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Attractors and their regularity for 2-D wave equations with nonlinear damping

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Abstract. We address the study of a weakly damped wave equation in space-dimension two, with a damping coefficient depending on the displacement. The equation is shown to generate a semigroup possessing a compact global attractor of optimal regularity, as well as an exponential attractor.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following wave equation with nonlinear damping:

$$\begin{cases} \partial_{tt}u + \sigma(u)\partial_t u - \Delta u + \varphi(u) = f, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{0.1}$$

Here, $f \in L^2(\Omega)$ is independent of time, while $\varphi \in C^2(\mathbb{R})$, with $\varphi(0) = 0$, fulfills

$$|\varphi''(u)| \leq c\big(1 + |u|^p\big), \qquad p \geq 0, \tag{0.2}$$

$$\varphi'(u) \geq -\ell, \qquad \ell \geq 0, \tag{0.3}$$

$$\liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda_1, \tag{0.4}$$

where $c \geq 0$ and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on $L^2(\Omega)$ with Dirichlet boundary conditions. Concerning the damping term, we assume that $\sigma \in C^1(\mathbb{R})$ with

$$\sigma(u) \geq \sigma_0 > 0, \tag{0.5}$$

$$|\sigma'(u)| \leq c[\sigma(u)]^{1-\delta}, \qquad \delta \in (0, 1], \tag{0.6}$$

for some $c \geq 0$. Note that (0.6) implies that (redefining the constant c)

$$|\sigma'(u)| \leq c\big(1 + |u|^q\big), \qquad q = \tfrac{1-\delta}{\delta}. \tag{0.7}$$

Equation (0.1) is a model for a vibrating membrane in a stratified viscous medium: the variable u represents the displacement from equilibrium, $\partial_t u$ is the velocity, whereas the term $\sigma(u)\partial_t u$ accounts for dynamical friction. Finally, $f - \varphi(u)$ corresponds to a (nonlinear) elastic force. Our main result reads as follows:

Theorem 0.1. *Equation* (0.1) *generates a semigroup* $S(t)$ *on* $H_0^1(\Omega) \times L^2(\Omega)$ *which possesses a (unique) compact global attractor* $\mathcal{A}$. *Moreover,* $\mathcal{A}$ *is a bounded subset of* $\big[H^2(\Omega) \cap H_0^1(\Omega)\big] \times H_0^1(\Omega)$, *and it coincides with the unstable set of the stationary points of* $S(t)$.

The same problem in space-dimension one has been considered in [6], where the existence of a strongly continuous semigroup possessing a regular compact attractor (and also exponential attractors) has been proven. Clearly, the analysis made in [6] took great advantage of the “goodëmbedding properties that hold in dimension one. Indeed, the result obtained there is optimal, and is valid also if condition (0.5) is replaced by the weaker requirement that $\sigma(u) > 0$ for every u (meaning that the density of the medium is allowed to vanish at infinity).

On the contrary, in dimension two we can no longer appeal to the continuous embedding $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ (which is false in dimensions greater than one). This introduces some difficulties, that can be overcome by means of a subtler analysis. The main ingredient is the use of a suitable decomposition of the solution, which has been shown to be very effective to prove asymptotic compactness for this kind of hyperbolic problems in the recent paper [10] (but see also [5, 8, 12]).

The three-dimensional case, for which we already established a well-posedness result (see Remark 1.6 below), is much harder, and requires the introduction of different techniques. We will address this issue in a forthcoming work.

Notation. We denote by $H_s = \mathrm{dom}\big[(-\Delta)^{s/2}\big]$, $s \in \mathbb{R}$, the scale of Hilbert spaces generated by $-\Delta$ with Dirichlet boundary conditions on $(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|)$. In particular,

$$H_{-1} = H^{-1}(\Omega), \qquad H_0 = L^2(\Omega), \qquad H_1 = H_0^1(\Omega), \qquad H_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

Then, we introduce the family of product Hilbert spaces

$$\mathcal{H}_s = H_{s+1} \times H_s,$$

endowed with the standard inner products and norms. Throughout the paper, we shall tacitly make use of the Poincaré, Young and Hölder inequalities, along with the continuous embedding $H_1 \hookrightarrow L^p(\Omega)$, for every $p \in [1,\infty)$. We shall also need the Gagliardo-Nirenberg interpolation inequality in dimension two, namely

$$\|z\|_{L^{2p}} \leq c\|z\|^{1/p}\|\nabla z\|^{1-1/p}, \qquad p \in [1,\infty). \tag{0.8}$$

The symbols c and Q will stand for a generic positive constant and a generic positive increasing function, respectively. Finally, for any given function $z(t)$, we write for short $\xi_z(t) = (z(t), \partial_t z(t))$.

We conclude the section recalling two Gronwall-type lemmas that will be used in the sequel.

Lemma 0.2. *Let $E : \mathcal{H}_0 \to \mathbb{R}$ satisfy*

$$\beta\|\zeta\|^2_{\mathcal{H}_0} - m \leq E(\zeta) \leq Q(\|\zeta\|_{\mathcal{H}_0}) + m, \qquad \forall \zeta \in \mathcal{H}_0,$$

for some $\beta > 0$ and $m \geq 0$. Let now $\xi \in C(\mathbb{R}^+, \mathcal{H}_0)$ be given. Suppose that the map $t \mapsto E(\xi(t))$ is continuously differentiable and fulfills the differential inequality

$$\frac{d}{dt}E(\xi(t)) + \varepsilon\|\xi(t)\|^2_{\mathcal{H}_0} \leq k,$$

for some $\varepsilon > 0$ and $k > 0$. Then

$$\|\xi(t)\|_{\mathcal{H}_0} \leq Q\big(k + m + \beta^{-1}\big), \qquad \forall t \geq t_0,$$

where $t_0 = Q(\|\xi(0)\|_{\mathcal{H}_0}) + Q(k)$.

Lemma 0.3. *Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t) + 2\varepsilon\Lambda(t) \leq h(t)\Lambda(t) + k,$$

where $\varepsilon > 0$, $k \geq 0$ and $\int_s^t h(\tau)d\tau \leq \varepsilon(t-s) + m$, for all $t \geq s \geq 0$ and some $m \geq 0$. Then,

$$\Lambda(t) \leq \Lambda(0)e^m e^{-\varepsilon t} + \frac{ke^m}{\varepsilon}, \qquad \forall t \geq 0.$$

We address the reader to [2] for the proof of Lemma 0.2, whereas Lemma 0.3 is obtained quite directly from the usual Gronwall lemma.

1. The Solving Semigroup

To begin our analysis, we prove

Theorem 1.1. *Equation* (0.1) *generates a semigroup* $S(t)$ *on the phase space* $\mathcal{H}_0$.

The proof of the theorem is carried out by means of a Galerkin approximation scheme. Existence is obtained exploiting the uniform energy estimate established in the next proposition, and then passing to the limit in a standard way.

Proposition 1.2. *For every* $t \geq 0$*, there holds*

$$\|\xi_u(t)\|_{\mathcal{H}_0} \leq Q(\|\xi_u(0)\|_{\mathcal{H}_0}) + Q(\|f\|).$$

Proof. Introduce the energy functional

$$E_0 = \|\xi_u\|^2_{\mathcal{H}_0} + 2\langle \Phi(u), 1\rangle - 2\langle f, u\rangle,$$

where

$$\Phi(u) = \int_0^u \varphi(y)dy.$$

From (0.4),

$$\|\nabla u\|^2 + 2\langle \Phi(u), 1\rangle \geq 2\beta\|\nabla u\|^2 - c, \tag{1.1}$$

for some $\beta > 0$. Thus, (0.2) entails

$$\beta\|\xi_u\|^2_{\mathcal{H}_0} - Q(\|f\|) \leq E_0 \leq Q(\|\xi_u\|_{\mathcal{H}_0}) + Q(\|f\|).$$

Multiplying (0.1) by $\partial_t u$, we find

$$\frac{d}{dt}E_0 + 2\langle \sigma(u)\partial_t u, \partial_t u\rangle = 0, \tag{1.2}$$

and the conclusion follows integrating on $(0, t)$. □

Integrating equality (1.2) on $(0, \infty)$, and using Proposition 1.2 and (0.5), we also obtain the existence of suitable dissipation integrals, namely,

Lemma 1.3. *There holds*

$$\sigma_0 \int_0^\infty \|\partial_t u(t)\|^2 dt \leq \int_0^\infty \langle \sigma(u(t))\partial_t u(t), \partial_t u(t)\rangle dt \leq Q(\|\xi_u(0)\|_{\mathcal{H}_0}) + Q(\|f\|).$$

Remark 1.4. Observe that E_0 is a Lyapunov function for $S(t)$.

Uniqueness is a consequence of the following continuous dependence result.

Proposition 1.5. *For every* $T > 0$ *and every* $R \geq 0$*, any two solutions* u^1 *and* u^2 *to equation* (0.1) *fulfill the estimate*

$$\|\xi_{u^1}(T) - \xi_{u^2}(T)\|^2_{\mathcal{H}_{-1}} \leq Q(R)e^{Q(R)T}\|\xi_{u^1}(0) - \xi_{u^2}(0)\|^2_{\mathcal{H}_{-1}},$$

for all initial data $\|\xi_{u^i}(0)\|_{\mathcal{H}_0} \leq R$.

Proof. Define $w(t) = \int_0^t u(\tau)d\tau$. Integrating (0.1) on $(0,t)$ yields

$$\partial_{tt}w(t) + \Sigma(u(t)) - \Delta w(t) = -\int_0^t \varphi(u(\tau))d\tau + \Sigma(u(0)) + \partial_t u(0) + tf,$$

where we put

$$\Sigma(u) = \int_0^u \sigma(y)dy.$$

Let now u^1, u^2 be two solutions to (0.1) with initial data $\|\xi_{u^i}(0)\|_{\mathcal{H}_0} \leq R$, and denote their difference by $\bar{u} = u^1 - u^2$. From the uniform estimate of Proposition 1.2,

$$\|\xi_{u^i}(t)\| \leq Q(R), \qquad \forall t \geq 0.$$

Then, the corresponding difference $\bar{w} = w^1 - w^2$ solves the equation

$$\partial_{tt}\bar{w} + \Sigma(u^1) - \Sigma(u^2) - \Delta\bar{w} = F + G, \tag{1.3}$$

where

$$F(t) = -\int_0^t \big[\varphi(u^1(\tau)) - \varphi(u^2(\tau))\big]d\tau$$

and

$$G = \Sigma(u^1(0)) - \Sigma(u^2(0)) + \partial_t\bar{u}(0).$$

The monotonicity of Σ implies that

$$\langle \Sigma(u^1) - \Sigma(u^2), \bar{u}\rangle \geq 0.$$

Hence, multiplying (1.3) by $\partial_t\bar{w} = \bar{u}$, we have

$$\frac{1}{2}\frac{d}{dt}\|\xi_{\bar{w}}\|^2_{\mathcal{H}_0} \leq \frac{d}{dt}\langle F, \bar{w}\rangle + \frac{d}{dt}\langle G, \bar{w}\rangle - \langle \partial_t F, \bar{w}\rangle.$$

Integrating on $(0,T)$, we are led to

$$\begin{aligned}\|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} &\leq \|\bar{u}(0)\|^2 + 2\langle F(T), \bar{w}(T)\rangle + 2\langle G, \bar{w}(T)\rangle - 2\int_0^T \langle \partial_t F(t), \bar{w}(t)\rangle dt \\ &\leq \frac{1}{2}\|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} + 4\|F(T)\|^2_{H_{-1}} + \|\bar{u}(0)\|^2 + 4\|G\|^2_{H_{-1}} \\ &\quad + 2\int_0^T \|\partial_t F(t)\|_{H_{-1}}\|\xi_{\bar{w}}(t)\|_{\mathcal{H}_0}dt.\end{aligned}$$

Using now the growth restrictions (0.2) and (0.7) on φ and σ, we get at once the controls

$$\begin{aligned}4\|F(T)\|^2_{H_{-1}} &\leq Q(R)T\int_0^T \|\bar{u}(t)\|^2 dt \leq Q(R)T\int_0^T \|\xi_{\bar{w}}(t)\|^2_{\mathcal{H}_0}dt, \\ \|\bar{u}(0)\|^2 + 4\|G\|^2_{H_{-1}} &\leq Q(R)\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}}, \\ \|\partial_t F(t)\|_{H_{-1}} &\leq Q(R)\|\bar{u}(t)\| \leq Q(R)\|\xi_{\bar{w}}(t)\|_{\mathcal{H}_0}.\end{aligned}$$

Therefore, the differential inequality turns into

$$\|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} \leq Q(R)\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}} + Q(R)(1+T)\int_0^T \|\xi_{\bar{w}}(t)\|^2_{\mathcal{H}_0}dt,$$

and from the Gronwall lemma we conclude that

$$\|\bar{u}(T)\|^2 \le \|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} \le Q(R)e^{Q(R)T}\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}}.$$

Finally, from (1.3) we read that

$$\|\partial_t \bar{u}\|_{H_{-1}} = \|\partial_{tt}\bar{w}\|_{H_{-1}} \le \|\Sigma(u^1) - \Sigma(u^2)\|_{H_{-1}} + \|\nabla \bar{w}\| + \|F\|_{H_{-1}} + \|G\|_{H_{-1}},$$

which, due to the above inequalities and to the immediate control

$$\|\Sigma(u^1) - \Sigma(u^2)\|_{H_{-1}} \le Q(R)\|\bar{u}\|,$$

furnishes

$$\|\partial_t \bar{u}(T)\|^2_{H_{-1}} \le Q(R)e^{Q(R)T}\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}}.$$

The proof is then completed. □

Remark 1.6. The very same argument applies in $\mathbb{R}^3$, clearly, provided that φ and σ satisfy suitable growth restrictions (precisely, cubic growth for φ and quadratic growth for σ). In that case, Proposition 1.2 holds as well.

2. Dissipativity

We now proceed to investigate the asymptotic properties of (0.1). We preliminarily show the existence of a bounded absorbing set $\mathbb{B}_0 \subset \mathcal{H}_0$

Proposition 2.1. *For every $R \ge 0$ there exists $t_0 = t_0(R)$ such that*

$$\|\xi_u(t)\|_{\mathcal{H}_0} \le Q(\|f\|), \qquad \forall t \ge t_0,$$

whenever $\|\xi_u(0)\|_{\mathcal{H}_0} \le R$.

Proof. For $\varepsilon \in (0,1)$ to be fixed later, we introduce the energy functional

$$E_\varepsilon = \|\xi_u\|^2_{\mathcal{H}_0} + 2\langle \Phi(u), 1\rangle - 2\langle f, u\rangle + 2\varepsilon\langle \Upsilon(u), 1\rangle + 2\varepsilon\langle \partial_t u, u\rangle,$$

with $\Phi(u)$ as in Proposition 1.2 and

$$\Upsilon(u) = \int_0^u y\sigma(y)dy.$$

Notice that, from (0.5), $\langle \Upsilon(u), 1\rangle \ge 0$. Thus, on account of (0.2), (0.7) and (1.1) we have the controls

$$\beta\|\xi_u\|^2_{\mathcal{H}_0} - Q(\|f\|) \le E_\varepsilon \le Q(\|\xi_u\|_{\mathcal{H}_0}) + Q(\|f\|), \tag{2.1}$$

for some $\beta > 0$, provided that ε is small enough. Multiplying (0.1) by $\partial_t u + \varepsilon u$, we find

$$\frac{d}{dt}E_\varepsilon + 2\varepsilon\|\nabla u\|^2 + 2\langle \sigma(u)\partial_t u, \partial_t u\rangle - 2\varepsilon\|\partial_t u\|^2 + 2\varepsilon\langle \varphi(u), u\rangle = 2\varepsilon\langle f, u\rangle.$$

Using (0.4), we have the estimate

$$2\varepsilon\|\nabla u\|^2 + 2\varepsilon\langle \varphi(u), u\rangle \ge 2\beta\varepsilon\|\nabla u\|^2 - c,$$

whereas (0.5) yields

$$2\langle \sigma(u)\partial_t u, \partial_t u\rangle - 2\varepsilon\|\partial_t u\|^2 \ge \beta\varepsilon\|\partial_t u\|^2,$$

if ε is small enough. Thus, estimating the right-hand side of the differential equality as

$$2\varepsilon\langle f, u\rangle \leq \beta\varepsilon\|\nabla u\|^2 + c\|f\|^2,$$

we end up with the inequality

$$\frac{d}{dt}E_\varepsilon + \beta\varepsilon\|\xi_u\|^2_{\mathcal{H}_0} \leq Q(\|f\|). \tag{2.2}$$

Fixing now the parameter ε in such a way that all the above relationships hold, the claim follows from Lemma 0.2. □

For further scopes, it is convenient to subsume Proposition 1.2 and Proposition 2.1 in the following unitary fashion.

Proposition 2.2. *For every $t \geq 0$, there holds*

$$\|\xi_u(t)\|_{\mathcal{H}_0} \leq Q(\|\xi_u(0)\|_{\mathcal{H}_0})e^{-t} + Q(\|f\|).$$

The next step is to demonstrate higher-order dissipativity.

Proposition 2.3. *For every $t \geq 0$, there holds*

$$\|\xi_u(t)\|_{\mathcal{H}_1} \leq Q(\|\xi_u(0)\|_{\mathcal{H}_1})e^{-\varepsilon_1 t} + Q(\|f\|),$$

for some $\varepsilon_1 > 0$ and some positive increasing function Q.

Proof. Leaning on the absorbing set $\mathbb{B}_0$, it is enough to prove that for every $R > 0$ there exists $\nu = \nu(R)$ such that

$$\|\xi_u(t)\|_{\mathcal{H}_1} \leq Q(\|\xi_u(0)\|_{\mathcal{H}_1})e^{-\nu t} + Q(R + \|f\|), \tag{2.3}$$

whenever $\|\xi_u(0)\|_{\mathcal{H}_0} \leq R$. Fix then $R > 0$ and choose $\|\xi_u(0)\|_{\mathcal{H}_0} \leq R$. From Proposition 2.2, we learn that

$$\|\xi_u(t)\|_{\mathcal{H}_0} \leq Q_R, \tag{2.4}$$

where we wrote for short $Q_R = Q(R+\|f\|)$. Setting $\eta = \partial_t u$, differentiation of (0.1) with respect to time yields

$$\partial_{tt}\eta + \sigma(u)\partial_t\eta + \sigma'(u)\eta^2 - \Delta\eta + \varphi'(u)\eta = 0.$$

Then, for $\varepsilon \in (0, 1)$ to be fixed later, we define the functional

$$\Lambda = \|\xi_\eta\|^2_{\mathcal{H}_0} + 2\varepsilon\langle\eta, \partial_t\eta\rangle,$$

which satisfies the inequalities

$$\frac{1}{2}\|\xi_\eta\|^2_{\mathcal{H}_0} \leq \Lambda \leq 2\|\xi_\eta\|^2_{\mathcal{H}_0}, \tag{2.5}$$

provided that ε is small enough. Multiplying the above equation by $\partial_t\eta + \varepsilon\eta$, we are led to

$$\begin{aligned}&\frac{d}{dt}\Lambda + 2\varepsilon\|\nabla\eta\|^2 + 2\langle\sigma(u)\partial_t\eta, \partial_t\eta\rangle + 2\langle\sigma'(u)\eta^2, \partial_t\eta\rangle - 2\varepsilon\|\partial_t\eta\|^2\\ &= -2\varepsilon\langle\sigma(u)\eta, \partial_t\eta\rangle - 2\varepsilon\langle\sigma'(u)\eta^2, \eta\rangle - 2\varepsilon\langle\varphi'(u)\eta, \eta\rangle - 2\langle\varphi'(u)\eta, \partial_t\eta\rangle.\end{aligned}$$

On account of (0.2), (0.7) and (2.4), the terms in the right-hand side are controlled as

$$-2\varepsilon\langle\sigma(u)\eta,\partial_t\eta\rangle \le \frac{\varepsilon}{3}\|\nabla\eta\|^2 + \varepsilon Q_R\|\partial_t\eta\|^2,$$
$$-2\varepsilon\langle\sigma'(u)\eta^2,\eta\rangle - 2\varepsilon\langle\varphi'(u)\eta,\eta\rangle \le \frac{\varepsilon}{3}\|\nabla\eta\|^2 + Q_R,$$

and, using (0.8),

$$\begin{aligned}-2\varepsilon\langle\varphi'(u)\eta,\partial_t\eta\rangle &\le Q_R\|\eta\|_{L^4}\|\partial_t\eta\| \\ &\le Q_R\|\nabla\eta\|^{1/2}\|\partial_t\eta\| \\ &\le \frac{\varepsilon}{3}\|\nabla\eta\|^2 + \varepsilon\|\partial_t\eta\|^2 + \frac{Q_R}{\varepsilon^2}.\end{aligned}$$

Therefore, we get

$$\frac{d}{dt}\Lambda + \varepsilon\|\nabla\eta\|^2 + 2\langle\sigma(u)\partial_t\eta,\partial_t\eta\rangle + 2\langle\sigma'(u)\eta^2,\partial_t\eta\rangle - \varepsilon(3+Q_R)\|\partial_t\eta\|^2 \le \frac{Q_R}{\varepsilon^2}.$$

We now turn to the terms in the left-hand side. We have

$$2\langle\sigma'(u)\eta^2,\partial_t\eta\rangle \ge -\langle\sigma(u)\partial_t\eta,\partial_t\eta\rangle - \langle[\sigma'(u)]^2[\sigma(u)]^{-1}\eta^2,\eta^2\rangle.$$

At this point, we fix $\varepsilon = \varepsilon(R)$ small enough such that (2.5) holds and

$$\sigma_0 - \varepsilon(3+Q_R) \ge \varepsilon.$$

Hence, using (0.5) and (2.5), we obtain the differential inequality

$$\frac{d}{dt}\Lambda + \frac{\varepsilon}{2}\Lambda \le \langle[\sigma'(u)]^2[\sigma(u)]^{-1}\eta^2,\eta^2\rangle + Q_R. \tag{2.6}$$

The last step is the control of the remaining term in the left-hand side. On account of (0.5) and (0.6), there is no loss of generality to assume $\delta < 1/2$. Note that, from (0.6),

$$\big([\sigma'(u)]^2[\sigma(u)]^{-1}\big)^{1/(1-2\delta)} \le c\sigma(u).$$

Thus, applying the Hölder inequality with exponents $\big(\frac{1}{1-2\delta},p_1,p_2\big)$, with $1/p_1 + 1/p_2 = 2\delta$, we get

$$\begin{aligned}\langle[\sigma'(u)]^2[\sigma(u)]^{-1}\eta^2,\eta^2\rangle &= \langle[\sigma'(u)]^2[\sigma(u)]^{-1}|\eta|^{2-4\delta},\eta^2|\eta|^{4\delta}\rangle \\ &\le c\langle\sigma(u)\eta,\eta\rangle^{1-2\delta}\|\eta\|^2_{L^{2p_1}}\|\eta\|^{4\delta}_{L^{4\delta p_2}}.\end{aligned}$$

Exploiting the interpolation inequality (0.8), we find the controls

$$\|\eta\|^2_{L^{2p_1}} \le c\|\eta\|^{2/p_1}\|\nabla\eta\|^{2-2/p_1} \le Q_R\|\nabla\eta\|^{2-2/p_1},$$

and

$$\|\eta\|^{4\delta}_{L^{4\delta p_2}} \le c\|\eta\|^{2/p_2}\|\nabla\eta\|^{4\delta-2/p_2} \le Q_R\|\nabla\eta\|^{4\delta-2/p_2}.$$

Since by (0.7) and (2.4) we have

$$\langle\sigma(u)\eta,\eta\rangle \le Q_R\|\nabla\eta\|,$$

applying (2.5) we conclude that

$$
\begin{aligned}
&\langle[\sigma'(u)]^2[\sigma(u)]^{-1}\eta^2,\eta^2\rangle\\
&\le Q_R\langle\sigma(u)\eta,\eta\rangle^{1-2\delta}\|\nabla\eta\|^2\\
&\le Q_R\langle\sigma(u)\eta,\eta\rangle^{1-2\delta}+Q_R\langle\sigma(u)\eta,\eta\rangle^{1-2\delta}\Lambda\\
&\le Q_R+\frac{\varepsilon}{8}\|\nabla\eta\|^2+\frac{\varepsilon}{8}\Lambda+Q_R\langle\sigma(u)\eta,\eta\rangle\Lambda\\
&\le Q_R+\frac{\varepsilon}{4}\Lambda+Q_R\langle\sigma(u)\eta,\eta\rangle\Lambda.
\end{aligned}
$$

Therefore, (2.6) turns into

$$\frac{d}{dt}\Lambda+\frac{\varepsilon}{4}\Lambda\le Q_R+Q_R\langle\sigma(u)\eta,\eta\rangle\Lambda.$$

Thanks to Lemma 1.3 (recall that $\eta=\partial_t u$), we are in a position to apply Lemma 0.3, which, together with (2.5), entail

$$\|\xi_\eta(t)\|_{\mathcal{H}_0}\le Q(\|\xi_\eta(0)\|_{\mathcal{H}_0})e^{-\nu t}+Q_R,$$

for some $\nu=\nu(R)>0$. By comparison with the original equation (0.1), we obtain the required inequality (2.3). □

3. Asymptotic Compactness: Proof of Theorem 0.1

In order to prove the existence of the global attractor and its regularity, we shall exploit a quite effective decomposition of the solution, which has been used in the recent paper [10]. This approach will allow us to prove the desired result without bootstrap arguments; thus, avoiding the use of fractional operators, that would require a rather delicate treatment (due to the presence of the nonlinear damping). First, using (0.2) and Proposition 2.2, we choose $\theta\ge\ell$ large enough such that the inequality

$$\frac{1}{2}\|\nabla z\|^2+(\theta-2\ell)\|z\|^2-\langle\varphi'(u(t))z,z\rangle\ge 0 \tag{3.1}$$

holds for every $z\in H_1$, every $t\ge 0$ and every solution $u(t)$ with $\xi_u(0)\in\mathbb{B}_0$. Then, we set

$$\psi(r)=\varphi(r)+\theta r.$$

Condition (0.2) still holds with ψ in place of φ, besides by virtue of (0.3),

$$\psi'(r)\ge 0. \tag{3.2}$$

We now consider initial data $\xi_u(0)$ belonging to the bounded absorbing set $\mathbb{B}_0$ produced by Proposition 2.1, and we decompose the corresponding solution to (0.1) into the sum $u=w+v$, where w and v solve the equations

$$\begin{cases}\partial_{tt}w+\sigma(w)\partial_t w-\Delta w+\psi(w)=\theta u+f,\\ \xi_w(0)=(0,0),\\ w_{|\partial\Omega}=0,\end{cases} \tag{3.3}$$

and

$$
(3.4)\qquad \begin{cases} \partial_{tt} v + \sigma(u)\partial_t v - \Delta v + (\sigma(u) - \sigma(w))\partial_t w + \psi(u) - \psi(w) = 0, \\ \xi_v(0) = \xi_u(0), \\ v_{|\partial\Omega} = 0. \end{cases}
$$

Till the end of the section, the generic constant $c \geq 0$ will depend only on the size of the absorbing set $\mathbb{B}_0$. Arguing exactly as in Proposition 1.2 and Proposition 2.1, we obtain the uniform bound

$$
(3.5)\qquad \|\xi_w(t)\|_{\mathcal{H}_0} \leq c, \qquad \forall t \geq 0.
$$

In addition, multiplying (3.3) by $\partial_t w$ and integrating on (s,t), thanks to Lemma 1.3 we readily see that

$$
(3.6)\qquad \sigma_0 \int_s^t \|\partial_t w(\tau)\|^2 d\tau \leq \int_s^t \langle \sigma(w(\tau))\partial_t w(\tau), \partial_t w(\tau)\rangle d\tau \leq \omega(t-s) + \frac{c}{\omega},
$$

for every $t \geq s \geq 0$ and every $\omega > 0$.

Lemma 3.1. *For every $t \geq 0$, we have that $\|\xi_w(t)\|_{\mathcal{H}_1} \leq c$.*

We leave to the reader the proof of Lemma 3.1, since it is basically a repetition of the proof of Proposition 2.3. Indeed, setting now $\eta = \partial_t w$, differentiation of (3.3) with respect to time entails

$$
\partial_{tt}\eta + \sigma(w)\partial_t \eta + \sigma'(w)\eta^2 - \Delta\eta + \psi'(w)\eta = \theta\partial_t u.
$$

Hence, the only difference here is that the initial data are null, and in the final differential inequality it will appear also the extra term $\|\partial_t u\|^2$ multiplied by the functional. Notice that Lemma 0.3 is needed in its full strength, since in this case we have dissipation integrals of the form (3.6).

Lemma 3.2. *For every $t \geq 0$ and some $\nu > 0$, we have that $\|\xi_v(t)\|_{\mathcal{H}_0} \leq ce^{-\nu t}$.*

Proof. For $\varepsilon \in (0,1)$ to be determined later, define

$$
\Lambda = \|\xi_v\|^2_{\mathcal{H}_0} + 2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle + 2\varepsilon\langle \partial_t v, v\rangle.
$$

On account of (0.3) and (3.1), together with the uniform bounds on $\|\nabla u\|$ and $\|\nabla w\|$, the functional Λ satisfies the inequalities

$$
(3.7)\qquad \frac{1}{4}\|\xi_v\|^2_{\mathcal{H}_0} \leq \Lambda \leq c\|\xi_v\|^2_{\mathcal{H}_0},
$$

provided that ε is small enough. Multiplying (3.4) by $\partial_t v + \varepsilon v$, we have the equality

$$
\begin{aligned}
&\frac{d}{dt}\Lambda + 2\varepsilon\|\nabla v\|^2 + 2\langle \sigma(u)\partial_t v, \partial_t v\rangle - 2\varepsilon\|\partial_t v\|^2 + 2\varepsilon\langle \psi(u) - \psi(w), v\rangle \\
&= 2\langle (\psi'(u) - \psi'(w))\partial_t w, v\rangle - \langle \psi''(u)\partial_t u, v^2\rangle - 2\langle (\sigma(u) - \sigma(w))\partial_t w, \partial_t v\rangle \\
&\quad - 2\varepsilon\langle (\sigma(u) - \sigma(w))\partial_t w, v\rangle - 2\varepsilon\langle (\sigma(u)\partial_t v, v\rangle.
\end{aligned}
$$

We now reconstruct Λ in the right-hand side. Indeed, it is easily seen that, for ε small enough,

$$\begin{aligned}
&2\varepsilon\|\nabla v\|^2 + 2\langle\sigma(u)\partial_t v, \partial_t v\rangle - 2\varepsilon\|\partial_t v\|^2 + 2\varepsilon\langle\psi(u) - \psi(w), v\rangle \\
&\geq \varepsilon\Lambda + \frac{\varepsilon}{2}\|\nabla v\|^2 + \sigma_0\|\partial_t v\|^2.
\end{aligned}$$

Therefore, we are led to the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\Lambda + \varepsilon\Lambda + \frac{\varepsilon}{2}\|\nabla v\|^2 + \sigma_0\|\partial_t v\|^2 \\
&\leq 2\langle(\psi'(u) - \psi'(w))\partial_t w, v\rangle - \langle\psi''(u)\partial_t u, v^2\rangle - 2\langle(\sigma(u) - \sigma(w))\partial_t w, \partial_t v\rangle \\
&\quad - 2\varepsilon\langle(\sigma(u) - \sigma(w))\partial_t w, v\rangle - 2\varepsilon\langle(\sigma(u)\partial_t v, v\rangle.
\end{aligned}$$

Then, we proceed to control the terms in the right-hand side. Regarding the first two, we have (cf. [10] where similar calculations appear)

$$2\langle(\psi'(u) - \psi'(w))\partial_t w, v\rangle - \langle\psi''(u)\partial_t u, v^2\rangle \leq \frac{\varepsilon}{4}\|\nabla v\|^2 + \frac{c}{\varepsilon}\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda.$$

Concerning the remaining terms, we have (cf. (0.7))

$$\begin{aligned}
-2\langle(\sigma(u) - \sigma(w))\partial_t w, \partial_t v\rangle &\leq c\langle(1 + |u|^q + |w|^q)|\partial_t w|^{1/2}|\partial_t w|^{1/2}|v|, |\partial_t v|\rangle \\
&\leq c\|\partial_t w\|^{1/2}\|\nabla v\|\|\partial_t v\| \\
&\leq c\|\partial_t w\|^{1/2}\Lambda \\
&\leq \frac{\varepsilon}{4}\Lambda + c\|\partial_t w\|^2\Lambda.
\end{aligned}$$

Similarly,

$$-2\varepsilon\langle(\sigma(u) - \sigma(w))\partial_t w, v\rangle \leq \frac{\varepsilon}{4}\Lambda + c\|\partial_t w\|^2\Lambda.$$

Finally,

$$-2\varepsilon\langle\sigma(u)\partial_t v, v\rangle \leq c\varepsilon\|\partial_t v\|\|\nabla v\| \leq \frac{\varepsilon}{4}\|\nabla v\|^2 + c\varepsilon\|\partial_t v\|^2.$$

Collecting the above inequalities, we end up with

$$\frac{d}{dt}\Lambda + \frac{\varepsilon}{2}\Lambda + (\sigma_0 - c\varepsilon)\|\partial_t v\|^2 \leq \frac{c}{\varepsilon}\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda.$$

At this point, we fix $\varepsilon > 0$ small enough such that the above conditions are satisfied and $\sigma_0 - c\varepsilon \geq 0$, so to obtain

$$\frac{d}{dt}\Lambda + \frac{\varepsilon}{2}\Lambda \leq c\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda.$$

In view of the integral estimates provided by Lemma 1.3 and (3.6), the conclusion follows by applying Lemma 0.3 along with (3.7). □

We can now conclude the

Proof of Theorem 0.1. Proposition 2.1 provides the existence of a bounded absorbing set $\mathbb{B}_0$, while Lemma 3.1 and Lemma 3.2 show that $S(t)\mathbb{B}_0$ is (exponentially) attracted by a bounded subset $\mathcal{C} \subset \mathcal{H}_1$. Hence, $\mathcal{C}$ is a compact attracting set. By standard arguments of the theory of dynamical systems (see e.g. [1, 9, 11]), we conclude that there exists a (unique) compact global attractor $\mathcal{A} \subset \mathcal{C}$. Since $S(t)$

possesses a Lyapunov function (cf. Remark 1.4), the attractor is the unstable set of the stationary points of $S(t)$. □

4. Exponential Attractors

We finally state a result on the existence of an exponential attractor.

Theorem 4.1. *The semigroup $S(t)$ possesses a regular exponential attractor, namely, a compact set $\mathcal{M} \subset \mathcal{H}_0$, bounded in $\mathcal{H}_1$ and of finite fractal dimension in $\mathcal{H}_0$, positively invariant for $S(t)$, and satisfying the following exponential attraction property:*

(EA) *There exist $\omega > 0$ such that*

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{M}) \leq Q(R)e^{-\omega t},$$

for every $\mathcal{B} \subset \mathcal{H}_0$ such that $\sup_{\zeta \in \mathcal{B}} \|\zeta\|_{\mathcal{H}_0} \leq R$.

Here, $\mathrm{dist}_{\mathcal{H}_0}$ denotes the usual Hausdorff semidistance in $\mathcal{H}_0$. As a byproduct, we have

Corollary 4.2. *The global attractor $\mathcal{A}$ of $S(t)$ has finite fractal dimension in $\mathcal{H}_0$.*

Proof. In the previous section we proved the existence of a bounded subset of $\mathcal{C}$ of $\mathcal{H}_1$ (we can assume without loss of generality that $\mathcal{C}$ is a closed ball of $\mathcal{H}_1$) such that

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{C}) \leq Q(R)e^{-\nu t},$$

for every $\mathcal{B} \subset \mathcal{H}_0$ with $\sup_{\zeta \in \mathcal{B}} \|\zeta\|_{\mathcal{H}} \leq R$. It is also apparent from Proposition 2.3 that, up to possibly enlarging $\mathcal{C}$, there is a time $t_{\mathcal{C}} \geq 0$ such that $S(t)\mathcal{C} \subset \mathcal{C}$, whenever $t \geq t_{\mathcal{C}}$. We now appeal to the following abstract result [3] (see also [7]).

Lemma 4.3. *Let there exist $t^\star \geq t_{\mathcal{C}}$ such that*

(C1) *The map $(t, z) \mapsto S(t)z : [t^\star, 2t^\star] \times \mathcal{C} \to \mathcal{C}$ is Lipschitz continuous when $\mathcal{C}$ is endowed with the $\mathcal{H}_0$-topology.*

(C2) *Setting $S = S(t^\star)$, there are $\gamma \in (0, \frac{1}{2})$ and $\Gamma \geq 0$ such that, for every $\zeta_1, \zeta_2 \in \mathcal{C}$,*

$$S\zeta_1 - S\zeta_2 = D(\zeta_1, \zeta_2) + K(\zeta_1, \zeta_2),$$

where

$$\|D(\zeta_1, \zeta_2)\|_{\mathcal{H}_0} \leq \gamma \|\zeta_1 - \zeta_2\|_{\mathcal{H}_0} \qquad \text{and} \qquad \|K(\zeta_1, \zeta_2)\|_{\mathcal{H}_1} \leq \Gamma \|\zeta_1 - \zeta_2\|_{\mathcal{H}_0}.$$

Then there exists a set $\mathcal{M} \subset \mathcal{C}$, closed and of finite fractal dimension in $\mathcal{H}_0$, positively invariant for $S(t)$, such that

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{C}, \mathcal{M}) \leq Me^{-\omega_0 t},$$

for some $\omega_0 > 0$ and $M \geq 0$.

Since $\mathcal{C}$ is positively invariant and bounded in $\mathcal{H}_1$, the nonlinearities become nonessential. Hence, the check of (C1)-(C2) is not different from the analogous case in space dimension one, previously treated in the paper [6] (to which we address the reader for the details). Thus, we obtain “almostthe the thesis of Theorem 4.1, in the sense that the basin of exponential attraction is $\mathcal{C}$, and not the whole space $\mathcal{H}_0$, as required.

To reach the conclusion, we have to appeal to the transitivity of the exponential attraction [4, Theorem 5.1]. Namely, if

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B},\mathcal{C}) \leq Q(R)e^{-\nu t} \qquad \text{and} \qquad \mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{C},\mathcal{M}) \leq Me^{-\omega_0 t},$$

then the desired property (EA) follows, provided that we can show the (locally) Lipschitz continuity

$$\|S(t)\zeta_1 - S(t)\zeta_2\|_{\mathcal{H}_0} \leq Ce^{Kt}\|\zeta_1 - \zeta_2\|_{\mathcal{H}_0},$$

where both C and K may depend (increasingly) on the norms of ζ_1, ζ_2. This seems out of reach. Nonetheless, a closer look to the proof of [4, Theorem 5.1] shows that in fact it is enough to prove the above continuity for $\zeta_1 \in \mathcal{H}_0$ and $\zeta_2 \in \mathcal{C}$, which is true and quite easy to demonstrate. Indeed, denoting by u^j the solution to (0.1) with initial data ζ_j, and by $\bar{u} = u^1 - u^2$, multiplying (0.1) by $\partial_t \bar{u}$ the only problematic term to control is

$$\langle \sigma(u^1)\partial_t u^1 - \sigma(u^2)\partial_t u^2, \partial_t \bar{u}\rangle.$$

But, due to (0.6),

$$\begin{aligned}\langle \sigma(u^1)\partial_t u^1 - \sigma(u^2)\partial_t u^2, \partial_t u\rangle &= \langle \sigma(u^1)\partial_t \bar{u}, \partial_t \bar{u}\rangle + \langle (\sigma(u^1) - \sigma(u^2))\partial_t u^2, \partial_t \bar{u}\rangle \\ &\geq \langle (\sigma(u^1) - \sigma(u^2))\partial_t u^2, \partial_t \bar{u}\rangle,\end{aligned}$$

and using the fact that $\partial_t u^2 \in H_1$ we easily get that

$$|\langle (\sigma(u^1) - \sigma(u^2))\partial_t u^2, \partial_t \bar{u}\rangle| \leq c\|\xi_{\bar{u}}\|^2_{\mathcal{H}_0},$$

for some c depending only on the size of the initial data. An application of the Gronwall lemma completes the argument. □

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