mathcal{U} \subset \mathcal{B}, \qquad \forall t \geq t_0.$$

A set $\mathcal{K} \subset \mathcal{X}$ is called *attracting* for $S(t)$ if, for any bounded set $\mathcal{U} \subset \mathcal{X}$,

$$\lim_{t\to\infty} \delta(S(t)\mathcal{U}, \mathcal{K}) = 0,$$

where δ is the Hausdorff semidistance in $\mathcal{X}$. Clearly, an absorbing set is attracting as well. A semigroup possessing a compact attracting set is said to be *asymptotically compact*. A relevant object which provides an accurate description of the longterm dynamics of $S(t)$ is the *global attractor*, namely, a compact set $\mathcal{A} \subset \mathcal{X}$ which is at the same time attracting and fully invariant for $S(t)$ (i.e. $S(t)\mathcal{A} = \mathcal{A}$ for every $t \in \mathbb{K}$). The global attractor, if it exists, is easily seen to be unique.

In the classical textbooks (cf. [1, 2, 3, 4, 6, 7, 8]), the existence of the global attractor $\mathcal{A}$ is usually established for asymptotically compact semigroups within the assumption that $S(t) \in C(\mathcal{X}, \mathcal{X})$ for every fixed $t \in \mathbb{K}$. In that case, $\mathcal{A}$ turns out to be ω-limit set of (any) absorbing set $\mathcal{B}$, defined as

$$\omega(\mathcal{B}) = \bigcap_{t\in\mathbb{K}} \overline{\bigcup_{\tau\geq t} S(\tau)\mathcal{B}}.$$

On the other hand, there are interesting situations arising from concrete differential problems where the related semigroup of solutions $S(t)$ *does not* fulfill such a strong continuity property. Typically, the convergence $x_n \to x$ in $\mathcal{X}$ might imply that $S(t)x_n \to S(t)x$ only in some weaker topology. Nonetheless, in this case, condition (S.3) is immediately seen to hold.

Remark 1. To the best of our knowledge, the only exception is the treatise [1] (see §1, Theorem 2.1), where, besides the standard results, the existence of a global attractor for a semigroup $S(t)$ lacking strong continuity is proved under the assumption that there exists a compact absorbing set $\mathcal{B}$ such that, for any $t \geq t_0$ (where t_0 is the entering time of $\mathcal{B}$ into itself),

$$\overline{S(t)\mathcal{E}} \subset S(t)\overline{\mathcal{E}}, \qquad \forall \mathcal{E} \subset \mathcal{B}$$

and $S(t)^{-1}y$ is a closed set, for every $y \in \mathcal{X}$. In fact, this is the same as requiring the continuity of the map $S(t) : \mathcal{B} \to \mathcal{B}$. However, we observe that compact absorbing sets generally appear when dealing with semigroups generated by parabolic equations, which exhibit an instantaneous regularization of the initial data, whereas they never occur in hyperbolic problems.

2. The Main Result

In order to state the main result, we first recall a definition. Given a bounded set $\mathcal{U} \subset \mathcal{X}$, the *Kuratowski measure of noncompactness* $\alpha(\mathcal{U})$ is defined as

$$\alpha(\mathcal{U}) = \inf \big\{ d : \mathcal{U} \text{ has a finite covering of balls of } \mathcal{X} \text{ of diameter less than } d \big\}.$$

We report some well-known properties of α (see e.g. [3, 7]).

- ◇ $\alpha(\mathcal{U}) = \alpha(\overline{\mathcal{U}})$.
- ◇ $\mathcal{U}_1 \subset \mathcal{U}_2$ implies that $\alpha(\mathcal{U}_1) \leq \alpha(\mathcal{U}_2)$.
- ◇ $\alpha(\mathcal{U}) = 0$ if and only if $\overline{\mathcal{U}}$ is compact.
- ◇ If $\{\mathcal{U}_t\}_{t\in\mathbb{K}}$ is a family of nonempty closed sets such that $\mathcal{U}_{t_1} \supset \mathcal{U}_{t_2}$ for $t_1 < t_2$ and $\lim_{t\to\infty} \alpha(\mathcal{U}_t) = 0$, then $\mathcal{U} = \bigcap_{t\in\mathbb{K}} \mathcal{U}_t$ is nonempty and compact.
- ◇ If $\{\mathcal{U}_t\}_{t\in\mathbb{K}}$ and $\mathcal{U}$ are as above, given any $t_n \to \infty$ and any $x_n \in \mathcal{U}_{t_n}$, there exist $x \in \mathcal{U}$ and a subsequence $x_{n_k} \to x$.

Theorem 2. *Assume the following hypotheses:*

(i) *there exists an absorbing set* $\mathcal{B} \subset \mathcal{X}$*;*
(ii) *there exists a sequence* $t_n \in \mathbb{K}$ *such that* $\lim_{n\to\infty} \alpha(S(t_n)\mathcal{B}) = 0$.

Then, $\omega(\mathcal{B})$ *is the global attractor of* $S(t)$*.*

Proof. We begin to show that $\omega(\mathcal{B})$ is compact and attracting. This part makes use only of (S.1)-(S.2). Owing to (i), let $t_0 \in \mathbb{K}$ be such that $S(t)\mathcal{B} \subset \mathcal{B}$, for all $t \geq t_0$. For $t \geq t_0 + t_n$, we have the inclusion

$$S(t)\mathcal{B} = S(t_n)S(t - t_n)\mathcal{B} \subset S(t_n)\mathcal{B}.$$

Thus, (ii) actually implies that $\alpha(S(t)\mathcal{B}) \to 0$ as $t \to \infty$. Besides, if $t \geq t_0$,

$$\mathcal{U}_t = \bigcup_{\tau \geq t} S(\tau)\mathcal{B} = \bigcup_{\tau \in \mathbb{K}} S(t - t_0)S(\tau + t_0)\mathcal{B} \subset \bigcup_{\tau \in \mathbb{K}} S(t - t_0)\mathcal{B} = S(t - t_0)\mathcal{B}.$$

Hence,

$$\lim_{t \to \infty} \alpha(\overline{\mathcal{U}_t}) = \lim_{t \to \infty} \alpha(\mathcal{U}_t) = 0.$$

Since the sets $\mathcal{U}_t$ are nested, we conclude that $\omega(\mathcal{B}) = \bigcap_{t \geq 0} \overline{\mathcal{U}_t}$ is nonempty and compact. Assume now that $\omega(\mathcal{B})$ is not attracting for $S(t)$. Then, there exist $\varepsilon > 0$ and sequences $x_n \in \mathcal{B}$ and $\tau_n \to \infty$ such that

$$\inf_{x \in \omega(\mathcal{B})} \|S(\tau_n)x_n - x\| \geq \varepsilon.$$

If $\tau_n \geq t_0$, it follows that $S(\tau_n)x_n \in \mathcal{U}_{\tau_n - t_0}$. Appealing to the properties of α, the sequence $S(\tau_n)x_n$ must have a cluster point in $\omega(\mathcal{B})$, which is a contradiction.

The next step is to prove that $\omega(\mathcal{B})$ is fully invariant for $S(t)$. To this end, we preliminarily observe that, since $\omega(\mathcal{B})$ is compact and attracting, given any sequences $x_n \in \mathcal{B}$ and $\tau_n \to \infty$, there exist $y \in \omega(\mathcal{B})$ such that $S(\tau_n)x_n \to y$ up to a subsequence. Indeed, from the attracting property of $\omega(\mathcal{B})$ we have that

$$\lim_{n \to \infty} \delta(S(\tau_n)\mathcal{B}, \omega(\mathcal{B})) = 0.$$

Thus, in particular,

$$\lim_{n \to \infty} \Big[\inf_{x \in \omega(\mathcal{B})} \|S(\tau_n)x_n - x\| \Big] = 0.$$

In other words, there is a sequence $y_n \in \omega(\mathcal{B})$ such that

$$\lim_{n \to \infty} \|S(\tau_n)x_n - y_n\| = 0.$$

Exploiting the compactness of $\omega(\mathcal{B})$, there exists $y \in \omega(\mathcal{B})$ and a sequence n_k such that $y_{n_k} \to y$ which, in turn, implies that

$$S(\tau_{n_k})x_{n_k} \to y.$$

Let then $x \in \omega(\mathcal{B})$. By the definition of ω-limit set, there exist $\tau_n \to \infty$ and $x_n \in \mathcal{B}$ satisfying

$$S(\tau_n)x_n \to x.$$

On the other hand, given any $t \in \mathbb{K}$, there exist $y_1, y_2 \in \omega(\mathcal{B})$ such that, up to subsequences,

$$S(\tau_n - t)x_n \to y_1$$

and

$$S(t)S(\tau_n)x_n = S(\tau_n + t)x_n \to y_2.$$

Since $S(t)S(\tau_n - t)x_n = S(\tau_n)x_n$, in light of (S.3), we conclude that

$$x = S(t)y_1, \qquad y_2 = S(t)x,$$

which yields the sought invariance property $S(t)\omega(\mathcal{B}) = \omega(\mathcal{B})$. $\square$

Although we wrote the result in great generality, in the applications, assumption (ii) is usually verified by proving the existence of a compact attracting set for $S(t)$.

If $S(t) \in C(\mathcal{X}, \mathcal{X})$, the attractor $\mathcal{A}$ provided by Theorem 2 is well-known to be connected (when $\mathcal{X}$ is either a Banach space or a complete metric space whose balls are connected). Without this continuity assumption, connectedness may fail to hold, as the following example (for a discrete semigroup) shows.

Example 3. let $\mathcal{X} = \ell^2(\mathbb{N})$ be the space of square summable sequences $x = \{x^\iota\}_{\iota \in \mathbb{N}}$. Denoting by e_κ ($\kappa \in \mathbb{N}$) the element of $\mathcal{X}$ such that $e_\kappa^\iota = \delta_{\iota\kappa}$, we introduce the function $\varphi : \mathbb{R}^+ \to \mathcal{X}$ as

$$\varphi(r) = \frac{1}{2}\big[e_0 + (\kappa + 1 - r)e_{\kappa+1} + (r - \kappa)e_{\kappa+2}\big], \qquad r \in [\kappa, \kappa + 1).$$

Note that φ maps continuously $\mathbb{R}^+$ into the unit ball of $\mathcal{X}$, and $[\varphi(r)]^0 = 1/2$ for every $r \in \mathbb{R}^+$. Besides, it is constructed in such a way not to have any cluster point as $r \to \infty$. Next, we consider a continuous decreasing cut-off function $\theta : \mathbb{R}^+ \to [0, 1]$ such that $\theta(r) = 1$ for $r \leq 1/4$ and $\theta(r) = 0$ for $r \geq 1/2$. Finally, we define the map $S : \mathcal{X} \to \mathcal{X}$ as

$$Sx = \begin{cases} \dfrac{1 - \theta(x^0)}{2}(e_0 + x) + \theta(x^0)\varphi(1/x^0), & \text{if } x^0 > 0, \\ -e_0, & \text{if } x^0 \leq 0, \end{cases}$$

and we set

$$S(t) = \underbrace{S \circ S \cdots \circ S}_{t \text{ times}}, \qquad t \in \mathbb{N}.$$

Naming

$$\mathcal{X}^+ = \{x \in \mathcal{X} : x^0 > 0\}, \qquad \mathcal{X}^- = \{x \in \mathcal{X} : x^0 < 0\}, \qquad \mathcal{X}^0 = \{x \in \mathcal{X} : x^0 = 0\},$$

we observe that $S\mathcal{X}^+ \subset \mathcal{X}^+$ and $S(\mathcal{X}^0 \cup \mathcal{X}^-) \subset \mathcal{X}^-$. Hence, we readily see that $S(t)$ is continuous on $\mathcal{X}^+ \cup \mathcal{X}^-$. It is then easy to conclude that $S(t)$ is a closed map. Indeed, the only case to check is when $x_n \in \mathcal{X}^+$, $x \in \mathcal{X}^0$ are such that $x_n \to x$ and $S(t)x_n \to y$, for some $y \in \mathcal{X}$. But the latter convergence cannot occur, since φ has no cluster points at infinity. On the other hand, the attractor of $S(t)$ consists of *two* points, precisely,

$$\mathcal{A} = \{e_0, -e_0\}.$$

Here, the assumptions of Theorem 2 are satisfied, since

$$\mathcal{K} = \{\beta e_0,\ \beta \in [-1, 1]\}$$

is a compact attracting set.

There are however some cases where we can recover the connectedness of $\mathcal{A}$ without requiring the continuity of $S(t)$.

Proposition 4. *Assume there exist a sequence $t_n \to \infty$ and a connected set $\mathcal{C} \supset \mathcal{A}$ such that $S(t_n)\mathcal{C}$ is relatively compact for every n. Then $\mathcal{A}$ is connected.*

Proof. For every fixed n, the map $S_n = S(t_n) : \mathcal{C} \to \mathcal{X}$ is continuous. Indeed, if $x_k \in \mathcal{C}$ converges to some $x \in \mathcal{C}$, then $S_n x_k$ belongs to a compact set and, owing to (S.3), its only cluster point is $S_n x$. If $\mathcal{A}$ is not connected, there are two disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\mathcal{A} \cap \mathcal{O}_j \neq \emptyset$ and $\mathcal{A} \subset \mathcal{O}_1 \cup \mathcal{O}_2$. For every integer n, The set $S_n\mathcal{C}$ is connected and $S_n\mathcal{C} \supset S_n\mathcal{A} = \mathcal{A}$, which implies that $S_n\mathcal{C} \cap (\mathcal{O}_1 \cup \mathcal{O}_2) \neq \emptyset$. Thus, we can select $y_n \in S_n\mathcal{C} \setminus (\mathcal{O}_1 \cup \mathcal{O}_2)$. Since $\mathcal{A}$ is compact and attracting, there exists $y \in \mathcal{A}$ such that, up to a subsequence, $y_n \to y$. On the other hand, $y \notin \mathcal{O}_1 \cup \mathcal{O}_2$, and so $y \notin \mathcal{A}$, leading to a contradiction. □

Remark 5. Note that the compact attracting set $\mathcal{K}$ of Example 3 is connected, but the image $S(t)\mathcal{K}$ is not relatively compact for any $t \geq 1$.

Collecting Theorem 2 and Proposition 4, we have

Corollary 6. *Let $S(t)$ have a connected compact attracting set $\mathcal{K}$. Assume also that $S(t)\mathcal{K} \subset \mathcal{K}$ for every t large enough. Then $S(t)$ possesses a connected global attractor.*

Corollary 6 is particularly useful. Indeed, in most concrete cases, the compact attracting set $\mathcal{K}$ is a ball of $\mathcal{Z}$, where $\mathcal{Z}$ is another Banach space compactly embedded into $\mathcal{X}$. Besides, it is often possible to prove an estimate of the form

$$\|S(t)z\|_{\mathcal{Z}} \leq Q(\|z\|_{\mathcal{Z}})\Psi(t) + C, \qquad \forall z \in \mathcal{Z},$$

where $C > 0$, Q is a positive increasing function and Ψ is a positive function vanishing at infinity. It is then clear that, up to possibly replacing $\mathcal{K}$ with a larger ball of $\mathcal{Z}$, the set $\mathcal{K}$ is attracting and fulfills the relation $S(t)\mathcal{K} \subset \mathcal{K}$ for t large enough.

3. An Application

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Consider the wave equation with nonlinear damping

$$\begin{cases} \partial_{tt}u + \sigma(u)\partial_t u - \Delta u + \varphi(u) = 0, \qquad t > 0, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u_{|\partial\Omega} = 0. \end{cases}$$

The function $\varphi \in C^2(\mathbb{R})$, with $\varphi(0) = 0$, fulfills

$$|\varphi''(u)| \leq c_1\big(1 + |u|^p\big), \qquad \varphi'(u) \geq -c_2, \qquad \liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda,$$

where $p, c_1, c_2 \geq 0$ and $\lambda > 0$ is the first eigenvalue of $-\Delta$ on $L^2(\Omega)$ with Dirichlet boundary conditions, while $\sigma \in C^1(\mathbb{R})$ is such that

$$\sigma(u) \geq \sigma_0 > 0, \qquad |\sigma'(u)| \leq c_2[\sigma(u)]^\nu,$$

for some $c_2 \geq 0$ and some $\nu < 1$. As shown in [5], this problem generates a semigroup $S(t)$ on the phase space $\mathcal{X} = H_0^1(\Omega) \times L^2(\Omega)$, which possesses a compact attracting set $\mathcal{K}$ satisfying the hypotheses of Corollary 6. However, for any two initial data $x_1, x_2 \in \mathcal{X}$ with $\|x_j\| \leq \varrho$, only a continuous dependence estimate of the form

$$\|S(t)x_1 - S(t)x_2\|_{\mathcal{W}} \leq ke^{kt}\|x_1 - x_2\|$$

for some $k = k(\varrho)$ is available, where $\mathcal{W} = L^2(\Omega) \times H^{-1}(\Omega)$. Hence, we have the weaker continuity $S(t) \in C(\mathcal{X}, \mathcal{W})$, which is enough in order for (S.3) to hold. Corollary 6 then yields the existence of a connected global attractor.

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