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Smooth attractors for strongly damped wave equations

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Abstract. This paper is concerned with the semilinear strongly damped wave equation
$$\partial_{tt}u - \Delta\partial_t u - \Delta u + \varphi(u) = f.$$
The existence of compact global attractors of optimal regularity is proved for nonlinearities φ of critical and supercritical growth.

1. Introduction

We consider the following initial and boundary value problem for a semilinear strongly damped wave equation on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_{tt}u - \Delta\partial_t u - \Delta u + \varphi(u) = f, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1.1}$$

Let us begin by mentioning some relevant physical applications where this kind of equation appears.

◇ In space dimensions one and two, (1.1) models the transversal vibrations of a homogeneous string and the longitudinal vibrations of a homogeneous bar, respectively, subject to viscous effects. The term $-\Delta\partial_t u$ indicates that the stress is proportional not only to the strain, as with the Hooke law, but also to the strain rate as in a linearized Kelvin-Voigt material.

◇ In the three-dimensional case, (1.1) describes the variation from the configuration at rest of a homogeneous and isotropic linearly viscoelastic solid with short memory, called of rate-type (see [6]), in presence of an external displacement-dependent force $f - \varphi(u)$. If the body is also subject to dynamical friction, the additional term $\beta\partial_t u$, with $\beta > 0$, appears in the left-hand side of the equation.

◇ For $\alpha > 0$ and $\beta \geq 0$, we have the perturbed sine-Gordon equation

$$\partial_{tt}u - \alpha\Delta\partial_t u - \Delta u + \sin u + \beta\partial_t u = f,$$

describing the evolution of the current u in a Josephson junction (see [15]). The parameters α and β correspond to loss effects, whereas f is the external current driving the device. Although in the present paper we consider for simplicity a nonlinearity independent of $\partial_t u$, we mention that the extra term $\beta\partial_t u$ (with β positive, or even slightly negative compared to α) does not affect at all the results that follow.

◇ Another interesting example reads

$$\partial_{tt}u - \alpha\Delta\partial_t u - \Delta u + |u|^\gamma u + \beta\partial_t u = f, \qquad \gamma \geq 0,$$

which is also a perturbed wave equation of Klein-Gordon type occurring in quantum mechanics.

$\diamond$ We finally mention an (integro-differential) equation arising in the theory of isothermal viscoelasticity that has recently attracted some attention, namely,

$$\partial_{tt}u - k(0)\Delta u - \int_0^\infty k'(s)\Delta u(t-s)ds + \varphi(u) = f,$$

where k is a convex decreasing smooth kernel such that $k(0) > k(\infty) > 0$, typically, $k(s) = k(\infty) + e^{-\varepsilon s}$, with $\varepsilon > 0$. The function $u(t)$ for $t \leq 0$ (the so-called past history) is a prescribed datum. Performing an integration by parts, the equation turns into

$$\partial_{tt}u - k(\infty)\Delta u - \int_0^\infty \big[k(s) - k(\infty)\big]\Delta\partial_t u(t-s)ds + \varphi(u) = f.$$

Of particular interest is the case where the system has a very rapidly fading memory: in the limiting situation when $k(s) - k(\infty)$ is the Dirac mass at zero, possibly multiplied by a positive constant α, we recover

$$\partial_{tt}u - \alpha\Delta\partial_t u - k(\infty)\Delta u + \varphi(u) = f.$$

Although this limiting procedure is formal, the "closeness" between the integro-differential problem and its limiting equation (1.1) can be estimated in a rigorous way (see [5] and the references therein).

Problem (1.1) has been investigated quite extensively by several authors in the last years (see e.g. [3, 4, 7, 9, 13, 16, 17, 19, 21]), with particular regard to its asymptotics. The global existence and dissipativity of *strong* solutions (belonging to the regular phase space $[H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)$) has been established in [13] (see also [14]), without any growth restriction on the nonlinearity φ. On the other hand, under the additional growth restriction

$$|\varphi'(u)| \leq c\big(1 + |u|^p\big), \qquad p \leq 4,$$

equation (1.1) is also well-posed in the natural energy phase space $H^1_0(\Omega) \times L^2(\Omega)$, i.e. the associated weak energy solution exists globally and is unique. The growth rate $p = 4$ is critical, since for $p > 4$ well-posedness in $H^1_0(\Omega) \times L^2(\Omega)$ is lost. The existence of a compact global attractor and its smoothness for *weak* energy solutions when $p = 4$ is a delicate question (in the subcritical case it is well known, and can be obtained by the standard bootstrapping technique, see e.g. [13]). Indeed, even the existence of a global attractor for that case has been achieved only quite recently in [4, 12, 17] and, to the best of our knowledge, the question of its additional regularity remained open. In particular, it was not clear whether or not the global attractor associated with weak energy solutions coincides with the analogous one for strong solutions. Clearly, this lack of regularity prevented a more accurate analysis of the longterm dynamics.

The aim of the present work is to give a positive answer to the above regularity question for the case $p = 4$. The main difficulty here is that the classical Babin-Vishik method [2] of proving the regularity of attractors (based on the nonlinear decomposition and the existence of a dissipation integral), successfully employed for the weakly damped hyperbolic equation, does not work for the strongly damped one. Indeed, the presence of the strong damping term $\Delta\partial_t u$ breaks the simple relation between the spatial regularity of $\partial_{tt}u$ and Δu in the wave equation, which is crucial in the Babin-Vishik technique. Thus, a new approach is required. We exploit the fact that, along with its hyperbolic properties, the considered equation unveils a peculiar parabolic nature. In particular, despite it does not possess a “completeßmoothing property on finite time-intervals (like parabolic equations), it exhibits a *partial* smoothing, precisely, of the functions $\partial_t u$ and $\partial_{tt} u$. Based on this rather simple observation, we reduce the analysis of the initial equation (1.1) to the following purely hyperbolic nonautonomous problem:

$$-\Delta\partial_t u - \Delta u + \varphi(u) = g, \tag{1.2}$$

with $g(t) = f - \partial_{tt} u(t)$, which is essentially simpler than (1.1) and possesses a convenient splitting for proving the regularity (see Section 4). Quite surprisingly, our analysis makes no use of complicated arguments, such as bootstrapping, fractional power operators or analytic semigroup theory. In fact, all we need are suitable energy estimates. Moreover, the method allows us also to treat (with very minor changes) the nonautonomous case where the external force f depends explicitly on t.

The paper is organized as follows. In Section 2, for the reader's convenience, we recall the standard theory of strong solutions, and we verify the existence of a compact global attractor in the regular phase space. In Section 3, we reduce the analysis of the asymptotic regularity of solutions to equation (1.1) to the analogous problem for the simplified equation (1.2), taking advantage of the partial smoothing property mentioned above. Finally, the asymptotic regularity of solutions for this auxiliary equation is studied in Section 4.

Notation. We denote by $H_s = \mathcal{D}((-\Delta)^{s/2})$, $s \in \mathbb{R}$, the scale of Hilbert spaces generated by $-\Delta$ with Dirichlet boundary conditions on $(L^2(\Omega), \langle\cdot,\cdot\rangle, \|\cdot\|)$. In particular,

$$H_0 = L^2(\Omega), \qquad H_1 = H_0^1(\Omega), \qquad H_2 = H^2(\Omega) \cap H_0^1(\Omega).$$

Then, we introduce the product Hilbert spaces

$$\mathcal{H} = H_1 \times H_0 \qquad \text{and} \qquad \mathcal{V} = H_2 \times H_0.$$

Naming $\lambda_1 > 0$ the first eigenvalue of $-\Delta$,

$$\|w\|_{H_s} = \|(-\Delta)^{s/2} w\| \geq {\lambda_1}^{-s/2}\|w\|, \qquad \forall s \geq 0, \forall w \in H_s.$$

We shall often make use, without explicit mention, of this inequality, as well as of the Young and the Hölder inequalities. Throughout the paper, c and Q stand for a generic positive constant and a generic positive increasing function, respectively, depending only on Ω (hence of λ_1) and φ. Moreover, for any function $z(t)$, we write for short $\xi_z(t) = (z(t), \partial_t z(t))$.

We also refer the reader to the classical texts [2, 10, 11, 18] for a detailed presentation of the theory of attractors for dynamical systems.

2. Strong Solutions and the Associated Global Attractor

General assumptions. We take $f \in H_0$ independent of time. Besides, we require $\varphi \in C^1(\mathbb{R})$, with $\varphi(0) = 0$, be such that

$$\liminf_{|r|\to\infty} \varphi'(r) > -\lambda_1, \qquad \forall r \in \mathbb{R}. \tag{2.1}$$

In particular, (2.1) implies that

$$\varphi'(r) \geq -\ell, \qquad \forall r \in \mathbb{R}. \tag{2.2}$$

for some $\ell \geq \lambda_1$.

Remark 2.1. Notice that no growth restrictions on φ are made.

We set

$$\Phi(u) = \int_\Omega \bigg(\int_0^{u(x)} \varphi(y)dy \bigg) dx,$$

which is easily seen to satisfy the inequalities

$$\Phi(u) \geq -\frac{\vartheta}{2}\|u\|^2 - c, \tag{2.3}$$

$$\langle \varphi(u), u\rangle \geq \Phi(u) - \frac{\vartheta}{2}\|u\|^2 - c \geq -\vartheta\|u\|^2 - c, \tag{2.4}$$

for some $\vartheta < \lambda_1$.

Under the above assumption, we have

Theorem 2.2. *Problem* (1.1) *generates a strongly continuous semigroup $S(t)$ on the phase space $\mathcal{V}$. Moreover, the following dissipative estimate holds:*

$$\|\xi_u(t)\|_{\mathcal{V}} \leq Q(\|\xi_u(0)\|_{\mathcal{V}})e^{-\nu t} + Q(\|f\|), \tag{2.5}$$

for every $t \geq 0$ and some $\nu > 0$.

Proof. We shall limit ourselves to show the above dissipative estimate. Then, the existence of a solution can be obtained in a standard way, by means of a Galerkin approximation scheme. As far as uniqueness is concerned, it follows quite directly noting that

$$\|\varphi(u^1) - \varphi(u^2)\| \leq Q(\|\Delta u^1\| + \|\Delta u^2\|)\|u^1 - u^2\|,$$

for all $u^1, u^2 \in H_2$, due to the continuous embedding $H_2 \hookrightarrow C(\overline{\Omega})$.

In order to prove (2.5), we need the following

Lemma 2.3. *Introducing the energy functional*

$$E(\xi_u(t)) = \|\xi_u(t)\|^2_{\mathcal{H}} + |\Phi(u(t))|,$$

we have the estimate

$$\|\xi_u(t)\|^2_{\mathcal{H}} + \int_t^\infty \|\nabla\partial_t u(\tau)\|^2 d\tau \leq cE(\xi_u(0))e^{-\varepsilon t} + Q(\|f\|),$$

for every $t \geq 0$ and some $\varepsilon > 0$.

Proof. Here, as well as in the sequel, we perform *formal* multiplications, which are justified within the Galerkin approximation scheme. Passing to the limit, the bounds that we find for the approximants continue to hold for the solution. Multiplying (1.1) by $\partial_t u$, we have

$$\frac{d}{dt}\big(\|\nabla u\|^2 + \|\partial_t u\|^2 + 2\Phi(u) - 2\langle f, u\rangle\big) + 2\|\nabla \partial_t u\|^2 = 0.$$

A further multiplication by εu, with $\varepsilon > 0$ to be determined later, yields

$$\frac{d}{dt}\big(\varepsilon\|\nabla u\|^2 + 2\varepsilon\langle \partial_t u, u\rangle\big) + 2\varepsilon\|\nabla u\|^2 - 2\varepsilon\|\partial_t u\|^2 + 2\varepsilon\langle \varphi(u), u\rangle - 2\varepsilon\langle f, u\rangle = 0.$$

Introducing the functional

$$\Lambda_0 = (1+\varepsilon)\|\nabla u\|^2 + \|\partial_t u\|^2 + 2\Phi(u) + 2\varepsilon\langle \partial_t u, u\rangle - 2\langle f, u\rangle,$$

we obtain, thanks to (2.4),

$$\frac{d}{dt}\Lambda_0 + \varepsilon\Lambda_0 + \|\nabla\partial_t u\|^2 + \Gamma \le c\varepsilon.$$

where

$$\Gamma = \varepsilon(1-\varepsilon)\|\nabla u\|^2 - \varepsilon\vartheta\|u\|^2 + (\lambda_1 - 3\varepsilon)\|\partial_t u\|^2 - 2\varepsilon^2\langle \partial_t u, u\rangle.$$

Thus, setting ε small enough such that $\Gamma \ge 0$, we end up with

$$\frac{d}{dt}\Lambda_0 + \varepsilon\Lambda_0 + \|\nabla\partial_t u\|^2 \le c\varepsilon, \tag{2.6}$$

and the Gronwall lemma yields

$$\Lambda_0(t) \le \Lambda_0(0)e^{-\varepsilon t} + c.$$

Due to (2.3), it is apparent that

$$\Lambda_0 \ge \varrho\|\xi_u\|^2_{\mathcal{H}} - Q(\|f\|),$$

for some $\varrho > 0$; besides,

$$\Lambda_0 \le cE(\xi_u) + Q(\|f\|).$$

Therefore, we are led to the inequality

$$\|\xi_u(t)\|^2_{\mathcal{H}} \le cE(\xi_u(0))e^{-\varepsilon t} + Q(\|f\|).$$

Finally, setting $\varepsilon = 0$ in (2.6), and integrating on (t, ∞), we prove the remaining part of the claim. $\square$

Consider now the functional

$$\Lambda_1 = \frac{1}{2}\|\Delta u\|^2 - \langle \partial_t u, \Delta u\rangle.$$

A multiplication of (1.1) by $-\Delta u$ together with (2.2) entail

$$\begin{aligned}\frac{d}{dt}\Lambda_1 + \Lambda_1 + \frac{1}{2}\|\Delta u\|^2 &= \|\nabla\partial_t u\|^2 - \langle \varphi'(u)\nabla u, \nabla u\rangle - \langle \partial_t u, \Delta u\rangle - \langle f, \Delta u\rangle \\ &\le \frac{1}{2}\|\Delta u\|^2 + c\big(\|\nabla u\|^2 + \|\nabla\partial_t u\|^2 + \|f\|^2\big).\end{aligned}$$

Hence, by the Gronwall lemma and Lemma 2.3, we find

$$\Lambda_1(t) \le \Lambda_1(0)e^{-t} + cE(\xi_u(0))e^{-\nu t} + Q(\|f\|),$$

for some $\nu > 0$. Here, we used the inequality

$$\int_0^t e^{-(t-\tau)}\|\nabla\partial_t u(\tau)\|^2 d\tau \leq e^{-t}\int_0^\infty \|\nabla\partial_t u(\tau)\|^2 d\tau + \int_0^t e^{-(t-\tau)}\Big(\int_\tau^\infty \|\nabla\partial_t u(s)\|^2 ds\Big)d\tau,$$

that is readily obtained integrating by parts. Applying again Lemma 2.3, and noting that

$$E(\xi_u(0)) \leq Q(\|\xi_u(0)\|_{\mathcal{V}}),$$

we get the desired dissipative estimate in $\mathcal{V}$. ☐

The main result of this section is

Theorem 2.4. *The semigroup $S(t)$ on $\mathcal{V}$ possesses a (unique) compact global attractor $\mathcal{A}_{\mathcal{V}} \subset \mathcal{V}$. Moreover, $\mathcal{A}_{\mathcal{V}}$ is a bounded subset of $H_2 \times H_2$.*

Proof. We decompose the solution u as

$$u = v + w + (-\Delta)^{-1} f,$$

where v, w are the solutions to the problems

$$\begin{cases} \partial_{tt} v - \Delta\partial_t v - \Delta v = 0, \\ \xi_v(0) = (u_0, u_1), \\ v_{|\partial\Omega} = 0, \end{cases}$$

and

$$\begin{cases} \partial_{tt} w - \Delta\partial_t w - \Delta w + \varphi(u) = 0, \\ \xi_w(0) = (0, 0), \\ w_{|\partial\Omega} = 0. \end{cases}$$

Then, we have

Lemma 2.5. *The inequality*

$$\|\xi_v(t)\|_{\mathcal{V}} \leq Q(\|\xi_u(0)\|_{\mathcal{V}})e^{-\nu t}$$

holds for every $t \geq 0$ and some $\nu > 0$.

Proof. Argue exactly as in Theorem 2.2 (with v in place of u), noting that now $\varphi = 0$ and $f = 0$. ☐

Lemma 2.6. *The inequality*

$$\|\xi_w(t)\|_{H_3 \times H_2} \leq Q(\|\xi_u(0)\|_{\mathcal{V}})e^{-\nu t} + Q(\|f\|).$$

holds for every $t \geq 0$ and some $\nu > 0$.

Proof. Thanks to estimate (2.5) and the assumption $\varphi(0) = 0$, we know that $\varphi(u(t)) \in H_1$ and

$$\|\nabla\varphi(u(t))\|^2 \leq Q(\|\xi_u(0)\|_{\mathcal{V}})e^{-2\nu t} + Q(\|f\|).$$

For $0 < \varepsilon < 1/(2\lambda_1 + 1)$ to be fixed, let us set

$$\Lambda_2 = (1+\varepsilon)\|\nabla\Delta w\|^2 + \|\Delta\partial_t w\|^2 - 2\varepsilon\langle\nabla\partial_t w, \nabla\Delta w\rangle.$$

Multiplying the equation by $\Delta^2\partial_t w + \varepsilon\Delta^2 w$, we have

$$\frac{d}{dt}\Lambda_2 + 2\varepsilon\|\nabla\Delta w\|^2 + 2(1-\varepsilon)\|\nabla\Delta\partial_t w\|^2 = 2\varepsilon\langle\nabla\varphi(u), \nabla\Delta w\rangle + 2\langle\nabla\varphi(u), \nabla\Delta\partial_t w\rangle.$$

Controlling the right-hand side as

$$2\varepsilon\langle\nabla\varphi(u), \nabla\Delta w\rangle + 2\langle\nabla\varphi(u), \nabla\Delta\partial_t w\rangle \le \varepsilon\|\nabla\Delta w\|^2 + \|\nabla\Delta\partial_t w\|^2 + c\|\nabla\varphi(u)\|^2,$$

we are led to the differential inequality

$$\frac{d}{dt}\Lambda_2(t) + \varepsilon\|\xi_w(t)\|^2_{H_3\times H_2} \le Q(\|\xi_u(0)\|_{\mathcal{V}})e^{-2\nu t} + Q(\|f\|).$$

Choosing ε small enough such that

$$\frac{1}{2}\Lambda_2 \le \|\xi_w(t)\|^2_{H_3\times H_2} \le 2\Lambda_2,$$

the claim follows from an application of the Gronwall lemma. □

Let us summarize the results we obtained. Theorem 2.2 gives the existence of a bounded absorbing set $\mathbb{B}_{\mathcal{V}} \subset \mathcal{V}$ for $S(t)$. Lemma 2.5 and Lemma 2.6 show that the solution ξ_u with initial data $\xi_u(0) \in \mathbb{B}_{\mathcal{V}}$ decomposes into the sum of a uniformly exponentially decaying term and a term belonging to $\zeta + \mathbb{B}_0$, where $\zeta = ((-\Delta)^{-1}f, 0) \in H_2 \times H_s$ for every s, and $\mathbb{B}_0$ is a ball of $H_3 \times H_2$. In particular, $\zeta + \mathbb{B}_0$ is a compact (exponentially) attracting set for $S(t)$ on $\mathcal{V}$. This, by standard methods of the theory of attractors, yields the existence of the global attractor $A_{\mathcal{V}} \in \zeta + \mathbb{B}_0$. □

Remark 2.7. Under our hypotheses, the obtained regularity of $\mathcal{A}_{\mathcal{V}}$ is optimal. On the other hand, it can be improved up to where the regularity of φ and f permit. Indeed, if f and φ are smoother, differentiating the equation with respect to t and arguing in a standard way (namely, applying the techniques above to the new equation), we can prove that the attractor is more regular as well. In particular, if $\varphi \in C^\infty(\mathbb{R})$ and $f \in C^\infty(\overline{\Omega})$, then $\mathcal{A}_{\mathcal{V}} \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$.

3. Asymptotic Regularity of Weak Energy Solutions

In this section, in addition to the previous conditions on f and φ, we assume that φ satisfies the (critical) growth condition

$$|\varphi''(r)| \le c\big(1 + |r|^3\big), \qquad \forall r \in \mathbb{R}. \tag{3.1}$$

With these further assumptions, $S(t)$ is a strongly continuous semigroup on the phase space $\mathcal{H}$ as well [3, 4]. The papers [4, 17] provide the existence of a global compact attractor $\mathcal{A}_{\mathcal{H}}$ for $S(t)$, but no regularity results are given. Here, we prove the following

Theorem 3.1. *The attractor $\mathcal{A}_{\mathcal{H}}$ of the semigroup $S(t)$ on $\mathcal{H}$ is a bounded subset of $H_2 \times H_1$.*

In particular, this means that $\mathcal{A}_{\mathcal{H}}$ and $\mathcal{A}_{\mathcal{V}}$ coincide, so that $\mathcal{A}_{\mathcal{H}}$ in fact inherits the regularity of $\mathcal{A}_{\mathcal{V}}$ (that is, $H_2 \times H_2$ at least).

Remark 3.2. Due to the obtained regularity, it is possible to prove the existence of an exponential attractor of finite fractal dimension for $S(t)$ on $\mathcal{H}$ (cf. [17]). This, in turn, implies that $\mathcal{A}_{\mathcal{H}}$ has finite fractal dimension.

Remark 3.3. In fact, it is possible to show that Theorem 3.1 still holds if we replace (2.1) by the slightly weaker condition (cf. [1])

$$\liminf_{|r|\to\infty} \frac{\varphi(r)}{r} > -\lambda_1, \qquad \forall r \in \mathbb{R}. \tag{3.2}$$

We will return on this later in Remark 4.4.

We establish the proof of the theorem by means of several lemmas. First, we notice that, due to the growth restriction (3.1), we have the inequality

$$E(\xi_u(0)) \le Q(\|\xi_u(0)\|_{\mathcal{H}}).$$

Thus, we can rephrase Lemma 2.3 more conveniently as

Lemma 3.4. *The inequality*

$$\|\xi_u(t)\|^2_{\mathcal{H}} + \int_t^\infty \|\nabla \partial_t u(\tau)\|^2 d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}}) e^{-\varepsilon t} + Q(\|f\|)$$

holds for every $t \ge 0$ and some $\varepsilon > 0$.

The next step amounts to finding suitable regularity for the time-derivatives of u.

Lemma 3.5. *For every $t > 0$, we have the inequality*

$$\min\{t,1\}\|\nabla \partial_t u(t)\|^2 + \int_0^t \min\{\tau,1\}\|\partial_{tt} u(\tau)\|^2 d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|).$$

Proof. Denote

$$\Lambda_3 = \|\nabla \partial_t u\|^2 + 2\langle \nabla u, \nabla \partial_t u\rangle + 2\langle \varphi(u), \partial_t u\rangle - 2\langle f, \partial_t u\rangle.$$

Observe that, from (3.1) and Lemma 3.4,

$$\frac{1}{2}\|\nabla \partial_t u\|^2 - Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|) \le \Lambda_3 \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|)\|\nabla \partial_t u\|^2. \tag{3.3}$$

Then, multiply (1.1) by $\partial_{tt} u$, to obtain

$$\frac{d}{dt}\Lambda_3 + 2\|\partial_{tt} u\|^2 = 2\|\nabla \partial_t u\|^2 + 2\langle \varphi'(u)\partial_t u, \partial_t u\rangle$$

Exploiting (3.1),

$$2\langle \varphi'(u)\partial_t u, \partial_t u\rangle \le 2\|\varphi'(u)\|_{L^{3/2}}\|\partial_t u\|^2_{L^6} \le c(1 + \|\nabla u\|^6)\|\nabla \partial_t u\|^2.$$

Therefore, using Lemma 3.4, we are led to

$$\frac{d}{dt}\Lambda_3 + 2\|\partial_{tt} u\|^2 \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|)\|\nabla \partial_t u\|^2. \tag{3.4}$$

Assume first that $t \in (0,1]$. Multiplying (3.4) by τ, and integrating in $d\tau$ on $[0,t]$, gives

$$t\Lambda_3(t) + 2\int_0^t \tau\|\partial_{tt} u(\tau)\|^2 d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|)\int_0^t \|\nabla \partial_t u(\tau)\|^2 d\tau + \int_0^t \Lambda_3(\tau) d\tau.$$

Using again Lemma 3.4 together with (3.3), we finally obtain the desired inequality

$$\frac{t}{2}\|\nabla\partial_t u(t)\|^2 + 2\int_0^t \tau\|\partial_{tt}u(\tau)\|^2 d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|), \tag{3.5}$$

which is exactly what we wanted for $t \in (0,1]$. If $t > 1$ we integrate (3.4) in $d\tau$ on $(1,t)$. This entails

$$\Lambda_3(t) + 2\int_1^t \|\partial_{tt}u(\tau)\|^2 d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|) + \Lambda_3(1)$$

Substituting (3.3) and (3.5) in the above inequality yields

$$\frac{1}{2}\|\nabla\partial_t u(t)\|^2 + 2\int_1^t \|\partial_{tt}u(\tau)\|^2 d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|). \tag{3.6}$$

The conclusion follows collecting (3.5)-(3.6) □

Lemma 3.6. *For every $t > 0$, we have the inequality*

$$\min\{t^2, 1\}\|\partial_{tt}u(t)\| \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|).$$

Proof. Set $q = \partial_t u$, and differentiate (1.1) with respect to time. This yields

$$\partial_{tt}q - \Delta\partial_t q - \Delta q + \varphi'(u)\partial_t u = 0.$$

We denote

$$\Lambda_4 = \|\nabla q\|^2 + \|\partial_t q\|^2.$$

Observe that, collecting Lemma 3.4 and Lemma 3.5, we know that

$$\int_0^t \min\{\tau, 1\}\Lambda_4(\tau) d\tau \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|). \tag{3.7}$$

Then, multiplying the above equation by $\partial_t q$, we obtain

$$\frac{d}{dt}\Lambda_4 + 2\|\nabla\partial_t q\|^2 = -2\langle\varphi'(u)\partial_t u, \partial_t q\rangle.$$

In a similar fashion as in the previous lemma, we estimate the right-hand side as

$$\begin{aligned}-2\langle\varphi'(u)\partial_t u, \partial_t q\rangle &\le 2\|\varphi'(u)\|_{L^{3/2}}\|\partial_t u\|_{L^6}\|\partial_t q\|_{L^6} \\ &\le 2\|\nabla\partial_t q\|^2 + Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|)\|\nabla\partial_t u\|^2.\end{aligned}$$

Therefore, we come to the differential inequality

$$\frac{d}{dt}\Lambda_4 \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|)\|\nabla\partial_t u\|^2. \tag{3.8}$$

If $t \in (0,1]$, we multiply (3.8) by τ^2, and integrate in $d\tau$ on $[0,t]$. This, on account of Lemma 3.4 and (3.7), yields

$$t^2\Lambda_4(t) \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|),$$

which in turn gives

$$t^2\|\partial_t q(t)\|^2 \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|). \tag{3.9}$$

If $t > 1$ we integrate (3.8) in $d\tau$ on $(1,t)$, to get

$$\|\partial_t q(t)\|^2 \le \Lambda_4(t) \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|) + \Lambda_4(1) \le Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|). \tag{3.10}$$

Putting together (3.9)-(3.10), the proof is finished. □

Conclusion of the proof of Theorem 3.1. Lemma 3.4 ensures the existence of a bounded absorbing set $\mathbb{B}_{\mathcal{H}} \subset \mathcal{H}$ for the semigroup $S(t)$ on $\mathcal{H}$. We now rewrite (1.1) as

$$-\Delta\partial_t u - \Delta u + \varphi(u) = h,$$

having set

$$h = f - \partial_{tt} u.$$

In view of Lemma 3.6, for a fixed $t_* > 0$ (for instance, $t_* = 1$),

$$\sup_{t \geq t_*} \|h(t)\| \leq Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|).$$

Therefore, applying Theorem 4.3 of the next section, and recalling Lemma 3.4, we learn that $u = v + w$, with

$$\|\nabla v(t)\| \leq Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|)e^{-\nu t} \qquad \text{and} \qquad \|\Delta w(t)\| \leq Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|),$$

for every $t \geq t_*$. Besides, from Lemma 3.5,

$$\|\nabla\partial_t u(t)\| \leq Q(\|\xi_u(0)\|_{\mathcal{H}} + \|f\|),$$

for every $t \geq t_*$. In particular, for all initial data $\xi_u(0) \in \mathbb{B}_{\mathcal{H}}$, we have that

$$\|\nabla v(t)\| \leq ce^{-\nu t}, \qquad \|\Delta w(t)\| \leq c, \qquad \|\nabla\partial_t u(t)\| \leq c,$$

for every $t \geq t_*$, where the constant $c \geq 0$ depends only on $\|f\|$ and the size of $\mathbb{B}_{\mathcal{H}}$. Hence, calling $\mathbb{B}_1$ the ball of $H_2 \times H_1$ of radius $c\sqrt{2}$, we conclude that

$$\mathrm{dist}_{\mathcal{H}}(S(t)\mathbb{B}_{\mathcal{H}}, \mathbb{B}_1) \leq ce^{-\nu t}, \qquad \forall t \geq t_*,$$

where $\mathrm{dist}_{\mathcal{H}}$ denotes the usual Hausdorff semidistance in $\mathcal{H}$. In other words, $\mathbb{B}_1$ is a compact (exponentially) attracting set. So, by standard arguments, the semigroup $S(t)$ acting on $\mathcal{H}$ possesses a compact global attractor $\mathcal{A}_{\mathcal{H}} \subset \mathbb{B}_1$. ☐

4. An Auxiliary Equation

We conclude the paper examining the following hyperbolic equation:

$$\begin{cases} -\Delta\partial_t u - \Delta u + \varphi(u) = h, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{4.1}$$

Here, the assumptions on φ are as in Section 3, whereas $h \in L^\infty([t_*, \infty), H_0)$, for some $t_* \in \mathbb{R}$. For every $u_0 \in H_1$, (4.1) possesses a unique solution $u \in C([t_*, \infty), H_1)$ such that $u(t_*) = u_0$. Besides, u continuously depends on the initial datum u_0 on every finite time-interval.

Remark 4.1. In fact, using maximal monotone operator theory, we can establish existence and continuous dependence results for equation (4.1) even without the growth restriction (3.1) (assuming only (2.1)), and the results that follow can be proven true also in that case. Indeed, rewriting the equation as

$$\partial_t u + Au = \ell(-\Delta)^{-1}u + (-\Delta)^{-1}h,$$

where $Au = u + (-\Delta)^{-1}\varphi(u) + \ell(-\Delta)^{-1}u$, with ℓ as in (2.2), the operator A is easily seen to be maximal monotone on H_1.

As we saw in the previous section, equation (4.1) reproduces the hyperbolic features of the original problem (1.1). We begin to establish a first uniform estimate.

Lemma 4.2. *The inequality*

$$\|\nabla u(t)\| \leq c\|\nabla u(t_*)\|e^{-\varepsilon(t-t_*)} + Q(\|h\|_{L^\infty([t_*,\infty),H_0)})$$

holds for every $t \geq t_*$ and some $\varepsilon > 0$.

Proof. A multiplication of (4.1) by u leads to

$$\frac{d}{dt}\|\nabla u\|^2 + 2\|\nabla u\|^2 + 2\langle \varphi(u), u\rangle = 2\langle h, u\rangle.$$

Using (2.4), we obtain the inequality

$$\frac{d}{dt}\|\nabla u\|^2 + 2\varepsilon\|\nabla u\|^2 \leq c\big(1 + \|h\|^2\big),$$

for some $\varepsilon > 0$, and the claim follows from the Gronwall lemma. □

The result we need is the following.

Theorem 4.3. *The solution u to (4.1) can be decomposed into the sum $u = v + w$, where*

$$\|\nabla v(t)\| \leq \|\nabla u(t_*)\|e^{-(t-t_*)}$$

and

$$\|\Delta w(t)\| \leq c\|\nabla u(t_*)\|e^{-\nu(t-t_*)} + Q(\|h\|_{L^\infty([t_*,\infty),H_0)}),$$

for every $t \geq t_*$ and some $\nu > 0$.

Proof. We define

$$\psi(r) = \varphi(r) + \ell r,$$

with ℓ as in (2.2). Thus, $\psi'(r) \geq 0$ for every $r \in \mathbb{R}$. Then, we consider the splitting $u = v + w$, where v and w are the solutions to the equations

$$\begin{cases} -\Delta\partial_t v - \Delta v + \psi(u) - \psi(w) = 0, \\ v(t_*) = u(t_*), \\ v_{|\partial\Omega} = 0, \end{cases} \tag{4.2}$$

and

$$\begin{cases} -\Delta\partial_t w - \Delta w + \psi(w) = \ell u + h, \\ w(t_*) = 0, \\ w_{|\partial\Omega} = 0. \end{cases} \tag{4.3}$$

Multiplying (4.2) by v, from the monotonicity of ψ we readily get the inequality

$$\frac{d}{dt}\|\nabla v\|^2 + 2\|\nabla v\|^2 \leq 0,$$

which entails

$$\|\nabla v(t)\| \leq \|\nabla u(t_*)\|e^{-(t-t_*)}, \qquad \forall t \geq t_*.$$

Next, we multiply (4.3) by $-\Delta w$. Appealing again to the monotonicity of ψ, we obtain the inequality

$$\begin{aligned}\frac{d}{dt}\|\Delta w\|^2 + 2\|\Delta w\|^2 &\leq -2\ell\langle u, \Delta w\rangle - 2\langle h, \Delta w\rangle \\ &\leq \|\Delta w\|^2 + c\big(\|u\|^2 + \|h\|^2\big).\end{aligned}$$

Therefore, using the control provided by Lemma 4.2, we conclude that, for every $t \geq t_*$,

$$\frac{d}{dt}\|\Delta w(t)\|^2 + \|\Delta w(t)\|^2 \leq c\|\nabla u(t_*)\|^2 e^{-2\varepsilon(t-t_*)} + Q(\|h\|_{L^\infty([t_*,\infty),H_0)}).$$

Applying the Gronwall lemma, we finally have

$$\|\Delta w(t)\| \leq c\|\nabla u(t_*)\| e^{-\nu(t-t_*)} + Q(\|h\|_{L^\infty([t_*,\infty),H_0)}),$$

for all $t \geq t_*$ and some $\nu > 0$. This finishes the proof. □

Remark 4.4. Theorem 4.3 establishes the existence of an exponentially attracting ball in H_2, which is all we need to complete the proof of Theorem 3.1. This fact can also be proved assuming condition (3.2) in place of (2.1) (in that case, the growth condition (3.1) is essential). The proof is more complicated, and it requires a different splitting of u along with an *ad hoc* argument, devised in [20], and the transitivity property of exponential attraction, introduced and proved in [8]. Once one has Theorem 4.3, arguing as in [17], all the results of Section 3 can be shown to hold.

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