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A quadrature method for the hypersingular integral equation on an interval

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1 Introduction

In this paper we consider the hypersingular integral equation on the interval

$$(Du)(t) := p.f. \int_0^1 \frac{u(\tau)}{|\tau - t|^2} d\tau = f(t), \quad 0 \leq t \leq 1, \tag{1}$$

where f is a given function and u is to be found. The integral in (1) is to be interpreted as a Hadamard finite part integral. For the definition of such a finite part integral we refer, e.g., to [7].

The hypersingular integral equation (1) results from a certain boundary integral method, which has attracted the attention of several mathematicians in recent years. In particular, we mention the paper [3] of Costabel and Stephan, where the Galerkin method for the hypersingular integral equation on polygons is studied, and the article [2] of Costabel, which gives a survey about several boundary integral operators on Lipschitz domains and investigates the Galerkin method for those. In the paper [13] of von Petersdorff and Stephan a multigrid method on graded meshes is considered for the hypersingular integral equation. In [1] (Sect.1.6 and 5.1) a quadrature method for the hypersingular integral equation on an interval is deduced and an error estimate is proved. The first regorous analysis of a fully discretized method for the hypersingular integral equation has been given by Kieser, Kleemann and Rathsfeld in [8]. There a very easy discretisation scheme is used to get a quadrature method for this equation on a smooth closed curve and stability and error estimates for this method are obtained.

Another approach is given by Erwin, Stephan in [6], where a collocation method using Chebyshev polynomials has been considered for the hypersingular integral equation on the interval. In [6] the operator D is considered as an operator acting between some spaces of Sobolev type, which are defined by means of Chebyshev polynomials.

In the present paper we shall propose a fully discretized quadrature method for the hypersingular integral equation on the interval (1). Because the solution of this equation has an end–point behaviour like $s^{\frac{1}{2}}(1-s)^{\frac{1}{2}}$ (see [6]) we carry out a refinement of the grid near the end points of the interval. To this end we perform a change of the variables $\tau = \gamma(\sigma)$, $t = \gamma(s)$ in the integral (1), where γ has an end–point behaviour like s^α. Transformations like that have been used already for some integral equations, for example, in the case of the Cauchy singular integral equation (see [12] and [14]) or in the case of boundary integral equations of the second kind for the harmonic Dirichlet problem in plane domains with corners (c.f. [9]). In the present paper the transformation $\gamma : [0,1] \to [0,1]$ is chosen like in [9]

$$\gamma(s) = \frac{[v(s)]^\alpha}{[v(s)]^\alpha + [v(1-s)]^\alpha} \quad , \qquad 0 \leq s \leq 1, \tag{2}$$

with

$$v(s) = \left(\frac{1}{\alpha} - \frac{1}{2}\right)(1-2s)^3 + \frac{1}{\alpha}(2s-1) + \frac{1}{2}, \qquad \alpha > \frac{3}{2}. \tag{3}$$

The function γ has an end–point behaviour like s^α near 0 and like $1-(1-s)^\alpha$ near 1. Note that the cubic polynomial v is chosen such that $v(0) = 0$, $v(1) = 1$, and $\gamma'(\frac{1}{2}) = 2$. The latter property ensures, roughly speaking, that one half of the grid points is equally distributed over the total interval, whereas the other half is accumulated towards the two end points.

Multiplying Equation (1) by $\gamma'(s)$, we get the transformed equation

$$p.f. \int_0^1 \frac{\gamma'(s)\gamma'(\sigma)}{|\gamma(\sigma)-\gamma(s)|^2} w(\sigma)d\sigma = g(s), \qquad 0 < s < 1, \tag{4}$$

with

$$w(s) := u(\gamma(s)), \qquad g(s) := f(\gamma(s))\gamma'(s).$$

Using the quadrature rule

$$\int_{-\infty}^{\infty} f(t)dt \sim \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} f(t_j)\frac{2}{n} \quad , \qquad t_k = \frac{k}{n}, \tag{5}$$

for n even and applying a kind of regularization to the finite part integral (cf. the next section for more details), we get the quadrature method

$$g(t_k) = \sum_{\substack{j=1 \\ j\equiv k+1 \bmod 2}}^{n-1} \frac{2}{n} \frac{\gamma'(t_j)\gamma'(t_k)}{|\gamma(t_j)-\gamma(t_k)|^2}\xi_j - \frac{n\pi^2}{2}\xi_k \quad , k = 1, \ldots, n-1. \tag{6}$$

The term $-\frac{n\pi^2}{2}\xi_k$ results from the mentioned regularization. A corresponding term occurs in the case of a closed curve (see [8]).

The paper is organized as follows. In Section 2 the quadrature method (6) is deduced.

In Section 3 the mapping properties of the approximate operators corresponding to (6) and the corresponding discretized spaces are investigated. In Section 4 the stability of the method is proved. Let us denote the matrix of the linear system of Equation (6) by A_n. The main point of the proof is that there holds $\langle -A_n\xi, \xi\rangle \sim \langle B_n\xi, \xi\rangle$ for all finite sequences $\xi = \{\xi_k\}_{k=1}^{n-1}$ uniformly with respect to n. Here B_n is the norm isomorphism of the regarded discrete spaces.

The error estimate is deduced in Section 5. Let f be sufficiently smooth such that $w = u \circ \gamma$ belongs to the Sobolev space H^s with $s < \frac{\alpha+1}{2}$. If $u_n = w_n \circ \gamma^{-1}$ (i.e. $u_n \circ \gamma = w_n$), where w_n is a high order interpolation of the approximate values $w_n(t_j) = \xi_j$, $j = 1, \ldots, n-1$, obtained by solving (6), then the Sobolev norm $\|u - u_n\|_{\frac{1}{2}}$ can be estimated by $Cn^{\frac{3}{2}+\epsilon-s}$ with ϵ sufficiently small.

In Section 6 another transformation is used, namely, a cos–transformation like that used for the numerical solution of first–kind integral equations with logarithmic kernel (c.f. Sect. 3.8 in [16]). The quadrature method derived with the help of this transformation is shown to be stable, too. Here the proof reduces to the case of the unit circle. The stability of this method is easier to prove than that of the method with γ defined by (2) and the order of the convergence is higher, since there is no bound for s from above. However the techniques used in Sections 2–5 and 7 apply to the case of more general integral equations on the interval or on the polygon provided the asymptotic behaviour of the solution near to the endpoints or corner points, respectively, is known.

In the Appendix some technical lemmas are proved.

Here we acknowledge the useful advices of A. Rathsfeld and thank him.

2 The discretisation of the hypersingular integral equation

Consider the hypersingular integral equation on the interval $I = [0, 1]$,

$$(Du)(t) := p.f. \int_0^1 \frac{u(\tau)}{|\tau - t|^2} d\tau = f(t),\ t \in I.$$

By [13], [2] and [3], the mapping

$$D : \tilde{H}_{\frac{1}{2}}(I) \longrightarrow H_{-\frac{1}{2}}(I)$$

is bijective and continuous. Here the space $\tilde{H}_{\frac{1}{2}}(I)$ is defined by

$$\tilde{H}_{\frac{1}{2}}(I) := \{u|_I : u \in H_{\frac{1}{2}}(\mathbb{R}), u|_{\mathbb{R}\setminus I} \equiv 0\}$$

and equipped with the norm of $H_{\frac{1}{2}}(\mathbb{R})$. The space $H_{-\frac{1}{2}}(I)$ is defined as the dual space of $\tilde{H}_{\frac{1}{2}}(I)$ with respect to the L_2-scalar product (see [3]).

Remark 2.1 *For $u \in H_{\frac{1}{2}}(\mathbb{R})$ with $u|_{\mathbb{R}\setminus I} \equiv 0$, there holds*

$$\| u \|_{\frac{1}{2}}^2 = \| u \|_{L_2}^2 + | u |_{\frac{1}{2}}^2$$

with

$$\begin{aligned} |u|_{\frac{1}{2}}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy \\ &= \int_0^1\int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy + 2\int_0^1 \frac{|u(x)|^2}{x(1-x)} dx, \end{aligned}$$

and $\| u \|_{\frac{1}{2}} \sim | u |_{\frac{1}{2}}$.

The proof is well known and not hard.

Remark 2.2 *Let $f \in C^\infty$ and u be the solution of Equation (1). Then there holds $u(t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} gl(t)$ with $gl(t)$ smooth.*

The statement of Remark 2.2 will be frequently used in the subsequent Sections and will be proved in the appendix. For the case $f \notin C^\infty$ a statement about the asymptotics can be found in [6].

In order to get a refinement of the grid near the end points of the interval I we will apply a transformation of coordinates. Consider the transformation function $\gamma : I \to I$ of R.Kress [9] given by (2) and (3). The condition $\alpha > \frac{4}{3}$ is necessary to guarantee the monotonicity of v. The stronger condition $\alpha > \frac{3}{2}$ will be needed in the proof of stability.

By [7] it is possible to apply the usual rules of transformation for the finite part integral in (1) if $t \in (0,1)$. Thus (1) is equivalent to (4). Now we set

$$h(s,\sigma) := \frac{|\sigma - s|^2 \gamma'(s)\gamma'(\sigma)}{|\gamma(\sigma)-\gamma(s)|^2}.$$

With this notation Equation (4) is equivalent to

$$Aw(s) := p.f. \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2} h(s,\sigma) d\sigma = g(s). \tag{7}$$

We shall deduce a quadrature method for Equation (7). To this end we use the well known quadrature rule (5). Obviously, there holds

$$p.f. \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2} h(s,\sigma) d\sigma = p.f. \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2} d\sigma + \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2}(h(s,\sigma)-1) d\sigma. \tag{8}$$

Now we continue the function w to a function on $\mathbb{R}$ by setting $w(t) := 0$ for all $t \notin [0,1]$. Note that w remains smooth since $u(s)$ has the end–point asymptotics $s^{\frac{1}{2}}(1-s)^{\frac{1}{2}}$ and s is replaced by $s = \gamma(\sigma)$ with $\gamma(\sigma) \sim \sigma^\alpha (1-\sigma)^\alpha$, α being sufficiently large. The first integral on the right–hand side of (8) is a finite part integral. At the point $s = t_k$, $1 \le k \le n-1$ we can compute it by using the regularisation

$$p.f.\int_0^1 \frac{w(\sigma)}{|\sigma - t_k|^2}d\sigma = p.f.\int_{-\infty}^{\infty} \frac{w(\sigma)}{|\sigma - t_k|^2}d\sigma$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} \frac{w(\sigma) - w(t_k) - w'(t_k)(\sigma - t_k)}{|\sigma - t_k|^2}d\sigma \\ &\quad + w(t_k)p.f.\int_{-\infty}^{\infty} \frac{1}{|\sigma - t_k|^2}d\sigma + w'(t_k)p.f.\int_{-\infty}^{\infty} \frac{\sigma - t_k}{|\sigma - t_k|^2}d\sigma. \end{aligned}$$

Now there holds

$$\begin{aligned} p.f.\int_{-\infty}^{\infty} \frac{1}{|\sigma - t_k|^2}d\sigma &= 0, \\ p.f.\int_{-\infty}^{\infty} \frac{1}{\sigma - t_k}d\sigma &= 0. \end{aligned}$$

Thus with $\xi_j = w(t_j), j = 1, \ldots, n-1$, and $\xi_j = 0, j \le 0$ or $j \ge n$, we obtain

$$\begin{aligned} p.f.\int_0^1 \frac{w(\sigma)}{|\sigma - t_k|^2}d\sigma \sim \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{1}{|t_j - t_k|^2}\frac{2}{n}\xi_j - \xi_k \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{1}{|t_j - t_k|^2}\frac{2}{n} \\ - w'(t_k) \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{t_j - t_k}{|t_j - t_k|^2}\frac{2}{n}. \end{aligned}$$

Now we compute the sums in the following way

$$\begin{aligned} \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{1}{|t_j - t_k|^2}\frac{2}{n} &= \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{2n}{|j-k|^2} = \sum_{\substack{j=1 \\ j \text{ odd}}}^{\infty} \frac{4n}{j^2}, \\ &= \sum_{j=1}^{\infty} \frac{4n}{(2j-1)^2} = 4n\frac{\pi^2}{8} = \frac{n\pi^2}{2}, \\ \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{t_j - t_k}{|t_j - t_k|^2}\frac{2}{n} &= \sum_{\substack{j=-\infty \\ j\equiv k+1 \bmod 2}}^{\infty} \frac{2}{j-k} = \sum_{\substack{j=-\infty \\ j \text{ odd}}}^{\infty} \frac{4}{j} = 0. \end{aligned}$$

Finally, we arrive at

$$p.f.\int_0^1 \frac{w(\sigma)}{|\sigma - t_k|^2}d\sigma \sim \sum_{\substack{j=1 \\ j\equiv k+1 \bmod 2}}^{n-1} \frac{1}{|t_j - t_k|^2}\frac{2}{n}\xi_j - \frac{n\pi^2}{2}\xi_k. \tag{9}$$

Now we consider the second part of the sum in (8). First define

$$l(s,\sigma) := \frac{h(s,\sigma) - 1}{|\sigma - s|^2}. \tag{10}$$

The function $l(t_k,\sigma)$ is continuous, because

$$\lim_{\sigma\to t_k} l(t_k,\sigma) = \frac{1}{6}\frac{\gamma'''(t_k)}{\gamma'(t_k)} - \frac{1}{4}\left(\frac{\gamma''(t_k)}{\gamma'(t_k)}\right)^2, \qquad k=1,\dots,n-1.$$

Thus

$$\int_0^1 \frac{w(\sigma)}{|\sigma-t_k|^2}\left(h(\sigma,t_k)-1\right)d\sigma = \int_0^1 w(\sigma)\; l(t_k,\sigma)d\sigma \sim$$

$$\sim \sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1} \frac{2}{n}\, l(t_k,t_j)\xi_j = \sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1} \frac{2}{n}\left(\frac{\gamma'(t_j)\gamma'(t_k)}{|\gamma(t_j)-\gamma(t_k)|^2} - \frac{1}{|t_j-t_k|^2}\right)\xi_j.$$

Combining this with (9), we get the quadrature method (6) for the approximate solution of (7).

Now we define an approximate solution w_n for w by

$$w_n = \sum_{j=1}^{n-1} \xi_j \phi_j^{(d)}(s), \tag{11}$$

where $\phi_j^{(d)}$ denotes the interpolation basis of piecewise polynomials of degree d, i.e. $\phi_j^{(d)}(t_k) = \delta_{k,j}$, $\phi_j^{(d)}$ is continuous and the restriction of $\phi_j^{(d)}$ to the interval $[\frac{dk}{n}, \frac{d(k+1)}{n}]$ is a polynomial of the degree d for $k=0,\dots,\frac{n}{d}-1$. Here we choose n such that $\frac{n}{d}\in I\!N$. If $d=1$ we write ϕ_j instead of $\phi_j^{(1)}$. There holds

$$\phi_j(s) = \begin{cases} n(s-\frac{j-1}{n}), & s\in[\frac{j-1}{n},\frac{j}{n}],\\ n(\frac{j-1}{n}-s), & s\in[\frac{j}{n},\frac{j+1}{n}],\\ 0, & else. \end{cases}$$

Remark 2.3 *The linear spline functions are the simplest splines with* $\phi_j\in\tilde H_{\frac12}$ *and* $\phi_j\circ\gamma^{-1}\in\tilde H_{\frac12}$.

The proof of $\phi_j\circ\gamma^{-1}\in\tilde H_{\frac12}$ can be found in the appendix.

Let us denote the matrix of the linear system of equations (6) by A_n, i.e.

$$A_n = n(h(t_k,t_j)a_{k-j})_{k,j=1}^{n-1}, \tag{12}$$

with

$$a_k = \begin{cases} -\frac{\pi^2}{2}, & k=0,\\ \frac{2}{|k|^2}, & k \text{ is odd},\\ 0, & else. \end{cases}$$

We shall interprete this matrix A_n as an operator from the discrete space of finite sequences equipped with the norm induced by $\tilde{H}_{\frac{1}{2},\alpha}$ into its dual space. The definition of these discrete spaces is given in the next section.

3 The discrete spaces $h_{\frac{1}{2},\alpha}$, $h_{-\frac{1}{2},\alpha}$

Proving the stability for the operator sequences A_n means that we have to show the invertibility of the operators A_n and the uniform boundedness of their inverses. The operators A_n have to be considered from the discrete space of finite sequences equipped with the norm induced by $\tilde{H}_{\frac{1}{2},\alpha}$ into its dual space. It will be easier to prove the stability by using some equivalent norms of these spaces. Before introducing these norms we mention some properties of the transformation γ. The properties 1.–8. are easy consequences of the definitions of v, γ and h. A comment on the proof of property 9 can be found in the appendix.

Remark 3.1 *There holds*

1. $v(0) = 0$, $v(1) = 1$, $v(\frac{1}{2}) = \frac{1}{2}$, $v(s) = 1 - v(1-s)$, *and v is strictly monotonically increasing.*

2. $v'(s) = -6(\frac{1}{\alpha} - \frac{1}{2})(1-2s)^2 + \frac{2}{\alpha}$, $v'(s) = v'(1-s)$, $v'(0) = v'(1) = 3 - \frac{4}{\alpha}$, $v'(\frac{1}{2}) = \frac{2}{\alpha}$, $v' \sim C$ *and v' is monotonically decreasing on* $[0, \frac{1}{2}]$ *and monotonically increasing on* $[\frac{1}{2}, 1]$

3. *If* $s \in [0, \frac{1}{2}]$ *then* $v(s) \sim s$. *If* $s \in [\frac{1}{2}, 1]$ *then* $1 - v(s) \sim 1 - s$.

4. $\gamma(0) = 0$, $\gamma(1) = 1$, $\gamma(\frac{1}{2}) = \frac{1}{2}$, $\gamma(s) = 1 - \gamma(1-s)$ *and γ is strictly monotonically increasing.*

5. $\gamma'(s) \sim s^{\alpha-1}$ *if* $s \in [0, \frac{1}{2}]$, $\gamma'(s) \sim (1-s)^{\alpha-1}$ *if* $s \in [\frac{1}{2}, 1]$ *and*
$$\gamma'(s) = \frac{\alpha v'(s)(v(s))^{\alpha-1}(v(1-s))^{\alpha-1}}{[(v(s))^\alpha + (v(1-s))^\alpha]^2} > 0, \quad 0 < s < 1.$$

6. *There holds* $\frac{\gamma'(s)s(1-s)}{\gamma(s)(1-\gamma(1-s))} \sim 1$. *Therefore if* $s \in [0, \frac{n}{2}]$ *then* $\frac{1}{n}\frac{\gamma'(s/n)}{\gamma(s/n)} \sim \frac{1}{s}$ *and if* $s \in [\frac{n}{2}, n]$ *then* $\frac{1}{n}\frac{\gamma'(s/n)}{1-\gamma(s/n)} \sim \frac{1}{n-s}$.

7. $$\gamma''(s) = \frac{\alpha v(s)^{\alpha-2} v(1-s)^{\alpha-2}[v''(s)v(s)v(1-s) + (\alpha-1)(1-2v(s))]}{[v(s)^\alpha + v(1-s)^\alpha]^2}$$
$$+\frac{2\alpha(\alpha-1)v(s)^{\alpha-1}v(1-s)^{\alpha-1}v'(s)^2(v(1-s)^{\alpha-1} - v(s)^{\alpha-1})}{[v(s)^\alpha + v(1-s)^\alpha]^3}$$
and if $s \in [0, \delta(\alpha)]$ *then* $\gamma'' > 0$, *if* $s \in [1-\delta(\alpha), 1]$ *then* $\gamma'' < 0$, *with* $\delta(\alpha) \in (0, \frac{1}{2}]$. *If* $\alpha \le \frac{86}{30}$, *then we can choose* $\delta(\alpha) = \frac{1}{2}$.

8. *If* $s \in \left[\frac{\delta(\alpha)}{2}, 1 - \frac{\delta(\alpha)}{2}\right]$ *then* $\gamma'(s) \sim 1$.

9. *There holds* $h(s,\sigma) \leq 1$ *and* $h(s,s) = 1$.

Now we study the mapping properties of the operators and the spaces, in which the operators are acting, more precisely. The operator D is a bijective and continuous mapping between $\tilde{H}_{\frac{1}{2}}(I)$ and $H_{-\frac{1}{2}}(I)$ which are dual spaces with respect to the L_2-scalar product. Then the operator A transformed according to (7) is a bijective and continuous mapping in the transformed spaces,

$$A : \tilde{H}_{\frac{1}{2},\alpha}(I) \longrightarrow H_{-\frac{1}{2},\alpha}(I)$$

Here the spaces $\tilde{H}_{\frac{1}{2},\alpha}(I)$ and $H_{-\frac{1}{2},\alpha}(I)$ are defined in a natural way by

$$\tilde{H}_{\frac{1}{2},\alpha}(I) := \{\phi : \phi(t) = \psi(\gamma(t)), \psi \in \tilde{H}_{\frac{1}{2}}(I)\},$$

$$H_{-\frac{1}{2},\alpha}(I) := \{\tilde{\phi} : \tilde{\phi}(t) = \tilde{\psi}(\gamma(t))\gamma'(t), \tilde{\psi} \in H_{-\frac{1}{2}}(I)\}$$

and the norms are given by $|\phi|_{\frac{1}{2},\alpha} := |\psi|_{\frac{1}{2}}$ and $|\tilde{\phi}|_{-\frac{1}{2},\alpha} := |\tilde{\psi}|_{-\frac{1}{2}}$, respectively.

Because of

$$\int_0^1 \phi(s)\phi_1(s)ds = \int_0^1 \phi(\gamma(s))\phi_1(\gamma(s))\gamma'(s)ds = \int_0^1 \psi(s)\psi_1(s)ds,$$

the spaces $\tilde{H}_{\frac{1}{2},\alpha}(I)$ and $H_{-\frac{1}{2},\alpha}(I)$ are dual with respect to the L_2-scalar product. The operator $A : w \mapsto g$ is mapping from $\tilde{H}_{\frac{1}{2},\alpha}(I)$ into the space dual with respect to the L_2-scalar product.

We shall consider the approximate operators A_n in discrete spaces, using the theory of Vainikko [18]. We define a system of discrete spaces $(E_n)_{n\in \mathbb{N}}$ by $E_n := \{\{\xi_j\}_{j=1}^{n-1}\}$ equipped with the norm

$$\|\{\xi_j\}_{j=1}^{n-1}\|_{E_n} = \|\sum_{j=1}^{n-1} \xi_j\phi_j\|_{\tilde{H}_{\frac{1}{2},\alpha}(I)}.$$

Let $P = (p_n)_{n\in \mathbb{N}}$ be a sequence of operators

$$p_n : \tilde{H}_{\frac{1}{2},\alpha}(I) \to E_n, \quad p_n(\phi) := \{\phi(t_j)\}_{j=1}^{n-1}. \tag{13}$$

For each fixed n there holds

$$\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n-1}|\xi_j|^2\right)^{\frac{1}{2}} \sim \|\sum_{j=1}^{n-1}\xi_j\phi_j\|_{L_2}.$$

Thus we denote the finite l_2–space by $l_2(n) := \{\{\xi_j\}_{j=1}^{n-1}\}$, and equip it with the norm

$$\|\{\xi_j\}_{j=1}^{n-1}\|_{l_2(n)} = \frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n-1}|\xi_j|^2\right)^{\frac{1}{2}},$$

and the scalar product

$$\langle\{\xi_j\}_{j=1}^{n-1},\{\eta_j\}_{j=1}^{n-1}\rangle = \frac{1}{n}\sum_{j=1}^{n-1}\xi_j\eta_j.$$

Now it make sense to define the second system of discrete spaces $(F_n)_{n\in\mathbb{N}}$ by $F_n := \{\{\xi_j\}_{j=1}^{n-1}\}$ equipped with the norm, which is dual to the norm of E_n with respect to the $l_2(n)$–scalar product. Analogously, we set $Q = (q_n)_{n\in\mathbb{N}}$ with

$$q_n : H_{-\frac{1}{2},\alpha}(I) \to F_n, \quad q_n(\tilde{\phi}) := \{\tilde{\phi}(t_j)\}_{j=1}^{n-1}. \tag{14}$$

The approximate operators A_n are mapping in theses dual discrete spaces

$$A_n : E_n \to F_n.$$

Theorem 3.1 *There holds*

$$\|\{\xi_j\}\|_{E_n}^2 \sim \sum_{l=1}^{\frac{n}{2}-1}\frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1\\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{|i-l|^2}(\xi_l-\xi_i)^2$$

$$=: \langle B_n\{\xi_j\},\{\xi_j\}\rangle_{l_2(n)} = \|\sqrt{B_n}\{\xi_j\}\|_{l_2(n)}^2 \tag{15}$$

where B_n is a positive selfadjoint matrix. Additionally, there holds

$$\|\{\xi_j\}\|_{l_2(n)} \sim \|\sqrt{B_n}\{\xi_j\}\|_{F_n}.$$

The proof of this theorem requires a lot of cumbersome technical computations. It can be found in the appendix.

4 The stability of the quadrature method

With the help of Theorem 3.1 we are able to show the stability of sequence A_n. Note that the matrices $-A_n$ and B_n have a similar structure and differ only in the main diagonal:

$$-A_n = n(-h(t_k,t_j)a_{k-j})_{k,j=1}^{n-1},$$

and

$$B_n = n(-h(t_k,t_j)a_{k-j})_{k,j=1}^{n-1} + n(c_k\delta_{k,j})_{k,j=1}^{n-1} + n(d_k\delta_{k,j})_{k,j=1}^{n-1}$$

$$+n(a_0\delta_{k,j})_{k,j=1}^{n-1},$$

where

$$c_k = \begin{cases} \frac{1}{k}, & k \le \frac{n}{2}-1 \\ \frac{1}{n-k}, & k \ge \frac{n}{2} \end{cases},$$

$$d_k = 2 \sum_{\substack{j=1 \\ j \not\equiv k \bmod 2}}^{n-1} h(t_k,t_j)\frac{1}{|k-j|^2}.$$

The mapping properties of $\sqrt{B_n}$ are a direct consequence of Theorem 3.1:

$$\sqrt{B_n} : E_n \to l_2(n),$$

$$\sqrt{B_n} : l_2(n) \to F_n$$

are isomorph mappings and their norms are independend of n.

Now the mapping $A_n : E_n \to F_n$ is invertible if and only if $A_n' := -A_n : E_n \to F_n$ is invertible. Furthermore, B_n is positive and selfadjoint. Thus $A_n' : E_n \to F_n$ is invertible if and only if $\sqrt{B_n}^{-1}A_n'\sqrt{B_n}^{-1}$ is invertible in $l_2(n)$. The last assertion is equivalent to the relation

$$\left\langle \sqrt{B_n}^{-1}A_n'\sqrt{B_n}^{-1}\xi, \xi \right\rangle_{l_2(n)} \sim \langle \xi, \xi\rangle_{l_2(n)}$$

for all $\xi = \{\xi_j\}_{j=1}^{n-1} \in l_2(n)$ which is equivalent to the following one

$$\langle A_n'\xi, \xi\rangle_{l_2(n)} \sim \langle B_n\xi, \xi\rangle_{l_2(n)} = \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2$$

$$+ \sum_{\substack{l,i=1 \\ i \not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{|\,l-i\,|^2}(\xi_l - \xi_i)^2.$$

Thus the stability of the quadrature method follows directly from the subsequent theorem.

Theorem 4.1 *There holds*

$$\langle A_n'\xi,\xi\rangle \sim \sum_{l=1}^{\frac{n}{2}-1}\frac{1}{l}\xi_l^2+\sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2+\sum_{\substack{l,i=1\\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{\mid l-i\mid^2}(\xi_l-\xi_i)^2.$$

Proof: Using $h(x,y)=h(y,x)$ and $a_{l-i}=a_{i-l}$ we obtain

$$\langle A_n'\xi,\xi\rangle_{l_2(n)} = -\sum_{l,i=1}^{n-1} h(\frac{l}{n},\frac{i}{n})a_{l-i}\xi_l\xi_i$$

$$\begin{aligned}
&= -\sum_{\substack{l,i=1\\ i\neq l}}^{n-1} h(\frac{l}{n},\frac{i}{n})a_{l-i}(\xi_l\xi_i-\frac{1}{2}\xi_l^2-\frac{1}{2}\xi_i^2)\\
&\quad -\sum_{l=1}^{n-1}\left(a_0+\frac{1}{2}\sum_{\substack{i=1\\ i\neq l}}^{n-1}h(\frac{l}{n},\frac{i}{n})a_{l-i}+\frac{1}{2}\sum_{\substack{i=1\\ i\neq l}}^{n-1}h(\frac{i}{n},\frac{l}{n})a_{i-l}\right)\xi_l^2\\
&= \sum_{\substack{l,i=1\\ i\neq l}}^{n-1} h(\frac{l}{n},\frac{i}{n})a_{l-i}\frac{1}{2}(\xi_l-\xi_i)^2-\sum_{l=1}^{n-1}\left(a_0+\sum_{\substack{i=1\\ i\neq l}}^{n-1}h(\frac{l}{n},\frac{i}{n})a_{l-i}\right)\xi_l^2\\
&= \sum_{\substack{l,i=1\\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{\mid l-i\mid^2}(\xi_l-\xi_i)^2+\sum_{l=1}^{n-1}R_l\xi_l^2,
\end{aligned}$$

with

$$R_l := \frac{\pi^2}{2}-2\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{\mid l-i\mid^2}.$$

It remains to show that $R_l\sim\frac{1}{l}+\frac{1}{n-l}$. Because of $h\le 1$ (see Sect.3) we have

$$\begin{aligned}
R_l &= \frac{\pi^2}{2}-2\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{\mid l-i\mid^2}\\
&= \sum_{\substack{i=-\infty\\ i\not\equiv l \bmod 2}}^{0}\frac{2}{(l-i)^2}+\sum_{\substack{i=n\\ i\not\equiv l \bmod 2}}^{\infty}\frac{2}{(l-i)^2}\\
&\qquad +\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1}\frac{2}{(l-i)^2}\left(1-h(\frac{l}{n},\frac{i}{n})\right)\ge 0.
\end{aligned}$$

Thus there holds

$$R_l\sim\frac{1}{l}+\frac{1}{n-l}+\sum_{\substack{i=1\\ i\not\equiv l \bmod 2}}^{n-1}\frac{2}{(l-i)^2}\left(1-h(\frac{l}{n},\frac{i}{n})\right).$$

What remains to show is the estimate

$$\tilde{R}_l := \sum_{\substack{i=1 \\ i \not\equiv l \bmod 2}}^{n-1} 2\, r_{l,i} \le C(\frac{1}{l}) + C(\frac{1}{n-l}),$$

$$r_{l,i} := \frac{1}{(l-i)^2}\left(1 - h(\frac{l}{n}, \frac{i}{n})\right).$$

Using the definition of the function h we get

$$r_{l,i} = \frac{\left(\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})\right)^2 - \gamma'(\frac{l}{n})\gamma'(\frac{i}{n})\left(\frac{l-i}{n}\right)^2}{\left(\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})\right)^2 (l-i)^2}.$$

For the denominator we have

$$\left(\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})\right)^2 (l-i)^2 = \left(\int_0^1 \gamma'(\frac{l}{n} + h(\frac{i-l}{n}))dh\right)^2 \frac{(l-i)^4}{n^2}.$$

Furthermore there holds

$$\gamma(\frac{l}{n}) - \gamma(\frac{i}{n}) = \int_0^1 \gamma'(\frac{i}{n} + h(\frac{l-i}{n}))dh\frac{l-i}{n}$$

$$\begin{aligned}
&= \gamma'(\frac{i}{n})(\frac{l-i}{n}) + \int_0^1 \left[\gamma'(\frac{i}{n} + h(\frac{l-i}{n})) - \gamma'(\frac{i}{n})\right] dh\frac{l-i}{n} \\
&= \gamma'(\frac{i}{n})(\frac{l-i}{n}) + \int_0^1 \int_0^1 \gamma''(\frac{i}{n} + uv(\frac{l-i}{n}))dvudu\left(\frac{l-i}{n}\right)^2,
\end{aligned}$$

and thus

$$\tilde{r}_{l,i} := \frac{\left(\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})\right)^2 - \gamma'(\frac{l}{n})\gamma'(\frac{i}{n})\left(\frac{l-i}{n}\right)^2}{\left(\frac{l-i}{n}\right)^4} = \tag{16}$$

$$\begin{aligned}
&= \frac{n}{l-i}\int_0^1 \int_0^1 \gamma'(\frac{l}{n})\gamma''(\frac{i}{n} + uv(\frac{l-i}{n})) \\
&\qquad\qquad -\gamma'(\frac{i}{n})\gamma''(\frac{l}{n} + uv(\frac{i-l}{n}))dvudu \\
&\quad -\int_0^1 \int_0^1 \gamma''(\frac{i}{n} + uv(\frac{l-i}{n}))dvudu \int_0^1 \int_0^1 \gamma''(\frac{l}{n} + uv(\frac{i-l}{n}))dvudu \\
&= \frac{\gamma'(\frac{l}{n}) - \gamma'(\frac{i}{n})}{\frac{l-i}{n}} \int_0^1 \int_0^1 \gamma''(\frac{i}{n} + uv(\frac{l-i}{n}))dvudu
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma'(\frac{i}{n})}{\frac{l-i}{n}}\int_0^1\int_0^1\left[\gamma''(\frac{i}{n}+uv(\frac{l-i}{n}))-\gamma''(\frac{l}{n}+uv(\frac{i-l}{n}))\right]dvudu\\
&-\int_0^1\int_0^1\gamma''(\frac{i}{n}+uv(\frac{l-i}{n}))dvudu\int_0^1\int_0^1\gamma''(\frac{l}{n}+uv(\frac{i-l}{n}))dvudu\\
=\;&\int_0^1\gamma''(\frac{l}{n}+h(\frac{i-l}{n}))dh\int_0^1\int_0^1\gamma''(\frac{i}{n}+uv(\frac{l-i}{n}))dvudu\\
&+\gamma'(\frac{i}{n})\int_0^1\int_0^1\int_0^1\gamma'''(\frac{l}{n}+uvw(\frac{i-l}{n}))dvuduwdw\\
&-\int_0^1\int_0^1\gamma''(\frac{i}{n}+uv(\frac{l-i}{n}))dvudu\int_0^1\int_0^1\gamma''(\frac{l}{n}+uv(\frac{i-l}{n}))dvudu.
\end{aligned}
$$

We arrive at

$$
r_{l,i}=\frac{1}{n^2}\frac{\tilde{r}_{l,i}}{\left(\int_0^1\gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh\right)^2}. \tag{17}
$$

Now we have to distinguish several cases. Choose a positive number $\epsilon<\frac{1}{2}$.

First let $l<n\epsilon$. There holds $\gamma'(x)\sim x^{\alpha-1}$ if $x\in[0,\frac{1}{2})$. Furthermore for those l and arbitrary i we have

$$
\int_0^1\gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh\geq C\left(\frac{l+i}{n}\right)^{\alpha-1}. \tag{18}
$$

To proof (18) we first remark, that if $i<l$ then

$$
\int_0^1\gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh\geq\int_0^{\frac{1}{2}}\gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh
$$

$$
\geq C\int_0^{\frac{1}{2}}\left(\frac{l}{n}+h(\frac{i-l}{n})\right)^{\alpha-1}dh\geq C\frac{1}{2}\left(\frac{l+i}{n}\right)^{\alpha-1}.
$$

If $i\geq l$ and $\frac{i}{n}\leq\frac{1}{2}$ then we have $\frac{l}{n}+h(\frac{i-l}{n})\leq\frac{1}{2}$ and

$$
\int_0^1\gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh\geq\int_{\frac{1}{2}}^1\gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh
$$

$$
\geq C\int_{\frac{1}{2}}^1\left(\frac{l}{n}+h(\frac{i-l}{n})\right)^{\alpha-1}dh\geq C\left(\frac{l+i}{n}\right)^{\alpha-1}.
$$

In the case when $i\geq l$ and $\frac{1}{2}\leq\frac{i}{n}\leq 1$, define the number h_1 by $h_1:=\frac{1-2\epsilon}{2-2\epsilon}<\frac{1}{2}$. For all $h<h_1$ we find

$$
\frac{l}{n}+h(\frac{i-l}{n})=(1-h)\frac{l}{n}+h\frac{i}{n}\leq(1-h)\epsilon+h=h(1-\epsilon)+\epsilon\leq\frac{1}{2}.
$$

Thus

$$\int_0^1 \gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh \geq \int_{\frac{h_1}{2}}^{h_1} \gamma'(\frac{l}{n}+h(\frac{i-l}{n}))dh$$

$$\geq C\int_{\frac{h_1}{2}}^{h_1} \left(\frac{l}{n}+h(\frac{i-l}{n})\right)^{\alpha-1} dh \geq C\left(\frac{h_1}{2}\right)^{\alpha}\left(\frac{l+i}{n}\right)^{\alpha-1},$$

because

$$\frac{l}{n}+h\left(\frac{i-l}{n}\right) \geq \frac{l+i}{n}\frac{h_1}{2}, \quad if \quad \frac{h_1}{2} \leq h \leq h_1.$$

So (18) is true for all i. Furthermore there holds

$$\gamma''(x) \leq Cx^{\alpha-2}, \qquad \gamma'''(x) \leq Cx^{\alpha-3},$$

and hence

$$\gamma''(\frac{l}{n}+h(\frac{i-l}{n})) \leq C\left(\frac{l+i}{n}\right)^{\alpha-2}, \gamma'''(\frac{l}{n}+h(\frac{i-l}{n})) \leq C\left(\frac{l+i}{n}\right)^{\alpha-3}$$

Using (16) and (17) we see that

$$r_{l,i} \leq \frac{1}{n^2}C\frac{\left(\frac{l+i}{n}\right)^{\alpha-2}\left(\frac{l+i}{n}\right)^{\alpha-2}+\left(\frac{l+i}{n}\right)^{\alpha-1}\left(\frac{l+i}{n}\right)^{\alpha-3}}{\left(\frac{l+i}{n}\right)^{2\alpha-2}} = \frac{1}{n^2}C\frac{n^2}{(l+i)^2} = \frac{C}{(l+i)^2}.$$

With the definition of $r_{l,i}$ and $\tilde{R}_l$ we arrive at

$$\tilde{R}_l = 2\sum_{\substack{i=1 \\ i\not\equiv l \bmod 2}}^{n-1} r_{l,i} \leq C\sum_{\substack{i=1 \\ i\not\equiv l \bmod 2}}^{n-1} \frac{1}{(l+i)^2} \leq C\frac{1}{l}.$$

The second case $l > n(1-\epsilon)$ can be reduced to the first case $l < n\epsilon$ with the help of the relations

$$\gamma(x) = 1-\gamma(1-x), \qquad \gamma'(x) = \gamma'(1-x),$$

$$\gamma''(x) = -\gamma''(1-x), \qquad \gamma'''(x) = \gamma'''(1-x).$$

In the third case $n\epsilon < l < n(1-\epsilon)$ it remains to show that $R_l \leq \frac{1}{n}C$, because in this case $l \sim n, n-l \sim n$. The assertion $R_l \leq \frac{1}{n}C$ is true if the function

$$\hat{R}_l(y) := \frac{(\gamma(x)-\gamma(y))^2-\gamma'(x)\gamma'(y)(x-y)^2}{(\gamma(x)-\gamma(y))^2(x-y)^2}$$

is integrable for fixed $x = \frac{l}{n}$. Now $\hat{R}_l(y)$ can be transformed analogously to $r_{l,i}$ and we get

$$\hat{R}_l(y) := \frac{\tilde{R}_l(y)}{\left(\int_0^1 \gamma'(x+h(y-x))dh\right)^2}$$

with

$$\tilde{R}_l(y) := \int_0^1 \gamma''(x+h(y-x))dh \int_0^1 \int_0^1 \gamma''(y+uv(x-y))dvudu$$

$$+\gamma'(y) \int_0^1 \int_0^1 \int_0^1 \gamma'''(x+uvw(y-x))dvuduwdw$$
$$-\int_0^1 \int_0^1 \gamma''(y+uv(x-y))dvudu \int_0^1 \int_0^1 \gamma''(x+uv(y-x))dvudu.$$

This term $\hat{R}_l(y)$ is integrable if the numerator $\tilde{R}_l(y)$ is integrable , because of

$$\int_0^1 \gamma'(x+h(y-x))dh \geq C, \qquad y \in [0,1].$$

For $y \in U_\delta(0)$ there holds

$$\gamma''(y) \sim y^{\alpha-2}, \gamma'''(y) \sim y^{\alpha-3}.$$

We see, that $\tilde{R}_l(y)$ is integrable if $y^{2\alpha-4}$ is integrable. Obviously this is true if $\alpha > \frac{3}{2}$. This completes the proof of Theorem 4.1. ∎

5 The convergence of the quadrature method

In this section we shall derive error estimations. First we remark that due to the definition there holds

$$\|u-u_n\|_{\frac{1}{2}} = \|w-w_n\|_{\frac{1}{2},\alpha}, \tag{19}$$

where u is the solution of (1), $w = u \circ \gamma$ is the solution of (7), w_n is the approximate solution of (7) defined by (6) and (11) and $u_n \circ \gamma = w_n$. Furthermore there holds $K_n^d = P_n^d p_n$, where $K_n^d \in L(\tilde{H}_{\frac{1}{2},\alpha})$ is the interpolation projector onto the continuous polynomial splines of degree d, $p_n : \tilde{H}_{\frac{1}{2},\alpha} \to E_n$ is the discretisation operator defined by (13) and $P_n^d : E_n \to \tilde{H}_{\frac{1}{2},\alpha}$ is the prolongation operator defined by $P_n^d\{\xi_j\}_{j=1}^{n-1} = \sum_{j=1}^{n-1} \xi_j \phi_j^{(d)}$. So Equation (11) is equivalent to $w_n = P_n^d \tilde{w}_n$ with $\tilde{w}_n = \{\xi_j\}_{j=1}^{n-1}$ defined by (6). Using the triangle inequality, we obtain

$$\|u-u_n\|_{\frac{1}{2}} = \|w-P_n^d\tilde{w}_n\|_{\frac{1}{2},\alpha} \leq \|w-K_n^d w\|_{\frac{1}{2},\alpha} + \|P_n^d(\tilde{w}_n - p_n w)\|_{\frac{1}{2},\alpha}. \tag{20}$$

First we estimate $\|K_n^d w - w\|_{\frac{1}{2},\alpha}$. To this end we use the following lemma:

Lemma 5.1 *For all $w \in \tilde{H}_{\frac{1}{2}}$ there holds $\|w\|_{\frac{1}{2},\alpha} \leq C\|w\|_{\frac{1}{2}}$.*

Proof: If $u \in \tilde{H}_{\frac{1}{2}}, u(\gamma(t)) = w(t)$ then $\|w\|_{\frac{1}{2},\alpha} = \|u\|_{\frac{1}{2}}$. Thus we get

$$\|u\|_{\frac{1}{2}}^2 = \int_0^1 \int_0^1 \frac{|u(x)-u(y)|^2}{|x-y|^2} dxdy + 2\int_0^1 \frac{|u(x)|^2}{x(1-x)} dx$$

$$= \int_0^1 \int_0^1 \frac{|w(x)-w(y)|^2}{|\gamma(x)-\gamma(y)|^2} \gamma'(x)\gamma'(y) dxdy + 2\int_0^1 \frac{|w(x)|^2\gamma'(x)}{\gamma(x)(1-\gamma(x))} dx$$
$$= \int_0^1 \int_0^1 \frac{|w(x)-w(y)|^2}{|x-y|^2} h(x,y) dxdy + 2\int_0^1 \frac{|w(x)|^2}{x(1-x)} \frac{\gamma'(x)x(1-x)}{\gamma(x)(1-\gamma(x))} dx$$
$$\leq C\left(\int_0^1 \int_0^1 \frac{|w(x)-w(y)|^2}{|x-y|^2} dxdy + 2\int_0^1 \frac{|w(x)|^2}{x(1-x)} dx\right) = C\|w\|_{\frac{1}{2}},$$

with the properties 6 and 9 of Remark 3.1. Thus

$$\|w\|_{\frac{1}{2},\alpha} = \|u\|_{\frac{1}{2}} \leq C\|w\|_{\frac{1}{2}}$$

and the lemma is proved. ∎

Due to [16], Sect.5.31 there holds

$$\|K_n^d w - w\|_{\frac{1}{2},\alpha} \leq \|K_n^d w - w\|_{\frac{1}{2}} \leq Cn^{\frac{1}{2}-s}\|w\|_s \tag{21}$$

if $w \in H^s$ and $1 \leq s \leq d+1$.

Now we consider the second term of the sum in (20).

Lemma 5.2 *There holds*

$$\|P_n^d \xi\|_{\frac{1}{2},\alpha} \leq C\|\xi\|_{E_n}.$$

for all $\xi = \{\xi_j\} \in E_n$ with a constant C independend of n.

The proof can be found at the end of the appendix.

Using this lemma, we get

$$\|P_n^d(\tilde{w}_n - p_n w)\|_{\frac{1}{2},\alpha} \leq C\|\tilde{w}_n - p_n w\|_{E_n}. \tag{22}$$

From the Theorems 3.1 and 4.1 it follows that the sequence $\{A_n\}_{n\in\mathbf{N}}$, $A_n \in L(E_n, F_n)$ converges stably to $A \in L(\tilde{H}_{\frac{1}{2},\alpha}, H_{-\frac{1}{2},\alpha})$ in the sense of Vainikko [18], i.e. $A_n \xrightarrow{PQ} A$ and there is a number n_0 such that $A_n^{-1} \in L(F_n, E_n)$ exists for all $n > n_0$ and the inverses are uniformly bounded.

Hereby we use the following notation: We write $A_n \xrightarrow{PQ} A$, if the convergence $\|q_n Ax - A_n x_n\|_{F_n} \to 0$ follows from $\|p_n x - x_n\| \to 0$. The uniformly boundedness of the inverses of A_n and the convergence $A_n \xrightarrow{PQ} A$ follows from $\langle \sqrt{B_n}^{-}1 A_n \sqrt{B_n}^{-}1 \xi, \xi \rangle \sim \langle \xi, \xi \rangle$ and the construction of A_n. Thus the assumptions of the convergence theorem in Sect.3 of [18] are fulfilled and there holds

$$\|\tilde{w}_n - p_n w\|_{E_n} \leq C \|A_n p_n w - q_n A w\|_{F_n}. \tag{23}$$

The operators A and A_n can be represented in the form $A = D + L$ and $A_n = D_n + L_n$ with

$$Dw(s) = \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2} d\sigma,$$

$$Lw(s) = (A - D)w(s) = \int_0^1 \frac{w(\sigma)}{|\sigma - s|^2}(h(s,\sigma) - 1) d\sigma = \int_0^1 w(\sigma) l(s,\sigma) d\sigma,$$

$$D_n = n \left(a_{k-j}\right)_{k,j=1}^{n-1}, \qquad L_n = A_n - D_n = n \left(a_{k-j}(h(t_k, t_j) - 1)\right)_{k,j=1}^{n-1}.$$

Obviously,

$$\|q_n A w - A_n p_n w\|_{F_n} \leq \|q_n D w - D_n p_n w\|_{F_n} + \|q_n L w - L_n p_n w\|_{F_n}. \tag{24}$$

Because we have to estimate the norm $\|\cdot\|_{F_n}$, but no explicit formula for that norm is available, we shall use the following lemma.

Lemma 5.3 *Let M be an arbitrary but fixed real positive number and let $\alpha < M$. Then there holds*

$$\|\psi\|_{F_n} \leq C \|\psi\|_{l_\infty}$$

for all $\psi = \{\psi_j\}_{j=1}^{n-1} \in l_\infty$.

Proof: Due to Sobolev's embedding theorem the mapping $E : H_{\frac{1}{2}} \to L_M$ is continuous. Thus by duality, the mapping $E : L_q \to H_{-\frac{1}{2}}$ is continuous for $q = \frac{M}{M-1}$. Let $g \in L_\infty$ be an arbitrary function. Using the definition of $H_{-\frac{1}{2},\alpha}$ we get

$$\|g\|_{-\frac{1}{2},\alpha} = \left\| \frac{g \circ \gamma^{-1}}{\gamma' \circ \gamma^{-1}} \right\|_{-\frac{1}{2}} \leq C \left\| \frac{g \circ \gamma^{-1}}{\gamma' \circ \gamma^{-1}} \right\|_{L_q} \leq C \|g \circ \gamma^{-1}\|_{L_\infty} \left\| \frac{1}{\gamma' \circ \gamma^{-1}} \right\|_{L_q}.$$

From the properties of γ we see that

$$\begin{aligned}\left\|\frac{1}{\gamma'\circ\gamma^{-1}}\right\|_{L_q}^q &= \int_0^1 \left|\frac{1}{\gamma'(\gamma^{-1}(t))}\right|^q dt = \int_0^1 \left|\frac{1}{\gamma'(t)}\right|^q \gamma'(t)dt \\ &= \int_0^1 \left|\frac{1}{\gamma'(t)}\right|^{q-1} dt = \int_0^1 \left|\frac{1}{\gamma'(t)}\right|^{\frac{1}{M-1}} dt \le C,\end{aligned}$$

since $\alpha < M$. We arrive at

$$\|g\|_{-\frac{1}{2},\alpha} \le C\|g\circ\gamma^{-1}\|_{L_\infty} = C\|g\|_{L_\infty}.$$

By duality we get

$$\|f\|_{L_1} \le C\|f\|_{\frac{1}{2},\alpha} \tag{25}$$

for arbitrary $f \in \tilde{H}_{\frac{1}{2},\alpha}$. Using the norm equivalence of Theorem 3.1 we find

$$\begin{aligned}\|\sum_{j=1}^{n-1}\xi_j\phi_j\|_{L_1} &\le C\|\sum_{j=1}^{n-1}\xi_j\phi_j\|_{\frac{1}{2},\alpha} \\ &= C\|\{\xi_j\}_{j=1}^{n-1}\|_{E_n} \\ &\le C\|\sqrt{B_n}\{\xi_j\}_{j=1}^{n-1}\|_{l_2(n)}.\end{aligned}$$

On the other side there holds

$$\|\sum_{j=1}^{n-1}\xi_j\phi_j\|_{L_1} \sim \frac{1}{n}\sum_{j=1}^{n-1}|\xi_j| =: \|\{\xi_j\}_{j=1}^{n-1}\|_{l_1(n)}.$$

Thus we get

$$\|\{\xi_j\}_{j=1}^{n-1}\|_{l_1(n)} \le C\|\sqrt{B_n}\{\xi_j\}_{j=1}^{n-1}\|_{l_2(n)}. \tag{26}$$

Furthermore we obtain

$$\begin{aligned}\|\psi\|_{F_n} &= \|\sqrt{B_n}^{-1}\psi\|_{l_2(n)} \le \sup_{\|\eta\|_{l_2(n)}\le 1}\langle\sqrt{B_n}^{-1}\psi,\eta\rangle_{l_2(n)} \\ &= \sup_{\|\sqrt{B_n}\zeta\|_{l_2(n)}\le 1}\langle\psi,\zeta\rangle_{l_2(n)} \le \sup_{\|\zeta\|_{l_1(n)}\le C}\langle\psi,\zeta\rangle_{l_2(n)} \le C\|\psi\|_{l_\infty}\end{aligned}$$

and the lemma is proved. ∎

Choose now a sufficiently large number M and assume $\alpha < M$. Remind that

$$l(\sigma,t) = \frac{h(\sigma,t)-1}{|\sigma-t|^2} = \frac{\gamma'(t)\gamma'(\sigma)}{|\gamma(t)-\gamma(\sigma)|^2} - \frac{1}{|\sigma-t|^2}.$$

Now we can estimate

$$
\begin{aligned}
\|q_n Lw - L_n p_n w\|_{F_n} &\le C\|q_n Lw - L_n p_n w\|_{l_\infty} \\
&= C\left\|\left\{\int_0^1 w(\sigma)l(\sigma,t_k)d\sigma - \frac{2}{n}\sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1} w(t_j)l(t_j,t_k)\right\}_{k=1}^{n-1}\right\|_{l_\infty} \\
&\le C\sup_{k=1,\dots,n-1}\left|\int_0^1 w(\sigma)l(\sigma,t_k)d\sigma - \frac{2}{n}\sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1} w(t_j)l(t_j,t_k)\right| \\
&\le Cn^{-s}\sup_{k=1,\dots,n-1}\|l(\cdot,t_k)w\|_{W_1^s} \le Cn^{-s}\sup_{0<t<1}\|l(\cdot,t)w\|_{W_1^s} \qquad (27)
\end{aligned}
$$

if $0 \le s < \frac{\alpha}{2}$ and $w \in W_1^s$. Here W_1^s denotes the Sobolev space of power 1 and order s (cf. Triebel [17]). The last estimate is true, because w and its derivativs up to the order $\frac{\alpha}{2}$ are periodic and the rectangle rule over a periodic interval approximates the integral of a function with arbitrarily high order. More exactly, there holds the following lemma, which can be found in [4], pp.109-110.

Lemma 5.4 *Let $s > 0$. If $f \in W_1^s$ and $f^{(r)}(0) = f^{(r)}(1)$ for $r \le s$, then*

$$\left|\int_0^1 f(t)dt - \sum_{j=0}^{n-1} f(t_j)\frac{1}{n}\right| \le Cn^{-s}\|f\|_{W_1^s}, \quad t_j = \frac{j}{n}.$$

The function $l(t,\sigma)w(\sigma)$ and its derivatives up to the order s are periodic functions of σ if $s < \frac{\alpha}{2}$, because $\left(\frac{\partial}{\partial\sigma}\right)^k l(t,\cdot)$ is bounded for any fixed $t \in (0,1)$, $w(\sigma) \sim \sigma^{\frac{\alpha}{2}}$ in a neighbourhood of 0, $w(\sigma) \sim (1-\sigma)^{\frac{\alpha}{2}}$ in a neighbourhood of 1, and thus $w^{(s)}(0) = w^{(s)}(1)$ if $s < \frac{\alpha}{2}$. Consequently we get Equation (27).

It remains to examine whether the norms $\|l(\cdot,t)w\|_{W_1^s}$ are uniformly bounded or not.

Lemma 5.5 *The mapping $\sigma \mapsto \sigma^{k+2}(1-\sigma)^{k+2}\left(\frac{\partial}{\partial\sigma}\right)^k l(\sigma,t)$ is uniformly bounded with respect to t for an arbitrary integer k.*

Proof: First let $k = 0$. Then we have

$$l(\sigma,t) = \begin{cases} -\frac{1}{\sigma^2}, & t = 0, \\ \frac{\gamma'(\sigma)\gamma'(t)}{|\gamma(\sigma)-\gamma(t)|^2} - \frac{1}{|\sigma-t|^2}, & t \ne 0;1, \\ -\frac{1}{(1-\sigma)^2}, & t = 1. \end{cases}$$

Using the Lagrange form of the remainder of the Taylor's series, we get

$$l(\sigma,t) \to \frac{1}{6}\frac{\gamma'''(\sigma)}{\gamma'(\sigma)} - \frac{1}{4}\left(\frac{\gamma''(\sigma)}{\gamma'(\sigma)}\right)^2 \text{ if } t \to \sigma \ne 0.$$

Obviously the function $(1-\sigma)^2\sigma^2 l(\sigma,t)$ is uniformly bounded with respect to t if $\sigma \notin U_\epsilon(0)\cup U_\epsilon(1)$. If $\sigma \in U_\epsilon(0)$ then $(1-\sigma)^2\sigma^2 l(\sigma,t)$ is uniformly bounded with respect to t, $|t-\sigma|>\delta$, because the denominater contains $t-\sigma$ and $\gamma(t)-\gamma(\sigma)$, γ is monotone and continuous and the numerator is bounded. If $\sigma \in U_\epsilon(0)$, $|t-\sigma|<\delta$ then $|l(\sigma,t)-\lim_{t\to\sigma} l(\sigma,t)|<C$ and $(1-\sigma)^2\sigma^2 l(\sigma,t)\le C$, since $\gamma'(\sigma)\sim\sigma^{\alpha-1}$, $\gamma''(\sigma)\sim\sigma^{\alpha-2}$, $\gamma'''(\sigma)\sim\sigma^{\alpha-3}$ and thus $\sigma^2\left(\frac{1}{6}\frac{\gamma'''(\sigma)}{\gamma'(\sigma)}-\frac{1}{4}\left(\frac{\gamma''(\sigma)}{\gamma'(\sigma)}\right)^2\right)\le Const.$ If $\sigma\in U_\epsilon(1)$ then the uniformly boundedness of $(1-\sigma)^2\sigma^2 l(\sigma,t)$ can be deduced analogously.
Now let $k=1$. Then there holds

$$\frac{\partial}{\partial\sigma}l(\sigma,t)=\begin{cases} -\frac{2}{\sigma^3}\,, & t=0,\\ \frac{\gamma''(\sigma)\gamma'(t)}{|\gamma(\sigma)-\gamma(t)|^2}-2\frac{\gamma'(\sigma)^2\gamma'(t)}{|\gamma(\sigma)-\gamma(t)|^3}+\frac{2}{|\sigma-t|^3}\,, & t\neq 0;1\\ -\frac{2}{(1-\sigma)^3}\,, & t=1.\end{cases}$$

Using the Lagrange form of the remainder of Taylor's series again we get

$$\frac{\partial}{\partial\sigma}l(\sigma,t)\to\frac{1}{3}\frac{\gamma^{(4)}(\sigma)}{\gamma'(\sigma)}-\frac{1}{2}\left(\frac{\gamma''(\sigma)}{\gamma'(\sigma)}\right)^2\quad\text{if } t\to\sigma\neq 0.$$

Combining this assertion with $\gamma'(\sigma)\sim\sigma^{\alpha-1}$, $\gamma''(\sigma)\sim\sigma^{\alpha-2}$, $\gamma'''(\sigma)\sim\sigma^{\alpha-3}$ and $\gamma^{(4)}(\sigma)\sim\sigma^{\alpha-4}$ for $\sigma\in U_\epsilon(0)$, we get the uniform boundedness of $(1-\sigma)^3\sigma^3\frac{\partial}{\partial\sigma}l(\sigma,t)$ analogously to the proof of the uniform boundedness of $(1-\sigma)^2\sigma^2 l(\sigma,t)$. By further differentiations of the formula for $l(\sigma,t)$ we get the assertion of the lemma. ∎

On the other hand the solution $w(\sigma)$ of Equation (7) has an end point behaviour like $w(\sigma)=\sigma^{\frac{\alpha}{2}}(1-\sigma)^{\frac{\alpha}{2}}\tilde{gl}(\sigma)$ with smooth $\tilde{gl}$ since the solution $u(\sigma)$ of Equation (1) can be written in the form $u(\sigma)=\sigma^{\frac{1}{2}}(1-\sigma)^{\frac{1}{2}}gl(\sigma)$ (see remark 2.2) and $w(\sigma)=u(\gamma(\sigma))$. This fact together with Lemma 5.5 implies that

$$\sup_{0<t<1}\|l(t,\cdot)w\|_{W_1^s}<\infty \tag{28}$$

for any s satisfying $0\le s<\frac{\alpha}{2}-1$. Indeed, consider, for example, the function $w(\sigma)=\sigma^{\frac{\alpha}{2}}$ and suppose s is an integer. Then

$$\begin{aligned}\left(\frac{\partial}{\partial\sigma}\right)^s\left(l(\sigma,t)\sigma^{\frac{\alpha}{2}}\right) &= \sum_{j=0}^{s}C_j\left(\frac{\partial}{\partial\sigma}\right)^j l(\sigma,t)\sigma^{\frac{\alpha}{2}-(s-j)}\\ &= \left(\sum_{j=0}^{s}C_j\sigma^{j+2}\left(\frac{\partial}{\partial\sigma}\right)^j l(\sigma,t)\right)\sigma^{\frac{\alpha}{2}-s-2}.\end{aligned}$$

The last function is integrable if $\frac{\alpha}{2}-s-2>-1$, i.e., $s<\frac{\alpha}{2}-1$. If $s>0$ is not an integer, then another straightforward argumentation including the special definition of the norm in W_1^s leads to the same result. Toghether with the estimate (27) we arrive at

$$\|q_n Lw - L_n p_n w\|_{F_n} \le Cn^{-s}, \qquad 0 \le s < \frac{\alpha}{2} - 1. \tag{29}$$

It remains to estimate $\|q_n Dw - D_n p_n w\|_{F_n}$. Using Sobolev's embedding theorem and Lemma 5.3, we get

$$\begin{aligned} \|q_n Dw - D_n p_n w\|_{F_n} &\le C\|q_n Dw - D_n p_n w\|_{l_\infty} \\ &\le C\|P_n q_n Dw - P_n D_n p_n w\|_{L_\infty} \\ &= C\|K_n Dw - P_n D_n p_n w\|_{L_\infty} \\ &\le C\|K_n^R D^R w - P_n^R D_n^R p_n^R w\|_{H_{\frac{1}{2}+\epsilon}} \end{aligned} \tag{30}$$

with

$$P_n(\{\xi_j\}_{j=1}^{n-1}) := \sum_{j=1}^{n-1} \xi_j \psi_j; \quad t_j = \frac{j}{n},$$

where ψ_j is the smoothest interpolation spline of order $\tilde{d}$ with respect to the partition t_j,

$$P_n^R(\{\xi_j\}_{j=-\infty}^{\infty}) := \sum_{j=-\infty}^{\infty} \xi_j \psi_j,$$

$$p_n^R f = q_n^R f = \{f(t_j)\}_{j=-\infty}^{\infty},$$

$$K_n := P_n p_n = P_n q_n, \qquad K_n^R := P_n^R p_n^R = P_n^R q_n^R,$$

$$D^R f(t) := p.f. \int_{-\infty}^{\infty} \frac{f(\sigma)}{|\sigma - t|^2} d\sigma,$$

$$D_n^R := n\,(a_{k-j})_{k,j=-\infty}^{\infty} = nC(a),$$

with

$$a_k = \begin{cases} -\frac{\pi^2}{2}, & k = 0, \\ \frac{2}{|k|^2}, & k \text{ is odd}, \\ 0, & \text{else}. \end{cases} \qquad a(t) = a(e^{is}) = -\pi|s|; \; -\pi < s \le \pi.$$

Here $C(a)$ denotes the convolution matrix generated by the Fourier coefficients of a. The spline order $\tilde{d}$ is only of technical importance in the proof. It can be choosen large enough. Here the estimate is true for arbitrary $\tilde{d}$. The operators K_n and K_n^R are the interpolation projectors onto the space of smoothest splines of order $\tilde{d}$. The last estimate in (30) is true, because $K_n Dw - P_n D_n p_n w$ is a projection of $K_n^R D^R w - P_n^R D_n^R p_n^R w$ in L_∞.

Lemma 5.6 *If $\frac{3}{2} < s \leq \tilde{d} + 2$ and if ϵ is arbitrarily small but fixed, then there is a constant $C > 0$ such that*

$$\|K_n^R D^R f - P_n^R D_n^R p_n^R f\|_{H_{\frac{1}{2}+\epsilon}} \leq C n^{\frac{3}{2}+\epsilon-s} \|f\|_{H_s}$$

for any $f \in H_s(R)$.

Proof: It is well known that the vector $\left(e^{-i\xi j}\right)_{j=-\infty}^{\infty}$; $(-\pi < \xi \leq \pi)$ is an eigenvector of the convolution operator $C(a) = (a_{k-j})_{k,j=-\infty}^{\infty}$ corresponding to the eigenvalue $a(e^{i\xi}) = -\pi|\xi|$. Furthermore, for $g^\xi(t) := e^{-i\xi t}$, we obtain

$$p_n^R g^\xi = \left\{e^{-i\xi\frac{j}{n}}\right\}_{j=-\infty}^{\infty}$$

and

$$C(a) p_n^R g^\xi = a(e^{i\frac{\xi}{n}}) p_n^R g^\xi.$$

Obviously we have also

$$P_n^R n C(a) p_n^R g^\xi = n a(e^{i\frac{\xi}{n}}) P_n^R p_n^R g^\xi = n a(e^{i\frac{\xi}{n}}) K_n^R g^\xi.$$

Let F denote the usual Fourier transform

$$(Ff)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{it\xi} dt.$$

Then there holds

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ff)(\xi) g^\xi(t) d\xi,$$

and therefore

$$\begin{aligned} P_n^R n C(a) p_n^R f &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ff)(\xi) \left(P_n^R n C(a) p_n^R g^\xi\right) d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (Ff)(\xi) n a(e^{i\frac{\xi}{n}}) K_n^R g^\xi d\xi. \end{aligned}$$

On the other hand there holds

$$(D^R f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\pi\xi \ \mathrm{sign}(\xi)(Ff)(\xi) g^\xi(t) d\xi$$

(c.f. [11]) and thus

$$(K_n^R D^R f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -\pi\xi \mathrm{sign}(\xi)(Ff)(\xi)(K_n^R g^\xi)(t) d\xi.$$

Consequently we get

$$P_n^R nC(a)p_n^R f - K_n^R D^R f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)(K_n^R g^\xi)(t)d\xi,$$

with $r_n(\xi) = na(e^{i\frac{\xi}{n}}) + \pi|\xi|$. There holds

$$|r_n(\xi)| \leq C\frac{|\xi|^l}{n^{l-1}} \qquad ; l \geq 1. \tag{31}$$

To see (31) we have to distinguish two cases. If $|\xi| < n\pi$ then $na(e^{i\frac{\xi}{n}}) = -\pi|\xi|$ and thus $r_n(\xi) = 0$. If $|\xi| \geq n\pi$ then $|na(e^{i\frac{\xi}{n}})| \leq \pi^2 < \pi\left(\frac{|\xi|}{n}\right)^l$ and $\pi|\xi| \leq \left(\frac{1}{\pi}\right)^{l-2}\frac{|\xi|^l}{n^{l-1}}$. Obviously there holds

$$\begin{aligned} \|P_n^R nC(a)p_n^R f \;&-\; K_n^R D^R f(t)\|_{H_{\frac{1}{2}+\epsilon}} \leq \\ &\leq \|(K_n^R - L_n^R)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)g^\xi(t)d\xi\|_{H_{\frac{1}{2}+\epsilon}} + \\ &\quad + \|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)g^\xi(t)d\xi\|_{H_{\frac{1}{2}+\epsilon}}, \end{aligned} \tag{32}$$

where L_n^R denotes the orthoprojection onto the spline space $lin\{\psi_j\}_{j=-\infty}^{\infty}$. Using (31), we get the following estimate for the second term of the sum

$$\begin{aligned} \|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \;r_n(\xi)\; (Ff)(\xi)g^\xi d\xi\|_{H_{\frac{1}{2}+\epsilon}} &\leq \|(Ff)(\xi)r_n(\xi)\sqrt{1+\xi^2}^{\frac{1}{2}+\epsilon}\|_{L_2} \\ &\leq Cn^{1-l}\|(Ff)(\xi)\sqrt{1+\xi^2}^{l+\frac{1}{2}+\epsilon}\|_{L_2} \\ &\leq Cn^{1-l}\|f\|_{H_{l+\frac{1}{2}+\epsilon}} \leq Cn^{\frac{3}{2}+\epsilon-s}\|f\|_{H_s}, \end{aligned} \tag{33}$$

with $l = s - \frac{1}{2} - \epsilon$.

From [16], Section 2, we see that for $\frac{1}{2} + \epsilon \leq s_1 \leq \tilde{d} + 1$ there holds

$$\begin{aligned} \|(K_n^R - L_n^R)\; \tfrac{1}{\sqrt{2\pi}} \;&\int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)g^\xi d\xi\|_{H_{\frac{1}{2}+\epsilon}} \leq \\ &\leq Cn^{\frac{1}{2}+\epsilon-s_1}\|\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} r_n(\xi)(Ff)(\xi)g^\xi d\xi\|_{H_{s_1}} \\ &\leq Cn^{\frac{1}{2}+\epsilon-s_1}\|(Ff)(\xi)r_n(\xi)\sqrt{1+\xi^2}^{s_1}\|_{L_2} \\ &\leq Cn^{\frac{1}{2}+\epsilon-s_1}\|(Ff)(\xi)\sqrt{1+\xi^2}^{s_1+1}\|_{L_2} \\ &\leq Cn^{\frac{1}{2}+\epsilon-s_1}\|f\|_{H_{s_1+1}} = Cn^{\frac{3}{2}+\epsilon-s}\|f\|_{H_s}, \end{aligned} \tag{34}$$

for all $s = s_1 + 1$ with $\frac{3}{2} < s \leq \tilde{d} + 2$. This completes the proof of the lemma. ∎

Note that the estimate (30) is true for arbitrary $\tilde{d}$ and using Lemma 5.6 we arrive at

$$\|q_n Dw - D_n p_n w\|_{F_n} \leq C n^{\frac{3}{2}+\epsilon-s} \|w\|_{H_s}, \tag{35}$$

if $\frac{3}{2} < s$ and $w \in H_s$. From the estimates (20), (21), (22), (23), (24) ,(29) and (35) we obtain the following theorem:

Theorem 5.1 *Let $0 < s < min\{d, \frac{\alpha}{2} - 1\}$. If u is the solution of Equation (1), w the solution of (7), w_n the solution of the quadrature equation (6) defined by (11) and $u_n := w_n \circ \gamma^{-1}$, then there holds*

$$\|u - u_n\|_{\frac{1}{2}} = \|w - w_n\|_{\frac{1}{2},\alpha} \leq C n^{-s}, \tag{36}$$

if $w \in H_{s+\frac{3}{2}+\epsilon}$ for some positive ϵ.

Remark 5.1 *If f is sufficiently smooth, then there holds $w \in H_s$ for $s < \frac{\alpha}{2}$.*

This is an easy consequence of the definition of w and Remark 2.2.

6 Another quadrature method for the hypersingular integral equation

In this section we shall propose another quadrature method for the Equation (1), using a cos–transformation. We shall proceed like in Sect 3.8. of [16], where the numerical solution of first–kind integral equations with logarithmic kernel is treated.

First recall the hypersingular integral equation :

$$(Du)(t) := \int_0^1 \frac{u(\tau)}{|\tau - t|^2} d\tau = f(t).$$

Now we change the variables with another transformation function γ_1:

$$\gamma_1(s) := \frac{1 - \cos \pi s}{2}; \qquad s \in [0, 1].$$

Similar to Section 2 we get that Equation (1) is equivalent to

$$Aw(s) := \int_0^1 \frac{\gamma_1'(s)\gamma_1'(\sigma)}{|\gamma_1(\sigma) - \gamma_1(s)|^2} w(\sigma) d\sigma = g(s), \tag{37}$$

with

$$w(s) := u(\gamma_1(s)), \qquad g(s) := f(\gamma_1(s))\gamma_1'(s).$$

There holds

$$\begin{aligned}
\frac{\gamma_1'(s)\gamma_1'(\sigma)}{|\gamma_1(\sigma)-\gamma_1(s)|^2} &= \frac{\pi^2}{4}\frac{\sin\pi\sigma\sin\pi s}{\left(\frac{\cos\pi s-\cos\pi\sigma}{2}\right)^2}\\
&= \frac{\pi^2}{4}\frac{\sin\pi\sigma\sin\pi s}{\sin^2\pi\frac{s-\sigma}{2}\sin^2\pi\frac{s+\sigma}{2}}\\
&= \frac{\pi^2}{4}\left(\frac{1}{\sin^2\pi\frac{s-\sigma}{2}}-\frac{1}{\sin^2\pi\frac{s+\sigma}{2}}\right).
\end{aligned}$$

Thus

$$Aw(s)=\int_0^1\frac{\pi^2}{4}\left(\frac{1}{\sin^2\pi\frac{s-\sigma}{2}}-\frac{1}{\sin^2\pi\frac{s+\sigma}{2}}\right)w(\sigma)d\sigma=g(s). \tag{38}$$

Analogously to Section 2 we can deduce the quadrature method for n even

$$\begin{aligned}
g(t_k) &= \sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1}\frac{2}{n}\frac{\gamma_1'(t_j)\gamma_1'(t_k)}{|\gamma_1(t_j)-\gamma_1(t_k)|^2}\xi_j-\frac{n\pi^2}{2}\xi_k\\
&= \sum_{\substack{j=1\\ j\equiv k+1 \bmod 2}}^{n-1}\frac{\pi^2}{2n}\left(\frac{1}{\sin^2\pi\frac{t_k-t_j}{2}}-\frac{1}{\sin^2\pi\frac{t_k+t_j}{2}}\right)\xi_j-\frac{n\pi^2}{2}\xi_k \quad ,
\end{aligned} \tag{39}$$

with $k=1,\dots,n-1$.

The kernel function $\frac{\gamma_1'(s)\gamma_1'(\sigma)}{|\gamma_1(\sigma)-\gamma_1(s)|^2}$ of A is 2-periodic and odd with respect to each variable over the interval $[-1,1]$. For real t, let H^t denote the Sobolev space of 2–periodic functions (distributions). We will especially be interested in the subspace H_o^t of odd functions,

$$H_o^t=\{f\in H^t : f(-s)=-f(s)\}.$$

There holds

$$(A_o w)|_{[0,1]}=A(w|_{[0,1]}), \qquad w\in H_o^t, \tag{40}$$

where

$$A_o w(s):=\frac{\pi^2}{4}\int_{-1}^1\frac{w(\sigma)}{\sin^2\pi\frac{s-\sigma}{2}}d\sigma. \tag{41}$$

This can be seen in the following way

$$\begin{aligned}
A_o w(s) &= \frac{\pi^2}{4}\int_{-1}^1\frac{w(\sigma)}{\sin^2(\pi\frac{s-\sigma}{2})}d\sigma\\
&= \frac{\pi^2}{4}\int_0^1\frac{w(\sigma)}{\sin^2(\pi\frac{s-\sigma}{2})}d\sigma+\frac{\pi^2}{4}\int_0^1\frac{w(-\sigma)}{\sin^2\pi\frac{s+\sigma}{2})}d\sigma\\
&= \frac{\pi^2}{4}\int_0^1\left(\frac{1}{\sin^2(\pi\frac{s-\sigma}{2})}-\frac{1}{\sin^2(\pi\frac{s+\sigma}{2})}\right)w(\sigma)d\sigma \;=\; Aw(s),
\end{aligned}$$

since w is odd.

An easy computation shows that A_o maps odd functions into odd functions and even functions into even functions.

We continue g to an odd function on $[-1,1]$ by $g(-s) := -g(s)$ and set $\xi_{-k} := -\xi_k$; $k = 1,\ldots,n-1$, which corresponds to an odd continuation of w. Then the quadrature method (39) is equivalent to

$$\begin{aligned} g(t_k) &= \sum_{\substack{j=1-n \\ j\equiv k+1 \bmod 2}}^{n-1} \frac{\pi^2}{2n} \frac{1}{\sin^2 \pi\frac{t_k-t_j}{2}} \xi_j - \frac{n\pi^2}{2}\xi_k, \\ 0 &= \xi_0. \end{aligned} \tag{42}$$

with $k = 1-n,\ldots,n-1$; $k \neq 0$. The restriction $k \neq 0$ can be omitted, because $g(0) = f(\gamma_1(0))\gamma_1'(0) = 0$ and the sum of the right–hand side of (42) is zero if $k = 0$, because of $\xi_j = -\xi_{-j}$.

Furthermore there holds

$$A_o w(s) = \frac{\pi^2}{4}\int_{-1}^{1} \frac{w(\sigma)}{\sin^2 \pi\frac{s-\sigma}{2}} d\sigma = \frac{\pi^2}{2}\int_{-1}^{1} \frac{w(\sigma)}{1-\cos\pi(s-\sigma)} d\sigma, \tag{43}$$

and thereby A_o is the hypersingular integral operator on the unit circle (c.f.[8]). Stability and error estimates for the following quadrature method applied to this hypersingular integral are proved in [8]:

$$\begin{aligned} g(t_k) &= \sum_{\substack{j=1-n \\ j\equiv k+1 \bmod 2}}^{n-1} \frac{\pi^2}{2n} \frac{1}{\sin^2 \pi\frac{t_k-t_j}{2}} \xi_j - \frac{n\pi^2}{2}\xi_k \\ &= \sum_{\substack{j=1-n \\ j\equiv k+1 \bmod 2}}^{n-1} \frac{\pi^2}{n} \frac{1}{1-\cos\pi(t_k-t_j)} \xi_j - \frac{n\pi^2}{2}\xi_k \\ 0 &= \sum_{j=1-n}^{n-1} \xi_j \end{aligned} \tag{44}$$

with $k = 1-n,\ldots,n-1$. In the papers of Kress [10] and Proessdorf and Saranen [15] it is shown that the product integration formula leads to the same quadrature method. The quadrature methods (42) and (44) differ from each other only by the one–dimensional functional, which guarantees the uniqueness of the solution. So we get the stability of (42) and of (39) by pertubation theorems (c.f. [11]). For the quadrature method (44) the following convergence estimate is proved in [8]

$$\|w-w_n\|_r \leq Cn^{r-s}\|w\|_s, \qquad \|w-w_n\|_{\frac{1}{2}} \leq Cn^{1-s}\|w\|_s, \tag{45}$$

provided $w \in H_s$ and $s > \frac{3}{2}$, $s \geq r \geq 1$. Repeating the arguments of [8] we get the same convergence estimate for (42) and for (39). Thus

$$\|u-u_n\|_{\frac{1}{2}} = \|w-w_n\|_{\frac{1}{2},\alpha} \leq \|w-w_n\|_{\frac{1}{2}} \leq Cn^{1-s}\|w\|_s,$$

if $w \in H_s$ and $s > \frac{3}{2}$.

7 Appendix

In this section we shall give the missing technical proofs.

7.1. First we shall prove that $\phi_k(\gamma^{-1}(t)) \in \tilde{H}_{\frac{1}{2}}(I)$. Obviously it remains to show $\phi_1(\gamma^{-1}(t)) \in \tilde{H}_{\frac{1}{2}}(I)$, that means

$$\int_0^{\gamma^{-1}(\frac{1}{n})} \int_0^{\gamma^{-1}(\frac{1}{n})} \frac{(n\gamma^{-1}(t) - n\gamma^{-1}(s))^2}{(t-s)^2} dtds + \int_0^{\gamma^{-1}(\frac{1}{n})} \frac{(n\gamma^{-1}(t))^2}{t} dt < \infty.$$

Using the properties of γ (see Remark 3.1), we obtain

$$\begin{aligned} \int_0^{\gamma^{-1}(\frac{1}{n})} \frac{(n\gamma^{-1}(t))^2}{t} dt &= \int_0^{\frac{1}{n}} \frac{(ns)^2}{\gamma(s)} \gamma'(s) ds \\ &= \int_0^1 \frac{1}{n} \frac{s^2}{\gamma(\frac{s}{n})} \gamma'(\frac{s}{n}) ds \\ &\sim \int_0^1 \frac{s^2}{s} ds = \frac{1}{2} < \infty. \end{aligned}$$

Furthermore there holds

$$\int_0^{\gamma^{-1}(\frac{1}{n})} \int_0^{\gamma^{-1}(\frac{1}{n})} \frac{(n\gamma^{-1}(t) - n\gamma^{-1}(s))^2}{(t-s)^2} dtds =$$

$$\begin{aligned} &= \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} n^2 \gamma'(s)\gamma'(t) \frac{(t-s)^2}{(\gamma(t) - \gamma(s))^2} dtds \\ &= \int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} n^2 h(s,t) dtds \le C, \end{aligned}$$

because h is bounded and hence the assertion is proved. ∎

7.2. Further we shall give an argumentation for property 9 in Remark 3.1, i.e. for the assertion $h(s,\sigma) \le 1$, $s, \sigma \in [0,1]$. In the case of $\tilde{\gamma}(s) = s^\alpha$ there holds

$$h(s,\sigma) = \frac{\alpha^2 s^{\alpha-1} \sigma^{\alpha-1} (s-\sigma)^2}{(s^\alpha - \sigma^\alpha)^2}.$$

Setting $x = \frac{s}{\sigma}$ we get

$$h(s,\sigma) = \tilde{h}(x) := \frac{\alpha^2 x^{\alpha-1}(1-x)^2}{(1-x^\alpha)^2} = \frac{x^{\alpha-1}}{\left(\int_0^1 (x + h(1-x))^{\alpha-1} dh\right)^2}.$$

On the other hand,

$$\int_0^1 (x+h(1-x))^{\alpha-1}dh =$$

$$\begin{aligned} &= \int_0^{\frac{1}{2}} (x+h(1-x))^{\alpha-1}dh + \int_{\frac{1}{2}}^1 (x+h(1-x))^{\alpha-1}dh \\ &= \int_{\frac{1}{2}}^1 \left((x+h(1-x))^{\alpha-1} + (x+(1-h)(1-x))^{\alpha-1}\right)dh. \end{aligned} \tag{46}$$

Using the well known inequality $a^z + b^z \geq 2(ab)^{\frac{z}{2}}$ with $z > 0$, we find that

$$\begin{aligned} &(x+h(1-x))^{\alpha-1} + (x+(1-h)(1-x))^{\alpha-1} \geq \\ &\qquad \geq 2(x+h(1-x))^{\frac{\alpha-1}{2}}(x+(1-h)(1-x))^{\frac{\alpha-1}{2}}. \end{aligned} \tag{47}$$

Furthermore there holds

$$(x+h(1-x))(x+(1-h)(1-x)) = x + h(1-h)(x^2-2x+1) \geq x,$$

and thus

$$(x+h(1-x))^{\frac{\alpha-1}{2}}(x+(1-h)(1-x))^{\frac{\alpha-1}{2}} \geq x^{\frac{\alpha-1}{2}}. \tag{48}$$

By (46), (47) and (48) it follows that

$$\int_0^1 (x+h(1-x))^{\alpha-1}dh \geq 2\int_{\frac{1}{2}}^1 x^{\frac{\alpha-1}{2}}dh = x^{\frac{\alpha-1}{2}},$$

and thus

$$h(s,\sigma) = \tilde{h}(x) = \frac{x^{\alpha-1}}{\left(\int_0^1 (x+h(1-x))^{\alpha-1}dh\right)^2} \leq 1.$$

■

For the transformation γ introduced in (2) the proof is more complicated.

7.3. Now we pass to the proof of Remark 2.2, i.e. we prove that the solution u of the hypersingular integral equation on the interval can be written in the form $u(t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}gl(t)$ with $gl \in C^\infty$. Like in Sect. 6 we transform Equation (1) setting $t = \gamma_1(s) = \frac{1-\cos(\pi s)}{2}$ to get Equation (37). The relation (41) holds provided $w \in \tilde{H}_0^t$ is odd. The mapping properties of operator $\dot{A}_0$ can be found in [8]. In particular we get $w \in C^\infty$ if $g \in C^\infty$, where g is defined on $[-1,1]$ by odd continuation. Now $g \in C^\infty$ follows from $f \in C^\infty$ in the following way:

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}t^k, \quad t \in [0,1],$$

and thus

$$g(t) = \gamma_1'(t) \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (\gamma_1(t))^k, \quad t \in [0,1].$$

Using that g and γ_1' are odd and γ_1 is even, we get

$$\begin{aligned} g(t) &= -\gamma_1'(-t) \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (\gamma_1(-t))^k \\ &= \gamma_1'(t) \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (\gamma_1(t))^k, \quad t \in [-1,0]. \end{aligned}$$

We arrive at $g \in C^\infty$. Using the definitions of γ_1 and g it can be seen that the odd continuation of g is 2–peridic with all derivativs. Thus $w \in C^\infty$. Now there holds

$$u(t) = w(\gamma_1^{-1}(t)) = w\left(\frac{1}{\pi} \arccos(1-2t)\right).$$

Because $u(0) = u(1) = 0$, w is odd and $\arccos(1-2t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} gl_1(t)$ with $gl_1 \in C^\infty$ the assertion $u(t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}} gl(t)$ is proved. ∎

7.4. Now we give the proof of the main theorem of Section 3. First we use the definitions of the norms in E_n and $\tilde{H}_{\frac{1}{2},\alpha}$ and the definition of the linear splines ϕ_j to evaluate $\|\{\xi_j\}\|_{E_n}$:

$$\begin{aligned} \|\{\xi_i\}\|_{E_n}^2 &= \int_0^1 \int_0^1 \frac{\left(\sum_{i=1}^{n-1} \xi_i \phi_i(x) - \sum_{l=1}^{n-1} \xi_l \phi_l(y)\right)^2}{(x-y)^2} h(x,y)\,dx\,dy \\ &\qquad + \int_0^1 \frac{\left(\sum_{l=1}^{n-1} \xi_l \phi_l(x)\right)^2}{\gamma(x)(1-\gamma(x))} \gamma'(x)\,dx \\ &= \sum_{i,l=0}^{n-1} \int_i^{i+1} \int_l^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2} h(\frac{x}{n}, \frac{y}{n})\,dx\,dy \\ &\qquad + \sum_{l=0}^{n-1} \int_l^{l+1} \frac{1}{n} \frac{\left(\xi_l(l+1-x) + \xi_{l+1}(x-l)\right)^2}{\gamma(\frac{x}{n})(1-\gamma(\frac{x}{n}))} \gamma'(\frac{x}{n})\,dx. \end{aligned} \tag{49}$$

with

$$s_{l,i}(x,y) := \xi_l(l+1-x) + \xi_{l+1}(x-l) - \xi_i(i+1-y) - \xi_{i+1}(y-i).$$

We set

$$S_{l,i} := \int_i^{i+1} \int_l^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2} h(\frac{x}{n}, \frac{y}{n})\,dx\,dy. \tag{50}$$

First we consider the second term of the sum (49). Because of $\frac{\gamma'(s)s(1-s)}{\gamma(s)(1-\gamma(s))} \sim 1$ (c.f. prop.6 of Remark 3.1) there holds

$$\int_l^{l+1} \frac{1}{n} \frac{\left(\xi_l(l+1-x)+\xi_{l+1}(x-l)\right)^2}{\gamma(\frac{x}{n})(1-\gamma(\frac{x}{n}))} \gamma'(\frac{x}{n}) dx \sim$$

$$\int_l^{l+1} \frac{\left(\xi_l(l+1-x)+\xi_{l+1}(x-l)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})} dx$$

Let $1 \leq l \leq \frac{n}{2}-1$. Since $1-\frac{x}{n} \sim 1$ if $\frac{x}{n} < \frac{1}{2}$, we have

$$\int_l^{l+1} \frac{\left(\xi_l(l+1-x)+\xi_{l+1}(x-l)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})} dx$$

$$\begin{aligned} &\sim \int_l^{l+1} \frac{1}{x} \left(\xi_l(l+1-x)+\xi_{l+1}(x-l)\right)^2 dx \\ &\sim \frac{1}{l} \int_0^1 \left(\xi_l(1-x)+\xi_{l+1}x\right)^2 dx \\ &= \frac{1}{3}\frac{1}{l}(\xi_l^2+\xi_{l+1}^2+\xi_l\xi_{l+1}) \sim \frac{1}{l}(\xi_l^2+\xi_{l+1}^2). \end{aligned}$$

If $l=0$ then

$$\int_0^1 \frac{\xi_1^2 x^2}{x(1-\frac{x}{n})} dx \sim \xi_1^2 \int_0^1 x dx \sim \xi_1^2.$$

Analogously we obtain

$$\int_l^{l+1} \frac{\left(\xi_l(l+1-x)+\xi_{l+1}(x-l)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})} dx \sim \frac{1}{n-l}(\xi_l^2+\xi_{l+1}^2),$$

if $\frac{n}{2} \leq l \leq n-1$. We arrive at

$$\sum_{l=0}^{n-1} \int_l^{l+1} \frac{1}{n} \frac{\left(\xi_l(l+1-x)+\xi_{l+1}(x-l)\right)^2}{\gamma(\frac{x}{n})(1-\gamma(\frac{x}{n}))} \gamma'(\frac{x}{n}) dx \sim \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2. \tag{51}$$

Now we investigate $S_{l,i}$. Without loss of generality let $i \leq l$. Before studying $S_{l,i}$ we prove the following lemma.

Lemma 7.1 *If $t,s \in [0,1]$, $1 \leq i,l \leq n-2$, $\quad |l-i| \geq 2$ then*

$$h\left(\frac{l}{n},\frac{i}{n}\right) \sim h\left(\frac{l+t}{n},\frac{i+s}{n}\right). \tag{52}$$

In order to prove this lemma we recall the definition of h and consider several terms of the product

$$h(x,y)=\frac{\gamma'(x)\gamma'(y)(x-y)^2}{(\gamma(x)-\gamma(y))^2}.$$

First let $1\le l,i\le \frac{n}{2}-1$, $\quad |l-i|\ge 2$. Furthermore assume $\frac{l+1}{n}<\delta(\alpha)$ with $\delta(\alpha)$ being choosen as in Remark 3.1 . By considering the denominator and using the monotonicity of γ, we see that

$$\begin{aligned}\frac{\gamma(\frac{l+t}{n})-\gamma(\frac{i+s}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{i}{n})} &\le \frac{\gamma(\frac{l+t}{n})-\gamma(\frac{i}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{i}{n})}\\ &\le 1+\frac{\gamma(\frac{l+t}{n})-\gamma(\frac{l}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{l-2}{n})}\\ &\le 1+\frac{1}{2}\frac{\gamma'(\frac{l+1}{n})}{\gamma'(\frac{l-2}{n})},\end{aligned}$$

because γ' is konvex in $[0,\delta(\alpha)]$. Further we have

$$\begin{aligned}\frac{\gamma'(\frac{l+1}{n})}{\gamma'(\frac{l-2}{n})} &\sim \frac{v(\frac{l+1}{n})^{\alpha-1}}{v(\frac{l-2}{n})^{\alpha-1}}\frac{v(1-\frac{l+1}{n})^{\alpha-1}}{v(1-\frac{l-2}{n})^{\alpha-1}}\left(\frac{v(\frac{l-2}{n})^\alpha+v(\frac{n-l+2}{n})^\alpha}{v(\frac{l+1}{n})^\alpha+v(\frac{n-l-1}{n})^\alpha}\right)^2\\ &\le \frac{v(\frac{l+1}{n})^{\alpha-1}}{v(\frac{l-2}{n})^{\alpha-1}}2^{\alpha-1}2^{2(\alpha-1)},\end{aligned}$$

because $v(x)\le 1$, $\quad v(1-\frac{l-2}{n})\ge\frac{1}{2}$ and $\left(\frac{1}{2}\right)^{\alpha-1}\le v(x)^\alpha+v(1-x)^\alpha\le 1$. Since $l\ge i+2\ge 3$ and using the properties of v (see Remark 3.1), we get

$$\frac{v(\frac{l+1}{n})}{v(\frac{l-2}{n})}\sim\frac{\frac{l+1}{n}}{\frac{l-2}{n}}=\frac{l+1}{l-2}\le 6.$$

We arrive at

$$\frac{\gamma(\frac{l+t}{n})-\gamma(\frac{i+s}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{i}{n})}\le C,$$

provided $l+1\le\delta(\alpha)n$. Let now $l\ge\delta(\alpha)n-1$. Then there holds

$$\begin{aligned}\frac{\gamma(\frac{l+t}{n})-\gamma(\frac{i+s}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{i}{n})} &\le \frac{\gamma(\frac{l+t}{n})-\gamma(\frac{i}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{i}{n})}\\ &\le 1+\frac{\gamma(\frac{l+t}{n})-\gamma(\frac{l}{n})}{\gamma(\frac{l}{n})-\gamma(\frac{l-2}{n})}\\ &\le 1+\frac{1}{2}\frac{\gamma'(\xi_1)}{\gamma'(\xi_2)}\le C,\end{aligned}$$

because $\gamma' \sim 1$ on the interval $(\frac{\delta(\alpha)}{2}, 1-\frac{\delta(\alpha)}{2})$.

To get the estimate from below we have to distinguish two cases again. If $i \leq \delta(\alpha)n - 2$ then

$$\begin{aligned}
\frac{\gamma(\frac{l+t}{n}) - \gamma(\frac{i+s}{n})}{\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})} &\geq \frac{\gamma(\frac{l}{n}) - \gamma(\frac{i+1}{n})}{\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})} \\
&= 1 + \frac{\gamma(\frac{i}{n}) - \gamma(\frac{i+1}{n})}{\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})} \\
&\geq 1 - \frac{\gamma(\frac{i+1}{n}) - \gamma(\frac{i}{n})}{\gamma(\frac{i+2}{n}) - \gamma(\frac{i}{n})} \\
&= \frac{\gamma'(\frac{i+1}{n})\frac{1}{n} + \gamma''(\xi_2)\frac{1}{2}\frac{1}{n^2}}{\gamma'(\frac{i+2}{n})\frac{2}{n} + \gamma''(\xi_1)\frac{2}{n^2}},
\end{aligned}$$

with $\xi_1 \in [\frac{i}{n}, \frac{i+2}{n}]$ and $\xi_2 \in [\frac{i+1}{n}, \frac{i+2}{n}]$. For the reciprocal value there holds

$$\frac{\gamma'(\frac{i+2}{n})\frac{2}{n} + \gamma''(\xi_1)\frac{2}{n^2}}{\gamma'(\frac{i+1}{n})\frac{1}{n} + \gamma''(\xi_2)\frac{1}{2}\frac{1}{n^2}} \leq 2 + \frac{2}{n}\frac{\gamma''(\xi_1)}{\gamma'(\frac{i+2}{n})} \leq C,$$

because $\frac{1}{n}\frac{\gamma''(x/n)}{\gamma'(x/n)}$ is bounded (see the formula of γ'' in Remark 3.1).

If $i \geq \frac{\delta(\alpha)n}{2}$ then

$$\begin{aligned}
\frac{\gamma(\frac{l+t}{n}) - \gamma(\frac{i+s}{n})}{\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})} &\geq \frac{\gamma(\frac{l}{n}) - \gamma(\frac{i+1}{n})}{\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})} \\
&= \frac{\gamma'(\xi_1)(l-i-1)}{\gamma'(\xi_2)(l-i)} \geq C,
\end{aligned}$$

with $\xi_1 \in [\frac{i+1}{n}, \frac{l}{n}]$, $\quad \xi_2 \in [\frac{i}{n}, \frac{l}{n}]$. The last estimate is true, because $\frac{l-i-1}{l-i} \geq \frac{1}{2}$ and $\gamma' \sim 1$ on the interval $(\frac{\delta(\alpha)}{2}, 1-\frac{\delta(\alpha)}{2})$. We arrive at

$$\left(\gamma(\frac{l}{n}) - \gamma(\frac{i}{n})\right)^2 \sim \left(\gamma(\frac{l+t}{n}) - \gamma(\frac{i+s}{n})\right)^2. \tag{53}$$

Furthermore there holds

$$\left(\frac{l}{n} - \frac{i}{n}\right)^2 \sim \left(\frac{l+t}{n} - \frac{i+s}{n}\right)^2, \tag{54}$$

because of $|l-i| \geq 2$. Now it remains to consider the term

$$\frac{\gamma'(\frac{l+t}{n})}{\gamma'(\frac{l}{n})} = \frac{\left(v(\frac{l+t}{n})\right)^{\alpha-1}\left(v(\frac{n-l-t}{n})\right)^{\alpha-1} v'(\frac{l+t}{n})}{\left(v(\frac{l}{n})\right)^{\alpha-1}\left(v(\frac{n-l}{n})\right)^{\alpha-1} v'(\frac{l}{n})}\left(\frac{f(\frac{l}{n})}{f(\frac{l+t}{n})}\right)^2,$$

with $f(t) := (v(t))^\alpha + (v(1-t))^\alpha$. Because of $\left(\frac{1}{2}\right)^{\alpha-1} \leq f(t) \leq 1$, $v' \sim 1$, $v(\frac{n-l}{n}), v(\frac{n-l-t}{n}) \sim 1$ and $v(\frac{l+t}{n}) \sim \frac{l+t}{n}$, $v(\frac{l}{n}) \sim \frac{l}{n}$ we get

$$\frac{\gamma'(\frac{l+t}{n})}{\gamma'(\frac{l}{n})} \sim \frac{(l+t)^{\alpha-1}}{l^{\alpha-1}}.$$

Obviously

$$1 \leq \frac{(l+t)^{\alpha-1}}{l^{\alpha-1}} \leq \frac{(l+1)^{\alpha-1}}{l^{\alpha-1}} \leq 2^{\alpha-1}.$$

We arrive at

$$\gamma'(\frac{l+t}{n}) \sim \gamma'(\frac{l}{n}). \tag{55}$$

With (53), (54) and (55) we obtain (52) in the case $1 \leq l, i \leq \frac{n}{2} - 1$, $|l-i| \geq 2$. The case $\frac{n}{2} \leq l, i \leq n-2$, $|l-i| \geq 2$ runs analogously.

Consider the last case $1 \leq l, i \leq n-2$, $|l-i| \geq 2$, $i \leq \frac{n}{2} - 1, l \geq \frac{n}{2}$. In this case there hold also (54) and (55). Using (53) separately for the intervals $(0, \frac{1}{2}]$ and $[\frac{1}{2}, 1)$, we have

$$\gamma(\frac{l+t}{n}) - \gamma(\frac{i+s}{n}) = \gamma(\frac{l+t}{n}) - \gamma(\frac{1}{2}) + \gamma(\frac{1}{2}) - \gamma(\frac{i+s}{n})$$

$$\sim \gamma(\frac{l}{n}) - \gamma(\frac{1}{2}) + \gamma(\frac{1}{2}) - \gamma(\frac{i}{n}) = \gamma(\frac{l}{n}) - \gamma(\frac{i}{n}).$$

So assertion (53) is true in this case, too. The formula (52) follows from (53), (54) and (55). This completes the proof of the lemma. ∎

Now we continue the proof of the theorem. First let $1 \leq l, i \leq n-2$, $|l-i| \geq 2$. Using Lemma 7.1, and the definition of $S_{l,i}$, we see

$$\begin{aligned} S_{l,i} &\sim h(\frac{l}{n}, \frac{i}{n}) \int_i^{i+1} \int_l^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2} dx dy \\ &\sim h(\frac{l}{n}, \frac{i}{n}) \frac{1}{(l-i)^2} \int_i^{i+1} \int_l^{l+1} s_{l,i}^2(x,y) dx dy, \end{aligned}$$

since $\frac{1}{(x-y)^2} \sim \frac{1}{(l-i)^2}$. With the definition of $s_{l,i}$ we compute

$$\int_i^{i+1} \int_l^{l+1} s_{l,i}^2(x,y) dx dy = \frac{1}{3} \left(\xi_l^2 + \xi_{l+1}^2 + \xi_i^2 + \xi_{i+1}^2 + \xi_l \xi_{l+1} + \xi_i \xi_{i+1}\right)$$

$$-\frac{1}{2} \left(\xi_i \xi_l + \xi_l \xi_{i+1} + \xi_{l+1} \xi_i + \xi_{i+1} \xi_{l+1}\right).$$

This is a quadratic form with zero in $\xi_i = \xi_l = \xi_{l+1} = \xi_{i+1}$ and therewith equivalent to the following quadratic form with zero in $\xi_i = \xi_l = \xi_{l+1} = \xi_{i+1}$:

$$\int_i^{i+1}\int_l^{l+1} s_{l,i}^2(x,y)dxdy \sim (\xi_l-\xi_i)^2 + (\xi_{l+1}-\xi_i)^2 +$$

$$+(\xi_l-\xi_{i+1})^2 + (\xi_{l+1}-\xi_{i+1})^2.$$

Thus we get for $1 \le l, i \le n-2, \quad |l-i| \ge 2$

$$S_{l,i} \sim h(\frac{l}{n},\frac{i}{n})\frac{1}{(l-i)^2}((\xi_l-\xi_i)^2 + (\xi_{l+1}-\xi_i)^2 \tag{56}$$

$$+(\xi_l-\xi_{i+1})^2 + (\xi_{l+1}-\xi_{i+1})^2).$$

Now we consider the second case $1 \le i, l \le n-2, \quad |l-i| < 2$. Before estimating $S_{l,i}$ in this case in more detail we prove the following lemma.

Lemma 7.2 *If $x, y \in [\frac{1}{n}, \frac{n-1}{n}]$, $|x-y| \le \frac{2}{n}$ then $h(x,y) \sim 1$.*

Without loss of generality let $x \ge y$.
If $x = y$ then there holds $h(x,y) = 1$ (see the definition of h).
Let now $x \ne y$. Then there holds

$$h(x,y) = \frac{\gamma'(x)\gamma'(y)(x-y)^2}{(\gamma(x)-\gamma(y))^2} = \frac{\gamma'(x)\gamma'(y)}{(\gamma'(\xi))^2},$$

with a $\xi \in [y,x]$. Because of $x, y \in [\frac{1}{n}, \frac{n-1}{n}]$, $|x-y| \le \frac{2}{n}$ there is an l with $1 \le l \le n-4$ such that

$$\frac{l}{n} \le y \le \xi \le x \le \frac{l+3}{n}.$$

Analogously to (55) we get

$$\gamma'(\frac{l+1}{n}) \sim \gamma'(x) \sim \gamma'(y) \sim \gamma'(\xi)$$

and we arrive at the assertion $h(x,y) \sim 1$. ∎

Using the definition of $S_{l,i}$ and the last lemma we obtain

$$S_{l,l} \sim (\xi_l - \xi_{l+1})^2 \tag{57}$$

provided $1 \leq i = l \leq n-2$.
Now let $2 \leq l = i+1 \leq n-2$. Using the definition of $S_{l,i}$ and the last lemma again, we get

$$
\begin{aligned}
S_{l,l-1} &\sim \int_{l-1}^{l}\int_{l}^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2}dxdy \\
&= \xi_l^2(3-4\ln 2) + (\xi_{l+1}^2 + \xi_{l-1}^2)(1-\ln 2) + \xi_{l-1}\xi_{l+1}(1-2\ln 2) \\
&\qquad + (\xi_l\xi_{l+1} + \xi_l\xi_{l-1})(-3+4\ln 2).
\end{aligned}
$$

This is a quadratic form with zero in $\xi_l = \xi_{l+1} = \xi_{l-1}$ and therewith equivalent to the following quadratic form with zero in $\xi_l = \xi_{l+1} = \xi_{l-1}$:

$$
S_{l,l-1} \sim (\xi_l - \xi_{l-1})^2 + (\xi_l - \xi_{l+1})^2 . \tag{58}
$$

Let $i = 0$. For the present consideration let $l \geq 2$. Then there holds $\frac{1}{(x-y)^2} \leq \frac{4}{l^2}$ for all $y \in [0,1], x \in [l, l+1]$. Using this property and $h(x,y) \leq 1$, we have

$$
\begin{aligned}
0 &\leq S_{l,0} \leq \int_0^1 \int_l^{l+1} \frac{(\xi_l(l+1-x) + \xi_{l+1}(x-l) - \xi_1 y)^2}{(x-y)^2} dxdy \\
&\leq \frac{4}{l^2}\int_0^1\int_0^1 (\xi_l(1-x) + \xi_{l+1}x - \xi_1 y)^2\, dxdy \\
&= \frac{4}{l^2}\left(\frac{1}{3}(\xi_l^2 + \xi_{l+1}^2 + \xi_1^2 + \xi_l\xi_{l+1}) - \frac{1}{2}(\xi_l\xi_1 + \xi_{l+1}\xi_1)\right) \\
&\leq \frac{C}{l^2}(\xi_l^2 + \xi_{l+1}^2 + \xi_1^2).
\end{aligned}
$$

Let now $i = 0$ and $l = 1$. Using $h(x,y) \leq 1$ again, we get

$$
\begin{aligned}
0 &\leq S_{1,0} \leq \int_0^1 \int_1^2 \frac{(\xi_l(2-x) + \xi_2(x-1) - \xi_1 y)^2}{(x-y)^2} dtds \\
&= \xi_1^2(3-4\ln 2) + \xi_2^2(1-\ln 2) + \xi_1\xi_2(-3+4\ln 2) \\
&\leq C(\xi_1^2 + \xi_2^2).
\end{aligned}
$$

We arrive at

$$
0 \leq \sum_{l=1}^{n-1} S_{l,0} \leq C\left(\sum_{l=1}^{n-1}\frac{1}{l}\xi_l^2\right) \leq C\left(\sum_{l=1}^{\frac{n}{2}-1}\frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2\right). \tag{59}
$$

Analogously we show that

$$
0 \leq \sum_{l=1}^{n-1} S_{l,n-1} \leq C\left(\sum_{l=1}^{n-1}\frac{1}{n-l}\xi_l^2\right) \leq C\left(\sum_{l=1}^{\frac{n}{2}-1}\frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2\right). \tag{60}
$$

From (49), (50), (51), (56), (57), (58) , (59) and (60) it follows that

$$\begin{aligned}
\|\{\xi_j\}\|^2_{E_n} &\sim \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{l=1}^{n-2}(\xi_l - \xi_{l+1})^2 \\
&\quad + \sum_{\substack{l,i=1 \\ |i-l|\geq 2}}^{n-2} h(\frac{l}{n},\frac{i}{n})\frac{1}{|i-l|^2}((\xi_l-\xi_i)^2 + (\xi_l - \xi_{i+1})^2 \\
&\qquad + (\xi_{l+1}-\xi_i)^2 + (\xi_{l+1}-\xi_{i+1})^2) \\
&\sim \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{l=1}^{n-1}(\xi_l - \xi_{l+1})^2 \\
&\quad + \sum_{\substack{l,i=1 \\ |i-l|\geq 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{|i-l|^2}(\xi_l-\xi_i)^2,
\end{aligned}$$

because for $|l-i|\geq 2$ there holds $\frac{1}{|l-i|^2} \sim \frac{1}{|l-i-1|^2} \sim \frac{1}{|l-i+1|^2}$ and $h(\frac{l}{n},\frac{i}{n}) \sim h(\frac{l+1}{n},\frac{i}{n}) \sim h(\frac{l}{n},\frac{i+1}{n})$. Using $h(\frac{l}{n},\frac{l+1}{n}) \sim 1$ we get

$$\begin{aligned}
\|\{\xi_j\}\|^2_{E_n} &\sim \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1 \\ i\neq l}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{|i-l|^2}(\xi_l-\xi_i)^2 \\
&\sim \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2 + \\
&\quad + \sum_{\substack{l,i=1 \\ i\neq l}}^{\frac{n}{2}-1} h(\frac{2l}{n},\frac{2i}{n})\frac{1}{|2i-2l|^2} q(\xi_{2l},\xi_{2l+1},\xi_{2i},\xi_{2i+1}),
\end{aligned}$$

with

$$\begin{aligned}
q(\xi_{2l},\xi_{2l+1},\xi_{2i},\xi_{2i+1}) &= (\xi_{2l}-\xi_{2i})^2 + (\xi_{2l}-\xi_{2i+1})^2 + (\xi_{2l+1}-\xi_{2i})^2 \\
&\quad + (\xi_{2l+1}-\xi_{2i+1})^2.
\end{aligned}$$

Obviously $q(\xi_{2l},\xi_{2l+1},\xi_{2i},\xi_{2i+1})$ is a quadratic form with zero in $\xi_{2l} = \xi_{2l+1} = \xi_{2i} = \xi_{2i+1}$ and thus equivalent to the quadratic form $(\xi_{2l}-\xi_{2i+1})^2 + (\xi_{2l+1}-\xi_{2i})^2 + (\xi_{2l}-\xi_{2l+1})^2 + (\xi_{2i}-\xi_{2i+1})^2$. Using this equivalence, we see that

$$\begin{aligned}
\|\{\xi_j\}\|^2_{E_n} &\sim \sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1 \\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{|i-l|^2}(\xi_l-\xi_i)^2 \\
&\quad + \sum_{l=1}^{n-2}\left(\sum_{i=1}^{n-2} h(\frac{l}{n},\frac{i}{n})\frac{1}{|i-l|^2}\right)(\xi_l-\xi_{l+1})^2.
\end{aligned}$$

Furthermore there holds

$$\sum_{i=1}^{n-2} h(\frac{l}{n},\frac{i}{n})\frac{1}{\mid i-l\mid^2} \le \frac{\pi^2}{3},$$

and we arrive at

$$\|\{\xi_j\}\|^2_{E_n} \sim \sum_{l=1}^{\frac{n}{2}-1}\frac{1}{l}\xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2 + \sum_{\substack{l,i=1\\ i\not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n},\frac{i}{n})\frac{1}{\mid i-l\mid^2}(\xi_l-\xi_i)^2,$$

which completes the proof of the theorem, because the second assertion follows immediately by duality. ∎

7.5. Last we give the proof of Lemma 5.2, i.e we show that

$$\|P_n^d\xi\|_{\frac{1}{2},\alpha} \le C\|\xi\|_{E_n}$$

for all $\xi = \{\xi_j\} \in E_n$. By definition of P_n^d and of the norm there holds

$$\begin{aligned}
\|P_n^d\xi\|^2_{\frac{1}{2},\alpha} &= \|\sum_{j=1}^{n-1}\xi_j\phi_j^d\|^2_{\frac{1}{2},\alpha} \\
&= \int_0^1\int_0^1 \frac{\left(\sum_{i=1}^{n-1}\xi_i\phi_i^{(d)}(x) - \sum_{l=1}^{n-1}\xi_l\phi_l^{(d)}(y)\right)^2}{(x-y)^2}h(x,y)dxdy \\
&\qquad + \int_0^1 \frac{\left(\sum_{l=1}^{n-1}\xi_l\phi_l^{(d)}(x)\right)^2}{\gamma(x)(1-\gamma(x))}\gamma'(x)dx \\
&= \sum_{i,l=0}^{n-1}\int_i^{i+1}\int_l^{l+1}\frac{s_{l,i}^2(x,y)}{(x-y)^2}h(\frac{x}{n},\frac{y}{n})dxdy \\
&\qquad + \sum_{l=0}^{n-1}\int_l^{l+1}\frac{1}{n}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{\gamma(\frac{x}{n})(1-\gamma(\frac{x}{n}))}\gamma'(\frac{x}{n})dx.
\end{aligned} \tag{61}$$

with

$$s_{l,i}(x,y) := \sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x) - \sum_{j\in D(i)}\xi_j\tilde{\phi}_j^{(d)}(y)$$

and $D(l) := \{d(l), d(l)+1,\dots,(l)+d\}$, where $d(l)$ is defined by $d(l) \in Z$, $d(l) \le l < d(l)+d$, $d(l) \equiv 0 \bmod d$. The interpolation polynomials $\tilde{\phi}_k^{(d)}$ are polynomials in $[d(k), d(k)+d]$ with $\tilde{\phi}_k^{(d)}(j) = \delta_{k,j}$. We set

$$S_{l,i} := \int_i^{i+1}\int_l^{l+1}\frac{s_{l,i}^2(x,y)}{(x-y)^2}h(\frac{x}{n},\frac{y}{n})dxdy. \tag{62}$$

First we consider the second term of the sum (61). Because of $\frac{\gamma'(s)s(1-s)}{\gamma(s)(1-\gamma(1-s))} \sim 1$ (property 6 in Remark 3.1), we have

$$\sum_{l=0}^{n-1}\int_l^{l+1}\frac{1}{n}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{\gamma(\frac{x}{n})(1-\gamma(\frac{x}{n}))}\gamma'(\frac{x}{n})dx \sim \sum_{l=0}^{n-1}\int_l^{l+1}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})}dx$$

Let $1 \le l \le \frac{n}{2}-1$. Then we obviously have

$$\begin{aligned}
\int_l^{l+1}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})}dx &\le C\int_l^{l+1}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{x}dx \\
&\le C\frac{1}{l}\int_l^{l+1}\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2dx \\
&\le C\frac{1}{l}\int_l^{l+1}\left(\sum_{k=0}^{d}\xi_{k+d(l)}\tilde{\phi}_k^{(d)}(x)\right)^2dx \\
&\le C\frac{1}{l}\sum_{k=0}^{d}\xi_{k+d(l)}^2. \qquad (63)
\end{aligned}$$

If $l=0$ then

$$\int_0^1\frac{\left(\sum_{k=1}^{d}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})}dx \le C\sum_{k=1}^{d}\xi_k^2,$$

because $\frac{1}{1-x/n} \le 2$ and $\frac{\tilde{\phi}_k^{(d)}(x)}{x}$ is a polynom of degree $d-1 \ge 1$ for $k=1,\ldots,d$ (c.f. definition of $\tilde{\phi}_k^{(d)}$). With (63) we get

$$\sum_{l=0}^{\frac{n}{2}-1}\int_l^{l+1}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{x(1-\frac{x}{n})}dx \le C\sum_{l=0}^{\frac{n}{2}}\frac{1}{l}\xi_l^2.$$

Analogously there holds

$$\sum_{l=\frac{n}{2}}^{n-1}\int_l^{l+1}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{\frac{x}{n}(n-x)}dx \le C\sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2.$$

and thus

$$\sum_{l=0}^{n-1}\int_l^{l+1}\frac{\left(\sum_{k\in D(l)}\xi_k\tilde{\phi}_k^{(d)}(x)\right)^2}{n\frac{x}{n}(1-\frac{x}{n})}dx \le C\left(\sum_{l=0}^{\frac{n}{2}}\frac{1}{l}\xi_l^2+\sum_{l=\frac{n}{2}}^{n-1}\frac{1}{n-l}\xi_l^2\right). \qquad (64)$$

Consider now the first term of the right–hand side of (61), which will be denoted by $\sum_{\substack{i,l=1 \\ l\not\equiv i \bmod 2}}^{n-1} S_{l,i}$. To estimate $S_{l,i}$ we distinguish several cases. First let $1 \le l, i \le n-2$, $|l-i| \ge 2$. Using the Lemma 7.1, we see that

$$\begin{aligned} S_{l,i} &\sim h(\frac{l}{n},\frac{i}{n}) \int_i^{i+1} \int_l^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2} dxdy \\ &\sim h(\frac{l}{n},\frac{i}{n}) \frac{1}{(l-i)^2} \int_i^{i+1} \int_l^{l+1} s_{l,i}^2(x,y) dxdy, \end{aligned}$$

since $\frac{1}{(x-y)^2} \sim \frac{1}{(l-i)^2}$. With the definition of $s_{l,i}$ we compute

$$\int_i^{i+1} \int_l^{l+1} s_{l,i}^2(x,y)dxdy =$$

$$= \int_{i-d(i)}^{i+1-d(i)} \int_{l-d(l)}^{l+1-d(l)} \left(\sum_{k=0}^{d} \xi_{k+d(l)} \tilde{\phi}_k^{(d)}(x) - \sum_{j=0}^{d} \xi_{j+d(i)} \tilde{\phi}_j^{(d)}(y) \right)^2 dxdy.$$

Because of

$$\tilde{\phi}_k^{(d)}(x) = \frac{\Pi_{\substack{j=0 \\ j\neq k}}^{d} (x-j)}{\Pi_{\substack{j=0 \\ j\neq k}}^{d} (k-j)},$$

and

$$\tilde{\phi}_0^{(d)} = 1 - \sum_{j=1}^{d} \tilde{\phi}_j^{(d)},$$

we get

$$\sum_{k=0}^{d} \xi_{k+d(l)} \tilde{\phi}_k^{(d)}(x) - \sum_{j=0}^{d} \xi_{j+d(i)} \tilde{\phi}_j^{(d)}(y) =$$

$$\sum_{k=1}^{d} (\xi_{k+d(l)} - \xi_{d(l)}) \tilde{\phi}_k^{(d)}(x) - \sum_{j=1}^{d} (\xi_{j+d(i)} - \xi_{d(l)}) \tilde{\phi}_j^{(d)}(y) + (\xi_{d(l)} - \xi_{d(i)}),$$

and therewith

$$\int_{i-d(i)}^{i+1-d(i)} \int_{l-d(l)}^{l+1-d(l)} \left(\sum_{k=0}^{d} \xi_{k+d(l)} \tilde{\phi}_k^{(d)}(x) - \sum_{j=0}^{d} \xi_{j+d(i)} \tilde{\phi}_j^{(d)}(y) \right)^2 dxdy \le$$

$$
\begin{aligned}
&\leq C((\xi_{d(l)} - \xi_{d(i)})^2 \\
&\quad + \sum_{k=1}^{d} (\xi_{k+d(l)} - \xi_{d(l)})^2 \int_{i-d(i)}^{i+1-d(i)} \int_{l-d(l)}^{l+1-d(l)} \left(\tilde{\phi}_k^{(d)}(x)\right)^2 dxdy \\
&\quad + \sum_{j=1}^{d} (\xi_{j+d(i)} - \xi_{d(l)})^2 \int_{i-d(i)}^{i+1-d(i)} \int_{l-d(l)}^{l+1-d(l)} \left(\tilde{\phi}_j^{(d)}(y)\right)^2 dxdy) \\
&\leq C\left((\xi_{d(l)} - \xi_{d(i)})^2 + \sum_{k=1}^{d} (\xi_{k+d(l)} - \xi_{d(l)})^2 + \sum_{j=1}^{d} (\xi_{j+d(i)} - \xi_{d(l)})^2 \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
S_{l,i} \leq Ch(\frac{l}{n}, \frac{i}{n}) \frac{1}{(l-i)^2} \Bigg((\xi_{d(l)} - \xi_{d(i)})^2 + \sum_{k=1}^{d} (\xi_{k+d(l)} - \xi_{d(l)})^2 + \\
+ \sum_{j=1}^{d} (\xi_{j+d(i)} - \xi_{d(l)})^2 \Bigg).
\end{aligned}
\tag{65}
$$

if $|l-i| \geq 2$, $1 \leq l, i \leq n-2$.

If $i = 0$, $l \geq 2$ then $\frac{1}{(x-y)^2} \leq \frac{4}{l^2}$ and thus

$$
S_{l,0} \leq \frac{C}{l^2} \int_0^1 \int_{l-d(l)}^{l+1-d(l)} \left(\sum_{k=0}^{d} \xi_{k+d(l)} \tilde{\phi}_k^{(d)}(x) - \sum_{j=1}^{d} \xi_j \tilde{\phi}_j^{(d)}(y) \right)^2 dxdy
$$

$$
\leq \frac{C}{l^2} \left(\sum_{k=0}^{d} \xi_{k+d(l)}^2 + \sum_{j=1}^{d} \xi_j^2 \right). \tag{66}
$$

Analogously there holds

$$
S_{l,n-1} \leq \frac{C}{(n-l)^2} \left(\sum_{k=0}^{d} \xi_{k+d(l)}^2 + \sum_{j=n-d}^{n-1} \xi_j^2 \right). \tag{67}
$$

Now we consider the second case $|l-i| < 2$. Using the property 9 in Remark 3.1, i.e. $h(x,y) \leq 1$, we get

$$
S_{l,i} \leq C \int_i^{i+1} \int_l^{l+1} \frac{s_{l,i}^2(x,y)}{(x-y)^2} dxdy,
$$

in this case. Using $\tilde{\phi}_0^{(d)}(x) = 1 - \sum_{k=1}^{d} \tilde{\phi}_k^{(d)}(x)$, we obtain

$$
S_{l,l} = \int_{l-d(l)}^{l+1-d(l)} \int_{l-d(l)}^{l+1-d(l)} \frac{\left(\sum_{k=0}^{d} \xi_{k+d(l)} \left(\tilde{\phi}_k^{(d)}(x) - \tilde{\phi}_k^{(d)}(y) \right) \right)^2}{(x-y)^2} dxdy
$$

$$
\begin{aligned}
&= \int_{l-d(l)}^{l+1-d(l)} \int_{l-d(l)}^{l+1-d(l)} \frac{\left(\sum_{k=1}^{d} \left(\xi_{k+d(l)} - \xi_{d(l)}\right) \left(\tilde{\phi}_k^{(d)}(x) - \tilde{\phi}_k^{(d)}(y)\right)\right)^2}{(x-y)^2} dx dy \\
&\leq C \left(\sum_{k=1}^{d} \left(\xi_{k+d(l)} - \xi_{d(l)}\right)^2 \right) \int_{l-d(l)}^{l+1-d(l)} \int_{l-d(l)}^{l+1-d(l)} \frac{\left(\tilde{\phi}_k^{(d)}(x) - \tilde{\phi}_k^{(d)}(y)\right)^2}{(x-y)^2} dx dy \\
&\leq C \left(\sum_{k=1}^{d} \left(\xi_{k+d(l)} - \xi_{d(l)}\right)^2 \right), \qquad (68)
\end{aligned}
$$

because for $k \in \{1, \dots, d\}$ there holds

$$
\tilde{\phi}_k^{(d)}(x) - \tilde{\phi}_k^{(d)}(y) = \sum_{k=1}^{d} c_k (x^k - y^k) = (x-y) \sum_{k=0}^{d-1} \left(c_k \sum_{j=0}^{k} x^j y^{k-j} \right),
$$

with $c_k \in \mathbb{C}$ some complex numbers.

Let now $2 \leq i = l+1 \leq n-2$. Additionally let $l+1 \not\equiv 0 \bmod (d)$. Then $d(l) = d(l+1)$. Using the definition of $S_{l,i}$ as well as Lemma 7.2 and the representation $\tilde{\phi}_0^{(d)}(x) = 1 - \sum_{k=1}^{d} \tilde{\phi}_k^{(d)}(x)$, we get analogously to the case $i = l$

$$
S_{l,l+1} = \int_{l-d(l)}^{l+1-d(l)} \int_{l+1-d(l)}^{l+2-d(l)} \frac{\left(\sum_{k=0}^{d} \xi_{k+d(l)} \left(\tilde{\phi}_k^{(d)}(x) - \tilde{\phi}_k^{(d)}(y)\right)\right)^2}{(x-y)^2} dx dy
$$

$$
\begin{aligned}
&= \int_{l-d(l)}^{l+1-d(l)} \int_{l+1-d(l)}^{l+2-d(l)} \frac{\left(\sum_{k=1}^{d} \left(\xi_{k+d(l)} - \xi_{d(l)}\right) \left(\tilde{\phi}_k^{(d)}(x) - \tilde{\phi}_k^{(d)}(y)\right)\right)^2}{(x-y)^2} dx dy \\
&\leq C \left(\sum_{k=1}^{d} \left(\xi_{k+d(l)} - \xi_{d(l)}\right)^2 \right). \qquad (69)
\end{aligned}
$$

Let now $2 \leq i = l+1 \leq n-2$ and $l+1 \equiv 0 \bmod (d)$. Then $d(l) = l+1-d$, $d(l+1) = l+1$ and we find

$$
S_{l,l+1} = \int_0^1 \int_{-1}^0 \frac{\left(\sum_{k=-d}^{0} \xi_{k+l+1} \tilde{\phi}_k^{(d)}(x) - \sum_{j=0}^{d} \xi_{j+l+1} \tilde{\phi}_j^{(d)}(y)\right)^2}{(x-y)^2} dx dy
$$

$$
\begin{aligned}
&= \int_0^1 \int_{-1}^0 \frac{\left(\sum_{k=-d}^{-1} (\xi_{k+l+1} - \xi_{l+1}) \tilde{\phi}_k^{(d)}(x) - \sum_{j=1}^{d} (\xi_{j+l+1} - \xi_{l+1}) \tilde{\phi}_j^{(d)}(y)\right)^2}{(x-y)^2} dx dy \\
&\leq C \Bigg(\sum_{k=-d}^{-1} (\xi_{k+l+1} - \xi_{l+1})^2 \int_0^1 \int_{-1}^0 \frac{\left(\tilde{\phi}_k^{(d)}(x)\right)^2}{(x-y)^2} dx dy \\
&\qquad + \sum_{j=1}^{d} (\xi_{j+l+1} - \xi_{l+1})^2 \int_0^1 \int_{-1}^0 \frac{\left(\tilde{\phi}_j^{(d)}(y)\right)^2}{(x-y)^2} dx dy \Bigg) \\
&\leq C \left(\sum_{k=-d}^{-1} (\xi_{k+l+1} - \xi_{l+1})^2 + \sum_{j=1}^{d} (\xi_{j+l+1} - \xi_{l+1})^2 \right), \qquad (70)
\end{aligned}
$$

because for $k \in \{1, \dots, d\}$ there holds

$$\int_0^1 \int_{-1}^0 \frac{\left(\tilde{\phi}_k^{(d)}(x)\right)^2}{(x-y)^2} dxdy \leq C \int_0^1 \int_{-1}^0 \frac{x^2}{(x-y)^2} dxdy \leq C$$

Using the estimates (65), (66), (67), (68), (69) and (70) and the fact that, analogously to the considerations in the proof of Theorem (3.1), quadratic formulas with the same zeros are equivalent, we arrive at

$$\begin{aligned} \sum_{i,l=0}^{n-1} S_{l,i} \leq\ & C\Big(\sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l} \xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l} \xi_l^2 \\ & + \sum_{\substack{l,i=1 \\ i \not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n}, \frac{i}{n}) \frac{1}{\mid i-l \mid^2} (\xi_l - \xi_i)^2 \Big). \end{aligned}$$

Together with (61) and (64) we get that

$$\|P_n^d \xi\|_{\frac{1}{2},\alpha} \leq C\Big(\sum_{l=1}^{\frac{n}{2}-1} \frac{1}{l} \xi_l^2 + \sum_{l=\frac{n}{2}}^{n-1} \frac{1}{n-l} \xi_l^2 + \sum_{\substack{l,i=1 \\ i \not\equiv l \bmod 2}}^{n-1} h(\frac{l}{n}, \frac{i}{n}) \frac{1}{\mid i-l \mid^2} (\xi_l - \xi_i)^2 \Big).$$

Finally from Theorem 3.1 the assertion $\|P_n^d \xi\|_{\frac{1}{2},\alpha} \leq C \|\xi\|_{E_n}$ follows.

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