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From synchronization to Lyapunov exponents and back

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Abstract

The goal of this paper is twofold. In the first part we discuss a general approach to determine Lyapunov exponents from ensemble- rather than time-averages. The approach passes through the identification of locally stable and unstable manifolds (the Lyapunov vectors), thereby revealing an analogy with generalized synchronization. The method is then applied to a periodically forced chaotic oscillator to show that the modulus of the Lyapunov exponent associated to the phase dynamics increases quadratically with the coupling strength and it is therefore different from zero already below the onset of phase-synchronization. The analytical calculations are carried out for a model, the generalized special flow, that we construct as a simplified version of the periodically forced Rössler oscillator.

1 Introduction

As soon as synchronization phenomena in chaotic systems have been discovered [1, 2], the standard tools of nonlinear dynamics have been implemented in order to clarify this phenomenon. This is particularly true for the Lyapunov exponents (LEs), [3] because they measure the degree of stability along different directions and are thus the natural candidates to quantify the degree of synchronization of different regimes. However, several subtleties have been immediately discovered. For instance, the negativity of the "transversal" LE is only a necessary condition for the stability of complete synchronization: (i) in low-dimensional systems, fluctuations of the finite-time LEs may render the synchronized regime unstable even when the "averageëxponent is negative[4]; (ii) in high dimensional systems, it has been ascertained that the propagation of finite-amplitude perturbations can sustain an unsynchronized regime, in spite of its linear stability[5, 6]. A still open problem concerns the behaviour of the LEs in the context of phase synchronization [7, 9] and more precisely of the exponent quantifying the stability of the phase dynamics. In fact, it is often claimed that this LE is the right order-parameter to characterize the onset of phase-synchronization: below the transition it is conjectured to be zero, while it is strictly negative above the transition [7, 10]. However, the situation is certainly less simple than initially believed because a negative exponent has been found also in correspondence of locking phenomena occurring below the onset of phase-synchronization[11]. It is therefore important to clarify analytically the stability of the dynamics along the "phase" direction: in the absence of coupling, this is a marginally stable direction and it is thus natural to expect some difficulties. Here, we develop a method that allows concluding that the LE

corresponding to the phase dynamics is different from zero (and possibly positive) as soon as the coupling is switched on and therefore even below the onset of phase-synchronization.

One of the main problems is the lack of analytical methods for determining even perturbatively the LEs. Some ideas have been put forward for the maximum exponent [12, 13], because almost any initial condition eventually grows with the maximum rate and no special care is required to tune the direction of the perturbation. However, very little is known for the other exponents, starting already from the second one. This is precisely what is needed to determine the stability of phase dynamics in the simplest system exhibiting phase synchronization, i.e. in a periodically forced chaotic attractor, where the first LE accounts for the overall instability of the chaotic dynamics. Here, we attack and solve the problem by developing a formalism to determine LEs as ensemble- rather than time-averages. Similar ideas have been already discussed by Ershov and Potapov [14], although they have not gone much beyond the level of formal statements. In fact, their method relies on the determination of the growth rates of hypervolumes of increasing dimension. While this idea proved very effective for the development of a powerful algorithm to compute the LEs[15], its ensemble-average extension has some limitations due to the difficulty of disentangling the various exponents. The advantage of our approach is that we are able to associate each non-degenerate LE to a field of local directions, the *Lyapunov vectors* (LVs). Roughly speaking, the ith LV is determined into two steps: the first one consists in iterating forward in time a hypervolume of dimension i in tangent space to identify the local orientation of the most expanding i directions (this is also considered in [14]); the second step consists in iterating backward a vector lying within such a hypervolume. As a result, a coordinate-independent LV can be determined: the LE is finally obtained by averaging the corresponding instantaneous expansion/contraction rate over the entire phase-space, according to the invariant measure.

An objective identification of LVs is particularly interesting in the study of the hydrodynamic behaviour of extended systems. In the last years, mostly as a consequence of the pioneering work of Posch and collaborators [16, 17], it has been discovered that in models of fluids (more in general in Hamiltonian systems) the directions corresponding to the smallest (in absolute value) LEs almost coincide with long-wavelength Fourier modes. This observation has in turn suggested that the Lyapunov analysis naturally leads to a hydrodynamic description without the need of introducing a suitable coarse graining. However, progress has been hindered by the lack of an absolute definition of the Lyapunov "modes", that have been mostly identified with the vectors arising from the implementation of the Gram-Schmidt orthogonalization procedure during a standard computation of the LEs. The only examples of a philososphy similar to that one outlined in the present paper concern the chronotopic Lyapunov approach [18] and the characterization of space-time chaos [19].

It is also interesting to notice that the problem of identifying the LVs is itself equivalent to a problem of (generalized) synchronization. In fact, the Lyapunov vectors are determined by integrating a skew-product system composed of the original nonlinear dynamics plus the

“forcedëvolution in tangent space. As a result, the direction of the LVs varies in a possibly singular way with the position in real space. However, this difficulty does not hinder the LE determination, which results from an average that is substantially insensitive to the presence of local singularities.

An analytic investigation of the stability of phase dynamics in a generic setup is an extremely difficult task because of the lack of structural stability of low-dimensional chaos. For this reason, it is convenient to consider suitable simplified models. The simplest system where phase-synchronization has been investigated is the so-called special flow[20]. This is basically a skew-product system, where the phase dynamics is forced by the chaotic amplitude dynamics. In this system, it is possible to estabilish analytically a certain number of results, because the phase evolution is basically unidimensional and there is no need to deal with the problem of identifying the direction of perturbations. In order to perform a more realistic analysis of phase synchronization, a suitable coupling between phase and amplitude dynamics has been added to the special flow [21]. Here, setting up a perturbative approach for the weakly forced Rössler oscillator, we show that the structure of the model proposed in Ref. [21] is quite similar to that one expected in generic chaotic systems, whenever the presence of strong dissipations allows eliminating the stable directions. Furthermore, in order to simplify the analytic treatment of the LEs, we focus our attention on a model that we call the generalized special flow (GSF), very similar to that one analyzed in Ref. [21] but characterized by a finite Markov partition. As a result, we find that the modulus of the second LE exponent increases quadratically with the coupling strength and its corresponding smallness justifies the claims often found in the literature that the second LE is equal to zero below the onset of phase-synchronization. In other words, we conclude that the LE is not the right order parameter to describe this transition.

More precisely, this paper is organized as follows. In the next section we introduce a general approach for the determination of Lyapunov exponents through an average over the invariant measure. In section 3 we present our case-study model, the GSF, deriving it as a discrete–time approximation of a periodically forced Rössler system. In sections 4 and 5 we illustrate the perturbation expansion for the second LE, the corresponding LV and the invariant measure. Finally some numerical results are presented in section 6, along with conclusions.

2 A general approach for the determination of Lyapunov vectors and Lyapunov exponents

In this section we discuss a method to determine Lyapunov exponents from suitable ensemble averages. It is easy to write down a formal meaningful definition, but the problem lies in translating it into a workable procedure. With reference to an N-dimensional discrete–time system, described by the mapping rule

$$\mathbf{x}_{t+1} = \mathbf{f}_d(\mathbf{x}_t) \qquad \mathbf{x} \in \mathcal{R}^N, \tag{1}$$

one can express the ith LE (as usual, LE are supposed to be ordered from the largest to the smallest one) as

$$\lambda^{(i)} = \frac{1}{2} \int d\mathbf{x} P(\mathbf{x}) \ln \left[\frac{||\partial_x \mathbf{f}_d \mathbf{V}^{(i)}(\mathbf{x})||^2}{||\mathbf{V}^{(i)}(\mathbf{x})||^2} \right] \tag{2}$$

where $P(\mathbf{x})$ is the corresponding invariant measure, $\partial_x \mathbf{f}_d$ is the Jacobian of the transformation, and the Lyapunov vector $\mathbf{V}^{(i)}(\mathbf{x})$ identifies the ith most expanding direction in $\mathbf{x}$.

With reference to a continuous–time system, ruled by the ordinary differential equation

$$\dot{\mathbf{x}} = \mathbf{f}_c(\mathbf{x}) \qquad \mathbf{x} \in \mathcal{R}^N. \tag{3}$$

the ith LE writes as

$$\lambda^{(i)} = \int d\mathbf{x} P(\mathbf{x}) \frac{\left[\partial_x \mathbf{f}_c \mathbf{V}^{(i)}(\mathbf{x})\right] \bullet \mathbf{V}^{(i)}(\mathbf{x})}{||\mathbf{V}^{(i)}(\mathbf{x})||^2} \tag{4}$$

where $\bullet$ denotes the scalar product.

Unless a clear procedure to determine the LV is given, Eqs. (2,4) are nothing but formal statements. As anticipated in the introduction, $\mathbf{V}^{(i)}(\mathbf{x})$ can be obtained by following a two-step procedure. We start with a generic set of i linearly independent vectors lying in the tangent space and let them evolve in time. This is the standard procedure to determine LEs, and it is well known that the hypervolume $\mathbf{Y}^{(i)}$ identified by such vectors contains for, large enough times, the i most expanding directions. Furthermore, with reference to the set of orthogonal cordinates obtained by implementing the Gram-Schmidt procedure, the component v_k of a generic vector $\mathbf{v}$ evolves according to the following differential equation (for the sake of simplicity, we refer to continuous–time systems),

$$\dot{v}_k = \sum_{j=k}^{i} \sigma_{k,j}(\mathbf{x}) v_j \qquad 1 \leq k \leq i \tag{5}$$

where, as shown in Ref. [14], $\sigma_{k,j}$ does not explicitely depend on time, but only through the position $\mathbf{x}$ in the phase space. As a result, the ith Lyapunov exponent can be formally expressed as the ensemble average of the local expansion rate $\sigma_{i,i}$, i.e.,

$$\lambda^{(i)} = \int d\mathbf{x} P(\mathbf{x}) \sigma_{i,i}(\mathbf{x}) \tag{6}$$

By comparing with Eq. (4), one finds the obvious equality

$$\sigma_{i,i} = \frac{\left[\partial_x \mathbf{f}_c \mathbf{V}^{(i)}(\mathbf{x})\right] \bullet \mathbf{V}^{(i)}(\mathbf{x})}{||\mathbf{V}^{(i)}(\mathbf{x})||^2} \tag{7}$$

In Sec. 4, where this formalism is applied to a phase-synchronization problem, we find that the only workable way to obtain an analytic expression for $\sigma_{i,i}$ passes through the determination of the direction of the corresponding LV vector $\mathbf{V}^{(i)}(\mathbf{x})$.

Let us now consider the backward evolution of a generic vector $\mathbf{V}^{(i)} \in \mathbf{Y}^{(i)}$. Its direction is identified by the $(i-1)$-dimensional vector

$$\mathbf{u} \equiv (u_1, u_2, \ldots, u_{i-1}) \tag{8}$$

where $u_k = v_k/v_i$. From the definition of $\mathbf{u}$ and from Eq. (5), one easily finds that the backward evolution follows the equation

$$\dot{u}_k = (\sigma_{i,i} - \sigma_{k,k})u_k - \sum_{j=k+1}^{i-1} \sigma_{k,j}(t)u_j - \sigma_{k,i} \qquad 1 \leq k < i \tag{9}$$

This is a cascade of skew-product linear stable equations (they are stable because the Lyapunov exponents are organized in descending order). The overall stability is basically determined by the smallest $(\sigma_{k,k} - \sigma_{i,i})$ that is obtained for $k = i-1$. It is, therefore, sufficient to turn our attention to the last $(i-1)$ component of the vector $\mathbf{V}$. Its equation has the following structure

$$\dot{u}(t) = \gamma u + \sigma(t) \tag{10}$$

where $\gamma = \lambda_i - \lambda_{i-1} < 0$ and we have dropped the subscript i for simplicity. The value of the direction u is obtained by integrating this equation. By neglecting the temporal fluctuations of γ (it is not difficult to include them, but this is not important for our final goal), the formal solution of Eq. (10) reads

$$u(\mathbf{x}(t)) = \int_{-\infty}^{t} e^{\gamma(t-\tau)}\sigma(\mathbf{x})\, d\tau \quad . \tag{11}$$

This equation does not simply tell us the value of u at time t, but the value of u when the trajectory sits in $\mathbf{x}(\mathbf{t})$. It is in fact important to investigate the dependence of u on $\mathbf{x}$. We proceed by determining the deviation $\delta_j u$ induced by a perturbation δx_j of $\mathbf{x}$ along the jth direction,

$$\delta_j u = \int_{-\infty}^{t} e^{\gamma(t-\tau)}\delta_j\sigma(\tau)\, d\tau \tag{12}$$

where, assuming a smooth dependence of σ on $\mathbf{x}$, (see below for a further discussion of this point),

$$\delta_j\sigma(\tau) \approx \sigma_x(\tau)\delta x_j(\tau) = \sigma_x(\tau)\delta x_j(t)e^{\lambda_j(t-\tau)} \quad . \tag{13}$$

(notice that the dynamics is flowing backward). If the Lyapunov exponent λ_j is negative, $\delta_j\sigma(\tau)$ decreases for $\tau \to -\infty$ and the integral over τ in Eq. (12) converges. As a result, $\delta_j u$ is proportional to δx_j, indicating that the direction of the LV is smooth along the jth direction. If λ_j is positive, $\delta_j\sigma(\tau)$ diverges, and below time t_0 where

$$\delta x_j(t)e^{\lambda_j(t-t_0)} = 1 \tag{14}$$

linearization breaks down. In this case, $\delta\sigma(\tau)$ for $\tau < t_0$ is basically uncorrelated with its "initial value" $\delta_j\sigma(t)$ and one can estimate $\delta_j u$, by limiting the integral to the range $[t_0, t]$

$$\delta_j u(t) = \delta x_j(t) \int_{t_0}^{t} d\tau e^{(\lambda_j+\gamma)(t-\tau)}\sigma_x(\tau) \tag{15}$$

where t_0 is given by Eq. (14). By bounding σ_x with constant functions and thereby performing the integral in Eq. (15), we finally obtain

$$\delta_j u(t) \approx \delta x_j(t) + \delta x_j(t)^{-\gamma/\lambda_j} \tag{16}$$

. The scaling behaviour is finally obtained as the smalles number between 1 and $-\gamma/\lambda_j$. If we now introduce the exponent η_j to identify the scaling behaviour of the deviation of the LV direction when the point of reference is moved along the jth direction in phase space, the results are summarized in the following way

$$\eta_j = \begin{cases} 1 & \text{for } \lambda_j \leq -\gamma \\ -\gamma/\lambda_j & \text{for } \lambda_j > -\gamma \end{cases} \tag{17}$$

The former case corresponds to a smooth behavior (the derivative is finite), while the latter one reveals a singular behaviour that is the signature of a generalized synchronization.

Although most of the assumptions made to derive the above equation are quite plausible (even though not rigorously proved), there is one point that needs to be more carefully checked: the smoothness of $\sigma(\mathbf{x})$. In the absence of a more careful analysis of this point, we can only claim that the above equation provides an upper bound to the true range of smoothness for the LV direction.

3 From the periodically forced Rössler system to the generalized special flow

The first model where phase synchronization has been explored is the forced Rössler oscillator [7]. In this section we derive a discrete-time mapping describing a forced Rössler system in the limit of weak coupling. We obtain what we call the Generalized Special Flow (GSF), because it extends a mapping previously introduced to characterize the onset of phase synchronization[20].

The starting set of ordinary differential equations is

$$\begin{aligned} \dot{x} &= -y - z + \varepsilon y \cos(\Omega t + \psi_0) \\ \dot{y} &= x + a_0 y - \varepsilon x \sin(\Omega t + \psi_0) \\ \dot{z} &= a_1 + z(x - a_2) \end{aligned} \tag{18}$$

where ψ_0 fixes the phase of the forcing term at time 0. It is convenient to introduce cylindrical coordinates, namely $\mathbf{u} = (\varphi, r, z)$, ($x = r\cos\phi$, $y = r\sin\phi$). For the future sake of clarity, let us denote with $\mathbf{S_c}$ the 3-dimensional space parametrized by such coordinates. The differential equation (18) writes as

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) + \varepsilon \mathbf{G}(\mathbf{u}, \Omega t + \psi_0) \tag{19}$$

where

$$\begin{aligned}
\mathbf{F} &= \left[1+\frac{z}{r}\sin\phi+\frac{a_0}{2}\sin 2\phi\,,\; a_0 r\sin^2\phi - z\cos\phi\,,\; a_1+z(r\cos\phi-a_2)\right] \qquad (20)\\
\mathbf{G} &= \left[-\sin^2\phi\cos(\Omega t+\psi_0)-\cos^2\phi\sin(\Omega t+\psi_0)\,,\; \frac{r}{\sqrt{2}}\sin 2\phi\cos(\Omega t+\psi_0+\pi/4)\,,\; 0\right]
\end{aligned}$$

Note that system (19) can be written in the equivalent autonomous form

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) + \varepsilon\mathbf{G}(\mathbf{u},\psi), \quad \dot{\psi} = \Omega,$$

where ψ denotes the phase of the forcing term.

We pass to a discrete-time description, by monitoring the system each time the phase ϕ is a multiple of 2π. In the new framework, the relevant variables are r, z, and ψ, all measured when the Poincaré section is crossed. The task is to determine the transformation mapping the state (r, z, ψ) onto (r', z', ψ').

In order to obtain the expression of the map, it is necessary to formally integrate the equations of motion from one to the next section. This can be done, by expanding around the unperturbed solution for $\varepsilon = 0$ (which must nevertheless be obtained numerically). The task is anyhow worth, because it allows determining the structure of the resulting map, which turns out to be (see appendix A)

$$\begin{aligned}
\psi' &= \psi + \langle T^{(0)}\rangle\,\Omega + A_1 + \varepsilon\,(B_1^c\cos\psi + B_1^s\sin\psi)\\
r' &= A_2 + \varepsilon\,(B_2^c\cos\psi + B_2^s\sin\psi) \qquad (21)\\
z' &= A_3 + \varepsilon\,(B_3^c\cos\psi + B_3^s\sin\psi)
\end{aligned}$$

where $\langle T^{(0)}\rangle$ is the average period of the unperturbed Rössler oscillator and A_m's and B_m's are functions of z and r. As it is shown in appendix A, they can be numerically determined by integrating the appropriate set of equations. Up to first order in ε, the structure of the model is fairly general as it is obtained for a generic periodically forced oscillator represented in cylindrical coordinates (as long the phase of the attractor can be unambiguously identified).

For the usual parameter values, the Rössler attractor is characterized by a strong contraction along one direction [22]. As a result, one can neglect the z dependence since this variable is basically a function of r, and thus write

$$\begin{aligned}
\psi' &= \psi + \langle T^{(0)}\rangle\,\Omega + A_1(r) + \varepsilon\,(B_1^c(r)\cos\psi + B_1^s(r)\sin\psi)\\
r' &= A_2(r) + \varepsilon\,(B_2^c(r)\cos\psi + B_2^s(r)\sin\psi) \qquad (22)
\end{aligned}$$

where all the functions can be obtained by integrating numerically the equations of motion of the single Rössler oscillator.[1]

[1] Strictly speaking, A and B functions in (21) and (22) are different (see appendix A). We use the same notations here to simplify the presentation.

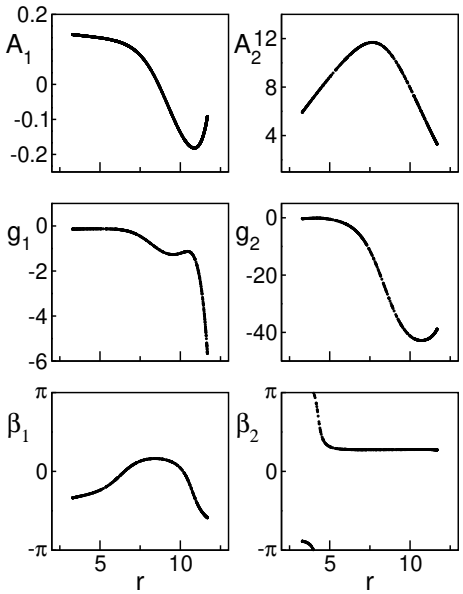

Figure 1: Numerically computed functions A_i, g_i and β_i $(i = 1, 2)$ for the Rössler oscillator. Rössler parameters have been chosen as in Ref. [20]: $a_0 = 0.2$, $a_1 = 1$ and $a_2 = 9$.

To simplify further manipulations, we finally recast equation (22) in the form

$$\begin{aligned} \psi' &= \psi + K + A_1(r) + \varepsilon g_1(r) \cos(\psi + \beta_1(r)) \\ r' &= A_2(r) + \varepsilon\ g_2(r) \cos(\psi + \beta_2(r)) \end{aligned} \tag{23}$$

where

$$\begin{aligned} B_i^c(r) &= g_i(r) \cos \beta_i(r) \\ B_i^s(r) &= -g_i(r) \sin \beta_i(r) \end{aligned} \tag{24}$$

for $i = 1, 2$. The parameter $K = \langle T^{(0)} \rangle\, \Omega - 2\pi$ represents the detuning between the original Rössler-system average frequency and the forcing frequency Ω.

The correctness of the scheme is confirmed in Fig. 1, where all the functions defining the model have been numerically obtained. The very fact that they all look as one-dimensional curves, confirms the conjecture that z-dependence can be neglected.

The GSF (23) generalizes the model introduced in Ref. [20], where the effect of the phase on the r dynamics was not included. This implies that the GSF looses the skew-product structure. This has important consequences on the orientation of the second Lyapunov vector that we determine in the next sections. Notice also that the GSF (23) generalizes and justifies the model invoked in Ref. [21].

In spite of the simplification introduced by removing the z variable, a rigorous treatment of Eq. (23) for generic functions g and β is still very difficult. A first obstacle may be

the lack of a finite Markov partition for the unperturbed system, which does not allow us expressing the second order correction to the LV in a closed form (see appendix B for details). A second obstacle is that the perturbation itself may and will in general destroy the Markov partition, making the invariant measure hardly accessible to a perturbative expansion. For both reasons, we restrict ourselves to considering specific A, g, and β functions which guarantee the existence of a finite Markov partitions in a finite range of the coupling constant. In the last section we shall comment on the possibility to extend our formalism to a more general setup.

For the sake of simplicity, we have decided to analyse the following model,

$$\begin{aligned} r' &= f(r) + 2\varepsilon c g(r)\cos(\psi+\alpha) \\ \psi' &= \psi + K + \Delta r + \varepsilon b \cos\psi \end{aligned} \tag{25}$$

where

$$f(r) = 1 - 2|r| \quad , \quad g(r) = r^2 - |r| \tag{26}$$

with $r \in [-1, 1]$. The tent-map choice for r ensures that $[-1, 0]$ and $[0, 1]$ are the two atoms of a Markov partion. Moreover, since $g(r)$ is equal to 0 for $r = 0$ and $r = \pm 1$, this remains true also when the perturbation is switched on. This is a key property that is necessary to perform a completely analytical treatment in the following sections.

In this two-dimensional setup, the formal expression of the ith LE (2) writes

$$\lambda^{(i)} = \frac{1}{2}\int_{-1}^{1} dr \int_{0}^{2\pi} d\psi P(r,\psi) \ln\left[\frac{||\mathbf{J}(r,\psi)\mathbf{V}^{(i)}(r,\psi)||^2}{||\mathbf{V}^{(i)}(r,\psi)||^2}\right] \tag{27}$$

and the Jacobian is

$$\mathbf{J}(r,\psi) = \begin{pmatrix} f_r(r) + 2\varepsilon c g_r(r)\cos(\psi+\alpha) & -2\varepsilon c g(r)\sin(\psi+\alpha) \\ \Delta & 1 - \varepsilon b \sin\psi \end{pmatrix} \tag{28}$$

where the subscript r denotes the derivative with respect to r. The computation of the Lyapunov exponent therefore, requires determining both the invariant measure $P(r,\psi)$ and the local direction of the Lyapunov vector $\mathbf{V}^{(i)}$.

4 A perturbative calculation of the second Lyapunov exponent

In this section we derive a perturbative expression for the second LE of the GSF (25), by expanding Eq. (27). One of the key ingredients is the second LV, whose direction can be identified by writing $\mathbf{V} = (V, 1)$ (for the sake of clarity, from now on, we omit the superscript $i = 2$ in $\mathbf{V}$ and λ, as we shall refer only to the second direction). Due to the

skew-product structure of the unperturbed map (25), the second LV is, for $\varepsilon = 0$, aligned along the ψ direction (i.e. $V = 0$). It is therefore natural to expand V in powers of ε

$$V \approx \varepsilon v_1(r,\psi) + \varepsilon^2 v_2(r,\psi) \tag{29}$$

Accordingly, the logarithm of the norm of $\mathbf{V}$ is

$$\ln ||\mathbf{V}||^2 = \ln(1+\varepsilon^2 v_1^2) = \varepsilon^2 v_1^2 \tag{30}$$

while its forward iterate writes as (including only those terms that contribute up to second order in the norm),

$$\mathbf{JV} = \begin{pmatrix} \varepsilon f_r(r) v_1 - 2c\varepsilon g(r)\sin(\psi+\alpha) \\ 1 + \varepsilon(\Delta v_1 - b\sin\psi) + \varepsilon^2 \Delta v_2) \end{pmatrix} \tag{31}$$

Notice that we have omitted the (r,ψ) dependence of v_1 and v_2 to keep the notation compact.

The Euclidean norm of the forward iterate is

$$||\mathbf{JV}||^2 = 1 + 2\varepsilon(\Delta v_1 - b\sin\psi) + \varepsilon^2\Big\{(\Delta v_1 - b\sin\psi)^2 + 2\Delta v_2 + [f_r(r)v_1 - 2cg(r)\sin(\psi+\alpha)]^2\Big\} \tag{32}$$

and its logarithm is

$$\ln ||\mathbf{JV}||^2 = 2\varepsilon(\Delta v_1 - b\sin\psi) - \varepsilon^2\Big\{(\Delta v_1 - b\sin\psi)^2 - 2\Delta v_2 - [f_r(r)v_1 - 2cg(r)\sin(\psi+\alpha)]^2\Big\} \tag{33}$$

We now proceed by formally expanding the invariant measure in powers of ε

$$P(r,\psi) \approx p_0(\psi) + \varepsilon p_1(r,\psi) + \varepsilon^2 p_2(r,\psi). \tag{34}$$

The determination of the p_i coefficients is presented in the next section, but here we anticipate that, as a consequence of the skew-product structure for $\varepsilon = 0$, the zeroth-order component of the invariant measure does not depend on the phase ψ. Moreover, because of the structure of the tent-map, p_0 is also independent of r, i.e. $p_0 = 1/4\pi$. The second Lyapunov exponent can thus be written as

$$\begin{aligned} \lambda \;=\; & \int_{-1}^{1} dr \int_0^{2\pi} d\psi \Big(\frac{1}{4\pi} + \varepsilon p_1(r,\psi)\Big)\Big\{2\varepsilon(\Delta v_1(r,\psi) - b\sin\psi) - \varepsilon^2\Big[(\Delta v_1(r,\psi) - b\sin\psi)^2 \\ & -2\Delta v_2(r,\psi) + [f_r(r)v_1(r,\psi) + 2cg(r)\sin(\psi+\alpha)]^2 + v_1^2(r,\psi)\Big]\Big\} + o(\varepsilon^2) \end{aligned} \tag{35}$$

As the variable ψ is a phase, it is not a surprise that some simplifications can be found by expanding the relevant functions into Fourier components. We start writing the first component of the invariant measure as

$$p_1(r,\psi) = \frac{1}{2\pi}\sum_n q_i(r) e^{in\psi} \tag{36}$$

We then turn our attention to the first order component $v_1(r,\psi)$ of the second LV (29). Due to the sinuosidal character of the forcing term in the GSF (25), it is easy to verify (see the next section) that $v_1(r,\psi)$ contains just the first Fourier component,

$$v_1(r,\psi) = c\Big[L(r)\sin(\psi+\alpha) + R(r)\cos(\psi+\alpha)\Big] \tag{37}$$

By now, inserting Eqs. (36,37) into Eq. (35) and performing the integration over ψ, we obtain

$$\begin{aligned}\lambda \;=\; & \varepsilon^2\int_{-1}^{1} dr\,\Bigg\{\Delta c\Bigg[q_1^r\Big[L(r)\sin\alpha + R(r)\cos\alpha\Big] - q_1^i\Big[L(r)\cos\alpha - R(r)\sin\alpha\Big]\Bigg] \\ & + bq_1^i - \frac{b^2}{8} + \Delta\frac{bc}{4}\Big[L(r)\cos\alpha - R(r)\sin\alpha\Big] + \frac{c^2}{8}(3-\Delta^2)\Big[L^2(r)+R^2(r)\Big] \\ & + \frac{c^2}{2}g^2(r) + c^2\frac{|r|}{r}g(r)L(r)\Bigg\} + \frac{\Delta I_2}{4\pi}\end{aligned} \tag{38}$$

where we have further decomposed $q_1(r)$ in its real and imaginary parts

$$q_1(r) = q_1^r(r) + iq_1^i(r) \tag{39}$$

and we have defined

$$I_2 := \int_{-1}^{1} dr \int_0^{2\pi} d\psi\, v_2(r,\psi), \tag{40}$$

which accounts for the contribution arising from the second order correction to the LV. This expansion shows that the highest-order contribution to the second Lyapunov exponent of the GSF scales quadratically with the perturbation amplitude. This is indeed a general result that does not depend on the particular choice of the functions used to define the GSF, but only on the skew-product structure of the unperturbed time evolution and on the validity of the expansion assumed in (34) (we shall comment on this last issue in the next section).

By inserting the expression for I_2 obtained in appendix B (see Eq. (93)) in Eq. (38), we finally obtain the perturbative expression for the second LE,

$$\begin{aligned}\lambda = \varepsilon^2\Bigg\{ & \frac{c^2}{30} - \frac{b^2}{4} + \int_{-1}^{1} dr\Bigg[bq_1^i(r) + \frac{c^2}{16}(6-\Delta^2)\Big[L^2(r)+R^2(r)\Big] \\ & + \Delta c q_1^r(r)\Big[L(r)\sin\alpha + R(r)\cos\alpha\Big] + \Delta c\left(\frac{b}{4} - q_1^i(r)\right)\Big[L(r)\cos\alpha - R(r)\sin\alpha\Big] \\ & + c^2\frac{|r|}{r}g(r)L(r) + \frac{\Delta c^2}{4}r\sin\left(\frac{\Delta(1-r)}{2}\right)\Big[L(r)\cos K - R(r)\sin K\Big]\Bigg]\Bigg\}\end{aligned} \tag{41}$$

Accordingly, the numerical value of the second LE can be obtained by performing integrals which involve the four functions $q_1^r(r)$, $q_1^i(r)$, $L(r)$, and $R(r)$, that are determined in the next section.

5 Determining the coefficients of the power expansion

After having more or less formally expanded the expression of the second LE in powers of the coupling strength ε in the previous section, now we show how the coefficients can be determined for both the invariant measure and the direction of the LV. Notice that the second part of the project passes through the implementation of the general ideas put forward in Sec. II.

5.1 The invariant measure

We start focusing our attention on the invariant measure $P(r,\psi)$ which can be computed as a fixed point of the Frobenius-Perron equation

$$P'(r',\psi') = \frac{P(r^-,\psi^-)}{|\det \mathbf{J}(r^-,\psi^-)|} + \frac{P(r^+,\psi^+)}{|\det \mathbf{J}(r^+,\psi^+)|} \tag{42}$$

where (r^-,ψ^-) and (r^+,ψ^+) are the two preimages of (r',ψ'). It is important to notice that our choice of the map guarantees that two solutions do exist in the whole rectangle $[-1,1]\times[0,2\pi]$ in a finite range of ε-values. This will be crucial for obtained exact expressions. As it has been shown in the previous section, we are interested in solving the above equation up to first order. Accordingly, we write

$$P(r',\psi') = p_0(r',\psi') + \frac{\varepsilon}{2\pi}\sum_n q_i(r)e^{in\psi} \tag{43}$$

where we have expanded the first order contribution as in Eq. (36). It is also necessary to expand the preimages

$$r^\pm = r_0^\pm + \varepsilon r_1^\pm \tag{44}$$

$$\psi^\pm = \psi_0^\pm + \varepsilon \psi_1^\pm \tag{45}$$

where

$$r_0^\pm = \pm\frac{1-r'}{2} \tag{46}$$

$$\psi_0^\pm = \psi' - K \mp \Delta\frac{1-r'}{2} \tag{47}$$

$$r_1^\pm = \mp c\frac{1-r'^2}{4}\cos(\psi_0^\pm+\alpha) \tag{48}$$

$$\psi_1^\pm = \pm c\Delta\frac{1-r'^2}{4}\cos(\psi_0^\pm+\alpha) - b\cos\psi_0^\pm \tag{49}$$

At zeroth order in ε, it is easy to see that the Frobenius-Perron equation (42) reduces to

$$p_0(r',\psi') = \frac{1}{2}\big(p_0(r_0^+,\psi_0^+) + p_0(r_0^-,\psi_0^-)\big) \tag{50}$$

whose solution is everywhere constant, as anticipated in section 4. By imposing the normalization condition, one obtains

$$p_0 = \frac{1}{4\pi}. \tag{51}$$

By then considering that

$$|\det \mathbf{J}(r,\psi)|^{-1} = \frac{1}{2}\big[1 + \varepsilon b \sin\psi + \varepsilon c\, \mathrm{sign} r (\Delta g(r)\sin(\psi+\alpha) + g_r(r)\cos(\psi+\alpha))\big] \tag{52}$$

and projecting Eq. (42) over its first Fourier component, we finally obtain a closed equation for $q_1(r)$

$$q_1(r') = \frac{\mathrm{e}^{-iK}}{2}\left[\mathrm{e}^{-i\Delta r_0^+}\left(q_1(r_0^+) - i\frac{b}{4} + \frac{c}{4}\mathrm{e}^{i\alpha} g_r(r_0^+) - i\frac{c}{4}\mathrm{e}^{i\alpha}\Delta g(r_0^+)\right) + \right. \tag{53}$$

$$\left. \mathrm{e}^{-i\Delta r_0^-}\left(q_1(r_0^-) - i\frac{b}{4} - \frac{c}{4}\mathrm{e}^{i\alpha} g_r(r_0^-) + i\frac{c}{4}\mathrm{e}^{i\alpha}\Delta g(r_0^-)\right)\right] \tag{54}$$

The structure of this equation is very similar to a Frobenius-Perron equation for a one-dimensional system. The dimensionality reduction has been made possible by exploting the skew-product structure of the unperturbed system. Considering also the simple expression of the preimages of r' (they have to be determined at zeroth order), the above equation can be accurately solved by implementing the standard method to solve a Frobenius-Perron equation (the only limit being imposed by the numerical round-off).

In Fig. 2 we have plotted the real and immaginary parts of q_1 for three different choices of the parameters b and c. In all cases, one can see a very smooth dependence, which thus suggests the possibility to obtain accurate fully analytic expressions by expanding polynomially $q_1(r)$. However, being more interested in testing the overall validity of the perturbative approach, we do not explore this possibility.

In fact, in order to test the general validity of the power expansion, we have numerically investigated three different GSFs, corresposinding to the following choices of the functions f and g: *i)* $f(r) = 1 - 2|r|$, $g(r) = r^2 - |r|$ as considered in (26); *ii)* $f(r) = 0.8 - 1.8|r|$ and $g(r) = 1/2$, for which there is no finite Markov partition; *iii)* $f(r) = 2(1 - 2\varepsilon c)(1 - |r|)$ and $g(r) = 1/2$, for which the finite Markov partition existing in the unperturbed case is destroyed as soon as the perturbation is switched on.

In order to compare such models, we have computed the deviation of the zero Fourier component of the invariant measure of the map (25) induced by a small finite coupling ε,

$$\langle P(r,\psi)\rangle = \int_{-1}^{1} dr \int_0^{2\pi} d\psi \Big[P(r,\psi)|_\varepsilon - P(r,\psi)|_{\varepsilon=0}\Big]. \tag{55}$$

As it can be clearly seen in Fig. 3, in the first two cases the linear term is even equal to zero, while relevant multiplicative logarithmic corrections are present in the third case. This "pathological" behaviour is induced by the fact that as soon as the coupling is switched on, an infinite series of discontinuities in the invariant measure suddenly arises in the vicinity of the former fixed point $r = -1$. It is, however, important to notice that no peculiarity is found in the more generic second case.

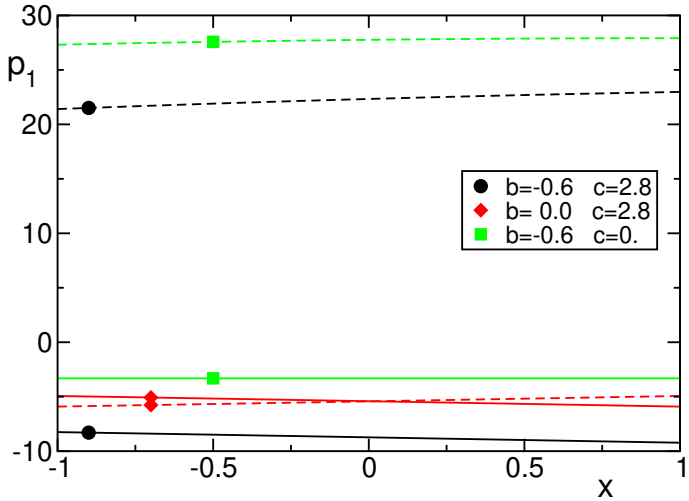

Figure 2: The first order contribution q_1 to the probability density for the parameter values: $\Delta = -0.18$, $\alpha = -\pi/4$, $K = 0.04\pi$. Solid and dashed lines refer to real and imaginary parts.

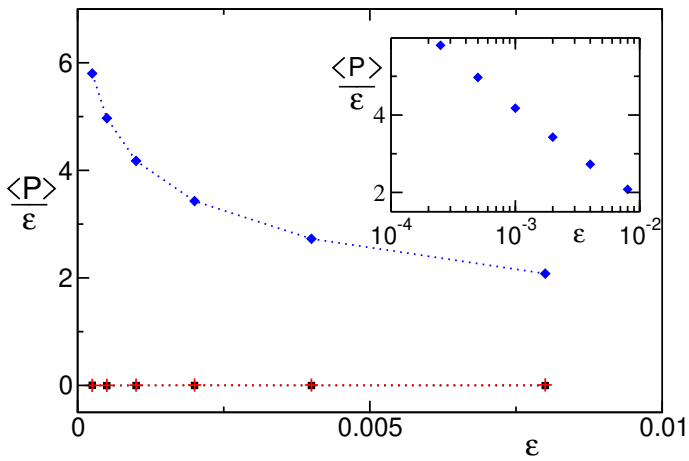

Figure 3: The deviation of the zero Fourier component of the invariant measure (see Eq.(55) of map (25) as a function of ε. Three different choices of f and g have been tested: i) $f(r) = 1-2|r|$ and $g(r) = r^2-|r|$ (crosses), ii) $f(r) = 0.8-1.8|r|$ and $g(r) = 1/2$ (squares), and iii) $f(r) = 2(1-2\varepsilon c)(1-|r|)$ and $g(r) = 1/2$ (diamonds). Parameter values have been fixed to $\Delta = -0.18$, $\alpha = -\pi/4$, $K = 0.04\pi$, $b = -0.6$ and c=2.8. Abscissas are divided by ε to better emphasize deviations from linear scaling. The logarithmic deviations dispalyed by the diamonds are emphasized in the inset.

5.2 The direction of the second Lyapunov vector

In this subsection we derive a self-consistent equation for the second LV. We start from Eq. (31), retaining only the relevant perturbation terms

$$\mathbf{JV} = \begin{pmatrix} [f_r(r) + 2\varepsilon c g_r(r)\cos(\psi+\alpha)]V - 2\varepsilon c g(r)\sin(\psi+\alpha) \\ \Delta V + 1 - \varepsilon b \sin\psi \end{pmatrix} \tag{56}$$

By computing the ratio between the components of $\mathbf{JV}$ we obtain the new value of the slope $V' = \varepsilon v_1' + \varepsilon v_2' + \ldots$,

$$v_1' + \varepsilon v_2' = \frac{\Big[f_r(r) + 2\varepsilon c g_r(r)\cos(\psi+\alpha)\Big](v_1 + \varepsilon v_2) - 2cg(r)\sin(\psi+\alpha)}{\Delta\varepsilon v_1 + (1 - \varepsilon b \sin\psi)}, \tag{57}$$

where we have again kept only the relevant terms (up to first order after dividing both sides by ε) and where v_1' and v_2' are both functions of the iterates r' and ψ',

$$r' = r_0' + 2\varepsilon c g(r)\cos(\psi+\alpha) \tag{58}$$

$$\psi' = \psi_0' + \varepsilon b \cos\psi \tag{59}$$

where $r_0' = f(r)$ and $\psi_0' = \psi + K + \Delta r$ are the unperturbed iterates. By replacing the expressions for r' and ψ' in Eq. (57), at leading order, we obtain

$$v_1'(f(r), \psi + K + \Delta r) = f_r(r) v_1(r,\psi) - 2cg(r)\sin(\psi+\alpha) \tag{60}$$

As anticipated in Sec. 2, this recursive relation can be solved by following backwards the dynamics of (r,ψ). It is worth stressing that the term $2cg(r)\sin(\psi+\alpha)$, i.e. the effect of the phase on the amplitude dynamics, acts as a source term in Eq. (60). In its absence, the latter equation would yield a trivial vanishing solution for v_1 (which in turn also implies $v_2 = 0$, as it can be appreciated in App. B). It is therefore the feedback of the phase on the amplitude dynamics that generates a nontrivial orientation of the perturbed second Lyapunov vector. Furthermore, the structure of the source term $2cg(r)\sin(\psi+\alpha)$ naturally suggests the Ansatz (37). By inserting it in Eq. (60) we obtain two recursive equations,

$$\begin{aligned} L(r) &= \mathrm{sign}(r)\left[\frac{1}{2}\Big(R(f(r))\sin(K+\Delta r) - L(f(r))\cos(K+\Delta r)\Big) - g(r)\right] \\ R(r) &= -\frac{1}{2}\mathrm{sign}(r)\left[(R(f(r))\cos(K+\Delta r) + L(f(r))\sin(K+\Delta r)\right] \end{aligned} \tag{61}$$

This equation can be solved numerically, by considering it as a recursive relation to be iterated backward in time until the fixed point solution is eventually attained and the functions L and R, computed with the desired precision. In Figs. 4 we can see some examples of how they look like.

>From the analysis carried on in Sec. 2 and in particular from Eq. (17), we see that the condition for a smooth behaviour of the direction V along the phase-direction is (noticing

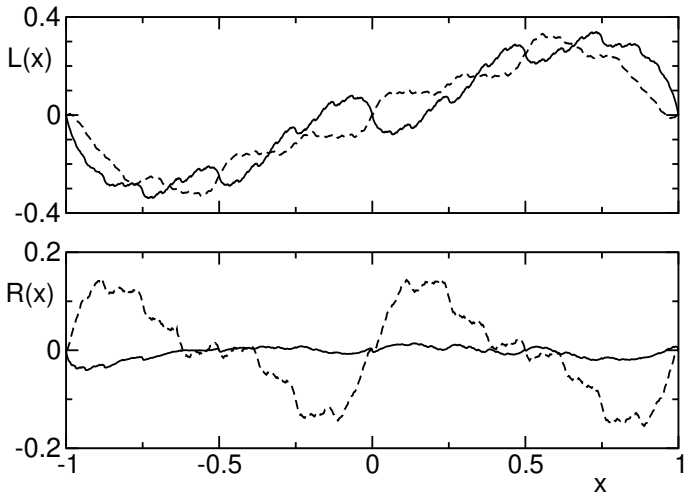

Figure 4: The functions $L(r)$ and $R(r)$ computed for $\Delta = -0.18$ and $K = 0$ (solid curve) and $K = -1$ (dashed line).

that here, $\gamma = \lambda_2 - \lambda_1$) is $\lambda_1 > 2\lambda_2$ that is certainly verified and this fully justifies the expansion in Fourier modes along such a direction. On the other hand, along the expanding direction r, the codition writes $\lambda_2 < 0$, that is only marginally verified. The apparent roughness exhibited by $L(r)$ and $R(r)$ can therefore be a manifestation of the expected non-complete smoothness. It is, however, also important to stress the role played by the functions we have specifically considered in the GSF. In fact, the tent map induces a discontinuity in the tangent space that propagates everywhere, though significantly squeezed. Luckily enough, as it can be appreciated in App. B, such singularities are integrated out when determining the leading contribution to the Lyapunov exponent which is therefore substantially insensitive and can be computed without much harm.

6 Numerical results and conclusions

In this paper we have introduced a novel approach to determine analytically the Lyapunov exponent and applied it to the specific case of a periodically forced chaotic oscillator described by a model (the generalized special flow) which is also introduced here starting from the specific case of the Rössler oscillator.

Given the many technical difficulties that is necessary to overcome in order to finally obtain the numerical value of the quadratic coefficient, it is wise to compare the analytic expression with the direct numerical computation of the second LE performed for small enough values of ε. In Fig. 5, the second order coefficient λ/ε^2 is determined from the analytic expression (41) and by directly simulating the GSF for ε-values in the range $[10^{-4}, 10^{-2}]$. The good agreement obtained for all K values confirm the correctness of the

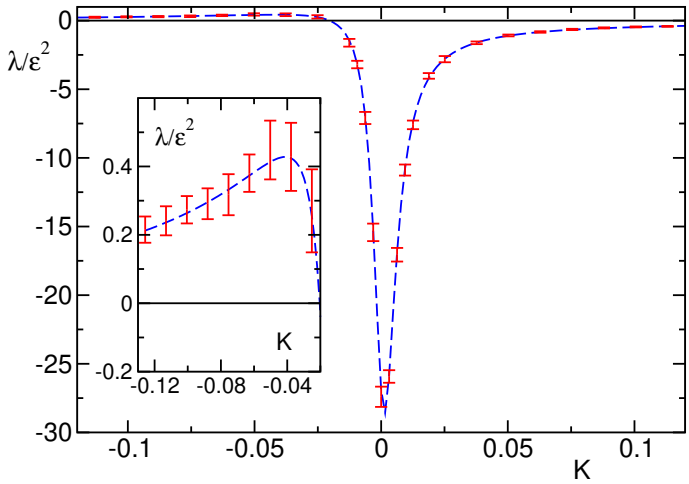

Figure 5: Second Lyapunov exponent for the Generalized Special Flow (25) as a function of detuning K. The dashed line represent the analytical result, while dots (with the bars indicating one standard deviation) results of direct numerical simulations of the GSF. Parameter values have been fixed to: $\Delta = -0.18$, $\alpha = -\pi/4$, $b = -0.6$, $c = 2.8$. Abscissas have been rescaled by a factor ε^2 to evidentiate the second order coefficient. The inset magnifies a part of the larger graph.

analytical calculations. The relative strong negative peak around $K = 0$ is a manifestation of a resonance phenomenon. The LE tends to be more negative when the forcing frequency is close to the average frequency of the chaotic attractor.

It is also important to stress that our results are valid for arbitrarily small ε, i.e. below the transition to phase-synchronization (if there is any) and therefore tells us that the LE corresponding to the phase dynamics is immediately different from zero, as soon as the coupling is switched on.

Another important point concerns the sign of the LE: naive considerations might suggest that the coupling tends to stabilize the phase dynamics and thereby to give a negative LE. However, the left tail in Fig. 5 (see also the inset) definitely shows a positive exponent. It is desirable to find some simple heuristic arguments to understand when and why the phase dynamics is stable, but this does not seem to be an easy task and is left as an open problem for future investigations.

The major difference between the GSF, we analyse in this manuscript and the special flow introduced in [20] is the term proportional to c in the equation for r in Eq. (25). Such a term prevents the possibility of further dimension reductions and requires setting up the machinery we have developed in this paper. It is therefore interesting to quantify its direct effect on the actual value of the LE. This can be simply done, by setting the other coupling term $b = 0$, an assignment that is basically complementary to what done in the standard special flow. The results, reported in Fig. 6 show a sort of "dispersive" behaviour for the

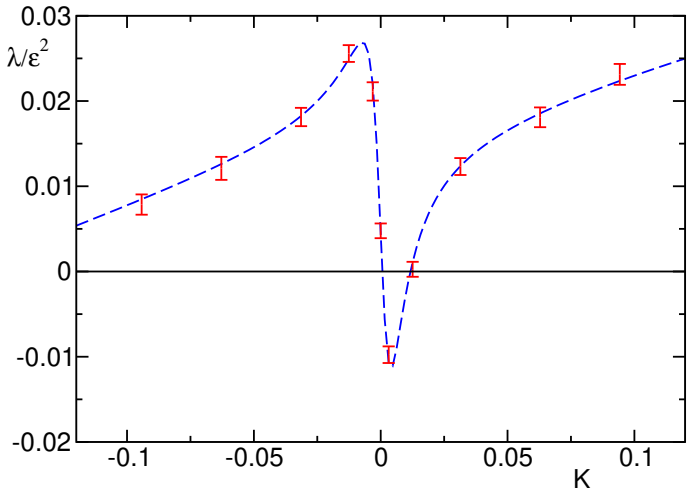

Figure 6: Second Lyapunov exponent for the Generalized Special Flow (25) as a function of detuning K. Symbols and parameters values are the same as in Fig. 5, except for $b = 0$.

LE which also tends to be positive. This suggests that the back coupling of the phase dynamics onto the amplitude evolution maybe responsible for an eventually positive LE.

While Eq. (41) cannot by any means capture the quantitative behavior of the original Rössler system, still the quadratic behaviour of the second LE seems to be a very general feature even though we can imagine that the lack of structural stability of generic oscillators may mask the overall behaviour with the presence of additional stability windows. We have therefore computed directly the second Lyapunov exponent for the periodically forced Rössler system chosing the same set of parameters ($a_0 = 0.2$, $a_1 = 1$ and $a_2 = 9$) considered in Ref. [20].

When the frequency of the periodic force is close to the natural frequency of the oscillator, $\nu_0 = 1.0158(1)$ for our choice of parameters, we are able to detect the quadratic behavior with a good accuracy, as demonstrated in Fig. 7. Since the coupling strengths we have reached are much below the onset of phase synchronization, as can bee seen in Ref. [20], these numerical results confirm our theoretical conclusions that the Lyapunov exponent corresponding to the phase dynamics deviates from zero as soon as the coupling is switched on. It would be now interesting to extend the analysis carried out in this paper to two coupled chaotic oscillators, perhaps by investigating suitable discrete-time models such as that one introduced in [23]. However, while one can presumably learn something on the phenomenon of phase-synchronization, we do not expect qualitative changes for the behaviour of the Lyapunov spectrum. This is supported by the recently reported quadratic growth of the fifth LE (the first one corresponding to phase dynamics) in a system of four coupled Rössler oscillators [24].

Altogether, the numerical and analytical results presented in this paper clearly show that the onset of phase-sinchronization is not signalled by the second LE (or, more generally,

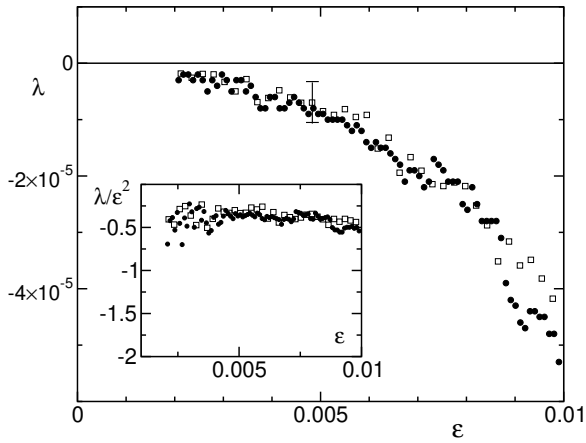

Figure 7: Second Lyapunov exponent for the periodically forced Rössler system close to the resonance. Full circles refer to a forcing frequency $\Omega = 1.007$ (the Rössler natural average frequency is $\nu_0 = 1.0158(1)$), while open squares correspond to $\Omega = 1$. Rössler parameters have been fixed as $a_0 = 0.2$, $a_1 = 1$, $a_2 = 9$, while the integration interval is about $t = 10^8$. The inset shows the quadratic relation between $\lambda^{(2)}$ and ε.

the LE associated to the phase dynamics) turning negative, but it is rather associated to a change in the structure of the dynamical attractor [21, 25] that is not directly related to the sign of the "phase exponent". On the other hand, the quadratic dependence on the coupling strength makes it difficult to numerically appreciate deviations from zero (especially because of the statistical fluctuations that necessarily affect numerical simulations) and explains why in earlier studies, the LE has been mistakenly regarded as a proper order parameter to characterize the transition to a phase-synchronized regime.

Another important point concerns the sign of the second Lyapunov exponent. In fact, it was formerly believed that phase chaos (i.e. a positive LE) can only occur in the presence of a specific structure of the underlying chaotic attractor (see e.g. [26]). On the other hand, our analytical results show that the second LE can be positive even in a context where no peculiar amplitude evolution has to be invoked. However, our approach does not give any physical insight about the expected sign of the LE. It will be certainly useful to find under which conditions a chaotic phase dynamics may arise.

Finally another major achievement of this paper is that Lyapunov exponents can be effectively determined from ensemble averages, passing through the determination of the local direction of the corresponding Lyapunov vectors. From a purely numerical point of view, there is no conceptual difficulty to applying this method for a more detailed characterization of high-dimensional chaos [19]. However, in the perspective of obtaining more general analytical results, it is desirable to go beyond systems with finite Markov partitions.

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A From continuous to discrete time

In this appendix we present the detailed calculations relative to the determination of the Poincaré mapping (21) for the periodically forced Rössler oscillator (18). Notice, however, that the methodology is quite general and is indeed applicable to a generic periodically perturbed system, as long as it can be written in the form (19).

Let us start by introducing some useful notations

$$\mathbf{U}(r,z,\psi;t) \equiv (\Phi(r,z,\psi;t), R(r,z,\psi;t), Z(r,z,\psi;t)) \tag{62}$$

denotes the phase point in $\mathbf{S_c}$ at time t of a trajectory started in $(0,r,z)$ at time 0 and with an initial phase of the forcing term equal to ψ (pay attention to the fact that the triple $(r,z,\psi) \in \mathbf{S_d}$). The crossing time with the Poincaré surface is determined by imposing that the phase Φ has increased by 2π, i.e.,

$$\Phi(r,z,\psi;T) = 2\pi. \tag{63}$$

As we are interested in the small coupling regime, we can expand $\mathbf{U}$ in powers of ε and retain just the first order term,

$$\mathbf{U}(t) = \mathbf{U}^{(0)}(t) + \varepsilon \mathbf{U}^{(1)}(t) \tag{64}$$

In particular, from Eq. (63), we obtain

$$2\pi = \Phi^{(0)}(T) + \varepsilon\Phi^{(1)}(T) = \Phi^{(0)}(T^{(0)}) + \varepsilon\Phi^{(1)}(T^{(0)}) + \varepsilon\frac{\partial\Phi^{(0)}}{\partial t}(T^{(0)})T^{(1)} \tag{65}$$

where we have expanded T as well, assuming that $T = T^{(0)} + \varepsilon T^{(1)}$. Since, $\Phi^{(0)} = 2\pi$, we conclude that

$$T^{(1)} = -\frac{\Phi^{(1)}(T^{(0)})}{f_1} \tag{66}$$

where $f_1 = \frac{\partial\Phi^{(0)}(T^{(0)})}{\partial t}$ is determined by the right-hand side of (19) with $\varepsilon = 0$, namely, it is the first component of $\mathbf{F}$.

It can be easily seen that the zeroth and first order components satisfy the differential equations

$$\dot{\mathbf{U}}^{(0)} = \mathbf{F}(\mathbf{U}^{(0)}) \tag{67}$$

$$\dot{\mathbf{U}}^{(1)} = \mathbf{DF}(\mathbf{U}^{(0)})\mathbf{U}^{(1)} + \mathbf{G}(\mathbf{U}^{(0)}, \Omega t + \psi) \tag{68}$$

where $\mathbf{DF}$ denotes the Jacobian of the velocity field $\mathbf{F}$ and we have introduced an explicit dependence on the phase ψ, as it changes in going from one to the next section. The equation for the first order correction can be formally solved,

$$\mathbf{U}^{(1)} = \int_0^{T^{(0)}} d\tau \mathbf{W}(T^{(0)},\tau)\mathbf{G}(\mathbf{U}^{(0)}(\tau), \Omega\tau + \psi) \tag{69}$$

where $\mathbf{W}(t,\tau)$ is the matrix of fundamental solutions of the equation $\dot{\mathbf{U}} = \mathbf{DF}(\mathbf{U}^{(0)})\mathbf{U}$. Since $\mathbf{G}$ contains only first harmonics in ψ, it can be decomposed into sine and cosine components,

$$\mathbf{G}(\mathbf{U}^{(0)}, \Omega\tau + \psi) = \mathbf{G}^c(\mathbf{U}^{(0)}, \Omega\tau)\cos\psi + \mathbf{G}^s(\mathbf{U}^{(0)}, \Omega\tau)\sin\psi \tag{70}$$

where

$$\mathbf{G}^c(\mathbf{U}^{(0)}, \Omega\tau) = \begin{pmatrix} -\sin^2\Phi^{(0)}\cos\Omega\tau - \cos^2\Phi^{(0)}\sin\Omega\tau \\ R^{(0)}\sin 2\Phi^{(0)}\cos(\Omega\tau + \pi/4)/\sqrt{2} \\ 0 \end{pmatrix} \tag{71}$$

and

$$\mathbf{G}^s(\mathbf{U}^{(0)}, \Omega\tau) = \begin{pmatrix} \sin^2\Phi^{(0)}\sin\Omega\tau - \cos^2\Phi^{(0)}\cos\Omega\tau \\ -R^{(0)}\sin 2\Phi^{(0)}\sin(\Omega\tau + \pi/4)/\sqrt{2} \\ 0 \end{pmatrix} \tag{72}$$

Accordingly, as it follows from (69), $\mathbf{U}^{(1)}$ can be decomposed in the same way

$$\mathbf{U}^{(1)} = \mathbf{M}^c(r,z)\cos\psi + \mathbf{M}^s(r,z)\sin\psi \tag{73}$$

where

$$\mathbf{M}^c(r,z) = \int_0^{T^{(0)}} d\tau \mathbf{W}(T^{(0)},\tau)\mathbf{G}^c(\mathbf{U}^{(0)}(\tau), \Omega\tau) \tag{74}$$

and a similar equation holds for $\mathbf{M}^s(r,z)$.

The first component of Eq. (73) gives $\Phi^{(1)}$. After substituting it into Eq. (66), we obtain,

$$T = T^{(0)}(r,z) - \frac{\varepsilon}{f_1}(M_1^c(r,z)\cos\psi + M_1^s(r,z)\sin\psi) \tag{75}$$

where the subscripts indicate once more the component of the vector. Accordingly, the new phase ψ' is

$$\psi' = \psi + \Omega T = \psi + \Omega T^{(0)}(r,z) - \frac{\varepsilon\Omega}{f_1}[M_1^c(r,z)\cos\psi + M_1^s(r,z)\sin\psi] \tag{76}$$

On the other hand, from the second and third components of Eq. (64) we obtain the new values r' and z' by also expanding the expression of T around $T^{(0)}$,

$$\mathbf{U}(T) = \mathbf{U}^{(0)}(T^{(0)}) + \varepsilon\mathbf{U}^{(1)}(T^{(0)}) + \varepsilon\mathbf{F}(\mathbf{U}^{(0)})T^{(1)} \tag{77}$$

Straightforward but tedious calculations lead to

$$\psi' = \psi + \langle T^{(0)}\rangle\Omega + A_1 + \varepsilon(B_1^c\cos\psi + B_1^s\sin\psi) \tag{78}$$

$$r' = A_2 + \varepsilon(B_2^c\cos\psi + B_2^s\sin\psi) \tag{79}$$

$$z' = A_3 + \varepsilon(B_3^c\cos\psi + B_3^s\sin\psi) \tag{80}$$

where $\langle T^{(0)} \rangle$ is the average period of the unperturbed Rössler oscillator. The functions A_i can be determined by integrating the unperturbed equations

$$\begin{aligned} A_1(r,z) &= \left[T^{(0)}(r,z) - \langle T^{(0)} \rangle\right] \Omega \\ A_2(r,z) &= R^{(0)}(r,z;T^{(0)}) \\ A_3(r,z) &= Z^{(0)}(r,z;T^{(0)}) \end{aligned} \tag{81}$$

while the functions B_i^c read as

$$\begin{aligned} B_1^c(r,z) &= -\Omega M_1^c(r,z)/f_1 \\ B_2^c(r,z) &= M_2^c(r,z) - f_2 M_1^c/f_1 \\ B_3^c(r,z) &= M_3^c(r,z) - f_3 M_1^c/f_1 \end{aligned} \tag{82}$$

and similar equations hold for the sine components.

B Second order contribution to the Lyapunov vector

In this appendix we derive a closed expression for the term I_2, which accounts for the contribution to the LE arising from second order-corrections to the LV direction (see Eq. (40)). To this pourpose, we have to consider all terms of order ε in Eq. (57), starting from $v_1'(r',\psi') \equiv v_1'(r_0',\psi_0') + \varepsilon \delta v_1'$, with

$$\begin{aligned} \delta v_1' = &+bc\Big[L(r_0')\cos(\psi_0'+\alpha) - R(r_0')\sin(\psi_0'+\alpha)\Big]\cos\psi \\ &+2c^2 g(r)\Big[L_r(r_0')\sin(\psi_0'+\alpha) + R_r(r_0')\cos(\psi_0'+\alpha)\Big]\cos(\psi+\alpha) \end{aligned} \tag{83}$$

where $r_0' = f(r)$ and $\psi_0' = \psi + K + \Delta r$ are the iterates of the unperturbed GSF as defined in section 5. In the following we shall not care about the possible lack of differentiability along the direction r for two reasons: *i)* we have verified that setting up a more accurate procedure leads to the same results, but its presentation would be more cumbersome; *ii)* the procedure is in itself correct, because in the end we are interested in the integral that is insensitive to the presence of singularities.

The recursive equation for the second order term writes as

$$v_2(r,\psi) = \frac{1}{f_r(r)} v_2'(r_0',\psi_0') + s(r,\psi) \tag{84}$$

where we have introduced the source term

$$s(r,\psi) = \frac{1}{f_r(r)}\Big[-2cg_r(r)\cos(\psi+\alpha)v_1(r,\psi) + \delta v_1' + v_1'(r_0',\psi_0')(\Delta v_1 - b\sin\psi)\Big] \tag{85}$$

and v_1' is given by Eq. (60).

Being interested in the integral I_2 (see Eq. (40)), we see that the integration over ψ can be easily performed since ψ_0' ranges over $[0, 2\pi]$. More delicate is the integral over r because of the folding of r_0'. However, one can still solve the problem by separately integrating over the negative and positive values of r, i.e. the two atoms of the finite Markov partition. By introducing the integral over negative r-values

$$I_2^- = \int_{-1}^{0} dr \int_0^{2\pi} d\psi \, v_2(r, \psi) \tag{86}$$

and analogously definining I_2^+, we obtain from Eq. (84)

$$\begin{aligned} I_2^- &= \frac{I_2^- + I_2^+}{4} + S \\ I_2^+ &= -\frac{I_2^- + I_2^+}{4} + S \end{aligned} \tag{87}$$

where S is the integral of $s(r, \psi)$.

$$S = \int_{-1}^{1} dr \int_0^{2\pi} d\psi \, s(r, \psi) \tag{88}$$

We thus eventually obtain

$$I_2 = I_2^- + I_2^+ = S. \tag{89}$$

It is now convenient to express $s(r, \psi)$ as a function of v_1' only,

$$s(r, \psi) = \frac{1}{f_r(r)} \Big\{ - b v_1' \sin\psi + \delta v_1' + \frac{\Delta v_1' - 2cg_r \cos(\psi + \alpha)}{f_r(r)} [v_1' + 2cg(r) \sin(\psi + \alpha)] \Big\} \tag{90}$$

Upon integrating over ψ, we obtain,

$$\begin{aligned} S = \pi c^2 \frac{\Delta}{4} \int_{-1}^{1} dy [L^2(y) + R^2(y)] \\ -2\pi c^2 \int_0^1 dr \, (r^2 - r) \sin(\Delta r) \Big[L_r (1 - 2r) \cos K - R_r (1 - 2r) \sin K \Big] \\ +\pi c^2 \Delta \int_0^1 dr \, (r^2 - r) \cos(\Delta r) \Big[L (1 - 2r) \cos K - R (1 - 2r) \sin K \Big] \\ +\pi c^2 \int_0^1 dr \, (1 - 2r) \sin(\Delta r) \Big[L (1 - 2r) \cos K - R (1 - 2r) \sin K \Big] \end{aligned} \tag{91}$$

After integrating by parts the integral involving L_r and R_r and rescaling the dummy variable, we finally arrive at the desired result:

$$I_2 = \pi c^2 \frac{\Delta}{4} \int_{-1}^{1} dy [L^2(y) + R^2(y)] + \tag{92}$$

$$\pi c^2 \int_{-1}^{1} dy \sin \frac{\Delta(1 - y)}{2} y \Big[L(y) \cos K - R(y) \sin K \Big] \tag{93}$$

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