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Large deviations for sums defined on a Galton–Watson process

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Abstract. In this paper we study the large deviation behavior of sums of i.i.d. random variables X_i defined on a supercritical Galton-Watson process Z. We assume the finiteness of the moments EX_1^2 and $EZ_1 \log Z_1$. The underlying interplay of the partial sums of the X_i and the lower deviation probabilities of Z is clarified. Here we heavily use lower deviation probability results on Z we recently published in [FW06].

Contents

1. Introduction and results

1.1. **Motivation.** Let $Z = (Z_n)_{n\geq 0}$ denote a *Galton-Watson process* with offspring law $\{p_k;\, k \geq 0\}$. We will assume that Z is *supercritical*: $m := \sum_{k=1}^{\infty} kp_k \in (1, \infty)$. As a rule we start with $Z_0 = 1$.

A basic task in statistical inference of Galton-Watson processes is the estimation of the offspring mean m. Let us recall at this place the well-known Lotka-Nagaev estimator Z_{n+1}/Z_n of m due to A.V. Nagaev [Nag67]. If $\sigma := (\mathbf{V}\text{ar}Z_1)^{1/2} \in (0, \infty)$, then for every $x \in \mathbb{R}$,

$$\lim_{n\uparrow\infty} \mathbf{P}\Big(m^{n/2}\big(\tfrac{Z_{n+1}}{Z_n} - m\big) < x;\ Z_n > 0\Big) \;=\; \int_0^{\infty} \Phi\big(\tfrac{xu^{1/2}}{\sigma}\big)\, w(u)\, \mathrm{d}u, \qquad (1)$$

where w denotes the density function of the a.s. limit variable $W := \lim_{n\uparrow\infty} m^{-n} Z_n$ restricted to $(0, \infty)$, and Φ is the standard normal distribution function,

$$\Phi(y) \;:=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \mathrm{e}^{-z^2/2}\, \mathrm{d}z, \qquad y \in \mathbb{R}. \qquad (2)$$

The study of the ratio Z_{n+1}/Z_n has attracted the attention of several researchers in recent years, since it can also be used for estimating important parameters such as the amplification rate and the initial size in a quantitative polymerase chain reaction experiment; see Jacob and Peccoud [JP96, JP98].

Fix $k \geq 0$. In a finer description of the Galton-Watson model, let $Z_n(k)$ denote the number of particles in the n^{th} generation having exactly k children. Then, on the event $\{Z_n > 0\}$, results for the estimator $\tilde{p}_k(n) := Z_n(k)/Z_n$ of p_k, which hold analogously to (1), had been provided by Pakes [Pak75, Theorems 5 and 6].

The mentioned results from [Nag67] and [Pak75] can be seen from a unified point of view as follows. Independently of Z, let $X = (X_n)_{n\geq 1}$ denote a family of *i.i.d. random variables with mean zero and variance in* $(0,\infty)$. Let $n \geq 0$. Put $S_n := X_1 + \ldots + X_n$. On the event $\{Z_n > 0\}$, the random variable

$$R_n := S_{Z_n}/Z_n \tag{3}$$

is well-defined. For convenience, *we agree* that an event involving R_n is always tacitly assumed to be included in $\{Z_n > 0\}$. For instance, $\mathbf{P}(R_n < x)$ means $\mathbf{P}\bigl(R_n < x;\ Z_n > 0\bigr)$ more carefully written. If now X_1 coincides in law with $Z_1 - m$, then, for n fixed, R_n coincides in law with $Z_{n+1}/Z_n - m$ on the event $\{Z_n > 0\}$. On the other hand, if X_1 takes on the value $1 - p_k$ with probability p_k (for k fixed) and $-p_k$ otherwise, then $R_n = \tilde{p}_k(n) - p_k$ in law on the event $\{Z_n > 0\}$ for fixed n.

Sums such as S_{Z_n} arise also in models of polymerase chain reactions with mutations, see Piau [Pia04].

From now on, we work with the more general meaning of R_n, based on (X, Z), as introduced in (3). Clearly, we have the following *strong law of large numbers:*

$$R_n \to 0 \quad \text{a.s. as} \quad n \uparrow \infty. \tag{4}$$

Moreover, using methods from [Nag67] and [Pak75], one can easily verify the following "*normal deviation probabilities*" for R_n:

$$\lim_{n\uparrow\infty} \mathbf{P}\bigl(m^{n/2} R_n < x\bigr) = \int_0^\infty \Phi\bigl(\tfrac{xu^{1/2}}{\sigma}\bigr)\, w(u)\, \mathrm{d}u, \qquad x \in \mathbb{R}, \tag{5}$$

where $\sigma := (\mathbf{E}X_1^2)^{1/2}$ from now on. Let $\varepsilon_n > 0$. In the case $\varepsilon_n m^{n/2} \to \infty$, this implies the following simple large deviation probabilities for R_n:

$$\lim_{n\uparrow\infty} \mathbf{P}(R_n \geq \varepsilon_n) = 0. \tag{6}$$

But the main task of large deviation theory is to determine the *rate* of this convergence. Clearly, one of the reasons to be interested in large deviation probabilities comes from statistical applications. Firstly, these probabilities describe the quality (error probabilities) of many tests. On the other hand, a question concerning the Bahadur efficiency of estimators leads also to the large deviation problem.

For the particular model $X_1 \stackrel{\mathcal{L}}{=} Z_1 - m$, the special case $\varepsilon_n \equiv \varepsilon$ is more or less studied in the literature. In fact, Athreya [Ath94] proved that if $p_0 = 0$, $p_1 > 0$, and $\mathbf{E}Z_1^{2\alpha+\delta} < \infty$ for some $\delta > 0$, where $\alpha \in (0,\infty)$ denotes the so-called Schröder constant [see (8) below], then

$$\lim_{n\uparrow\infty} m^{\alpha n}\, \mathbf{P}\bigl(|R_n| \geq \varepsilon\bigr) \quad \text{exists finitely.} \tag{7}$$

On the other hand, using asymptotic properties of harmonic moments of Z_n, Ney and Vidyashankar [NV03] found the rate of $\mathbf{P}\bigl(|R_n| \geq \varepsilon\bigr)$ under the weaker assumption that $\mathbf{P}(Z_1 \geq j) \sim a j^{1-\eta}$ as $j \uparrow \infty$, for some $\eta > 2$ and $a > 0$. The same authors proved in [NV04] a version of a large deviation principle for R_n conditioned on $Z_n \geq v_n$ with numbers $v_n \to \infty$; see also Rouault [Rou00].

The *purpose of the present paper* is to study the rate of convergence of large deviation probabilities of $R_n \geq \varepsilon_n$ in the more interesting case $\varepsilon_n \to 0$ as $n \uparrow \infty$

(working with our more general setting of R_n). For this we heavily relay on results on lower deviation probabilities of Z, we recently derived in [FW06]. In the next subsection we briefly recall what we need from that paper.

Note that large deviation probabilities in the case $\varepsilon_n \to 0$ are needed, for instance, for testing two close hypotheses, i.e. when the distance between the hypotheses tends to zero as the size of the sample gets larger and larger.

1.2. **Lower deviation probabilities for Z.** We start with recalling the following basic notation, reflecting a crucial dichotomy for supercritical Galton-Watson processes.

Definition 1 (**Schröder and Böttcher case**)**.** For our supercritical offspring distribution we distinguish between the *Schröder* and the *Böttcher* case, in dependence on whether $p_0 + p_1 > 0$ or $= 0$, respectively. $\diamond$

Write f for the generating function of our supercritical offspring law: $f(s) = \sum_{j\geq 0} p_j s^j$, $0 \leq s \leq 1$. Let q denote the extinction probability of Z,

$$\text{set } \gamma := f'(q), \quad \text{and define } \alpha \text{ by } \quad \gamma = m^{-\alpha}. \tag{8}$$

Note that $\gamma \in [0,1)$ and $\alpha \in (0,\infty]$. Obviously, we are in the Schröder case if and only if $\gamma > 0$, if and only if $\alpha < \infty$. In this case, α is said to be the *Schröder constant.* We also need the following notion.

Definition 2 (**Type** (d,μ))**.** We say the offspring distribution *is of type* (d,μ), if $d \geq 1$ is the greatest common divisor of the set $\{j-\ell : j \neq \ell,\ p_j p_\ell > 0\}$, and $\mu \geq 0$ is the minimal j for which $p_j > 0$. $\diamond$

In the present paper, (d,μ) *always refers* to the type of our offspring law. Recall that in the Böttcher case, $\mu = \min\{k \geq 0 : p_k > 0\} \geq 2$. Here the *Böttcher constant* $\beta \in (0,1)$ is defined by $\mu = m^\beta$.

We also *always assume* that the moment $\mathbf{E} Z_1 \log Z_1$ is finite. Under this moment condition, the results of [FW06] can be specialized to the following two propositions.

Proposition 3 (**Schröder case**)**.** *In the Schröder case, for $k_n \leq m^n$ satisfying $k_n \to \infty$, we have*

$$\sup_{k\in[k_n,m^n] \text{ with } k\equiv\mu\,(\mathrm{mod}\,d)} \left| \frac{m^n}{d\,w\big(k/m^n\big)}\,\mathbf{P}(Z_n = k) \ - \ 1 \right| \underset{n\uparrow\infty}{\longrightarrow} 0 \tag{9}$$

and

$$\sup_{k\in[k_n,m^n]} \left| \frac{\mathbf{P}(0 < Z_n \leq k)}{\mathbf{P}\big(0 < W < k/m^n\big)} \ - \ 1 \right| \underset{n\uparrow\infty}{\longrightarrow} 0. \tag{10}$$

Proposition 4 (**Böttcher case**)**.** *Suppose the Böttcher case. Then there exist positive constants B_1 and B_2 such that for all $k_n \geq \mu^n$ with $k_n = o(m^n)$ as $n \uparrow \infty$,*

$$-B_1 \ \leq \ \liminf_{n\uparrow\infty} \, (k_n/m^n)^{\beta/(1-\beta)} \log \mathbf{P}(Z_n \leq k_n) \tag{11a}$$

$$\leq \ \limsup_{n\uparrow\infty} (k_n/m^n)^{\beta/(1-\beta)} \log \mathbf{P}(Z_n \leq k_n) \ \leq \ -B_2\,. \tag{11b}$$

The inequalities (11) *remain true if $\mathbf{P}(Z_n \leq k_n)$ is replaced by $m^n\,\mathbf{P}(Z_n = k_n)$, provided that $k_n \equiv \mu^n\,(\mathrm{mod}\,d)$.*

In order to explain the influence of lower deviation probabilities of Z_n on $R_n = S_{Z_n}/Z_n$, we look at the decomposition,

$$\mathbf{P}(R_n \geq \varepsilon_n) \;=\; \sum_{k=1}^{\infty} \mathbf{P}(Z_n = k)\, \mathbf{P}(S_k \geq \varepsilon_n k). \tag{12}$$

Thus, in order to find the asymptotics of $\mathbf{P}(R_n \geq \varepsilon_n)$, we need to determine the range of values of k, which give the main contribution in decomposition (12). As we will see, this depends on parameters of the offspring law (as α, for instance) and, on the other hand, on the tail behavior of X_1. Here we mention several possibilities. If k is of order m^n (the regime of normal deviations for Z_n) and $\varepsilon_n^2 m^n \to \infty$, then $\varepsilon_n k$ is in the domain of large deviations of S_k. On the other hand, if k is of order ε_n^{-2} (regime of normal deviations for S_k), then k is in the domain of lower deviations for Z_n. And finally, if $k/m^n \to 0$ and $\varepsilon_n^2 k \to \infty$, then simultaneously we have lower deviations for Z_n and large deviations for S_k.

1.3. **Large deviations in the Schröder case.** Recall that we always assume $\mathbf{E}Z_1 \log Z_1 < \infty$ and $\mathbf{E}X_1^2 < \infty$. As usual, we set $X_1^+ := X_1 \vee 0$.

Theorem 5 (**Schröder under a** $(1+\alpha)$**-moment condition on** X_1). *Suppose the Schröder case and that*

$$\mathbf{E}(X_1^+)^{1+\alpha} \;<\; \infty \tag{13}$$

[*with* $\alpha \in (0,\infty)$ *the Schröder constant from* (8)]. *Moreover, assume that* $\varepsilon_n \to 0$ *and* $\varepsilon_n^2 m^n \to \infty$ *as* $n \uparrow \infty$*. Then*

$$0 \;<\; V_* \Gamma_\alpha \;\leq\; \liminf_{n\uparrow\infty}\; \varepsilon_n^{2\alpha}\, m^{\alpha n}\, \mathbf{P}(R_n \geq \varepsilon_n) \tag{14a}$$

$$\leq\; \limsup_{n\uparrow\infty}\, \varepsilon_n^{2\alpha}\, m^{\alpha n}\, \mathbf{P}(R_n \geq \varepsilon_n) \;\leq\; V^*\Gamma_\alpha \;<\; \infty, \tag{14b}$$

where

$$V_* \;:=\; \liminf_{u\downarrow 0}\; u^{1-\alpha} w(u), \quad V^* \;:=\; \limsup_{u\downarrow 0}\; u^{1-\alpha} w(u) \tag{15}$$

and

$$\Gamma_\alpha \;:=\; \frac{2^{\alpha-1}\,\Gamma(\alpha+1/2)}{\alpha\sqrt{\pi}}\;\sigma^{2\alpha}. \tag{16}$$

Of course, here Γ refers to the Gamma function.

Next we recall some known facts on the asymptotic behavior of supercritical Galton-Watson processes in the Schröder case. With q and γ introduced in the beginning of Subsection 1.2 and with f_n denoting the iterates of f, the following limit exists:

$$\lim_{n\uparrow\infty} \frac{f_n(s)-q}{\gamma^n} \;=:\; \mathsf{S}(s) \;=:\; \sum_{j=0}^{\infty} \nu_j s^j, \qquad 0 \leq s < 1. \tag{17}$$

Hence,

$$\lim_{n\uparrow\infty} \gamma^{-n}\, \mathbf{P}(Z_n = k) \;=\; \nu_k\,, \qquad k \geq 1. \tag{18}$$

The Schröder constant $\alpha < \infty$ describes the behavior of the density function $w(u)$ as $u \downarrow 0$. In fact, according to Biggins and Bingham [BB93], there is a continuous, positive multiplicatively periodic function V such that

$$u^{1-\alpha} w(u) \;=\; V(u) + o(1) \quad \text{as } u \downarrow 0. \tag{19}$$

The function V in (19) can be replaced by a (positive) constant V_0 if and only if

$$\mathsf{S}\big(\varphi(h)\big) = V_0 h^{-\alpha}, \qquad h \geq 0, \tag{20}$$

where φ denotes the Laplace transform of the limit random variable W (cf. Asmussen and Hering [AH83, p. 96]. In this case, $V^* = V_*$ in Theorem 5, and we get the following conclusion.

Corollary 6 (**Schröder under an additional regularity of** Z)**.** *If* (20) *holds, then under the assumptions of Theorem* 5*,*

$$\lim_{n\uparrow\infty} \varepsilon_n^{2\alpha}\, m^{\alpha n}\, \mathbf{P}(R_n \geq \varepsilon_n) \;=\; V_0\, \Gamma_\alpha \tag{21}$$

[*with* Γ_α *from* (16)].

Under the assumptions of Theorem 5, the sum at the right hand side of (12) is determined by those values of k which are of order ε_n^{-2}. As we already mentioned, this corresponds to lower deviations of Z and normal deviations of S_k. But what happens if moment condition (13) fails? We are able to answer this question under some regularity of the tail probabilities of X_1. For this purpose, we say that X_1 *has a tail of index* θ, if for some constant $a > 0$,

$$\mathbf{P}(X_1 \geq x) \;\sim\; a\, x^{-\theta} \quad \text{as } x \uparrow \infty. \tag{22}$$

(Here the involved constant is always denoted by a.)

Theorem 7 (**Schröder under heavier tails concerning** X_1)**.** *Suppose that* $1 < \alpha < \infty$ *and that* X_1 *has a tail of index* $\theta \in (2, 1+\alpha)$*. Define* $\varkappa := (1+\alpha-\theta)/(2\alpha-\theta)$*.*

(a) *If* $\varepsilon_n m^{\varkappa n} \to 0$*, but* $\varepsilon_n^2 m^n \to \infty$*, then statements* (14) *remain valid.*

(b) *If* $\varepsilon_n m^{\varkappa n} \to \infty$*, then*

$$\lim_{n\uparrow\infty} \varepsilon_n^{\theta}\, m^{(\theta-1)n}\, \mathbf{P}(R_n \geq \varepsilon_n) \;=\; a\, I_\theta, \tag{23}$$

where

$$I_\theta \;:=\; \frac{1}{\Gamma(\theta-1)} \int_0^\infty \varphi(v)\, v^{\theta-2}\, \mathrm{d}v. \tag{24}$$

(c) *If* $\varepsilon_n m^{\varkappa n} \to \tau^{-1} \in (0,\infty)$*, then*

$$\tau^{2\alpha} V_* \Gamma_\alpha + \tau^{\theta} a\, I_\theta \;\leq\; \liminf_{n\uparrow\infty}\, m^{\alpha(\theta-2)n/(2\alpha-\theta)}\, \mathbf{P}(R_n \geq \varepsilon_n) \tag{25}$$
$$\leq\; \limsup_{n\uparrow\infty} m^{\alpha(\theta-2)n/(2\alpha-\theta)}\, \mathbf{P}(R_n \geq \varepsilon_n) \;\leq\; \tau^{2\alpha} V^* \Gamma_\alpha + \tau^{\theta} a\, I_\theta$$

[*with* V_*, V^* *from* (15) *and* Γ_α *from* (16)].

Large deviations as in part (b) have a structure, different from that in Theorem 5. Here the main contribution comes from normal deviations of Z_n and large deviations of S_n. In part (c) we have a combination of regimes appearing in (a) and (b).

Remark 8 (**Critical value of** θ)**.** Theorems 5 and 7 leave open the case that X_1 has a tail of index $\theta = \alpha+1$. Our methods allow to prove that (under $1 < \alpha < \infty$) part (a) holds, if $\varepsilon_n n^{1/(\alpha-1)} \to 0$. On the other hand, if $\varepsilon_n n^{1/(\alpha-1)} \to \infty$, then

$$\lim_{n\uparrow\infty} n^{-1}\, \varepsilon_n^{1+\alpha}\, m^{\alpha n}\, \mathbf{P}(R_n \geq \varepsilon_n) \;=\; a\, J_\alpha \tag{26}$$

where

$$J_\alpha \;:=\; \frac{1}{\Gamma(\alpha)} \int_1^m \mathsf{S}\big(\varphi(v)\big)\, v^{\alpha-1}\, \mathrm{d}v. \tag{27}$$

Finally, if $\varepsilon_n n^{1/(\alpha-1)} \to \tau^{-1} \in (0,\infty)$, then a similar statement as in (c) is true.$\diamond$

1.4. **Large deviations in the Böttcher case.** As well-known, in the Böttcher case the following limit

$$\lim_{n\uparrow\infty} \big(f_n(s)\big)^{(\mu^{-n})} \;=:\; \mathsf{B}(s), \qquad 0 \le s \le 1, \tag{28}$$

exists, is positive and continuous. From this it follows that in general $f_n(s)$ does not converge as $n \uparrow \infty$. But taking logarithms, we have

$$\lim_{n\uparrow\infty} \mu^{-n} \log f_n(s) \;=\; \log \mathsf{B}(s). \tag{29}$$

On the other hand, our result on lower deviations in the Böttcher case (Proposition 4) is also only for log-scaled probabilities. These two facts explain the use of a logarithmic scaling in our following theorem.

Theorem 9 (**Böttcher under light tails concerning** X_1). *Assume the Böttcher case, that* $\mathbf{E}\mathrm{e}^{h|X_1|}$ *is finite for some* $h > 0$, *and that* $\varepsilon_n \to 0$ *as well as* $\varepsilon_n^2 m^n \to \infty$ *as* $n \uparrow \infty$. *Then*

$$\mu \,\log \mathsf{B}\big(\varphi(1/2\sigma^2)\big) \;\le\; \liminf_{n\uparrow\infty}\; \varepsilon_n^{-2\beta} m^{-\beta n} \log \mathbf{P}(R_n \ge \varepsilon_n) \tag{30a}$$

$$\le\; \limsup_{n\uparrow\infty}\; \varepsilon_n^{-2\beta} m^{-\beta n} \log \mathbf{P}(R_n \ge \varepsilon_n) \;\le\; \mu^{-1} \log \mathsf{B}\big(\varphi(1/2\sigma^2)\big). \tag{30b}$$

If, additionally, $\varepsilon_n = m^{-\lambda_n/2}$ *for integers* $\lambda_n \to \infty$ *with* $\lambda_n = o(n)$ *as* $n \uparrow \infty$, *then*

$$\lim_{n\uparrow\infty} \varepsilon_n^{-2\beta} m^{-\beta n} \log \mathbf{P}(R_n \ge \varepsilon_n) \;=\; \log \mathsf{B}\big(\varphi(1/2\sigma^2)\big). \tag{31}$$

According to this theorem, the main contribution to $\mathbf{P}(R_n \ge \varepsilon_n)$ comes from lower deviations of Z_n and large deviations of S_n. In order to explain this heuristically, we note that by Proposition 4 there exist (positive and finite) constants $c_1 \ge c_2$ such that

$$\exp\!\big[-c_1\, (k/m^n)^{-\beta/(1-\beta)}\big] \;\le\; m^n\, \mathbf{P}(Z_n = k) \;\le\; \exp\!\big[-c_2\, (k/m^n)^{-\beta/(1-\beta)}\big]. \tag{32}$$

On the other hand (for details see the proof of Theorem 9 in Subsection 3.3 below),

$$\exp[-c_3\, \varepsilon_n^2 k] \;\le\; \mathbf{P}(S_k \ge \varepsilon_n k) \;\le\; \exp[-c_4\, \varepsilon_n^2 k] \tag{33}$$

for some $c_3 \ge c_4$. Then, roughly speaking,

$$\mathbf{P}(R_n \ge \varepsilon_n) \;\sim\; m^{-n} \sum_{k=\mu^n}^{\infty} \exp\!\big[-a\, (k/m^n)^{-\beta/(1-\beta)} - b\, \varepsilon_n^2 k\big] \tag{34}$$

with $a, b > 0$. Obviously, the value of this sum is determined, in a sense, by the maximal summand. It can now easily be seen, that the function

$$g(u) \;:=\; a\, (u/m^n)^{-\beta/(1-\beta)} + b\, \varepsilon_n^2 u, \qquad u > 0, \tag{35}$$

achieves its minimum at $u_* := c\, \varepsilon_n^{-2(1-\beta)} m^{n\beta}$ [with c we always denote a (positive, finite) constant which might change its value from place to place], and, consequently,

$$g(u_*) \;=\; c\, \varepsilon_n^{2\beta} m^{n\beta}. \tag{36}$$

This is in line with the normalizing sequence in Theorem 9 (except a constant factor).

If we put formally $\alpha = \infty$ in the conditions in Theorem 7(b), then (23) should hold under the condition $\varepsilon_n m^{n/2} \to \infty$, since $\varkappa \to 1/2$ as $\alpha \uparrow \infty$. But we prove it only under a slightly stronger condition on ε_n :

Theorem 10 (**Böttcher under heavier tails concerning** X_1). *Suppose the Böttcher case and that X_1 has a tail of index $\theta > 2$. If $\varepsilon_n m^{n/2} n^{-1/2\beta} \to \infty$, then* (23) *is true.*

There is the same "philosophy" behind Theorem 10 as it is behind Theorem 7(b). The main influence of normal deviations explains also the independence of (23) of the parameters α and β. Note also that in the special case $\varepsilon_n \equiv \varepsilon$, Theorem 7(b) was proved in [NV03].

Remark 11 (**Possible generalizations**). Many conditions in our results are too restrictive, but allow us to make proofs slightly shorter and clearer. Here we mention some (almost evident) generalizations of our theorems.

(a) It is possible to prove versions of Theorems 5 and 7 for X_1 from the domain of attraction of a stable law of any index.
(b) Theorems 7 and 10 can be generalized to the case $\mathbf{P}(X_1 \geq x) = L(x) x^{-\theta}$ with some L, slowly varying at infinity.
(c) We conjecture that condition $\mathbf{E} Z_1 \log Z_1 < \infty$ can be dropped in all of our theorems. In fact, we need it only for inequality (39) below, taken from Theorem II.4.2 of Athreya and Ney [AN72]. But it should be possible to prove this bound for all supercritical Galton-Watson processes.
(d) In [NV04], $\mathbf{P}\big(Z_n \geq \varepsilon_n\, ;\, Z_n \geq v_n\big)$ is considered with $v_n \to \infty$ and $\varepsilon_n \equiv \varepsilon$. Our methods allow to deal with the case $v_n = o(m^n)$ and $\varepsilon_n \to 0$. ◇

Remark 12 (**On critical Galton-Watson processes**). For the moment, suppose that the Galton-Watson process Z is critical, that is, $m = 1$. Furthermore, assume that $B := \mathbf{E} Z_1^2 - 1 \in (0, \infty)$. Then, analogously to (5),

$$\lim_{n\uparrow\infty} \mathbf{P}\big(n^{1/2} R_n < x \,\big|\, Z_n > 0\big) \;=\; \frac{2}{B} \int_0^\infty \Phi\big(\tfrac{x u^{1/2}}{\sigma}\big)\, \mathrm{e}^{-2u/B}\, \mathrm{d}u. \tag{37}$$

For the proof of this convergence in the two special cases of X_1 as mentioned in Subsection 1.1, see [Nag67] and [Pak75], respectively. From (37) we find that for critical processes the domain of large deviations is defined by the relation $\varepsilon_n^2 n \to \infty$ as $n \uparrow \infty$. The special case $\varepsilon_n \equiv \varepsilon$ was treated by Athreya and Vidyashankar [AV97]. If now $\varepsilon_n \to 0$ and $\varepsilon_n^2 n \to \infty$, then

$$\lim_{n\uparrow\infty} \varepsilon_n^2 n\, \mathbf{P}\big(R_n \geq \varepsilon_n \,\big|\, Z_n > 0\big) \;=\; \frac{\sigma^2}{B}\,. \tag{38}$$

Actually, (38) is similar to the statement of Theorem 5 in the case $\alpha = 1$ and if m^n replaced by the order n of $\mathbf{E}\{Z_n \,|\, Z_n > 0\}$. Also, the proof of (38) is close to the proof of Theorem 5 in the case $\alpha = 1$. There are only two differences. First, instead of (39) below, we have to use $\mathbf{P}\big(Z_n = k \,\big|\, Z_n > 0\big) \leq c\, n^{-1}$, which is derived in S.V. Nagaev and Wachtel [NW05]. Second, we have to use the local limit theorem for critical Galton-Watson processes instead of Proposition 3. For the proof of this local limit theorem under a second moment assumption, see [NW05]. ◇

2. Auxiliary results

In this section we prepare for the proofs of our main results.

2.1. **Separate considerations.**

As a first step, we state two bounds for local probabilities of our supercritical Galton-Watson process Z (satisfying $\mathbf{E}Z_1 \log Z_1 < \infty$).

Lemma 13 (**Local probabilities of** Z)**.** *We have*

$$\mathbf{P}\big(Z_n = k \,\big|\, Z_0 = \ell\big) \;\le\; c\,\frac{\ell}{k}\,, \qquad k, \ell, n \ge 1. \tag{39}$$

Moreover, in the Schröder case,

$$\mathbf{P}\big(Z_n = k \,\big|\, Z_0 = 1\big) \;\le\; c\,\frac{k^{\alpha-1}}{m^{\alpha n}}\,, \qquad k, n \ge 1. \tag{40}$$

Proof. For aperiodic ($d = 1$) offspring laws the proof of inequality (39) is given in [AN72, Theorem II.4.2]. The proof in the remaining case $d > 1$ can be carried out similarly.

In proving (40) it is sufficient to assume that $k \le m^n$, otherwise (40) follows from (39). Under the present condition $\mathbf{E}Z_1 \log Z_1 < \infty$, formula (151) from [FW06] with $N = \ell_0 := 1 + [1/\alpha]$ and $j = n - a_k$ where $a_k := \min\{j \ge 1 : m^j \ge k\}$ reads as

$$\sum_{\ell=\ell_0}^{\infty} \mathbf{P}(Z_{n-a_k} = \ell)\,\mathbf{P}\big(Z_{a_k} = k \,\big|\, Z_0 = \ell\big) \;\le\; \frac{c}{m^{a_k}}\, f_{n-a_k}(\mathrm{e}^{-\delta}). \tag{41}$$

It follows from (17) that the right hand side is bounded by $c\,m^{-a_k}\gamma^{n-a_k}$. Since

$$k \;\le\; m^{a_k} \;\le\; mk \quad \text{and} \quad \gamma = m^{-\alpha}, \tag{42}$$

we get the bound

$$\sum_{\ell=\ell_0}^{\infty} \mathbf{P}(Z_{n-a_k} = \ell)\,\mathbf{P}\big(Z_{a_k} = k \,\big|\, Z_0 = \ell\big) \;\le\; c\,\frac{k^{\alpha-1}}{m^{\alpha n}}\,. \tag{43}$$

If $\ell_0 = 1$, then the proof of (40) is complete, since the left hand side in (43) equals $\mathbf{P}(Z_n = k)$. Assume now that $\ell_0 \ge 2$. From (39) it follows that

$$\sum_{\ell=1}^{\ell_0-1} \mathbf{P}(Z_{n-a_k} = \ell)\,\mathbf{P}\big(Z_{a_k} = k \,\big|\, Z_0 = \ell\big) \;\le\; c\,\frac{\ell_0}{k} \sum_{\ell=1}^{\ell_0-1} \mathbf{P}(Z_{n-a_k} = \ell). \tag{44}$$

By (18), $\lim_{n\uparrow\infty} \gamma^{-n}\,\mathbf{P}(Z_n = \ell) = \nu_\ell < \infty$, for every fixed ℓ. Hence,

$$\sum_{\ell=1}^{\ell_0-1} \mathbf{P}(Z_{n-a_k} = \ell) \;\le\; c\,\gamma^{n-a_k} \tag{45}$$

for all $n \ge 1$. Using again (42), we get

$$\sum_{\ell=1}^{\ell_0-1} \mathbf{P}(Z_{n-a_k} = \ell)\,\mathbf{P}\big(Z_{a_k} = k \,\big|\, Z_0 = \ell\big) \;\le\; c\,\frac{k^{\alpha-1}}{m^{\alpha n}}\,. \tag{46}$$

This completes the proof. □

The following lemma contains two versions of the so-called Fuk-Nagaev inequality for tail probabilities of sums of i.i.d. variables. Recall that we assumed that X_1 is centered and has positive finite variance.

Lemma 14 (**Fuk-Nagaev inequality**). *For* $k \geq 1,\ \varepsilon_n > 0,\ n \geq 1,\ r > 1,$ *and* $t \geq 2,$

$$\mathbf{P}(S_k \geq \varepsilon_n k) \ \leq \ k\,\mathbf{P}\big(X_1 \geq r^{-1}\varepsilon_n k\big) + (\mathrm{e}\, r\sigma^2)^r\, \varepsilon_n^{-2r}\, k^{-r}, \tag{47}$$

and

$$\begin{aligned}\mathbf{P}(S_k \geq \varepsilon_n k) \ &\leq k\,\mathbf{P}\big(X_1 \geq r^{-1}\varepsilon_n k\big) + \exp\Big[-\frac{2}{(t+2)^2\,\mathrm{e}^t\,\sigma^2}\ \varepsilon_n^2 k\Big] \\ &\quad + \Big(\frac{(t+2)\, r^{t-1}\,\mathbf{E}\big\{X_1^t;\ 0 \leq X_1 \leq \varepsilon_n k\big\}}{t\,\varepsilon_n^t\, k^{t-1}}\Big)^{tr/(t+2)}.\end{aligned} \tag{48}$$

Proof. By (1.56) and (1.23) in S.V. Nagaev [Nag79], for all $u, v > 0,$

$$\mathbf{P}(S_k \geq u) \ \leq \ k\,\mathbf{P}(X_1 \geq v) + \mathrm{e}^{u/v}\Big(\frac{\sigma^2 k}{uv}\Big)^{u/v} \tag{49}$$

and

$$\begin{aligned}\mathbf{P}(S_k \geq u) \ &\leq k\,\mathbf{P}(X_1 \geq v) + \exp\Big[-\frac{2u^2}{(t+2)^2\,\mathrm{e}^t\,\sigma^2}\Big] \\ &\quad + \Big(\frac{(t+2)\,k\,\mathbf{E}\big\{X_1^t;\ 0 \leq X_1 \leq v\big\}}{t\,u\,v^{t-1}}\Big)^{tu/(t+2)v}.\end{aligned} \tag{50}$$

Putting here $u = \varepsilon_n k$ and $v = u/r,$ we get (47) and (48), finishing the proof. □

Remark 15 (**On the case** $\varepsilon_n \equiv \varepsilon$). Here we prove a one-sided version of (7) concerning our general R_n, assuming the Schröder case and that $\mathbf{E}(X_1^+)^{1+\alpha} < \infty$. Take any $\varepsilon > 0$ and set $g_n(k) := m^{\alpha n}\,\mathbf{P}(Z_n = k)\,\mathbf{P}(S_k \geq \varepsilon k)$. From estimate (40) we get, for all $n, k \geq 1,$ the inequality $g_n(k) \leq c\,\tilde{g}(k),$ where $\tilde{g}(k) := k^{\alpha-1}\,\mathbf{P}(S_k \geq \varepsilon k)$. Next we show that $\tilde{g}(k)$ is summable in k. Letting $\varepsilon_n = \varepsilon$ and $r = \alpha + 1$ in (47), we see that for all $k \geq 1,$

$$\tilde{g}(k) \ \leq \ k^{\alpha}\,\mathbf{P}\big(X_1 \geq \varepsilon k/(1+\alpha)\big) + c\,\varepsilon^{-2-2\alpha} k^{-2}. \tag{51}$$

But the summability of $k^{\alpha}\,\mathbf{P}(X_1 \geq ck)$ with some (hence all) positive c is equivalent to the finiteness of $\mathbf{E}(X_1^+)^{1+\alpha}$, and we get the claimed summability of $\tilde{g}(k)$.

On the other hand, it follows from (18) that for every fixed k,

$$\lim_{n\uparrow\infty} g_n(k) \ = \ \nu_k\,\mathbf{P}(S_k \geq \varepsilon k). \tag{52}$$

Therefore, by dominated convergence,

$$\lim_{n\uparrow\infty} \sum_{k=1}^{\infty} g_n(k) \ = \ \sum_{k=1}^{\infty} \nu_k\,\mathbf{P}(S_k \geq \varepsilon k). \tag{53}$$

Recalling the definition of $g_n(k)$ and using (12), we obtain

$$\lim_{n\uparrow\infty} m^{\alpha n}\,\mathbf{P}(R_n \geq \varepsilon_n) \ = \ \sum_{k=1}^{\infty} \nu_k\,\mathbf{P}(S_k \geq \varepsilon k), \tag{54}$$

yielding the wanted one-sided version. ◇

2.2. **Interplay between the two competing forces.** In the next four lemmas we prove bounds for different parts of the sum at the right hand side of (12).

Lemma 16 (**A tail estimate**). *Assume* X_1 *has a tail of index* $\theta > 2$. *Then*

$$\sum_{k\geq m^n} \mathbf{P}(Z_n = k)\,\mathbf{P}(S_k \geq \varepsilon_n k) \tag{55}$$
$$\leq\ c\Big(\varepsilon_n^{-\theta}\, m^{-(\theta-1)n} + (\varepsilon_n^2 m^n)^{-1}\exp\big[-c\,\varepsilon_n^2\, m^n\big]\Big), \qquad \varepsilon_n > 0,\ \ n\geq 1.$$

Proof. Letting $t = \theta + 1$ and $r = (t+2)/t$ in (48), and using that X_1 has a tail of index $\theta > 2$, we get the bound

$$\mathbf{P}(S_k \geq \varepsilon_n k)\ \leq\ c\left(\varepsilon_n^{-\theta}\, k^{-(\theta-1)} + \frac{\mathbf{E}\big\{X_1^{\theta+1};\, X_1 \in [0,\varepsilon_n k]\big\}}{\varepsilon_n^{\theta+1}\, k^{\theta}}\right) + \exp[-c\,\varepsilon_n^2 k]. \tag{56}$$

Clearly, under (22),

$$\mathbf{E}\big\{X_1^{\theta+1};\ X_1 \in [0,x]\big\}\ \sim\ a\,\theta x \quad \text{as } x\uparrow\infty. \tag{57}$$

Thus,

$$\mathbf{E}\big\{X_1^{\theta+1};\ X_1 \in [0,x]\big\}\ \leq\ c\,x, \qquad x \geq 1. \tag{58}$$

On the other hand, if $x \leq 1$,

$$\mathbf{E}\big\{X_1^{\theta+1};\ X_1 \in [0,x]\big\}\ \leq\ x^{\theta+1}\,\mathbf{P}(X_1 \in [0,x])\ \leq\ x. \tag{59}$$

Therefore,

$$\mathbf{E}\big\{X_1^{\theta+1};\ X_1 \in [0,x]\big\}\ \leq\ c\,x, \qquad x \geq 0. \tag{60}$$

Applying this to the expectation in (56), we get

$$\mathbf{P}(S_k \geq \varepsilon_n k)\ \leq\ c\,\varepsilon_n^{-\theta}\, k^{-(\theta-1)} + \exp[-c\,\varepsilon_n^2\, k]. \tag{61}$$

Moreover, combining this bound with (39) gives

$$\sum_{k\geq m^n} \mathbf{P}(Z_n = k)\,\mathbf{P}(S_k \geq \varepsilon_n k)\ \leq\ c\,\varepsilon_n^{-\theta}\sum_{k\geq m^n} k^{-\theta} + \sum_{k\geq m^n} k^{-1}\exp[-c\,\varepsilon_n^2 k]. \tag{62}$$

Obviously,

$$\sum_{k\geq m^n} k^{-\theta}\ \leq\ c\,m^{-(\theta-1)n}. \tag{63}$$

On the other hand,

$$\sum_{k\geq m^n} k^{-1}\exp[-c\,\varepsilon_n^2 k]\ \leq\ m^{-n}\sum_{k\geq m^n}\exp[-c\,\varepsilon_n^2 k]$$
$$\leq\ c\,(\varepsilon_n^2 m^n)^{-1}\exp\big[-c\,\varepsilon_n^2\, m^n\big]. \tag{64}$$

Substituting (63) and (64) into (62) finishes the proof. □

Lemma 17 (**Another tail estimate**). *Suppose the Schröder case, let* X_1 *satisfy the moment condition* (13), *and let* $\varepsilon_n \to 0$. *Then*

$$\limsup_{n\uparrow\infty}\, \varepsilon_n^{2\alpha}\, m^{\alpha n} \sum_{k\geq A/\varepsilon_n^2} \mathbf{P}(Z_n = k)\,\mathbf{P}(S_k \geq \varepsilon_n k)\ \leq\ \frac{c}{A}\,, \qquad A \geq 1. \tag{65}$$

Proof. Combining (40) and (47) with $r=\alpha+1$ gives

$$
\begin{aligned}
&m^{\alpha n}\sum_{k\,\geq A/\varepsilon_n^2}\mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\geq\varepsilon_n k)\\
&\leq\; c\Big(\sum_{k\,\geq A/\varepsilon_n^2}k^{\alpha}\,\mathbf{P}\big(X_1\geq(\alpha+1)^{-1}\varepsilon_n k\big)\;+\;\varepsilon_n^{-2(\alpha+1)}\sum_{k\,\geq A/\varepsilon_n^2}k^{-2}\Big). \qquad (66)
\end{aligned}
$$

Note that

$$
\varepsilon_n^{-2(\alpha+1)}\Big(\sum_{k\,\geq A/\varepsilon_n^2}k^{-2}\Big)\;\leq\;\frac{c}{A}\,\varepsilon_n^{-2\alpha},\qquad n>0,\;\;\varepsilon_n>0,\;\;A\geq 1. \qquad (67)
$$

On the other hand, to bound the first sum at the right hand side in (66), note first that

$$
\int_{k-1}^{k}u^{\alpha}\,\mathbf{P}(X_1\geq(\alpha+1)^{-1}\varepsilon_n u)\,\mathrm{d}u\;\geq\;(k-1)^{\alpha}\,\mathbf{P}\big(X_1\geq(\alpha+1)^{-1}\varepsilon_n k\big),\qquad k\geq 1.
$$

This inequality can be continued by using $k-1\geq k/2$ for $k\geq 2$. Summing up gives for $\varepsilon_n^2\leq 1/2$,

$$
\begin{aligned}
\sum_{k\,\geq A/\varepsilon_n^2}k^{\alpha}\,\mathbf{P}\big(X_1\geq(\alpha+1)^{-1}\varepsilon_n k\big)&\leq\; c\int_{A/\varepsilon_n^2-1}^{\infty}u^{\alpha}\,\mathbf{P}\big(X_1\geq(\alpha+1)^{-1}\varepsilon_n u\big)\,\mathrm{d}u\\
&\leq\; c\,\varepsilon_n^{-\alpha-1}\int_{(A-\varepsilon_n^2)/(\alpha+1)\varepsilon_n}^{\infty}v^{\alpha}\,\mathbf{P}(X_1\geq v)\,\mathrm{d}v. \qquad (68)
\end{aligned}
$$

Recall that we assumed the moment condition (13) and that $\varepsilon_n\to 0$. Then the integral in (68) converges to zero as $n\uparrow\infty$, uniformly in $A\geq 1$. In particular, under $\alpha\geq 1$, (68) is of order $o(\varepsilon_n^{-2\alpha})$, uniformly in $A\geq 1$. On the other hand, if $\alpha<1$ and since $\mathbf{E}X_1^2<\infty$,

$$
\begin{aligned}
&\int_{(A-\varepsilon_n^2)/(\alpha+1)\varepsilon_n}^{\infty}v^{\alpha}\,\mathbf{P}(X_1\geq v)\,\mathrm{d}v \qquad (69)\\
&\leq\; c\,\frac{\varepsilon_n^{1-\alpha}}{(A-\varepsilon_n^2)^{1-\alpha}}\int_{(A-\varepsilon_n^2)/(\alpha+1)\varepsilon_n}^{\infty}v\,\mathbf{P}(X_1\geq v)\,\mathrm{d}v\;=\;o(\varepsilon_n^{1-\alpha})\;=\;o(\varepsilon_n^{-2\alpha})
\end{aligned}
$$

as $n\uparrow\infty$, uniformly in $A\geq 1$. Thus, for each $\alpha<\infty$ we have

$$
\sup_{A\geq 1}\sum_{k\,\geq A/\varepsilon_n^2}k^{\alpha}\,\mathbf{P}\big(X_1\geq(\alpha+1)^{-1}\varepsilon_n k\big)\;=\;o(\varepsilon_n^{-2\alpha})\quad\text{as } n\uparrow\infty. \qquad (70)
$$

In particular,

$$
\limsup_{n\uparrow\infty}\,\varepsilon_n^{2\alpha}\sum_{k\,\geq A/\varepsilon_n^2}k^{\alpha}\,\mathbf{P}\big(X_1\geq(\alpha+1)^{-1}\varepsilon_n k\big)\;\leq\;\frac{c}{A}\,,\qquad A\geq 1. \qquad (71)
$$

Combining (66), (67), and (71) gives the claim in the lemma. □

Lemma 18 (Initial part). *In the Schröder case,*

$$
\sum_{1\,\leq k\,\leq\,\delta/\varepsilon_n^2}\mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\geq\varepsilon_n k)\;\leq\;c\,\delta^{\alpha}\varepsilon_n^{-2\alpha}\,m^{-\alpha n}, \qquad (72)
$$

$\delta>0,\;\;\varepsilon_n>0,\;\;n\geq 1.$

Proof. It follows from (40) that

$$\begin{aligned}\sum_{1\,\le\, k\,\le\,\delta/\varepsilon_n^2} \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k) \ &\le \sum_{1\,\le\, k\,\le\,\delta/\varepsilon_n^2} \mathbf{P}(Z_n=k)\\ &\le\ \frac{c}{m^{\alpha n}}\sum_{1\,\le\, k\,\le\,\delta/\varepsilon_n^2} k^{\alpha-1}\ \le\ c\,\delta^{\alpha}\varepsilon_n^{-2\alpha}\,m^{-\alpha n},\end{aligned}\tag{73}$$

finishing the proof. □

Lemma 19 (**A central part and another initial part estimate**). *Suppose* $1<\alpha<\infty$ *and that* X_1 *has a tail of index* $\theta\in(2,1+\alpha)$. *Then*

$$\sum_{A/\varepsilon_n^2\,\le\, k\,\le\,\delta m^n} \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k)\tag{74}$$

$$\le\ c\left(\delta^{1+\alpha-\theta}\varepsilon_n^{-\theta}m^{-(\theta-1)n}+A^{-1}\,\varepsilon_n^{-2\alpha}\,m^{-\alpha n}\right),$$

$A\ge 1,\ \ \delta>0,\ \ \varepsilon_n>0,\ \ n\ge 1,$ *and*

$$\sum_{1\,\le\, k\,\le\,\delta m^n} \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k)\tag{75}$$

$$\le\ c\left(\delta^{1+\alpha-\theta}\varepsilon_n^{-\theta}m^{-(\theta-1)n}+\varepsilon_n^{-2\alpha}\,m^{-\alpha n}\right),\qquad \delta>0,\ \ \varepsilon_n>0,\ \ n\ge 1.$$

Proof. Combining (40) and (47) with $r=\alpha+1$ gives

$$\sum_{A/\varepsilon_n^2\,\le\, k\,\le\,\delta m^n} \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k)\tag{76}$$

$$\le\ c\,m^{-\alpha n}\Bigg(\sum_{A/\varepsilon_n^2\,\le\, k\,\le\,\delta m^n} k^{\alpha}\,\mathbf{P}\big(X_1\ge(\alpha+1)^{-1}\varepsilon_n k\big)+\varepsilon_n^{-2(\alpha+1)}\sum_{A/\varepsilon_n^2\,\le\, k\,\le\,\delta m^n} k^{-2}\Bigg).$$

From (67),

$$\varepsilon_n^{-2(\alpha+1)}\sum_{A/\varepsilon_n^2\,\le\, k\,\le\,\delta m^n} k^{-2}\ \le\ \frac{c}{A}\,\varepsilon_n^{-2\alpha}.\tag{77}$$

On the other hand, since X_1 has a tail of index $\theta\in(2,1+\alpha)$,

$$\begin{aligned}\sum_{A/\varepsilon_n^2\,\le\, k\,\le\,\delta m^n} k^{\alpha}\,\mathbf{P}\big(X_1\ge(\alpha+1)^{-1}\varepsilon_n k\big)\ &\le\ c\,\varepsilon_n^{-\theta}\sum_{1\,\le\, k\,\le\,\delta m^n} k^{\alpha-\theta}\\ &\le\ c\,\varepsilon_n^{-\theta}\delta^{1+\alpha-\theta}m^{(1+\alpha-\theta)n}.\end{aligned}\tag{78}$$

Combine (76)–(78) to get (74).

Putting $A=1$ in (74) and $\delta=1$ in (72), we obtain (75), finishing the proof. □

Recall that (μ,d) refers to the type of the offspring law, $\alpha\in(0,\infty)$ to the Schröder constant, and that X_1 is assumed to have a finite variance. For $0<\delta<1<A<\infty$, consider

$$\Sigma_n(\delta,A)\ :=\ \sum_{\delta/\varepsilon_n^2\,\le\, k\,\le\,A/\varepsilon_n^2} \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k).\tag{79}$$

Lemma 20 (**Another central part estimate**). *Suppose the Schröder case, that* $\varepsilon_n \to 0$, *and that* $\varepsilon_n^2 m^n \to \infty$. *Then for all* $0 < \delta < 1 < A < \infty$,

$$\begin{aligned} V_* \int_\delta^A u^{\alpha-1}\, \overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u \;&\le\; \liminf_{n\uparrow\infty} \varepsilon_n^{2\alpha}\, m^{\alpha n}\, \Sigma_n(\delta, A) \\ \le\; \limsup_{n\uparrow\infty} \varepsilon_n^{2\alpha}\, m^{\alpha n}\, \Sigma_n(\delta, A) \;&\le\; V^* \int_\delta^A u^{\alpha-1}\, \overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u \end{aligned} \tag{80}$$

with V_* *and* V^* *defined in* (15), *and where* $\overline{\Phi}(x) := 1 - \Phi(x)$.

Proof. In view of (9) in Proposition 3 with $k_n = \delta/\varepsilon_n^2$,

$$\Sigma_n(\delta, A) = (1 + o(1))d \sum_{k\in H(\delta,A)} m^{-n}\, w\big(\tfrac{k}{m^n}\big)\, \mathbf{P}(S_k \ge \varepsilon_n k) \quad \text{as } n\uparrow\infty \tag{81}$$

with $H(\delta, A) := \big\{k \in [\delta/\varepsilon_n^2,\, A/\varepsilon_n^2] :\, k \equiv \mu \pmod{d}\big\}$. Clearly,

$$\begin{aligned} V_*(n) \sum_{k\in H(\delta,A)} \frac{k^{\alpha-1}}{m^{\alpha n}}\, \mathbf{P}(S_k \ge \varepsilon_n k) \;&\le\; \sum_{k\in H(\delta,A)} m^{-n}\, w\big(\tfrac{k}{m^n}\big)\, \mathbf{P}(S_k \ge \varepsilon_n k) \\ &\le\; V^*(n) \sum_{k\in H(\delta,A)} \frac{k^{\alpha-1}}{m^{\alpha n}}\, \mathbf{P}(S_k \ge \varepsilon_n k), \end{aligned} \tag{82}$$

where we set

$$V_*(n) \;:=\; \inf_{u\le A/\varepsilon_n^2 m^n} u^{1-\alpha} w(u), \quad V^*(n) \;:=\; \sup_{u\le A/\varepsilon_n^2 m^n} u^{1-\alpha} w(u). \tag{83}$$

By the central limit theorem,

$$\sup_{k\in H(\delta,A)} \Big|\mathbf{P}(S_k \ge \varepsilon_n k) - \overline{\Phi}\big(\sqrt{\varepsilon_n^2 k}/\sigma\big)\Big| \to 0 \quad \text{as } n\uparrow\infty. \tag{84}$$

Hence, as $n\uparrow\infty$,

$$\begin{aligned} \sum_{k\in H(\delta,A)} k^{\alpha-1}\, \mathbf{P}(S_k \ge \varepsilon_n k) \;&=\; \big(1 + o(1)\big) \sum_{k\in H(\delta,A)} k^{\alpha-1}\, \overline{\Phi}\big(\sqrt{\varepsilon_n^2 k}/\sigma\big) \\ &=\; \varepsilon_n^{-2\alpha}\big(1 + o(1)\big) \sum_{k\in H(\delta,A)} (\varepsilon_n^2 k)^{\alpha-1}\, \overline{\Phi}\big(\sqrt{\varepsilon_n^2 k}/\sigma\big)\, \varepsilon_n^2 \\ &=\; d^{-1}\varepsilon_n^{-2\alpha}\big(1 + o(1)\big) \int_\delta^A u^{\alpha-1}\, \overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u. \end{aligned} \tag{85}$$

Substituting (85) into (82) and noting that we have $V_*(n) \to V_*$ and $V^*(n) \to V^*$ as $n\uparrow\infty$ by our velocity assumption on ε_n, we obtain (80). □

Finally, we compute the limit, as $\delta \downarrow 0$ and $A\uparrow\infty$, of the integral from (80).

Lemma 21 (**A moment formula for the Gaussian law**). *For* $0 < \alpha < \infty$,

$$\int_0^\infty u^{\alpha-1}\, \overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u \;=\; \frac{2^{\alpha-1}\,\Gamma(\alpha + 1/2)}{\alpha\sqrt{\pi}}\, \sigma^{2\alpha} \;=\; \Gamma_\alpha\,. \tag{86}$$

Proof. Substituting $v = \sqrt{u}/\sigma$, we have

$$\int_0^\infty u^{\alpha-1}\,\overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u \;=\; 2\sigma^{2\alpha}\int_0^\infty v^{2\alpha-1}\,\overline{\Phi}(v)\,\mathrm{d}v$$

$$=\; 2\sigma^{2\alpha}\int_0^\infty \mathrm{d}v\; v^{2\alpha-1}\int_v^\infty \mathrm{d}t\;\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-t^2/2} \;=\; 2\,\frac{\sigma^{2\alpha}}{\sqrt{2\pi}}\int_0^\infty \mathrm{d}t\;\mathrm{e}^{-t^2/2}\int_0^t \mathrm{d}v\; v^{2\alpha-1}$$

$$=\; \frac{\sigma^{2\alpha}}{\alpha\sqrt{2\pi}}\int_0^\infty t^{2\alpha}\,\mathrm{e}^{-t^2/2}\,\mathrm{d}t. \tag{87}$$

Substituting now $v = t^2/2$, the chain of equalities can be continued with

$$=\; \frac{2^{\alpha-1}\sigma^{2\alpha}}{\alpha\sqrt{\pi}}\int_0^\infty v^{\alpha-1/2}\,\mathrm{e}^{-v}\,\mathrm{d}v \;=\; \frac{2^{\alpha-1}\,\Gamma(\alpha+1/2)}{\alpha\sqrt{\pi}}\;\sigma^{2\alpha}, \tag{88}$$

which equals Γ_α from (16). The proof is finished. □

3. Proof of the main results

3.1. **Proof of Theorem 5.** We will use decomposition (12). Combining Lemmas 18, 20, and 17, and using that δ and A are arbitrary, we see that

$$V_*\int_0^\infty u^{\alpha-1}\,\overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u \;\le\; \liminf_{n\uparrow\infty}\,\varepsilon_n^{2\alpha}\,m^{\alpha n}\sum_{k=1}^\infty \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k) \tag{89}$$

$$\le\; \limsup_{n\uparrow\infty}\,\varepsilon_n^{2\alpha}\,m^{\alpha n}\sum_{k=1}^\infty \mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k) \;\le\; V^*\int_0^\infty u^{\alpha-1}\,\overline{\Phi}(\sqrt{u}/\sigma)\,\mathrm{d}u.$$

With Lemma 21 the proof is finished. □

3.2. **Proof of Theorem 7.** Let $\varepsilon_n = o(m^{-\varkappa n})$. Then under the assumptions in the theorem,

$$\varepsilon_n^{-\theta}m^{-(\theta-1)n} + (\varepsilon_n^2 m^n)^{-1} \;=\; o(\varepsilon_n^{-2\alpha}\,m^{-\alpha n}). \tag{90}$$

From this relation, Lemma 16, and (74) with $\delta = 1$, we get

$$\limsup_{n\uparrow\infty}\,\varepsilon_n^{2\alpha}\,m^{\alpha n}\sum_{k\,\ge A/\varepsilon_n^2}\mathbf{P}(Z_n=k)\,\mathbf{P}(S_k\ge\varepsilon_n k) \;\le\; \frac{c}{A}\,, \qquad A\ge 1. \tag{91}$$

Theorem 7(a) follows from this bound, (72), and (80).

We turn now to the proof of parts (b) and (c). It is known (see for example Borovkov [Bor00]), that if $\mathbf{P}(X_1 \ge x)$ is regularly varying as $x\uparrow\infty$ with index $\theta > 2$, then for every sequence $a_k \to \infty$,

$$\lim_{k\uparrow\infty}\;\sup_{x:\;x\,\ge\,a_k(k\log k)^{1/2}}\left|\frac{\mathbf{P}(S_k\ge x)}{k\,\mathbf{P}(X_1\ge x)} - 1\right| \;=\; 0. \tag{92}$$

Note that if $\delta > 0$, $k \ge \delta m^n$, and $\varepsilon_n \ge \delta m^{-\varkappa n}$, then $\varepsilon_n \ge \delta^{1+\varkappa}k^{-\varkappa}$. Hence,

$$\frac{\varepsilon_n k}{(k\log k)^{1/2}} \;\ge\; \delta^{1+\varkappa}\,\frac{k^{1/2-\varkappa}}{(\log k)^{1/2}}\,. \tag{93}$$

Since $0 < \varkappa < 1/2$, the right hand side goes to infinity as $k \uparrow \infty$, and we will use it as a_k. Thus, applying (92) gives, as $n \uparrow \infty$,

$$\sum_{k>\delta m^n} \mathbf{P}(Z_n = k)\, \mathbf{P}(S_k \ge \varepsilon_n k) \;=\; \big(1+o(1)\big) \sum_{k>\delta m^n} k\, \mathbf{P}(Z_n = k)\, \mathbf{P}(X_1 \ge \varepsilon_n k)$$
$$= \big(1+o(1)\big)\, a\, \varepsilon_n^{-\theta} \sum_{k>\delta m^n} k^{-(\theta-1)}\, \mathbf{P}(Z_n = k), \tag{94}$$

where in the second step we used that X_1 has a tail of index $\theta \in (2, 1+\alpha)$. By (40) we have

$$\sum_{1\le k\le \delta m^n} k^{-(\theta-1)}\, \mathbf{P}(Z_n = k) \;\le\; c\, m^{-\alpha n} \sum_{1\le k\le \delta m^n} k^{\alpha-\theta} \;\le\; c\, m^{-(\theta-1)n}\, \delta^{1+\alpha-\theta}.$$

By Theorem 1 of [NV03], for $\theta - 1 < \alpha$, we have $\mathbf{E}\big\{Z_n^{-(\theta-1)};\, Z_n > 0\big\} \sim I_\theta\, m^{-(\theta-1)n}$ as $n \uparrow \infty$, with I_θ defined in (24). Hence, for all sufficiently large n,

$$\Big| \sum_{k>\delta m^n} k^{-(\theta-1)}\, \mathbf{P}(Z_n = k) - I_\theta\, m^{-(\theta-1)n} \Big| \;\le\; c\, m^{-(\theta-1)n}\, \delta^{1+\alpha-\theta}. \tag{95}$$

Combining (94) and (95), we have the bound

$$\limsup_{n\uparrow\infty} \Big| \varepsilon_n^{\theta}\, m^{(\theta-1)n} \sum_{k>\delta m^n} \mathbf{P}(Z_n = k)\, \mathbf{P}(S_k \ge \varepsilon_n k) - a\, I_\theta \Big| \;\le\; c\, \delta^{1+\alpha-\theta}. \tag{96}$$

If $\varepsilon_n m^{\varkappa n} \to \infty$, then, obviously, $\varepsilon_n^{-2\alpha}\, m^{-\alpha n} = o(\varepsilon_n^{-\theta}\, m^{-(\theta-1)n})$. Therefore, by estimate (75),

$$\limsup_{n\uparrow\infty} \varepsilon_n^{\theta}\, m^{(\theta-1)n} \sum_{1\le k\le \delta m^n} \mathbf{P}(Z_n = k)\, \mathbf{P}(S_k \ge \varepsilon_n k) \;\le\; c\, \delta^{1+\alpha-\theta}. \tag{97}$$

Part (b) follows from (96) and (97) by letting $\delta \downarrow 0$.

Finally, under $\varepsilon_n \sim \tau^{-1} m^{-\varkappa n}$, part (c) follows from (72), (80), (86), (74), and (96). The proof is finished altogether. □

3.3. **Proof of Theorem 9.** It follows from the assumed finiteness of an exponential moment of X_1, see e.g. Lemma III.5 in Petrov [Pet75], that for every $\delta \in (0,1)$ there exists $h_\delta > 0$ such that

$$\mathbf{E} e^{hX_1} \;\le\; e^{\sigma^2(1+\delta)h^2/2}, \qquad |h| \le h_\delta\,. \tag{98}$$

Thus, we may use the well-known Bernstein inequality, see Theorem III.15 in [Pet75]. This gives, for all $k \ge 1$ and $\varepsilon_n \le h_\delta$,

$$\mathbf{P}(S_k \ge \varepsilon_n k) \;\le\; \exp\Big[-(1-\delta)\,\frac{\varepsilon_n^2 k}{2\sigma^2}\Big]. \tag{99}$$

Therefore,

$$\mathbf{P}(R_n \ge \varepsilon_n) \;\le\; f_n\Big(\exp\Big[-(1-\delta)\,\frac{\varepsilon_n^2}{2\sigma^2}\Big]\Big) \quad \text{if } \varepsilon_n \le h_\delta\,. \tag{100}$$

We may also assume that $\varepsilon_n \le 1/m$. Set $r_n := \max\{k \ge 1 : m^k \le \varepsilon_n^{-2}\}$. Then,

$$m^{-r_n-1} \;<\; \varepsilon_n^2 \;\le\; m^{-r_n}. \tag{101}$$

The left hand inequality together with the monotonicity of f_n gives

$$f_n\Big(\exp\Big[-(1-\delta)\,\frac{\varepsilon_n^2}{2\sigma^2}\Big]\Big) \;\le\; f_n\Big(\exp\Big[-(1-\delta)\,\frac{m^{-r_n-1}}{2\sigma^2}\Big]\Big). \tag{102}$$

Bounds (100), (102), and the right hand inequality in (101) imply

$$\varepsilon_n^{-2\beta} m^{-n\beta} \log \mathbf{P}(R_n \geq \varepsilon_n) \;\leq\; \mu^{-n+r_n} \log f_n\Big(\exp\Big[-(1-\delta)\,\frac{m^{-r_n-1}}{2\sigma^2}\Big]\Big), \quad (103)$$

where we used $\mu = m^\beta$. Since $r_n \to \infty$, by the Kesten-Stigum theorem for supercritical Galton-Watson processes,

$$\lim_{n\uparrow\infty} f_{r_n+1}\Big(\exp\Big[-(1-\delta)\,\frac{m^{-r_n-1}}{2\sigma^2}\Big]\Big) \;=\; \varphi\big((1-\delta)/2\sigma^2\big). \quad (104)$$

On the other hand, from the assumption $\varepsilon_n^2 m^n \to \infty$ and the right hand inequality in (101) it follows that $n - r_n \to \infty$. Therefore, by (28) we have for $s \in [0,1]$,

$$\lim_{n\uparrow\infty} \mu^{-n+r_n+1} \log f_{n-r_n-1}(s) \;=\; \log \mathsf{B}(s). \quad (105)$$

By the continuity of B, combining (104) and (105) we obtain

$$\lim_{n\uparrow\infty} \mu^{-n+r_n+1} \log f_n\Big(\exp\Big[-(1-\delta)\,\frac{m^{-r_n-1}}{2\sigma^2}\Big]\Big) \;=\; \log \mathsf{B}\big(\varphi\big((1-\delta)/2\sigma^2\big)\big). \quad (106)$$

Now (30b) follows from (103) and (106) letting $\delta \downarrow 0$.

In order to prove (30a) we will exploit the following version of Kolmogorov's inequality: For $0 < \delta < 1$ fixed, there exists a constant $D \in (0,\infty)$ such that

$$\mathbf{P}(S_k \geq \varepsilon_n k) \;\geq\; \exp\Big[-(1+\delta)\,\frac{\varepsilon_n^2 k}{2\sigma^2}\Big], \qquad k > D/\varepsilon_n^2, \quad n \geq 1. \quad (107)$$

See Statulevicius [Sta66]. Using (107) we obtain

$$\begin{aligned} \mathbf{P}(R_n \geq \varepsilon_n) \;&\geq\; \sum_{k > D/\varepsilon_n^2} \mathbf{P}(Z_n = k)\, \exp\Big[-(1+\delta)\,\frac{\varepsilon_n^2 k}{2\sigma^2}\Big] \\ &\geq\; f_n\Big(\exp\Big[-(1+\delta)\,\frac{\varepsilon_n^2}{2\sigma^2}\Big]\Big) - \mathbf{P}(Z_n \leq D/\varepsilon_n^2). \end{aligned} \quad (108)$$

Clearly, if $D/\varepsilon_n^2 < \mu^n$, then $\mathbf{P}(Z_n \leq D/\varepsilon_n^2) = 0$, and we pass directly to statement (112) below. Otherwise, it follows from Proposition 4 that

$$\mathbf{P}(Z_n \leq D/\varepsilon_n^2) \;\leq\; \exp\Big[-c\, D^{-\beta/(1-\beta)}\, (\varepsilon_n^2 m^n)^{\beta/(1-\beta)}\Big]. \quad (109)$$

From (108), (109), and the left hand inequality in (101), we have

$$\mathbf{P}(R_n \geq \varepsilon_n) \;\geq\; f_n\Big(\exp\Big[-(1+\delta)\,\frac{m^{-r_n}}{2\sigma^2}\Big]\Big) - \exp\Big[-c\,(\varepsilon_n^2 m^n)^{\beta/(1-\beta)}\Big]. \quad (110)$$

Analogously to (106),

$$\lim_{n\uparrow\infty} \mu^{-n+r_n} \log f_n\Big(\exp\Big[-(1+\delta)\,\frac{m^{-r_n}}{2\sigma^2}\Big]\Big) \;=\; \log \mathsf{B}\big(\varphi\big((1+\delta)/2\sigma^2\big)\big). \quad (111)$$

By the left hand inequality of (101), $\mu^{n-r_n} \leq m^\beta (\varepsilon_n^2 m^n)^\beta$. Therefore, from the limit statement (111) we see that the second term at the right hand side of estimate (110) is negligible compared with the first term there, i.e.

$$\mathbf{P}(R_n \geq \varepsilon_n) \;\geq\; f_n\Big(\exp\Big[-(1+\delta)\,\frac{m^{-r_n}}{2\sigma^2}\Big]\Big)(1+o(1)). \quad (112)$$

Thus, using the left hand inequality in (101), we get the bound

$$\varepsilon_n^{-2\beta} m^{-n\beta} \log \mathbf{P}(R_n \geq \varepsilon_n) \;\geq\; \mu^{-n+r_n+1} \log f_n\Big(\exp\Big[-(1+\delta)\,\frac{m^{-r_n}}{2\sigma^2}\Big]\Big) + o(1). \quad (113)$$

Since δ is arbitrary, combining (113) and (111) completes the proof of (30a).

In the derivation of (112) from (108) we learned that the second term at the right hand side of (108) is small compared with the first term there. Thus, from (108) together with (100) we get

$$f_n\Big(\exp\Big[-(1+\delta)\,\frac{\varepsilon_n^2}{2\sigma^2}\Big]\Big)\,\big(1+o(1)\big) \;\le\; \mathbf{P}(R_n \ge \varepsilon_n) \qquad (114)$$
$$\le\; f_n\Big(\exp\Big[-(1-\delta)\,\frac{\varepsilon_n^2}{2\sigma^2}\Big]\Big).$$

Hence, if $\varepsilon_n^2 = m^{-\lambda_n}$ then (31) follows from these inequalities and (111) replacing there r_n by λ_n, and finally letting $\delta \downarrow 0$. Altogether, the proof of Theorem 9 is complete. □

3.4. **Proof of Theorem 10.** With B_2 from Proposition 4, and $\theta > 2$ the tail index of X_1, define $k_n := m^n/\log^{(1-\beta)/\beta} m^{2n\theta/B_2}$. Then by Proposition 4, for all sufficiently large n,

$$\mathbf{P}(Z_n \le k_n) \;\le\; \exp\big[-(B_2/2)(k_n/m^n)^{-\beta/(1-\beta)}\big] \;=\; m^{-\theta n}. \qquad (115)$$

Hence, for these n,

$$\sum_{k\le k_n} \mathbf{P}(Z_n = k)\,\mathbf{P}(S_k \ge \varepsilon_n k) \;\le\; \mathbf{P}(Z_n \le k_n) \;\le\; m^{-\theta n} \qquad (116)$$

and

$$\sum_{k\le k_n} k^{-(\theta-1)}\,\mathbf{P}(Z_n = k) \;\le\; \mathbf{P}(Z_n \le k_n) \;\le\; m^{-\theta n}. \qquad (117)$$

It is easy to verify that

$$\frac{\varepsilon_n k_n}{(k_n \log k_n)^{1/2}} \;=\; \big(c+o(1)\big)\,\varepsilon_n m^{n/2} n^{-1/2\beta} \quad \text{as } n\uparrow\infty. \qquad (118)$$

By our assumption in the theorem, the right hand side converges to infinity. Then, we can use (92) with $a_k := \varepsilon_n (k/\log k)^{1/2}$ to obtain

$$\sum_{k>k_n} \mathbf{P}(Z_n = k)\,\mathbf{P}(S_k \ge \varepsilon_n k) = \big(1+o(1)\big) \sum_{k>k_n} k\,\mathbf{P}(Z_n = k)\,\mathbf{P}(X_1 \ge \varepsilon_n k)$$
$$= \big(1+o(1)\big)\,a\,\varepsilon_n^{-\theta} \sum_{k>k_n} k^{-(\theta-1)}\,\mathbf{P}(Z_n = k) \quad \text{as } n\uparrow\infty. \qquad (119)$$

Theorem 1 of [NV03] and (117) yield

$$\sum_{k>k_n} k^{-(\theta-1)}\,\mathbf{P}(Z_n = k) \;=\; I_\theta\, m^{-(\theta-1)n}\,\big(1+o(1)\big) \quad \text{as } n\uparrow\infty. \qquad (120)$$

Substituting this into (119) and combining with (116) completes the proof. □

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