

ON THE EXISTENCE  
OF  
UNBIASED MONTE CARLO ESTIMATORS

PETER MATHÉ

ABSTRACT. For many typical instances where Monte Carlo methods are applied attempts were made to find unbiased estimators, since for them the Monte Carlo error reduces to the statistical error. These problems usually take values in the scalar field. If we study vector valued Monte Carlo methods, then we are confronted with the question whether there can exist unbiased estimators. This problem is apparently new. Below it is settled precisely. Partial answers are given, indicating relations to several classes of linear operators in Banach spaces.

1. INTRODUCTION, NOTATION

In many practical applications the program designer is confronted with the "curse of dimensionality", an exponential dependence on the dimension, which is inherent in most error estimates provided by classical numerical analysis, see e.g. [TWW88] for a sample of typical numerical problems and the respective error estimates.

Often this can be overcome by choosing Monte Carlo methods, i.e., numerical methods involving random parameters in the computational process, see [HH64] for an excellent, by now classical treatment on the applicability of Monte Carlo methods. Within the classical theory one prefers unbiased Monte Carlo estimators, since they are self-focusing if the numerical simulation is repeated. Such unbiased estimators are often hard to find, see [KW86, ENS89].

In statistics, the problem whether there are unbiased estimates (in statistical sense) has also been attractive, recently. While it has been known since 1956 that there cannot exist such unbiased estimates for density estimation, see [Ros56], advantage has been made by [LB93]. They developed a machinery which enables to prove that there are no unbiased  $l$ -informative estimates for singular problems. Within our framework, the notion of  $l$ -informativity corresponds to the requirement of finite errors. On the other hand, all problems studied below are not singular in their sense. This may express that our arguments are completely different, indicating geometric properties of the target space, where the error is measured.

Below we are concerned with *vector valued Monte Carlo methods* for the randomized approximation of *linear* mappings. The classical results extend easily from real valued

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Monte Carlo methods to a finite number of them, if we gather unbiased estimates for each component. The problem is how far one can proceed by this: Are there infinite dimensional problems, which admit unbiased Monte Carlo estimators?

As an illustration let us briefly and informally introduce the following

*Example 1.* Let  $f$  be any periodic function defined on the interval  $[0, 2\pi]$ , which has a (generalized) derivative in  $\tilde{L}_2(0, 2\pi)$ , the space of all periodic square integrable functions. It is well known that any such function expands into a Fourier series (converging in  $\tilde{L}_2(0, 2\pi)$ ), which means, that we have

$$f(x) = \sum_{k=1}^{\infty} \gamma_k(f) e^{-ikx}, \quad x \in [0, 2\pi], \quad (1)$$

with Fourier coefficients

$$\gamma_k(f) := \frac{1}{2\pi} \int_0^{2\pi} f(\xi) e^{-ik\xi} d\xi. \quad (2)$$

For such functions an unbiased Monte Carlo estimator (using Fourier coefficients) can be obtained in the following (trivial) way by choosing a member of the Fourier series at random. Precisely, let  $p_k := \frac{6}{\pi^2 k^2}$ ,  $k \in \mathbb{N}$ , and choose  $\frac{\pi^2 k^2}{6} \gamma_k(f) e^{-ikx}$ ,  $k \in \mathbb{N}$ , with probability  $p_k$ , respectively. The average performance of this method yields the Fourier series of  $f$ , provided the random variable

$$k \longrightarrow \frac{\pi^2 k^2}{6} \gamma_k(f) e^{-ikx}$$

taking values in  $\tilde{L}_2(0, 2\pi)$  is integrable. But we even have the stronger square integrability from

$$\sum_{k=1}^{\infty} p_k \frac{\pi^4 k^4}{36} |\gamma_k(f)|^2 \|e^{-ikx}\|_2^2 = \frac{\pi^2}{6} \sum_{k=1}^{\infty} k^2 |\gamma_k(f)|^2 < \infty, \quad (3)$$

for functions with square integrable derivative, see [Pie87].

The above example can be considered as a special instance of approximating a diagonal mapping in the space  $l_2$  of square summable sequences. To see this we switch from the spaces of functions to the respective spaces of Fourier coefficients in the following way. Assign any function  $f$  the sequence  $x_k := k\gamma_k(f)$ ,  $k \in \mathbb{N}$ . Then the approximation of  $f$  is replaced by the approximation of  $(\gamma_k(f))_{k \in \mathbb{N}}$ . The differentiability assumption ensures that  $\sum_{k=1}^{\infty} |x_k|^2 < \infty$ . This means that we assign every sequence  $(x_k)_{k \in \mathbb{N}}$  the sequence  $(\frac{1}{k}x_k)_{k \in \mathbb{N}}$ , corresponding to a diagonal mapping  $D_\sigma : l_2 \rightarrow l_2$ .

Summarizing we switched from the random approximation of functions to the random approximation of a diagonal mapping between Hilbert spaces. Within this framework a theory of stochastic numerical methods is available, see [Mat91, Hei94]. The basic notion will be the notion of a (linear) Monte Carlo method. Throughout the paper we restrict ourselves to linear Monte Carlo methods only. Let  $X$  and  $Y$  be Banach spaces. Denote by  $\mathcal{L}(X, Y)$  the space of all bounded linear operators and by  $\mathfrak{F}(X, Y)$  the subspace of all operators of finite rank, cf. [Pie80, Pie87] for notation from the theory of operators in Banach spaces. By  $\mathfrak{F}^k(X, Y)$  we denote the subset of operators of rank at most  $k$ . Corresponding to [Mat91, Hei94, Mat94] we propose the following

**Definition 1.** A triple

$$\mathcal{P} := ([\Omega, \mathcal{F}, P], u, k)$$

is called a (*linear*) *Monte Carlo method*, if

1.  $[\Omega, \mathcal{F}, P]$  is a probability space.
2.  $u: \Omega \rightarrow \mathfrak{F}(X, Y)$  is such that the mapping  $\Phi: X_0 \times \Omega \rightarrow Y$ , defined by

$$\Phi(x, \omega) := (u(\omega))(x), \quad x \in X, \quad \omega \in \Omega,$$

is product measurable into  $Y$  and the set  $\{(u(\omega))(x), \quad x \in X, \quad \omega \in \Omega\}$  is a separable subset in  $Y$ .

3. The cardinality function  $k: \Omega \rightarrow \mathbb{N}$  is a measurable natural number, for which

$$u_\omega := u(\omega) \in \mathfrak{F}^{k(\omega)}(X, Y), \quad \omega \in \Omega.$$

*Remark 1.* For linear Monte Carlo methods as introduced above we could directly assign  $k(\omega) := \text{rank}(u(\omega))$ , since this would result in an appropriate measurable cardinality function. However, for general classes of methods such assignment would not be meaningful, such that we kept the definition with the more general choice of cardinality function, see [Hei94, Mat94].

For such Monte Carlo methods we can assign the cardinality

$$\text{MC-card}(\mathcal{P}) := \int_{\Omega} k(\omega) dP(\omega).$$

By definition, for every  $x \in X$  the mapping

$$\omega \longrightarrow u_\omega(x)$$

is a Radon random variable in  $Y$ , see [LT91] for spaces of Banach space valued random variables. Therefore the error of any Monte Carlo method  $\mathcal{P} := ([\Omega, \mathcal{F}, P], u, k)$  for an operator  $S: X \rightarrow Y$  at input  $x \in X$  may be defined as

$$e(S, \mathcal{P}, x) := \left( \int_{\Omega} \|S(x) - u_\omega(x)\|_Y^2 dP(\omega) \right)^{1/2},$$

while the overall performance is given by

$$e(S, \mathcal{P}) := \sup_{\|x\|_X \leq 1} e(S, \mathcal{P}, x).$$

We agree to denote by

$$a_n^{mc}(S) := \inf \left\{ \sup_{\|x\|_X \leq 1} \left( \int_{\Omega} \|S(x) - u_\omega(x)\|_Y^2 dP(\omega) \right)^{1/2}, \quad \text{MC-card}(\mathcal{P}) < n \right\}$$

the *n*th *Monte Carlo approximation number* of the linear operator  $S: X \rightarrow Y$ , see [Mat91] for more information on that topic. We shall make use of the following submultiplicativity property

$$a_n^{mc}(RST) \leq \|R\| a_n^{mc}(S) \|T\|, \quad (4)$$

whenever the product is correctly defined.

If a Monte Carlo method  $\mathcal{P}$  has a finite error for some linear operator  $S$ , i.e.,  $e(S, \mathcal{P}) < \infty$ , then the function

$$S_{\mathcal{P}}(x) := \int_{\Omega} u_{\omega}(x) dP(\omega)$$

exists and denotes the respective expectation. From now on we make the assumption that

$$\sup_{\|x\|_X \leq 1} \int_{\Omega} \|u_{\omega}(x)\|_Y^2 dP(\omega) < \infty,$$

ensuring that  $e(0, \mathcal{P}) < \infty$ , where 0 denotes the zero operator. It is the aim of this paper to study properties of  $S_{\mathcal{P}}$ .

**Definition 2.** An operator  $S \in \mathcal{L}(X, Y)$  admits an unbiased Monte Carlo method if there is a Monte Carlo method  $\mathcal{P}$  with

$$\text{MC-card}(\mathcal{P}) < \infty, \tag{1}$$

$$\sup_{\|x\|_X \leq 1} \int_{\Omega} \|u_{\omega}(x)\|_Y^2 dP(\omega) < \infty \tag{2}$$

and

$$S(x) = S_{\mathcal{P}}(x), \quad x \in X. \tag{3}$$

If  $S \in \mathcal{L}(X, Y)$  admits an unbiased Monte Carlo method then we let

$$u(S) := \inf \left\{ \sup_{\|x\|_X \leq 1} \left( \int_{\Omega} \|u_{\omega}(x)\|_Y^2 dP(\omega) \right)^{1/2}, \quad \mathcal{P} \text{ is unbiased for } S \right\}. \tag{5}$$

This turns into a norm and we have  $\|S: X \rightarrow Y\| \leq u(S)$ . As explained above, property (2) is equivalent to the statement that  $\mathcal{P}$  has a finite error for some bounded operator acting between  $X$  and  $Y$ . Within this framework it is immediate that any finite rank operator  $L$  admits an unbiased Monte Carlo method by letting  $\mathcal{P}$  be choosing  $L$  with probability 1. However, there are different ways of representing a given finite dimensional mapping, leading to different Monte Carlo methods.

*Example 2.* Let  $\text{Id}^m : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be the identity and  $e_j$ ,  $j = 1, \dots, m$ , denote the unit vector basis in  $\mathbb{R}^m$ .

1. (trivial) representation:

$$\text{Id}^m(x) = \frac{1}{m} \sum_{j=1}^m (mx_j) e_j, \quad x = (x_1, \dots, x_m).$$

2. (nuclear) representation:

$$\text{Id}^m(x) = \frac{1}{2^m} \sum_{\varepsilon_1, \dots, \varepsilon_m = \pm 1} \left( \sum_{j=1}^m \varepsilon_j x_j \right) \left( \sum_{j=1}^m \varepsilon_j e_j \right), \quad x = (x_1, \dots, x_m).$$

where  $\varepsilon_1, \dots, \varepsilon_m$  denotes a Bernoulli sequence, see Remark 2 below.

Observe that it is much more elaborate to find unbiased Monte Carlo methods with prescribed properties. We will not turn to that problem.

2. DIAGONAL MAPPINGS  $D_\sigma : l_2 \rightarrow l_q$ ,  $1 \leq q \leq \infty$ 

We are going to study diagonal mappings in sequence spaces. To make things precise we need to introduce the Lorentz sequence spaces, cf. [Pie87, 2.1]. For any sequence  $x = (\xi_j)_{j \in \mathbb{N}}$  of real numbers, which is convergent to 0 we assign by  $s_n(x)$  the non-increasing rearrangement (in modulus) of  $x$ . The (real) Lorentz space  $l_{r,w}$  consists of all sequences  $x$  for which the sequence  $(n^{1/r-1/w} s_n(x))_{n \in \mathbb{N}}$  belongs to  $l_w$  equipped with the norm arising from this identification. Explicitly, if  $1 \leq w < \infty$  then we let

$$\|x\|_{r,w} := \left( \sum_{j=1}^{\infty} [n^{1/r-1/w} s_n(x)]^w \right)^{1/w}$$

while for  $w = \infty$  we put

$$\|x\|_{r,\infty} := \sup_{n \in \mathbb{N}} \{n^{1/r} s_n(x)\}.$$

Observe that we can identify for any  $1 \leq p < \infty$  the classical sequence spaces  $l_p$  with  $l_{p,p}$ , while we denote by  $c_0$  the space  $l_{\infty,\infty}$  and by  $l_\infty$  the space of all bounded sequences equipped with the supremum norm.

We begin our study with the consideration of a specific class of diagonal operators.

Given any sequence  $\sigma = (\sigma_j)_{j \in \mathbb{N}}$  of real numbers, let us consider the mapping  $x = (x_j)_{j \in \mathbb{N}} \rightarrow (\sigma_j x_j)_{j \in \mathbb{N}}$ . Denote this mapping by  $D_\sigma$ . For choices of  $1 \leq p, q \leq \infty$  the operator  $D_\sigma$  acts continuously from  $l_p$  to  $l_q$  if and only if the diagonal  $\sigma$  belongs to  $l_r$  with  $\frac{1}{r} = \max\{0, \frac{1}{q} - \frac{1}{2}\}$ .

The simple construction outlined in the introductory section implies the existence of unbiased Monte Carlo methods for  $D_\sigma : l_2 \rightarrow l_2$  whenever  $\sigma \in l_2$ , which is more than required for continuity. This is typical and we shall derive necessary conditions later.

However, the class of operators admitting an unbiased Monte Carlo method can be enlarged for other spaces, employing a more involved construction. This is provided in the following

**Theorem 1.** *A diagonal mapping  $D_\sigma : l_2 \rightarrow l_q$  admits an unbiased Monte Carlo method if  $\sigma \in l_q$ . Moreover we have  $u(D_\sigma : l_2 \rightarrow l_q) \leq \|\sigma\|_q$ .*

Before proving the theorem we need a preparatory lemma. Let  $\varepsilon^m$  denote the uniform distribution on the extreme points of  $B_\infty^m$ , i.e., all vectors  $\omega \in \{+1, -1\}^m$ . Thus the basic probability space is  $[\{+1, -1\}^m, \mathcal{F}^m, \varepsilon^m]$ . For computations involving  $\varepsilon^m$  the following lemma is useful.

**Lemma 1.** *Let  $m > 1$ .*

1. *For all  $1 \leq j, k \leq m$  we have*

$$\int_{\{+1, -1\}^m} \langle \omega, e_j \rangle \langle \omega, e_k \rangle d\varepsilon^m(\omega) = \begin{cases} 1 & , \text{if } j = k \\ 0 & , \text{else} \end{cases}.$$

2. *For all  $a \in \mathbb{R}^m$  we have*

$$\left( \int_{\{+1, -1\}^m} |\langle \omega, a \rangle|^2 d\varepsilon^m(\omega) \right)^{1/2} = \|a\|_2.$$

*Proof.* The second statement is a consequence of the first one. To prove the first topic, let us observe that for  $\omega \in \{+1, -1\}^m$  and  $1 \leq j \leq m$  the number  $\langle \omega, e_j \rangle$  is  $\pm 1$ . This implies  $\int_{\{+1, -1\}} \langle \omega, e_j \rangle^2 d\varepsilon^m(\omega) = 1$ . If  $j \neq k$  then

$$\text{card}\{\omega \in \{+1, -1\}^m, \langle \omega, e_j \rangle \langle \omega, e_k \rangle = 1\} = \\ \text{card}\{\omega \in \{+1, -1\}^m, \langle \omega, e_j \rangle \langle \omega, e_k \rangle = -1\},$$

which is equal to  $2^{m-1}$ , and from which the proof can be completed.  $\square$

*Remark 2.* The above lemma is well known. The reader familiar with probability theory will recognize that the distribution of  $\langle \omega, a \rangle$  is the distribution of the sum  $\sum_{j=1}^m \varepsilon_j \langle e_j, a \rangle$ , where the  $\varepsilon_j$  are independent numbers taking values  $+1$  and  $-1$  with equal probability, i.e., they form a Bernoulli sequence.

*Proof of Theorem 1.* The proof is constructive. We are going to design a concrete Monte Carlo method of cardinality 1. Let  $\Omega^m := \{+1, -1\}^m$  equipped with  $\varepsilon^m$ , introduced above as probability. Denote by  $\Omega := \bigcup_{m \in \mathbb{N}} \Omega^m$  the disjoint union, equipped with the  $\sigma$ -algebra  $\mathcal{F}$ , generated from the sequence  $\mathcal{F}^m$ . A probability  $P$  will be given as mixture of  $\varepsilon^m$  in the following way.

Observe, that for  $\varepsilon > 0$  we can find a decreasing sequence  $(\tau_m)_{m \in \mathbb{N}}$ , such that  $\tau_1 = 1$  and  $\sigma_m = \tau_m \beta_m$ ,  $m \in \mathbb{N}$ ,  $\lim_{m \rightarrow \infty} \tau_m = 0$  and  $\sum_{j=1}^{\infty} |\beta_j|^q \leq (1 + \varepsilon)^q \sum_{j=1}^{\infty} |\sigma_j|^q$ . For a proof of this fact we refer to [Pie80, 8.6.4]. Let  $p_m := \tau_m - \tau_{m+1}$ ,  $m \in \mathbb{N}$ . Then we have  $p_m \geq 0$  and  $\sum_{m=1}^{\infty} p_m = \tau_1 = 1$ , which means that this sequence gives rise to a probability  $P$  by letting  $P := \sum_{m=1}^{\infty} p_m \varepsilon^m$ . (We implicitly extended the probabilities  $\varepsilon^m$  to all  $\Omega$ .) So far we have defined a probability space  $[\Omega, \mathcal{F}, P]$ . Now, define a mapping  $u: \Omega \rightarrow \mathfrak{F}(l_2, l_q)$  by

$$u_\omega(x) := \left( \sum_{j=1}^m \omega_j \langle x, e_j \rangle \right) \left( \sum_{j=1}^m \omega_j \beta_j e_j \right), \quad \omega = (\omega_1, \dots, \omega_m), \quad x \in l_2.$$

It is readily checked that  $\mathcal{P} = ([\Omega, \mathcal{F}, P], u, 1)$  is a linear Monte Carlo method of cardinality 1. Moreover, for  $x \in l_2$  we have by Lemma 1

$$\begin{aligned} \int_{\Omega} \|u_\omega(x)\|_q^2 dP(\omega) &= \sum_{m=1}^{\infty} p_m \int_{\Omega^m} \|u_\omega(x)\|_q^2 d\varepsilon^m(\omega) \\ &= \sum_{m=1}^{\infty} p_m \int_{\Omega^m} \left\| \sum_{j=1}^m \omega_j \langle x, e_j \rangle \right\|^2 \left\| \sum_{j=1}^m \omega_j \beta_j e_j \right\|_q^2 d\varepsilon^m(\omega) \\ &\leq \sum_{m=1}^{\infty} p_m \|x\|_2^2 \left( \sum_{j=1}^m |\beta_j|^q \right)^{2/q} \\ &\leq (1 + \varepsilon)^2 \|x\|_2^2 \left( \sum_{j=1}^{\infty} |\sigma_j|^q \right)^{2/q}, \end{aligned}$$

such that  $u(D_\sigma: l_2 \rightarrow l_q) \leq \left( \sum_{j=1}^{\infty} |\sigma_j|^q \right)^{1/q}$ , provided  $\mathcal{P}$  was unbiased for  $D_\sigma$ . But this is true, since

$$T(x) := \int_{\Omega} u_\omega(x) dP(\omega), \quad x \in l_2,$$

is well-defined and equal to

$$\begin{aligned}
 T(x) &= \sum_{m=1}^{\infty} p_m \int_{\Omega^m} \left( \sum_{j=1}^m \omega_j \langle x, e_j \rangle \right) \left( \sum_{j=1}^m \omega_j \beta_j e_j \right) d\varepsilon^m(\omega) \\
 &= \sum_{m=1}^{\infty} p_m \sum_{j=1}^m \beta_j \langle x, e_j \rangle e_j \\
 &= \sum_{j=1}^{\infty} \beta_j \left( \sum_{m=j}^{\infty} p_m \right) \langle x, e_j \rangle e_j \\
 &= \sum_{j=1}^{\infty} \beta_j \tau_j \langle x, e_j \rangle e_j \\
 &= D_{\sigma}(x),
 \end{aligned}$$

again by Lemma 1. The proof of the theorem is complete.  $\square$

Analogously we could prove

**Corollary 1.** *A diagonal mapping  $D_{\sigma}: l_2 \rightarrow l_{\infty}$  admits an unbiased Monte Carlo method provided the diagonal tends to 0.*

All operators considered so far have been compact. So the question arises, whether this is typical and leads to the following

**Open Problem .** *Does the identity  $\text{Id}: l_2 \rightarrow l_{\infty}$  admit an unbiased Monte Carlo method?*

Below we shall turn to the question whether there are necessary conditions for the existence of unbiased Monte Carlo methods. Such conditions will be expressed in terms of the Monte Carlo approximation numbers, introduced above. Therefore we are looking for a reformulation of Theorem 1. To do this we need the following result, which points at the relation of the ordinary approximation numbers to the Monte Carlo counterpart, see [Mat91, Lemma 5]. Topics (2) and (3), below provide the typical instances where the Monte Carlo and the (ordinary) approximation numbers differ.

**Proposition 1.** *There is a constant  $C < \infty$  such that for all  $m, n \in \mathbb{N}$  we have*

1.

$$a_n^{mc}(\text{Id}: l_2^m \rightarrow l_q^m) \leq C \frac{m^{1/q}}{n^{1/q'}},$$

if  $1 \leq q \leq 2$  and  $\frac{1}{q'} = 1 - \frac{1}{q}$ ,

2.

$$a_n^{mc}(\text{Id}: l_2^m \rightarrow l_q^m) \leq C \frac{m^{1/q}}{n^{1/2}},$$

if  $2 \leq q < \infty$  and

3.

$$a_n^{mc}(\text{Id}: l_2^m \rightarrow l_{\infty}^m) \leq C \sqrt{\frac{\log(1+m)}{n}}.$$

For the convenience of the reader we shall provide a proof, different from the one given previously in [Mat91, Hei94], thereby using the construction outlined above. We need a geometric property of Banach spaces.

**Definition 3.** A Banach space  $Y$  has type  $p$ ,  $1 \leq p \leq 2$ , if there is a constant  $C < \infty$ , such that for all probability spaces  $[\Omega, \mathcal{F}, P]$ ,  $k \in \mathbb{N}$  and independent random elements  $\Upsilon_1, \dots, \Upsilon_k$  in  $L_2(\Omega, \mathcal{F}, P, Y)$ , for which  $\int_{\Omega} \Upsilon_j(\omega) dP(\omega) = 0$ ,  $j = 1, \dots, k$ , we have

$$\left( \int_{\Omega} \left\| \sum_{j=1}^k \Upsilon_j(\omega) \right\|_Y^2 dP(\omega) \right)^{1/2} \leq C \left( \sum_{j=1}^k \int_{\Omega} \|\Upsilon_j(\omega)\|_Y^p dP(\omega) \right)^{1/p}. \quad (7)$$

The smallest constant satisfying the above inequality shall be called the *type- $p$ -constant* and is denoted by  $T_p(Y)$ .

*Remark 3.* The notion of a type of a Banach space was originally introduced in [MP76], where this definition was given in terms of Rademacher sequences. But, as can be seen easily, it can be extended to the above situation, albeit the type- $p$ -constant is different by a factor of at most 2, see [LT91, Chapt. 9.2], but also [HJ74]. It can be seen that  $T_2(\mathbb{R}) = 1$ . Moreover, every finite-dimensional Banach space has type  $p$ ,  $1 \leq p \leq 2$ , although the respective constant may depend on the dimension, cf. [TJ88, Ch. 1, §4]. However, we have

$$T_q(l_q^m) \leq C, \quad \text{if } 1 \leq q \leq 2$$

and

$$T_2(l_q^m) \leq C \begin{cases} 1 & , \text{if } 2 \leq q < \infty \\ \sqrt{\log 1 + m} & , \text{if } q = \infty \end{cases}$$

for some universal constant  $C < \infty$ .

Now we are prepared to provide the proof of Proposition 1, following the one given in [Mat94].

*Proof of Proposition 1.* We shall carry out the estimates only for  $2 \leq q < \infty$ . The other cases follow the same lines. By Theorem 1 there is an unbiased Monte Carlo method  $\mathcal{P} = ([\Omega, \mathcal{F}, P], u, 1)$  with  $u(\text{Id}: l_2^m \rightarrow l_q^m) \leq m^{1/q}$ . The sample mean

$$\mathcal{P}^n = ([\Omega^n, \mathcal{F}^n, P^n], v, n)$$

of  $n$  independent copies of  $\mathcal{P}$ , which is defined by

$$v_{\omega^n}(x) := \frac{1}{n} \sum_{j=1}^n u_{\omega_j}(x), \quad x \in l_2^m, \quad \omega^n := (\omega_1, \dots, \omega_n) \in \Omega^n,$$



provides another unbiased Monte Carlo method, this time of cardinality  $n$ , which has an error at  $x \in l_2^m$ ,  $\|x\|_2 \leq 1$ , of

$$e(\text{Id}: l_2^m \rightarrow l_q^m, \mathcal{P}^n, x) = \left( \int_{\Omega} \left\| x - \frac{1}{n} \sum_{j=1}^n u_{\omega_j}(x) \right\|_q^2 dP^n(\omega^n) \right)^{1/2} \quad (8)$$

$$= \frac{1}{n} \left( \int_{\Omega} \left\| \sum_{j=1}^n (x - u_{\omega_j}(x)) \right\|_q^2 dP^n(\omega^n) \right)^{1/2} \quad (9)$$

$$\leq \frac{1}{\sqrt{n}} T_2(l_q^m) e(\text{Id}: l_2^m \rightarrow l_q^m, \mathcal{P}). \quad (10)$$

Since

$$e(\text{Id}: l_2^m \rightarrow l_q^m, \mathcal{P}) \leq \|\text{Id}: l_2^m \rightarrow l_q^m\| + u(\text{Id}: l_2^m \rightarrow l_q^m),$$

the proof can be completed in case  $2 \leq q < \infty$ .  $\square$

Now we are able to provide the reformulation of Theorem 1 in terms of the respective Monte Carlo approximation numbers  $a_n^{mc}(D_\sigma: l_2 \rightarrow l_q)$ , as promised above.

**Corollary 2.** *Let  $1 \leq q < \infty$  and put  $r := \max\{2, q'\}$ . If*

$$(a_n^{mc}(D_\sigma: l_2 \rightarrow l_q))_{n \in \mathbb{N}} \in l_{r,q}$$

*then the mapping  $D_\sigma: l_2 \rightarrow l_q$  admits an unbiased Monte Carlo method.*

*Proof.* To simplify the proof we shall assume that the diagonal is nonnegative and non-increasing.

Since the following diagram is commutative (where  $R^m$  assigns every vector in  $l_q$  the vector with the first  $m$  coordinates,  $J^m$  denotes the natural embedding and  $D_\sigma^m$  is the corresponding restriction),

$$\begin{array}{ccc} l_2 & \xrightarrow{D_\sigma} & l_q \\ J^m \uparrow & & \downarrow R^m \\ l_2^m & \xrightarrow{D_\sigma^m} & l_q^m \end{array}$$

and  $\|J^m\| \leq 1$ ,  $\|R^m\| \leq 1$ , we have by inequality (4) the estimate

$$a_n^{mc}(D_\sigma^m) \leq a_n^{mc}(D_\sigma).$$

Moreover, we have

$$\sigma_m a_n^{mc}(\text{Id}_{2,q}^m: l_2^m \rightarrow l_q^m) \leq a_n(D_\sigma^m), \quad (11)$$

which can be seen as follows, cf. also [Pie87, 2.9.3]. Without loss of generality we may assume  $\sigma_m > 0$ . Since

$$\text{Id}_{2,q}^m = (D_\sigma^m: l_2^m \rightarrow l_q^m) \cdot (D_\sigma^m: l_2^m \rightarrow l_2^m)^{-1}$$

and

$$\|(D_\sigma^m : l_2^m \rightarrow l_2^m)^{-1}\| = \sigma_m^{-1}$$

the estimate (11) follows immediately.

Letting  $m = 2n - 1$  and inserting the results from Proposition 1 we see that the sequence  $(\sigma_{2n-1})_{n \in \mathbb{N}}$  belongs to  $l_q$  whenever the assumptions from Corollary 2 are fulfilled. But this is equivalent to  $(\sigma_n)_{n \in \mathbb{N}} \in l_q$ , see e.g. [Pie87, Prop. 2.1.9].  $\square$

We turn to the question whether there are non-trivial necessary conditions to be imposed on an operator in order to admit an unbiased Monte Carlo method. In terms of the Monte Carlo approximation numbers a fairly general condition can be given, provided the target space has some type  $p$ . Indeed, the sample mean construction from the proof of Proposition 1 implies

**Theorem 2.** *Suppose the Banach space  $Y$  has type  $p$ ,  $p > 1$ . If an operator  $S : X \rightarrow Y$  admits an unbiased Monte Carlo method then necessarily  $(a_n^{mc}(S))_{n \in \mathbb{N}} \in l_{p', \infty}$ .*

Taking into account the behavior of the Monte Carlo approximation numbers of the diagonal mappings and applying a technique similar to the one of Corollary 2 this transfers to

**Corollary 3.** *Let  $1 \leq q < \infty$ . If  $D_\sigma : l_2 \rightarrow l_q$  admits an unbiased Monte Carlo method then  $\sigma \in l_{q, \infty}$ .*

*Remark 4.* A look at the necessary and sufficient conditions proves that there is only a small gap left. Since the arguments to prove the necessary conditions are very rough we conjecture that the sufficient conditions are sharp.

On the class of operators acting between Hilbert spaces we immediately obtain

**Theorem 3.** *Any Hilbert-Schmidt operator (acting between Hilbert spaces) admits an unbiased Monte Carlo method. Conversely, if an operator between Hilbert spaces admits an unbiased Monte Carlo method then necessarily the sequence of singular numbers belongs to  $l_{2, \infty}$ .*

Indeed, The norm  $u$ , as defined in (5), is easily seen to be unitarily invariant, such that the result for diagonal operators implies the respective result for arbitrary ones using the Schmidt - representation, see [Pie87, 2.11.4]. The class of Hilbert - Schmidt operators corresponds to the class of operators having square summable singular numbers, see [Pie80, 15.5.5], these corresponding to diagonal mappings having square summable diagonal.

### 3. SOME RELATIONS TO OPERATOR IDEALS

As could be seen in Theorem 3, Hilbert - Schmidt operators admit an unbiased Monte Carlo method. The class of such operators is closely related to the theory of operator ideals as developed in [Pie80]. The first important (though simple) observation is

**Theorem 4.** *The class of all operators admitting an unbiased Monte Carlo method turns into a normed operator ideal by letting  $S \rightarrow u(S)$  be the norm as defined in equation (5).*

This immediately implies

**Corollary 4.** *Nuclear operators admit an unbiased Monte Carlo method.*

We shall introduce this class of operators below. The assertion of the corollary follows from the fact that the ideal of the nuclear operators is the smallest normed ideal. However, a direct proof of the corollary could also be given using the trivial unbiased representation known from the introductory example.

Next we shall see that the class of operators admitting an unbiased Monte Carlo method can be enlarged considerably in many situations. We shall exemplify this by introducing the ideal of  $(r, p, q)$ -nuclear operators.

Let  $0 < r \leq \infty$ ,  $1 \leq p, q \leq \infty$  with  $1 + 1/r \geq \frac{1}{p} + \frac{1}{q}$ . Denote by  $\mathfrak{N}_{(r,p,q)}$  the ideal of  $(r, p, q)$ -nuclear operators, i.e., an operator  $S \in \mathcal{L}(X, Y)$  is  $(r, p, q)$ -nuclear, if there is a representation

$$Sx = \sum_{j=1}^{\infty} \sigma_j \langle x, a_j \rangle y_j, \quad x \in X,$$

with  $a_j \in X'$  and  $y_j \in Y$  satisfying

$$\|\sigma\|_r < \infty$$

$$w_{q'}((a_j)_{j \in \mathbb{N}}) := \sup_{\|x''\| \leq 1} \left( \sum_{j=1}^{\infty} |\langle x'', a_j \rangle|^{q'} \right)^{1/q'} < \infty$$

and

$$w_{p'}((y_j)_{j \in \mathbb{N}}) := \sup_{\|b\| \leq 1} \left( \sum_{j=1}^{\infty} |\langle b, y_j \rangle|^{p'} \right)^{1/p'} < \infty.$$

Denote the respective quasi norm by

$$n_{(r,p,q)}(S) := \inf \|\sigma\|_r w_{q'}((a_j)_{j \in \mathbb{N}}) w_{p'}((y_j)_{j \in \mathbb{N}}),$$

where the infimum is taken over all possible representations, see [Pie80, 18.1] for more details. Especially we obtain as  $\mathfrak{N}_{(1,1,1)}$  the ideal of nuclear operators mentioned above. The ideal of  $(r, p, q)$ -nuclear operators can be characterized by a factorization property, see [Pie80, 18.1.3]. Especially, every  $(q, q, 2)$ -nuclear operator factors through a diagonal mapping from  $l_2$  to  $l_q$  with diagonal belonging to  $l_q$ . (In case  $q = \infty$  the diagonal converges to 0.)

Together with Theorem 1 and Corollary 1 this provides

**Corollary 5.** *Let  $1 \leq q < \infty$ . Every  $(q, q, 2)$ -nuclear operator  $S$  admits an unbiased Monte Carlo method. Moreover we have*

$$u(S) \leq n_{(q,q,2)}(S).$$

The largest ideal of this type which admits unbiased Monte Carlo methods is obtained as  $\mathfrak{N}_{(\infty, \infty, 2)}$ .

For operators acting between Hilbert spaces  $H$  and  $K$  it is readily checked that

$$\mathfrak{N}_{(\infty, \infty, 2)}(H, K) = \mathfrak{N}_{(2, 2, 2)}(H, K)$$

and we obtain the Hilbert - Schmidt operators once more, see [Pie80, 18.5.4].

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