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Global and exponential attractors for 3-D wave equations with displacement dependent damping

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Abstract. A weakly damped wave equation in the three-dimensional (3-D) space with a damping coefficient depending on the displacement is studied. This equation is shown to generate a dissipative semigroup in the energy phase space, which possesses finite-dimensional global and exponential attractors in a slightly weaker topology.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following initial-boundary value problem for a weakly damped wave equation with a nonlinear damping coefficient:

$$\begin{cases} \partial_{tt}u + \sigma(u)\partial_t u - \Delta u + \varphi(u) = f, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1.1}$$

The external force $f \in L^2(\Omega)$ is independent of time, while the nonlinearity $\varphi \in C^2(\mathbb{R})$, with $\varphi(0) = 0$, is subject to the conditions

$$|\varphi''(u)| \leq c\big(1 + |u|\big), \tag{1.2}$$

$$\varphi'(u) \geq -c, \tag{1.3}$$

$$\liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda_1, \tag{1.4}$$

where $c \geq 0$ and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on $L^2(\Omega)$ with Dirichlet boundary conditions. As far as the damping coefficient is concerned, we assume that $\sigma \in C^1(\mathbb{R})$ with

$$\sigma(u) \geq \sigma_0 > 0, \tag{1.5}$$

$$|\sigma'(u)| \leq c(1 + |u|). \tag{1.6}$$

Problem (1.1) is related to the semilinear reaction-diffusion equation with memory, where the classical Fourier's constitutive law is replaced by the Gurtin-Pipkin's one [15], namely,

$$\partial_t u - \int_0^\infty k(s)\Delta u(t-s)ds + \varphi(u) = f. \tag{1.7}$$

Here, the *memory kernel* k is a positive decreasing function, and $u(t)$ is supposed to be a given datum for $t \leq 0$, where it need not fulfill the equation. If k is of exponential type, that is, $k(s) = \varepsilon^{-1}\exp[-s/\varepsilon]$, where $\varepsilon > 0$ is the *relaxation time*, then (1.7) can be transformed into the partial differential equation

$$\varepsilon\partial_{tt}u + [1 + \varepsilon\varphi'(u)]\partial_t u - \Delta u + \varphi(u) = f,$$

simply by adding (1.7) and its time-derivative multiplied by ε. Note that, for ε small enough, the function $1+\varepsilon\varphi'(u)$ fulfills assumptions (1.5)-(1.6). This equation is quite interesting from a physical viewpoint, for instance, to describe the flow of viscoelastic fluids (see [5, 13, 16, 19]). Wave equations with a nonlinear damping of the form $\sigma(\partial_t u)$ have also been considered in the literature (see [9, 10, 11, 12]). However, the nonlinearity $\sigma(u)\partial_t u$ considered in the present work produces an equation which is essentially different. Indeed, the term $\sigma(u)\partial_t u$ is much more difficult to handle than $\sigma(\partial_t u)$, and the methods developed to treat the latter nonlinearity simply do not work in our situation. In particular, in contrast to the previous theory where the Lipschitz continuity in the phase space is immediate and the further regularity of solutions easily achievable (at least under suitable growth restrictions), with the term $\sigma(u)\partial_t u$ one does not even have the plain continuity of the semigroup in the phase space, and any additional regularity seems to be out of reach, unless σ is constant.

The asymptotic behavior of this kind of dissipative systems is well described by the existence of a global attractor, namely, the unique invariant compact sets which (uniformly) attracts bounded sets of initial data [1, 21]. The global attractor, however, does not provide an actual control of the convergence rate of trajectories and might be unstable with respect to perturbations. A more suitable object to have an effective control on the longterm dynamics is the exponential attractor [6, 7]. Contrary to the global one, the exponential attractor is not unique (thus, in some sense, is an artificial object), and it is only semi-invariant. However, it has the advantage of being stable with respect to perturbations, and it provides an exponential convergence rate which can be explicitly computed.

The 1-D and 2-D analogues of (1.1) have been investigated in the recent papers [14, 20], where the existence of strongly continuous semigroups possessing global and exponential attractors of optimal regularity has been proven. The analysis in the one-dimensional case [14] heavily leant on the embedding $H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$, which is false in higher dimensions. In fact, in [14] the strong positivity condition (1.5) can even be weakened, requiring in place that $\sigma(u) > 0$ for every u. On the contrary, in the two-dimensional case, considered in [20], one can no longer appeal to the continuous embedding $H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$, and the analysis becomes more complicated. Nevertheless, the above embedding is "almosttrue, in the sense that $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$ for all $p < \infty$. Thus, 2-D looks like a border (critical) case and, using sharp interpolation inequalities along with a rather delicate splitting of the equation into an exponentially decaying and a compact part, it is possible to develop a complete theory, which includes global existence of strong solutions, regularity and finite-dimensionality of the attractors, existence of exponential attractors.

The present work is focused on the analysis of the most relevant three-dimensional case. In contrast to the former situations, the embedding $H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$ is now far from being true, and the equation becomes "supercritical", showing many features similar to those of wave equations with fast growing nonlinearities (see e.g. [23]). In particular, it seems extremely difficult to verify the global existence of more regular solutions (even in the case of globally bounded φ and σ), and essential problems with the energy equality arise. However, the main difference with respect to the case of a fast growing φ and a constant σ is uniqueness. Indeed, exploiting some monotonicity arguments, it is possible (analogously to the 2-D case [20]) to establish the uniqueness of solutions, and even the Lipschitz continuity of the semigroup associated with (1.1) in a weaker energy space. We should remark that the strategy adopted to treat the 3-D case is quite different from the one employed for the corresponding 1-D and 2-D cases. Indeed, here we cannot obtain the existence of regular (exponentially) attracting sets. So, we first prove the existence of a weak attractor, and, in the case when σ is globally bounded, we obtain the existence of the strong global attractor via the *energy equality method*. Finally, if the growths of the functions φ and σ are slightly slower than in (1.2) and (1.6), we construct an exponential attractor which (exponentially) attracts bounded subsets of the phase space in a weaker topology, and which has finite fractal dimension there. The construction used here is, in fact, some modification/generalization of the so-called method of *l-trajectories*, which is known to be very effective for problems with lack of regularity (see [4, 17, 18, 24] and references therein). Thus, we succeed in constructing a finite-dimensional exponential attractor (in a weaker topology) in spite of the lack of compactness of the global attractor in the original topology. Still, in the 3-D case some questions remain open, such as the regularity and the finite dimensionality of the strong global attractor.

The paper is organized as follows. The existence and uniqueness of appropriate weak energy solutions of (1.1) and their dissipativity is verified in Section 2, where we also prove the energy equality under the additional assumption that $\sigma(u)$ is globally bounded. In Section 3 we establish the existence of a weak attractor and, when the energy equality holds, the existence of a strong global attractor. Section 4 deals with exponential attractors.

Notation. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm in $L^2(\Omega)$. Naming, for $s \in \mathbb{R}$, $H_s = \text{domain}\big[(-\Delta)^{s/2}\big]$ (with Dirichlet boundary conditions) we introduce the Hilbert spaces $\mathcal{H}_s = H_{s+1} \times H_s$, endowed with the usual inner products and norms. Throughout the paper, the symbols Q and c will stand for a *generic* monotonically increasing positive function and a *generic* positive constant, respectively. We shall tacitly make use of the Poincaré, Young and Hölder inequalities, along with the continuous embedding $H_1 \hookrightarrow L^p(\Omega)$, for every $p \in [1,6]$. Also, we shall employ the following functionals, related with φ and σ, namely,

$$\Phi(u) = \int_0^u \varphi(y)dy, \qquad \Sigma(u) = \int_0^u \sigma(y)dy, \qquad \Upsilon(u) = \int_0^u y\sigma(y)dy.$$

Finally, for any given function $u(t)$, we write for short $\xi_u(t) = (u(t), \partial_t u(t))$.

We conclude the section by reporting two technical results which will be needed in the course of the investigation. The first one is a modified version of the Gronwall lemma (see [2] for a proof).

Lemma 1.1. *Let $E : \mathcal{H}_0 \to \mathbb{R}$ satisfy*

$$\beta\|\zeta\|^2_{\mathcal{H}_0} - m \le E(\zeta) \le Q(\|\zeta\|_{\mathcal{H}_0}) + m, \qquad \forall \zeta \in \mathcal{H}_0,$$

for some $\beta > 0$ and $m \ge 0$. Let now $\xi \in C(\mathbb{R}^+, \mathcal{H}_0)$ be given. Suppose that the map $t \mapsto E(\xi(t))$ is continuously differentiable and fulfills the differential inequality

$$\frac{d}{dt}E(\xi) + \varepsilon\|\xi\|^2_{\mathcal{H}_0} \le k,$$

for some $\varepsilon > 0$ and $k > 0$. Then, there is $t_0 = Q(\|\xi(0)\|_{\mathcal{H}_0} + k) \ge 0$ such that

$$\|\xi(t)\|_{\mathcal{H}_0} \le Q\big(k + m + \beta^{-1}\big), \qquad \forall t \ge t_0.$$

Lemma 1.2. *Let $\psi \in L^{3+\delta}(\Omega)$, $\delta > 0$, be such that $\nabla\psi \in L^{3/2}(\Omega)$. Then, the multiplication by ψ is well-defined as an operator from $H_{-\varrho}$ to H_{-1} for some $\varrho = \varrho(\delta) \in (0, 1/2)$, and the following estimate holds:*

$$\|\psi u\|_{H_{-1}} \le c\|u\|_{H_{-\varrho}}, \qquad \forall u \in H_{-\varrho},$$

for some $c = c(\|\psi\|_{L^{3+\delta}}, \|\nabla\psi\|_{L^{3/2}})$.

Proof. It is simpler to verify the equivalent conjugate inequality $\|\psi v\|_{H_\varrho} \le c\|v\|_{H_1}$, for every $v \in H_1$. Indeed, due to the assumptions on ψ and the embedding $H_1 \hookrightarrow L^6(\Omega)$, we have

$$\|\psi v\|_{L^p(\Omega)} + \|\nabla(\psi v)\|_{L^{6/5}} \le c\|v\|_{H_1},$$

with $\frac{1}{p} = \frac{1}{3+\delta} + \frac{1}{6} < \frac{1}{2}$. Then, the required estimate is an immediate corollary of the interpolation inequality

$$\|w\|_{H_\varrho} \le c\|w\|_{L^p}^{1-\varrho}\|\nabla w\|_{L^{6/5}}^{\varrho}, \qquad \forall w \in H_\varrho,$$

where $\frac{1}{2} = \frac{1-\varrho}{p} + \frac{5\varrho}{6}$ or, equivalently, $\varrho = \frac{\delta}{3+2\delta}$ (see e.g. [22]). □

2. Well-Posedness and Dissipativity

To begin our analysis, we recall the definition of a weak energy solution.

Definition 2.1. A function $u(t)$ is a weak energy solution to (1.1) if, for any $T > 0$,

$$\xi_u \in L^\infty([0, T], \mathcal{H}_0), \qquad \sqrt{\sigma(u)}\,\partial_t u \in L^2([0, T], H_0),$$

and (1.1) holds in the sense of distributions.

Due to the growth restrictions on φ and σ, it is apparent that $\varphi(u) \in L^2([0, T], H_0)$ and $\sigma(u)\partial_t u \in L^2([0, T], H_{-1})$. Consequently, $\partial_{tt} u \in L^2([0, T], H_{-1})$ and equation (1.1) is understood as an equality in $L^2([0, T], H_{-1})$. Moreover, standard arguments show that $\xi_u \in C_{\rm w}([0, T], \mathcal{H}_0)$. Hence, the initial conditions are well defined.

The next proposition proves the Lipschitz continuity in a weaker energy space. As a byproduct, we obtain the uniqueness of a weak energy solution. This Lipschitz continuity turns out to be the main technical tool in our analysis of the equation.

Proposition 2.2. *Let u^1 and u^2 be two weak energy solutions to* (1.1)*. Then, for every $t > 0$, the following estimate holds:*

$$\|\xi_{u^1}(t) - \xi_{u^2}(t)\|_{\mathcal{H}_{-1}} \le ce^{ct}\|\xi_{u^1}(0) - \xi_{u^2}(0)\|_{\mathcal{H}_{-1}},$$

where $c \ge 0$ depends only on the energy norms of the initial data $\xi_{u^1}(0)$ and $\xi_{u^2}(0)$.

Proof. The argument is the same as the analogous one developed in [20] for the 2-D case. However, in order to make this paper self-contained, we report it in full detail. Let u^1, u^2 be two weak solutions to (1.1) such that $\|\xi_{u^j}(t)\|_{\mathcal{H}_0} \le R$ for every $t \in [0, T]$, for some $R \ge 0$, and denote $\bar u = u^1 - u^2$. Defining $w^j(t) = \int_0^t u^j(\tau)d\tau$ and $\bar w = w^1 - w^2$, integrating equation (1.1) for u^j ($j = 1, 2$) on $[0, t]$ and taking the difference yields

$$\partial_{tt}\bar w + \Sigma(u^1) - \Sigma(u^2) - \Delta\bar w = F + G, \tag{2.1}$$

where we put

$$F(t) = -\int_0^t \big[\varphi(u^1(\tau)) - \varphi(u^2(\tau))\big]d\tau, \qquad G = \Sigma(u^1(0)) - \Sigma(u^2(0)) + \partial_t\bar{u}(0).$$

Note that, on account of (1.2) and (1.6), all the terms of (2.1) belong at least to $L^2([0,T], H_{-1})$. Hence, their product with $\partial_t\bar{w} = \bar{u} \in L^\infty([0,T], H_1)$ is well defined. Taking this product, and observing that $\langle \Sigma(u^1) - \Sigma(u^2), \bar{u}\rangle \geq 0$, we get

$$\frac{1}{2}\frac{d}{dt}\|\xi_{\bar{w}}\|^2_{\mathcal{H}_0} \leq \frac{d}{dt}\langle F, \bar{w}\rangle + \frac{d}{dt}\langle G, \bar{w}\rangle - \langle \partial_t F, \bar{w}\rangle.$$

Integrating on $[0,T]$, we are led to

$$\begin{aligned}\|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} &\leq \|\bar{u}(0)\|^2 + 2\langle F(T), \bar{w}(T)\rangle + 2\langle G, \bar{w}(T)\rangle - 2\int_0^T \langle \partial_t F(t), \bar{w}(t)\rangle dt \\ &\leq \frac{1}{2}\|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} + 4\|F(T)\|^2_{H_{-1}} + \|\bar{u}(0)\|^2 + 4\|G\|^2_{H_{-1}} \\ &\quad + 2\int_0^T \|\partial_t F(t)\|_{H_{-1}}\|\xi_{\bar{w}}(t)\|_{\mathcal{H}_0} dt.\end{aligned}$$

Using now the growth restrictions (1.2) and (1.6) on φ and σ, we easily see that

$$\begin{aligned}4\|F(T)\|^2_{H_{-1}} &\leq Q(R)T\int_0^T \|\bar{u}(t)\|^2 dt \leq Q(R)T\int_0^T \|\xi_{\bar{w}}(t)\|^2_{\mathcal{H}_0}dt, \\ \|\bar{u}(0)\|^2 + 4\|G\|^2_{H_{-1}} &\leq Q(R)\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}}, \\ \|\partial_t F(t)\|_{H_{-1}} &\leq Q(R)\|\bar{u}(t)\| \leq Q(R)\|\xi_{\bar{w}}(t)\|_{\mathcal{H}_0}.\end{aligned}$$

Therefore, we end up with

$$\|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} \leq Q(R)\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}} + Q(R)(1+T)\int_0^T \|\xi_{\bar{w}}(t)\|^2_{\mathcal{H}_0}dt,$$

and from the Gronwall lemma we conclude that

$$\|\bar{u}(T)\|^2 \leq \|\xi_{\bar{w}}(T)\|^2_{\mathcal{H}_0} \leq Q(R)e^{Q(R)T}\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}}.$$

Finally, from (2.1), we read that

$$\|\partial_t\bar{u}\|_{H_{-1}} = \|\partial_{tt}\bar{w}\|_{H_{-1}} \leq \|\Sigma(u^1) - \Sigma(u^2)\|_{H_{-1}} + \|\nabla\bar{w}\| + \|F\|_{H_{-1}} + \|G\|_{H_{-1}},$$

which, due to the above estimates and the inequality $\|\Sigma(u^1) - \Sigma(u^2)\|_{H_{-1}} \leq Q(R)\|\bar{u}\|$, furnishes

$$\|\partial_t\bar{u}(T)\|^2_{H_{-1}} \leq Q(R)e^{Q(R)T}\|\xi_{\bar{u}}(0)\|^2_{\mathcal{H}_{-1}}.$$

To complete the proof it is enough to note that, if $\|\xi_{u^j}(0)\|_{\mathcal{H}_0} \leq R$, then the uniform estimates provided in the subsequent Theorem 2.3 ensure that $\|\xi_{u^j}(t)\|_{\mathcal{H}_0} \leq Q(R)$ for all $t > 0$. $\square$

The next theorem provides the existence of a weak solution together with the dissipative estimate.

Theorem 2.3. *There exists a (unique) weak energy solution of* (1.1). *This solution satisfies the dissipative estimate*

$$\|\xi_u(t)\|_{\mathcal{H}_0} \le Q(\|\xi_u(0)\|_{\mathcal{H}_0})e^{-t} + Q(\|f\|), \tag{2.2}$$

and the energy inequality

$$\begin{aligned}(2.3)\qquad &\|\xi_u(t)\|^2_{\mathcal{H}_0} + 2\langle\Phi(u(t)),1\rangle - 2\langle f,u(t)\rangle + 2\int_0^t \langle\sigma(u(\tau))\partial_t u(\tau),\partial_t u(\tau)\rangle d\tau \\ &\le \|\xi_u(0)\|^2_{\mathcal{H}_0} + 2\langle\Phi(u(0)),1\rangle - 2\langle f,u(0)\rangle.\end{aligned}$$

Proof. We give below the formal derivation of the a priori estimates (2.2) and (2.3), which can be justified in a standard way via a Galerkin approximation scheme. Indeed, (2.3) can be *formally* obtained multiplying (1.1) by $\partial_t u$ and integrating on $[0,t]\times\Omega$. In order to verify (2.2), for $\varepsilon\in(0,1)$ to be fixed later, we introduce the energy functional

$$E_\varepsilon = E_\varepsilon(\xi_u) = \|\xi_u\|^2_{\mathcal{H}_0} + 2\langle\Phi(u),1\rangle + 2\varepsilon\langle\Upsilon(u),1\rangle + 2\varepsilon\langle\partial_t u,u\rangle - 2\langle f,u\rangle. \tag{2.4}$$

Notice that, from (1.5), $\langle\Upsilon(u),1\rangle\ge 0$. Thus, on account of (1.2), (1.6) and the inequality

$$\|\nabla u\|^2 + 2\langle\Phi(u),1\rangle \ge 2\gamma\|\nabla u\|^2 - c, \qquad \gamma>0, \tag{2.5}$$

which follows from the dissipativity assumption (1.4), we have the controls

$$\beta\|\xi_u\|^2_{\mathcal{H}_0} - Q(\|f\|) \le E_\varepsilon \le Q(\|\xi_u\|_{\mathcal{H}_0}) + Q(\|f\|),$$

for some $\beta\in(0,1)$, provided that ε is small enough. Multiplying (1.1) by $\partial_t u+\varepsilon u$, we find

$$\frac{d}{dt}E_\varepsilon + 2\varepsilon\|\nabla u\|^2 + 2\langle\sigma(u)\partial_t u,\partial_t u\rangle - 2\varepsilon\|\partial_t u\|^2 + 2\varepsilon\langle\varphi(u),u\rangle = 2\varepsilon\langle f,u\rangle.$$

Using (1.4) and (1.5), we have the estimate

$$\begin{gathered}2\varepsilon\|\nabla u\|^2 + 2\varepsilon\langle\varphi(u),u\rangle \ge 2\beta\varepsilon\|\nabla u\|^2 - c,\\ 2\langle\sigma(u)\partial_t u,\partial_t u\rangle - 2\varepsilon\|\partial_t u\|^2 \ge \beta\varepsilon\|\partial_t u\|^2,\end{gathered}$$

if ε is small enough. Thus, estimating the right-hand side of the differential equality as

$$2\varepsilon\langle f,u\rangle \le \beta\varepsilon\|\nabla u\|^2 + c\|f\|^2,$$

we end up with the inequality

$$\frac{d}{dt}E_\varepsilon + \beta\varepsilon\|\xi_u\|^2_{\mathcal{H}_0} \le Q(\|f\|).$$

Fixing now the parameter ε in such a way that all the above relationships hold, we deduce from Lemma 1.1 that, for every $R\ge 0$, there exists $t_0=t_0(R)$ such that

$$\|\xi_u(t)\|_{\mathcal{H}_0} \le Q(\|f\|), \qquad \forall t\ge t_0,$$

whenever $\|\xi_u(0)\|_{\mathcal{H}_0}\le R$. Together with (2.3) and (2.5), this gives estimate (2.2) and finishes the proof of the theorem. □

Thus, equation (1.1) generates a dissipative semigroup $S(t)$ in the phase space $\mathcal{H}_0$ which is locally Lipschitz continuous in the $\mathcal{H}_{-1}$ metric.

Corollary 2.4. *The weak energy solution of* (1.1) *possesses the dissipation integrals*

$$\sigma_0\int_0^\infty \|\partial_t u(t)\|^2 dt \le \int_0^\infty \langle \sigma(u(t))\partial_t u(t), \partial_t u(t)\rangle dt \le Q(\|\xi_u(0)\|_{\mathcal{H}_0}) + Q(\|f\|).$$

Indeed, this follows immediately by passing to the limit $t \to \infty$ in (2.3) and using (2.5).

Remark 2.5. It is worth emphasizing that we cannot *directly* multiply equation (1.1) by $\partial_t u$, since the terms of (1.1) belong to $L^2([0,T], H_{-1})$, whereas $\partial_t u \in L^\infty([0,T], H_0)$ only. In order to overcome this obstacle, one usually works with the Galerkin approximate equations (which are smooth, so that this multiplication makes sense) and verify estimates (2.2) and (2.3) first for the approximate Galerkin solutions $u_N(t)$. Then, passing to the limit $N \to \infty$, one obtains the required inequalities for the limit solution u (see [1] for details). However, this limit procedure gives only the energy *inequality* (2.3). In contrast to this, the energy *equality* is a more delicate fact that should be verified independently (usually, stronger assumptions on the equation are required). We prove that under the assumption that $\sigma(u)$ is uniformly bounded.

Corollary 2.6. *Assume that $\sigma(u) \le c$ for every $u \in \mathbb{R}$. Then, strict equality in* (2.3) *holds, namely,*

$$\begin{aligned} \|\xi_u(t)\|^2_{\mathcal{H}_0} + 2\langle \Phi(u(t)), 1\rangle - 2\langle f, u(t)\rangle + 2\int_0^t \langle \sigma(u(\tau))\partial_t u(\tau), \partial_t u(\tau)\rangle d\tau \\ = \|\xi_u(0)\|^2_{\mathcal{H}_0} + 2\langle \Phi(u(0)), 1\rangle - 2\langle f, u(0)\rangle. \end{aligned} \tag{2.6}$$

Proof. Let $P_N : H_0 \to P_N H_0$ be the orthogonal projection in H_0 onto the first N eigenvectors of the Laplacian (equipped with Dirichlet boundary conditions), and let u be the weak energy solution to (1.1). The function $u_N = P_N u$ obviously satisfies

$$\partial_{tt} u_N + P_N(\sigma(u)\partial_t u) - \Delta u_N + P_N\varphi(u) = P_N f.$$

Multiplying this equation by $\partial_t u_N$, and integrating over $[0,t]\times\Omega$, we get

$$\begin{aligned} &\|\xi_{u_N}(t)\|^2_{\mathcal{H}_0} - 2\langle f, u_N(t)\rangle + 2\int_0^t \langle \sigma(u(\tau))\partial_t u(\tau), \partial_t u_N(\tau)\rangle d\tau + 2\int_0^t \langle \varphi(u(\tau)), \partial_t u_N(\tau)\rangle d\tau \\ &= \|\xi_{u_N}(0)\|^2_{\mathcal{H}_0} - 2\langle f, u_N(0)\rangle. \end{aligned}$$

We need now to pass to the limit $N \to \infty$ in this equality. Since, by definition, $\xi_u(t) \in \mathcal{H}_0$ for all t, then $\xi_{u_N}(t) \to \xi_u(t)$ and $\xi_{u_N}(0) \to \xi_u(0)$ in $\mathcal{H}_0$. Hence, the passage to the limit is immediate for all the terms except the two integral ones appearing in the left-hand side. For the first, we use the fact that $\sigma(u)\partial_t u \in L^2([0,T], H_0)$ (here the global boundedness of $\sigma(u)$ is needed) and $\partial_t u_N \to \partial_t u$ in that space. Finally, for the second one, we note that $\varphi(u) \in L^2([0,T], H_0)$ (due to the growth restrictions) and, consequently,

$$\int_0^t \langle \varphi(u(\tau)), \partial_t u_N(\tau)\rangle d\tau \to \int_0^t \langle \varphi(u(\tau)), \partial_t u(\tau)\rangle d\tau = \langle \Phi(u(t)), 1\rangle - \langle \Phi(u(0)), 1\rangle,$$

which proves (2.6). ☐

The energy equality (2.6) can be rewritten in a more convenient differential form. Indeed, introducing the energy functional

$$E_0 = E_0(\xi_u) = \|\xi_u\|^2_{\mathcal{H}_0} + 2\langle \Phi(u), 1\rangle - 2\langle f, u\rangle,$$

the integral equality (2.6) is equivalent to the fact that the function $E_0(\xi_u(t))$ is absolutely continuous as a function of t and satisfies almost everywhere

$$\frac{d}{dt}E_0 + 2\langle \sigma(u)\partial_t u, \partial_t u\rangle = 0. \tag{2.7}$$

This differential energy equality is crucial for the existence of a strong global attractor, as we will see in the next section.

3. Weak and Strong Global Attractors

We now proceed to investigate the asymptotic properties of (1.1), using the notion of a global attractor. We begin with the attractor in a weak topology.

Definition 3.1. A set $\mathcal{A} \subset \mathcal{H}_0$ is a weak global attractor of the semigroup $S(t)$ associated with equation (1.1) if

(i) $\mathcal{A}$ is *weakly* compact in $\mathcal{H}_0$;
(ii) $\mathcal{A}$ is strictly invariant, that is, $S(t)\mathcal{A} = \mathcal{A}$;
(iii) $\mathcal{A}$ attracts in the weak topology the images of all bounded subsets of $\mathcal{H}_0$, namely, for every bounded subset $\mathcal{B}$ of $\mathcal{H}_0$ and every neighborhood $\mathcal{O}$ of $\mathcal{A}$ in the weak topology of $\mathcal{H}_0$, there exists $T = T(\mathcal{B}, \mathcal{O}) \geq 0$ such that $S(t)\mathcal{B} \subset \mathcal{O}$, for every $t \geq T$.

In particular, the attraction in the weak topology of $\mathcal{H}_0$ implies the attraction in the *strong* topology of $\mathcal{H}_{-1}$.

The next proposition gives the existence of such a weak attractor.

Proposition 3.2. *The semigroup $S(t)$ associated with the wave equation* (1.1) *possesses a weak global attractor $\mathcal{A}$ in the sense of Definition 3.1. As usual, this attractor is generated by all complete bounded trajectories of* (1.1)*, that is, $\mathcal{A} = \mathcal{K}_{|t=0}$, where $\mathcal{K}$ is the set of all weak energy solutions $u(t)$ which are defined for all $t \in \mathbb{R}$ and bounded in the $\mathcal{H}_0$-norm.*

Proof. Due to the dissipative estimate (2.2), the ball $\mathcal{B}_0 = \big\{\xi \in \mathcal{H}_0,\ \|\xi\|_{\mathcal{H}_0} \leq R\big\}$ for a sufficiently large radius R is an absorbing set for $S(t)$ in $\mathcal{H}_0$. Obviously, this ball is compact in the weak topology of $\mathcal{H}_0$. Thus, $S(t)$ possesses a weakly compact absorbing set. On the other hand, due to Proposition 2.2, for every fixed $t \geq 0$, the map $S(t)$ is continuous on $\mathcal{B}_0$ in the $\mathcal{H}_{-1}$-topology and, consequently, it is continuous in the weak topology of $\mathcal{H}_0$ as well. The existence of a weak global attractor follows now from the classical attractor's existence theorem (see e.g. [3]). $\square$

By means of standard energy methods (cf. [21]), we now show the existence of a strong global attractor in the case where the energy equality (2.6) holds (for instance, when $\sigma(u)$ is globally bounded).

Theorem 3.3. *Assume that the energy equality* (2.6) *holds for every weak energy solution. Then, the semigroup $S(t)$ possesses a global attractor in the strong topology of $\mathcal{H}_0$ (which, obviously, coincides with the weak attractor $\mathcal{A}$ constructed in the previous proposition).*

Proof. To prove the existence of the strong global attractor, it is sufficient to verify that for every sequences $\xi_{u_n}(0) \in \mathcal{B}_0$ and $t_n \to \infty$ the associated sequence $\xi_{u_n}(t_n)$ is precompact in $\mathcal{H}_0$ (see e.g. [1]). Let then $\xi_{u_n}(0) \in \mathcal{B}_0$ and $t_n \to \infty$ be arbitrary. Without loss of generality, due to the previous proposition, we may assume that $\xi_{u_n}(t_n) \to \xi = (\xi_1, \xi_2)$ *weakly* in $\mathcal{H}_0$ for some $\xi \in \mathcal{A}$. The proof is finished if we show that $\xi_{u_n}(t_n) \to \xi$ *strongly* in $\mathcal{H}_0$. To this end, we use the simple observation that in a Hilbert space the weak convergence together with the convergence of the norms imply the strong convergence. Thus, we are left to prove that $\|\xi_{u_n}(t_n)\|_{\mathcal{H}_0} \to \|\xi\|_{\mathcal{H}_0}$. To reach this aim, we shall use suitable energy equalities. The basic energy equality (2.7) allows us to multiply *directly* equation (1.1) by $\partial_t u + \varepsilon u$. Indeed, the problematic multiplication by $\partial_t u$ is justified by (2.7), whereas the multiplication by u is allowed since $u \in L^\infty([0,T], H_1)$. Multiplying the initial equation (1.1) by this term, after simple manipulations, we deduce the equality

$$\frac{d}{dt} E_\varepsilon(\xi_u) + 2\varepsilon E_\varepsilon(\xi_u) + 2\langle [\sigma(u) - 2\varepsilon]\partial_t u, \partial_t u\rangle = L_\varepsilon(\xi_u), \tag{3.1}$$

with $E_\varepsilon(\xi_u)$ as in (2.4), where we put

$$L_\varepsilon(\xi_u) = -2\varepsilon\langle \varphi(u), u\rangle + 4\varepsilon\langle \Phi(u), 1\rangle + 4\varepsilon^2\langle \Upsilon(u), 1\rangle + 4\varepsilon^2\langle u, \partial_t u\rangle - 2\varepsilon\langle f, u\rangle,$$

and we fix $\varepsilon > 0$ enough small such that $\sigma(u) - 2\varepsilon > 0$. Then, the weak convergence of $\xi_{u_n}(t_n)$ to $\xi = (\xi_1, \xi_2)$, together with the growth restrictions on φ and σ and the compactness of the embedding $H_1 \hookrightarrow L^4(\Omega)$, lead to

$$\langle \Phi(u_n(t_n)), 1\rangle \to \langle \Phi(\xi_1), 1\rangle, \quad \langle \Upsilon(u_n(t_n)), 1\rangle \to \langle \Upsilon(\xi_1), 1\rangle, \quad \langle u_n(t_n), \partial_t u_n(t_n)\rangle \to \langle \xi_1, \xi_2\rangle.$$

Hence, in order to verify the required convergence of the norms, it is sufficient to check the convergence of the *energy* functionals

$$E_\varepsilon(\xi_{u_n}(t_n)) \to E_\varepsilon(\xi).$$

To this aim, we introduce the shifted functions $\tilde{u}_n(t) = u_n(t + t_n)$, which solve

$$\begin{cases} \partial_{tt}\tilde{u}_n + \sigma(\tilde{u}_n)\partial_t \tilde{u}_n - \Delta \tilde{u}_n + \varphi(\tilde{u}_n) = f, \\ \tilde{u}_n(-t_n) = u_n(0), \quad \partial_t \tilde{u}_n(-t_n) = \partial_t u_n(0), \end{cases}$$

and $E_\varepsilon(\xi_{\tilde{u}_n}(0)) = E_\varepsilon(\xi_{u_n}(t_n))$. Since $\xi_{u_n}(0) \in \mathcal{B}_0$, the dissipative estimate (2.2) implies that the solutions $\xi_{\tilde{u}_n}(t)$ are uniformly bounded in $L^\infty([-t_n, \infty), \mathcal{H}_0)$. Moreover, $\xi_{\tilde{u}_n}$ being precompact in $C_{\rm loc}(\mathbb{R}, \mathcal{H}_{-1})$, without loss of generality we may assume that, for every $t \in \mathbb{R}$, $\xi_{\tilde{u}_n}(t)$ converges weakly in $\mathcal{H}_0$ to some solution $\xi_u(t) \in \mathcal{A}$ (here we have implicitly used the fact that $t_n \to \infty$). Obviously, $\xi_u(0) = \xi$. So, we need to prove that

$$E_\varepsilon(\xi_{\tilde{u}_n}(0)) \to E_\varepsilon(\xi_u(0)). \tag{3.2}$$

First, note that the established weak convergence $\xi_{\tilde u_n}(t)\to\xi_u(t)$ and the boundedness in $\mathcal{H}_0$ together with the compact embedding $H_1\hookrightarrow L^4(\Omega)$ imply

$$
\begin{aligned}
&L_\varepsilon(\xi_{\tilde u_n}(t))\to L_\varepsilon(\xi_u(t)), \qquad \forall t\in\mathbb{R},\\
&|L_\varepsilon(\xi_{\tilde u_n}(t))|\le c.
\end{aligned}
\tag{3.3}
$$

Integrating now the energy equality (3.1) for $\tilde u_n(t)$ on $[-t_n,0]$, we get

$$
\begin{aligned}
E_\varepsilon(\xi_{\tilde u_n}(0))&+2\int_{-t_n}^0 e^{2\varepsilon t}\langle[\sigma(\tilde u_n(t))-2\varepsilon]\partial_t\tilde u_n(t),\partial_t\tilde u_n(t)\rangle dt\\
&=E_\varepsilon(\xi_{\tilde u_n}(-t_n))e^{-2\varepsilon t_n}+\int_{-t_n}^0 e^{2\varepsilon t}L_\varepsilon(\xi_{\tilde u_n}(t))dt.
\end{aligned}
$$

Furthermore, using (3.3) and the fact that that $t_n\to\infty$ and $\xi_{\tilde u_n}(-t_n)$ remains bounded, we deduce from the last equality that

$$
\lim_{n\to\infty}\left(E_\varepsilon(\xi_{\tilde u_n}(0))+2\int_{-t_n}^0 e^{2\varepsilon t}\langle[\sigma(\tilde u_n(t)-2\varepsilon]\partial_t\tilde u_n(t),\partial_t\tilde u_n(t)\rangle dt\right)=\int_{-\infty}^0 e^{2\varepsilon t}L_\varepsilon(\xi_u(t))dt.
$$

Comparing this result with the analogous energy equality for the limit solution $\xi_u(t)$, we conclude that

$$
\begin{aligned}
\lim_{n\to\infty}&\left(E_\varepsilon(\xi_{\tilde u_n}(0))+2\int_{-t_n}^0 e^{2\varepsilon t}\langle[\sigma(\tilde u_n(t))-2\varepsilon]\partial_t\tilde u_n(t),\partial_t\tilde u_n(t)\rangle dt\right)\\
&=E_\varepsilon(\xi_u(0))+2\int_{-\infty}^0 e^{2\varepsilon t}\langle[\sigma(u(t))-2\varepsilon]\partial_t u(t),\partial_t u(t)\rangle dt.
\end{aligned}
\tag{3.4}
$$

On the other hand, since $\|\xi\|_H\le\liminf_{n\to\infty}\|\xi_n\|_H$ for any weakly convergent sequence $\xi_n\to\xi$ in a reflexive space H, we have

$$
E_\varepsilon(\xi_u(0))\le\liminf_{n\to\infty}E_\varepsilon(\xi_{\tilde u_n}(0))
$$

and

$$
\int_{-\infty}^0 e^{2\varepsilon t}\langle[\sigma(u(t))-2\varepsilon]\partial_t u(t),\partial_t u(t)\rangle dt\le\liminf_{n\to\infty}\int_{-t_n}^0 e^{2\varepsilon t}\langle[\sigma(\tilde u_n(t)-2\varepsilon]\partial_t\tilde u_n(t),\partial_t\tilde u_n(t)\rangle dt.
$$

Indeed, the term $\langle[\sigma(\tilde u_n)-2\varepsilon]\partial_t\tilde u_n,\partial_t\tilde u_n\rangle$ can be written in the form $\|\partial_t\Psi(\tilde u_n)\|^2$, with $\Psi(v)=\int_0^v\sqrt{\sigma(y)-2\varepsilon}\,dy$. It remains to note that the last two inequalities, together with (3.4), imply the required energy convergence (3.2). $\square$

4. Finite-Dimensionality and Exponential Attractors in the Subcritical Case

In the final section, we prove our main theorem on the existence of the weak exponential attractor in the subcritical case. To be more precise, in addition to (1.2)-(1.6), we require that

$$
|\sigma(u)|+|\varphi'(u)|\le c(1+|u|^{2-\delta}),
\tag{4.1}
$$

for some $\delta\in(0,2]$.

Our construction of an exponential attractor is based on the following abstract result.

Proposition 4.1. *Let H, V, V_1 be Banach spaces such that the embedding $V_1 \hookrightarrow V$ is compact. Let B be a closed bounded subset of H, and let $\mathbb{S} : B \to B$ be a map. Assume also that there exists a uniformly Lipschitz continuous map $\mathbb{T} : B \to V_1$, i.e.*

$$\|\mathbb{T}b_1 - \mathbb{T}b_2\|_{V_1} \le L\|b_1 - b_2\|_H, \qquad \forall b_1, b_2 \in B,$$

for some $L \ge 0$, such that

$$\|\mathbb{S}b_1 - \mathbb{S}b_2\|_H \le \vartheta\|b_1 - b_2\|_H + K\|\mathbb{T}b_1 - \mathbb{T}b_2\|_V, \qquad \forall b_1, b_2 \in B, \tag{4.2}$$

for some $\vartheta < 1/2$ and $K \ge 0$. Then, there exists a (discrete) exponential attractor $\mathcal{M}_{\mathrm{d}} \subset B$ which satisfies the following properties:

(i) *semi-invariance: $\mathbb{S}\mathcal{M}_{\mathrm{d}} \subset \mathcal{M}_{\mathrm{d}}$;*
(ii) *compactness: $\mathcal{M}_{\mathrm{d}}$ is compact in H;*
(iii) *exponential attraction: $\mathrm{dist}_H(\mathbb{S}^n B, \mathcal{M}_{\mathrm{d}}) \le Ce^{-\omega n}$ for all $n \in \mathbb{N}$ and for some $\omega > 0$ and $C \ge 0$, where dist_H denotes the standard Hausdorff semidistance between sets in H;*
(iv) *finite-dimensionality: $\mathcal{M}_{\mathrm{d}}$ has finite fractal dimension in H.*

Moreover, the constants ω, C and the fractal dimension of $\mathcal{M}_{\mathrm{d}}$ can be explicitly expressed in terms of L, K, ϑ, $\|B\|_H$ and the Kolmogorov's κ-entropy of the compact embedding $V_1 \hookrightarrow V$, for some $\kappa = \kappa(L, K, \vartheta)$.

We recall that the Kolmogorov's κ-entropy of the compact embedding $V_1 \hookrightarrow V$ is the logarithm of the minimum number of balls of radius κ in V necessary to cover the unit ball of V_1.

The proof of this proposition in the particular instance when $H = V_1$ and $\mathbb{T}$ is the identity map is given in [7]. The general proof repeats word by word this particular case and so thus omitted (see also [4, 8]).

We are now ready to state and prove

Theorem 4.2. *Assuming* (4.1) *in addition to the general hypotheses, the semigroup $S(t)$ associated with* (1.1) *possesses a weak exponential attractor $\mathcal{M}$ in the following sense:*

(i) *$\mathcal{M}$ is bounded in $\mathcal{H}_0$ and compact in $\mathcal{H}_{-1}$;*
(ii) *$\mathcal{M}$ is semi-invariant: $S(t)\mathcal{M} \subset \mathcal{M}$, $t \ge 0$;*
(iii) *$\mathcal{M}$ attracts the images of bounded (in $\mathcal{H}_0$) subsets exponentially in the metric of $\mathcal{H}_{-1}$, i.e. there exist $\omega > 0$ and a monotone function Q such that, for every bounded set $\mathcal{B} \subset \mathcal{H}_0$,*

$$\mathrm{dist}_{\mathcal{H}_{-1}}(S(t)\mathcal{B}, \mathcal{M}) \le Q(\|\mathcal{B}\|_{\mathcal{H}_0})e^{-\omega t}, \qquad \forall t \ge 0.$$

(iv) *$\mathcal{M}$ has the finite fractal dimension in $\mathcal{H}_{-1}$.*

Proof. We first recall that, due the dissipative estimate (2.2), the semigroup $S(t)$ possesses an absorbing ball $\mathcal{B}_0$ in the phase space $\mathcal{H}_0$. Thus, it is sufficient to construct the exponential attractor for the restriction of this semigroup on $\mathcal{B}_0$ only.

In order to apply Proposition 4.1 to our situation, we need to verify the proper estimate for the difference of solutions, which is done in the following lemma.

Lemma 4.3. *Let the above assumptions hold, and let u^1 and u^2 be two weak energy solutions of* (1.1) *such that $\xi_{u^j}(0) \in \mathcal{B}_0$. Then*

$$\|\xi_{u^1}(t) - \xi_{u^2}(t)\|_{\mathcal{H}_{-1}} \le M e^{-\nu t} \|\xi_{u^1}(0) - \xi_{u^2}(0)\|_{\mathcal{H}_{-1}} + K \|u^1 - u^2\|_{L^2([0,t],H_{-\varrho})}. \tag{4.3}$$

for some $\nu > 0$, $\varrho \in (0, 1/2)$, $M \ge 0$ and $K \ge 0$, all independent of t and u^j.

Proof. For $\alpha \in (0,1)$ to be fixed later, let $v^j(t) = \int_0^t e^{-\alpha(t-\tau)} u^j(\tau) d\tau$. Then, $u^j = \partial_t v^j + \alpha v^j$ and $v^j(0) = 0$. Multiplying the equations for $u^j(\tau)$ by $e^{-\alpha(t-\tau)}$ and integrating in τ over $[0,t]$, after simple transformations, we arrive at

$$\partial_{tt} v^j(t) + \Sigma(u^j(t)) - \Delta v^j(t) + \int_0^t e^{-\alpha(t-\tau)} \varphi_\alpha(u^j(\tau)) d\tau = e^{-\alpha t} R(\xi_{u^j}(0)) + \frac{1}{\alpha}(1 - e^{-\alpha t}) f,$$

where we set

$$\varphi_\alpha(w) = \varphi(w) - \alpha \Sigma(w) \qquad \text{and} \qquad R(\xi_{u^j}(0)) = \partial_t u^j(0) - \alpha u^j(0) + \Sigma(u^j(0)).$$

Then, the difference $\bar{v} = v^1 - v^2$ solves

$$\begin{aligned} &\partial_{tt} \bar{v}(t) + [\Sigma(u^1(t)) - \Sigma(u^2(t))] - \Delta \bar{v}(t) + \int_0^t e^{-\alpha(t-\tau)} [\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau))] d\tau \\ &= e^{-\alpha t} [R(\xi_{u^1}(0)) - R(\xi_{u^2}(0))]. \end{aligned} \tag{4.4}$$

Multiplying the equation by $\partial_t \bar{v} + \alpha \bar{v} = u^1 - u^2 \in L^\infty(\mathbb{R}^+, H_1)$, and noting that $\Sigma'(w) \ge \sigma_0$, with standard computations, we get

$$\frac{d}{dt} E(\xi_{\bar{v}}(t)) + 2\alpha \|\nabla \bar{v}(t)\|^2 + 2(\sigma_0 - \alpha) \|\partial_t \bar{v}(t)\|^2 + 4\alpha\sigma_0 \langle \bar{v}(t), \partial_t \bar{v}(t) \rangle \le J(t), \tag{4.5}$$

with

$$\begin{aligned} E(\xi_v(t)) = \|\xi_{\bar{v}}(t)\|^2_{\mathcal{H}_0} &+ 2\alpha \langle \bar{v}(t), \partial_t \bar{v}(t) \rangle - 2e^{-\alpha t} \langle R(\xi_{u^1}(0)) - R(\xi_{u^2}(0)), \bar{v}(t) \rangle \\ &+ 2 \int_0^t e^{-\alpha(t-\tau)} \langle \varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau)), \bar{v}(t) \rangle d\tau, \end{aligned}$$

and

$$\begin{aligned} J(t) = 2\langle \varphi_\alpha(u^1(t)) - \varphi_\alpha(u^2(t)), \bar{v}(t) \rangle &- 4\alpha \int_0^t e^{-\alpha(t-\tau)} \langle \varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau)), \bar{v}(t) \rangle d\tau \\ &+ 4\alpha e^{-\alpha t} \langle R(\xi_{u^1}(0)) - R(\xi_{u^2}(0)), \bar{v}(t) \rangle. \end{aligned}$$

Besides, analogously to Proposition 2.2, we have

$$\|R(\xi_{u^1}(0)) - R(\xi_{u^2}(0))\|_{H_{-1}} \le c \|\xi_{u^1}(0) - \xi_{u^2}(0)\|_{\mathcal{H}_{-1}}.$$

Consequently, the function J in the right-hand side of (4.5) can be estimated via

$$\begin{aligned} J(t) \le c\Big[e^{-\alpha t} \|\xi_{u^1}(0) - \xi_{u^2}(0)\|^2_{\mathcal{H}_{-1}} &+ \|\varphi_\alpha(u^1(t)) - \varphi_\alpha(u^2(t))\|^2_{H_{-1}} \\ &+ \int_0^t e^{-\alpha(t-\tau)} \|\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau))\|^2_{H_{-1}} d\tau \Big] + \alpha \|\xi_{\bar{v}}(t)\|^2_{\mathcal{H}_0}, \end{aligned}$$

for some $c = c(\alpha) \ge 0$. On the other hand, it is clear that the quadratic form (with respect to $\bar{v}$ and $\partial_t \bar{v}$) in the left-hand side of (4.5) is positively defined if α is small enough. Thus, fixing a suitable α and using the above estimates, we transform (4.5) into

$$\begin{aligned}
&\frac{d}{dt}E(\xi_{\bar v}(t)) + \nu E(\xi_{\bar v}(t)) \\
&\le c\Big[e^{-\alpha t}\|\xi_{u^1}(0) - \xi_{u^2}(0)\|^2_{\mathcal{H}_{-1}} + \|\varphi_\alpha(u^1(t)) - \varphi_\alpha(u^2(t))\|^2_{H_{-1}} \\
&\quad + \int_0^t e^{-\alpha(t-\tau)}\|\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau))\|^2_{H_{-1}} d\tau\Big],
\end{aligned}$$

for some strictly positive $\nu < \alpha/2$. Applying the Gronwall lemma, we infer

$$\|\xi_{\bar v}(t)\|^2_{\mathcal{H}_0} \le ce^{-\nu t}\|\xi_{u^1}(0) - \xi_{u^2}(0)\|^2_{\mathcal{H}_{-1}} + c\int_0^t e^{-\nu(t-\tau)}\|\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau))\|^2_{H_{-1}} d\tau,$$

where c and ν are independent of t and ξ_{u^j}. Finally, as in Proposition 2.2, we can express the $\mathcal{H}_{-1}$-norm of $\xi_{u^1}(t) - \xi_{u^2}(t)$ in terms of the $\mathcal{H}_0$-norm of $\xi_{v^1}(t) - \xi_{v^2}(t)$, using equation (4.4), deducing that

$$\|\xi_{u^1}(t) - \xi_{u^2}(t)\|^2_{\mathcal{H}_{-1}} \le ce^{-\nu t}\|\xi_{u^1}(0) - \xi_{u^2}(0)\|^2_{\mathcal{H}_{-1}} + c\int_0^t e^{-\nu(t-\tau)}\|\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau))\|^2_{H_{-1}} d\tau.$$

In order to complete the lemma, we only need to verify that

$$\|\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau))\|_{H_{-1}} \le c\|u^1(\tau) - u^2(\tau)\|^2_{H_{-\varrho}} \tag{4.6}$$

for some $\varrho \in (0, 1/2)$ and some $c \ge 0$, both independent of $u^j(\tau)$. Indeed,

$$\varphi_\alpha(u^1(\tau)) - \varphi_\alpha(u^2(\tau)) = \psi(\tau)(u^1(\tau) - u^2(\tau))$$

with $\psi(\tau) = \int_0^1 \varphi'_\alpha(su^1(\tau) + (1-s)u^2(\tau))ds$. Moreover, using (1.2), (1.6) and the fact that the $u^j(\tau)$ are uniformly bounded in H_1, we find that $\nabla\psi(\tau)$ is uniformly bounded in $L^{3/2}(\Omega)$ and, from assumption (4.1), we obtain also that $\psi(\tau)$ is uniformly bounded in $L^{3+\delta}(\Omega)$. Thus, Lemma 1.2 entails (4.6). □

It is now not difficult to finish the proof of the theorem, using the abstract scheme of Proposition 4.1. As usual, we first construct the exponential attractor $\mathcal{M}_{\mathrm{d}}$ of the discrete map $S(T_*)$ on $\mathcal{B}_0$ (the above constructed absorbing ball in $\mathcal{H}_0$), for a sufficiently large T_*. Indeed, it follows from the dissipative estimate (2.2) that $S(T_*) : \mathcal{B}_0 \to \mathcal{B}_0$, provided that T_* is large enough. Then, we apply Proposition 4.1 on the set $B = \mathcal{B}_0$ with $H = \mathcal{H}_{-1}$ and $\mathbb{S} = S(T_*)$, with T_* large enough so that $\mathcal{B}_0$ is invariant and, in addition, $Me^{-\nu T_*} = \vartheta < 1/2$ (see (4.3)). Besides, with reference to Proposition 4.1, let

$$V_1 = \big\{u \in L^2([0,T_*], H_0),\ \partial_t u \in L^2([0,T_*], H_{-1})\big\} \ \Subset\ V = L^2([0,T_*], H_{-\varrho}).$$

Finally, define the operator $\mathbb{T} : \mathcal{B}_0 \to V_1$ to be the solving operator for (1.1) on the time-interval $[0, T_*]$, namely,

$$\mathbb{T}\xi_u(0) = u.$$

Due to Proposition 2.2, we have the global Lipschitz continuity of $\mathbb{T}$ from $\mathcal{B}_0$ to V_1, and Lemma 4.3 gives us the basic estimate (4.2) for the map $S(T_*)$. Therefore, the assumptions of Proposition 4.1 are verified and, consequently, the map $S(T_*)$ possesses an exponential attractor $\mathcal{M}_{\mathrm{d}}$ on $\mathcal{B}_0$.

The required exponential attractor for the semigroup $S(t)$ (with continuous time) can be now constructed by the standard formula

$$\mathcal{M} = \bigcup_{t\in[0,T_*]} S(t)\mathcal{M}_{\mathrm{d}}.$$

Indeed, since the H_{-1}-norm of $\partial_{tt}u$ and the H_0-norm of $\partial_t u$ are globally bounded if $\xi_u(0) \in \mathcal{B}_0$, then the semigroup $S(t)$ is also uniformly Lipschitz continuous with respect to $t \in [0, T_*]$ in the $\mathcal{H}_{-1}$-norm. Then, the finite-dimensionality of $\mathcal{M}$ follows from the analogous finite-dimensionality of $\mathcal{M}_{\mathrm{d}}$, and the remaining properties of $\mathcal{M}$ are immediate. This completes the proof of Theorem 4.2. □

Remark 4.4. Since an exponential attractor always contains the global one, the theorem implies, in particular, that the fractal dimension of the global attractor $\mathcal{A}$ of Proposition 3.2 is finite in $\mathcal{H}_{-1}$ as well. In fact, due to interpolation, this dimension is finite in $\mathcal{H}_{-\alpha}$ for every $\alpha < 0$.

Some open questions. In contrast to the one and the two-dimensional cases (where we have a complete theory, due to [14] and [20]), the situation with the 3-D case remains essentially less clear. In particular, the following important questions remain open:

- Global existence of *strong* solutions (belonging to $\mathcal{H}_1$) and the $\mathcal{H}_1$-regularity of the attractor $\mathcal{A}$.
- Finite-dimensionality of the global attractor in the critical case ($\delta = 0$ in (4.1)).
- Energy equality and compactness of the global attractor in the original topology of $\mathcal{H}_0$ when $\sigma(u)$ is not globally bounded.

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