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Quasilinear Parabolic Systems with Mixed Boundary Conditions

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Abstract

In this paper we investigate quasilinear systems of reaction-diffusion equations with mixed Dirichlet-Neumann bondary conditions on non smooth domains. Using techniques from maximal regularity and heat-kernel estimates we prove existence of a unique solution to systems of this type.

Contents

1 Introduction **2**

2 Preliminaries **4**

3 Main result **5**

4 Examples **8**

5 Tools for the proof of Theorem 3.1 **11**

6 Proof of the main result **16**

7 Appendix **18**

1 Introduction

The theory of quasilinear parabolic systems has many applications to evolution problems in natural sciences, see e.g. [2], [1], [4], [5], [19], [9], [30] and [38]. In this paper we investigate in particular systems of reaction-diffusion equations with mixed Dirichlet-Neumann boundary conditions on non-smooth domains $\Omega \subset \mathbb{R}^n$ for $n = 2, 3$ of the form

$$\begin{aligned} u_k' - div(G_k(v)\mu_k \nabla v_k) &= R_k(t, v, \nabla v), \qquad t \in (T_0, T), x \in \Omega, \\ u_k &= b_k\, F_k(v_k), \qquad t \in [T_0, T), x \in \Omega, \\ \nu \cdot \mu_k \nabla v_k &= 0, \qquad t \in [T_0, T), x \in \Gamma_N, \\ v_k &= \phi_k, \qquad t \in [T_0, T), x \in \Gamma_D, \\ v_k(T_0) &= v_{0k}, \qquad x \in \Omega. \end{aligned} \tag{1.1}$$

Here $v = (v_1, \ldots, v_m)$, $\mu_k \in L^\infty(\Omega, M_{n\times n})$ are diffusion coefficients, $b_k \in L^\infty(\Omega)$ reference densities and R_k, G_k, F_k denote the reaction, diffusion and superposition terms for $k \in \{1, \ldots, m\}$.

In many concrete problems which are described as a system of the form (1.1), the underlying domain is non-smooth and the coefficient functions b_k and μ_k are discontinuous. We therefore aim for minimal smoothness assumptions on the boundary $\partial\Omega$ of Ω, the coefficient functions b_k and μ_k as well as on the interface between the Neumann boundary part Γ_N of $\partial\Omega$ and the Dirichlet boundary part $\Gamma_D = \partial\Omega \setminus \Gamma_N$. More precisely, we generally assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain (see [23]) and $\Omega \cup \Gamma_N$ is regular in the sense of Gröger (see [24]). Our approach includes reaction terms R_k which depend discontinously on time t, which is important in many examples (see [38], [25], [30]), in particular in the control theory of parabolic equations. Alternatively, the reader should think e.g. of a manufacturing process for semiconductors, where at a certain moment light is switched on/off and, of course, parameters in the chemical process change abruptly. Note that the original formulation of the evolution equation in terms of balance laws takes the form (see [36, Chap. 21], see also [4])

$$\frac{\partial}{\partial t}\int_{\Omega'} u_k\,dx + \int_{\partial\Omega'} \nu\cdot j_k\,d\sigma = \int_{\Omega'} R_k\,dx \quad ; \quad j_k = j_k(v) = G_k(v)\mu_k\nabla v_k \tag{1.2}$$

where Ω' stands for any (Lipschitzian) subdomain of Ω. Within the variational theory of weak solutions, however, the indicator functions of the subdomains are not admissible test functions. Therefore the integral formulation (1.2) is equivalent to the above evolution equation only if the weak solutions have some additional regularity. It is the main advantage of the present concept that the divergence of the corresponding current $j_k(v)$ indeed is a function, not only a distribution. In a strict sense, only this justifies the application of Gauss' theorem to calculate the normal components of the currents over boundaries of suitable subdomains. Moreover, the fact $div\, j_k \in L^p$ is also of importance for the numerical treatment of (1.1), as the formulation (1.2) is the basis of finite volume methods (see [17]) – namely in the sense of local balances.
Global existence results for (1.1) cannot be expected within such a general approach (see e.g. [16] or [5] and the references therein, see also [27]), and are thus outside the scope of this paper.
In contrast to many papers where existence and uniqueness results for quasilinear parabolic systems are based on the construction of an appropriate evolution operator (see e.g. [1]), our approach relies heavily on maximal L^p-estimates for the linear part of (1.1). In fact, after rewriting equation (1.1) as an abstract evolution equation in $L^p(\Omega)^m$ of the form

$$\begin{aligned} w' - H(t,w)\big(div(\mu\nabla w)\big) &= S(t,w) \\ w(T_0) &= v_0 - \phi(T_0), \end{aligned} \tag{1.3}$$

our strategy to solve (1.3) follows the approach of Clément and Li [9] and Prüss [34]. The advantage in the given situation (1.1) is that subtle techniques from harmonic analysis as well as heat-kernel methods can be used to prove the central L^p-estimates of the linear part. In order to apply these methods in our situation one needs embedding properties of certain interpolation spaces between the domain of the L^p-realization of the underlying elliptic operators and $L^p(\Omega)$ into $W^{1,2p}(\Omega)$. This embedding property rests on the assumption that the operators formally defined by

$$-\nabla \cdot \mu_k \nabla + 1 : W^{1,q}_{\Gamma_N}(\Omega) \to W^{-1,q}_{\Gamma_N}(\Omega)$$

provide topological ismorphisms for some $q > n$. Note that this assumption is in fact fulfilled for many geometric constellations and coefficient functions; see Section 4.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and assume that $n = 2$ or $n = 3$. Denote by $\Gamma \subset \partial\Omega$ an open subset of $\partial\Omega$. For $1 < q < \infty$ we define $W^{1,q}_\Gamma(\Omega)$ as the closure of

$$\{\psi|_\Omega : \psi \in C_c^\infty(\mathbb{R}^n),\ \operatorname{supp}\psi \cap (\partial\Omega\backslash\Gamma) = \emptyset\}.$$

in the Sobolev space $W^{1,q}(\Omega)$. If $q = 2$, we write $H^1(\Omega)$ or $H^1_\Gamma(\Omega)$ instead of $W^{1,2}(\Omega)$ or $W^{1,2}_\Gamma(\Omega)$. Of course, if $\Gamma = \emptyset$, then $W^{1,q}_\Gamma(\Omega) = W^{1,q}_0(\Omega)$. Moreover, throughout this work we always suppose that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger ([24]), this means: for all $x \in \partial\Omega$ there exist open sets $U_x, V_x \subset \mathbb{R}^n$ and a bi-Lipschitz transform Ψ_x from U_x onto V_x such that $x \in U_x$, $\Psi_x(x) = 0$ and $\Psi_x(U_x \cap (\Omega \cup \Gamma_N))$ coincides with one of the sets

$$\begin{aligned}
E_1 &:= \{x \in \mathbb{R}^n : \max_{l=1\dots,n} |x_l| < 1, x_n < 0\},\\
E_2 &:= \{x \in \mathbb{R}^n : \max_{l=1,\dots,n} |x_l| < 1, x_n \le 0\},\\
E_3 &:= \{x \in E_2 : x_n < 0 \text{ or } x_1 > 0\}.
\end{aligned}$$

It is not hard to see that every Lipschitz domain and also its closure is regular in the sense of Gröger, the corresponding model sets are then E_1 or E_2, respectively, see [23]. Moreover, if $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $\partial\Omega \setminus \Gamma_N$ is the finite union of (non-degenerate) closed arc pieces from the boundary, then $\Omega \cup \Gamma_N$ is regular in the sense of Gröger. It is also known (see [20], Satz 1. 103 or [21]) that if $\Omega \cup \Gamma_N$ is regular in the sense or Gröger, then one has the following coincidence:

$$W^{1,q}_{\Gamma_N}(\Omega) = \{\psi \in W^{1,q}(\Omega) : \operatorname{tr}\psi = 0 \text{ a.e. on } \partial\Omega\backslash\Gamma_N\}. \tag{2.1}$$

Finally, for $k \in \{1, \dots, m\}$, let $\mu_k \in L^\infty(\Omega, M_{n\times n})$, where $M_{n\times n}$ denotes the set of all real, symmetric $n \times n$ matrices. Suppose that additionally

$$\inf_{x\in\Omega} \inf_{|\varsigma|=1} \mu_k(x)\varsigma \cdot \varsigma > 0. \tag{2.2}$$

For a closed subspace $V \subseteq H^1(\Omega)$ such that $H_0^1(\Omega) \subseteq V$ we define the form $a_k : V \times V \to \mathbb{R}$ by

$$a_k(u, v) := -\int_\Omega \mu_k \nabla u \cdot \nabla v \,\mathrm{d}x, \quad u, v \in V.$$

The form induces a continuous mapping $\mathcal{A}_k : V \to V'$ such that

$$a_k(u, v) = (\mathcal{A}_k u|v), \quad u, v \in V. \tag{2.3}$$

Here, for $v \in L^2(\Omega)$, $f_v(u) := (v|u)_{L^2}$ defines an element $f_v \in V'$ and $v \mapsto f_v : L^2(\Omega) \to V'$ defines a continuous injection. In the following, we identify v with f_v. We then define the operator A_k as

$$D(A_k) := \{u \in V : \exists f \in L^2(\Omega), a_k(u, \phi) = (f|\phi)\ \forall \phi \in V\} \tag{2.4}$$

$$A_k u := f. \tag{2.5}$$

It is well known that A_k generates an analytic semigroup on $L^2(\Omega)$ which is positivity preserving. Furthermore, this semigroup extends to a C_0-semigroup of contractions on $L^p(\Omega)$ for all $1 < p < \infty$, see [22]. The realization of its generator in L^p is denoted by A_k^p.

3 Main result

We start this section by giving precise assumptions on the coefficients and functions being involved in problem (1.1). In order to do so, let $0 \le T_0 < T_1$ and set $J := (T_0, T_1)$. For $k \in \{1, \dots, m\}$ let $\mu_k \in L^\infty(\Omega, M_{n\times n})$ and assume that (2.2) is satisfied.

Moreover, let for every $k \in \{1...m\}$ the functions $b_k \in L^\infty(\Omega; \mathbb{R})$ be bounded from below by some positive constant.

We assume the following for all $k \in \{1 \dots, m\}$

Op) There exists $p > \frac{n}{2}$ such that each $\mathcal{A}_k - Id$ is a topological isomorphism from $W^{1,2p}_{\Gamma_N}(\Omega)$ onto $W^{-1,2p}_{\Gamma_N}(\Omega)$. For all what follows we fix a number $r > \frac{4p}{2p-n}$.

Su) There exists $f_k \in C^2(\mathbb{R})$, positive, with strictly positive derivative, such that F_k is the superposition operator induced by f_k.

Ga) The mapping $G_k : \left(W^{1,2p}(\Omega)\right)^m \to W^{1,2p}(\Omega)$ is locally Lipschitz.

Gb) For any ball in $\left(W^{1,2p}(\Omega)\right)^m$ there exists $\delta > 0$ such that $G_k(u) \geq \delta$ for all u from this ball.

Ra) The function $R_k : J \times \left(W^{1,2p}(\Omega)\right)^m \to L^p(\Omega)$ is of Caratheodory type, i. e. $R_k(\cdot, u)$ is measurable for all $u \in \left(W^{1,2p}(\Omega)\right)^m$ and $R_k(t, \cdot)$ is continuous for a.a. $t \in J$.

Rb) $R_k(\cdot, 0) \in L^r(J, L^p(\Omega))$ and for $\beta > 0$ there exists $g_\beta \in L^r(J)$ such that

$$\|R_k(t,u) - R_k(t,\tilde{u})\|_{L^p} \leq g(t)\|u - \tilde{u}\|_{W^{1,2p}}, \quad t \in J$$

provided $\max(\|u\|_{W^{1,2p}}, \|\tilde{u}\|_{W^{1,2p}}) \leq \beta$.

BC) $\phi_k \in C(\overline{J}; W^{1,2p}(\Omega)) \cap W^{1,r}(J; L^p(\Omega))$ and $A_k\phi_k(t) = 0$ for all $t \in J$.

IC) $v_{0k} - \phi_k(T_0) \in (L^p(\Omega), D(A_k^p))_{1-\frac{1}{r},r}$.

The assumptions imply that the system (1.1) may be (formally) rewritten as a quasilinear system of the form

$$\begin{aligned} w_k' - H_k(t,w)A_k w_k &= T_k(t,w), \ k = 1, \ldots, m \\ w(T_0) &= v_0 - \phi(T_0), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} &T_k(t,w) := (b_k f_k'(w_k + \phi_k(t)))^{-1}[\nabla G_k(w + \phi(t)) \cdot [\mu_k \nabla(w_k + \phi_k(t))]] \\ &+Q_k(t,w) - \tfrac{\partial \phi_k}{\partial t}(t) \end{aligned} \tag{3.2}$$

with

$$H_k(t,z) := \frac{G_k(z+\phi(t))}{b_k\, f_k'(z_k+\phi_k(t))}, \quad t \in J,\ z \in \left(W^{1,2p}(\Omega)\right)^m \tag{3.3}$$

$$Q_k(t,z) := \frac{R_k(t,z+\phi(t))}{b_k\, f_k'(z_k+\phi_k(t))}, \quad t \in J,\ z \in \left(W^{1,2p}(\Omega)\right)^m \tag{3.4}$$

We are now in the position to state the main result of this paper.

3.1 Theorem. *Let $1 < r, p < \infty$ such that $r > \frac{4p}{2p-n}$, where $n \in \{2,3\}$. Assume that the assumptions (Op), (Su), (Ga), (Gb), (Ra), (Rb), (BC) and (IC) are satisfied. Then there exists a unique local solution $w = (w_1, \ldots, w_m)$ for equation (3.1) on an interval $I = (T_0, T)$ satisfying*

$$w_k \in W^{1,r}(I; L^p(\Omega)) \cap L^r(I; D(A_k)), \quad k \in \{1, \ldots, m\}. \tag{3.5}$$

3.2 Corollary. *Each w_k is Hölder continuous simultaneously in space and time.*

Some remarks at this point are in order.

3.3 Remarks. a) We refer to section 4 for precise geometric and smoothness conditions implying the validity of Assumption (Op).

b) Besides the exponential, a typical example for a function f satisfying assumption Su) is the Fermi-Dirac distribution function

$$f(t) := \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\sqrt{s}}{1+e^{s-t}}\, ds.$$

c) Suppose that v_k coincides on Γ_D with a function $\phi \in C^1(J, W^{1,2p}(\Omega))$. Then there exists ϕ_k satisfying Assumption BC).

d) Note that Condition (BC) implies $\nu \cdot \mu_k \nabla \phi_k = 0$ on Γ_N. This, together with the property (3.5) yields the Neumann boundary condition for v_k on Γ_N, see [18], [8].

4 Examples

Consider Ω and Γ_N, the subset of $\partial\Omega$ on which the Neumann boundary condition is prescribed. In this section we describe geometric configurations for which the above Theorem 3.1 holds true. Furthermore, we present concrete examples of mappings G_k and reaction terms R_k fitting in our framework.

We start with a result, due to Gröger [24], which completely covers the two-dimensional case.

4.1 Proposition. *Assume that $\Omega \cup \Gamma_N$ is regular in the sense of Gröger. Then there exists $q > 2$ such that $\mathcal{A}_k - Id$ is a topological isomorphism from $W^{1,q}_{\Gamma_N}(\Omega)$ onto $W^{-1,q}_{\Gamma_N}(\Omega)$.*

Admissable three-dimensional settings may be described as follows.

4.2 Proposition. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Then there exists $q > 3$ such that $\mathcal{A}_k - Id$ is a topological isomorphism from $W^{1,q}_{\Gamma_N}(\Omega)$ onto $W^{-1,q}_{\Gamma_N}(\Omega)$ provided there is a finite localization of Ω and Γ_N such that the localized sets satisfy one of the following conditions:*

i) Ω has a Lipschitz boundary (see [23]), $\Gamma_N = \emptyset, \mu_k \equiv 1$.

ii) Ω has a Lipschitz boundary, $\Gamma_N = \partial\Omega, \mu_k \equiv 1$.

iii) Ω is a three dimensional Lipschitzian polyhedron, $\Gamma_N = \emptyset$. There are hyperplanes $\mathcal{H}_1 ... \mathcal{H}_n$ in $\mathbb{R}^3$ which meet at most in a vertex of the polyhedron such that the coefficient function μ_k is constantly a real, symmetric, positive definite 3×3 matrix on each of the connected components of $\Omega \setminus \cup_{l=1}^n \mathcal{H}_l$. Moreover, for every edge on the boundary, induced by a hetero interface $\mathcal{H}_l$, the angles between the outer boundary plane and the hetero interface do not exceed π and at most one of them may equal π.

iv) Ω has a Lipschitz boundary. $\Gamma_N = \emptyset$ or $\Gamma_N = \partial\Omega$. $\Omega_\circ \subset \Omega$ is another domain which is C^1 and which does not touch the boundary of Ω. $\mu_k|_{\Omega_\circ} \in BUC(\Omega_\circ)$ and $\mu_k|_{\Omega \setminus \bar{\Omega}_\circ} \in BUC(\Omega \setminus \bar{\Omega}_\circ)$.

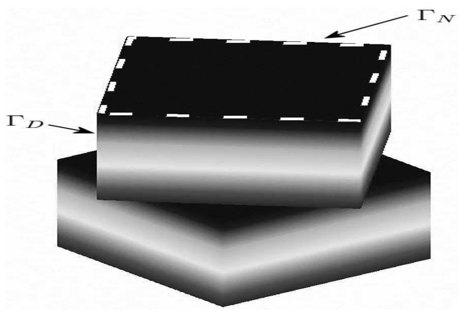
v) *Ω has a Lipschitz boundary. $\Gamma_N = \emptyset$. $\Omega_\circ \subset \Omega$ is a Lipschitz domain, such that $\partial\Omega_\circ \cap \Omega$ is a C^1 surface and $\partial\Omega$ and $\partial\Omega_\circ$ meet suitably. $\mu_k|_{\Omega_\circ} \in BUC(\Omega_\circ)$ and $\mu_k|_{\Omega\setminus\bar{\Omega}_\circ} \in BUC(\Omega \setminus \bar{\Omega}_\circ)$.*

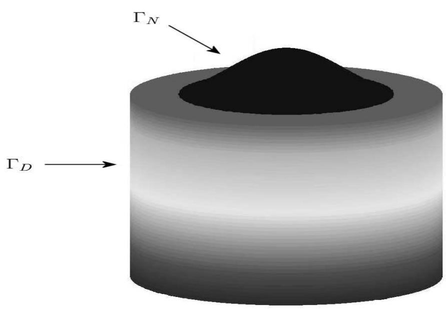
vi) *Ω is a convex polyhedron, $\overline{\Gamma_N} \cap (\partial\Omega\backslash\Gamma_N)$ is a finite union of line segments, $\mu_k \equiv 1$.*

vii) *Ω is a bounded domain with Lipschitz boundary. Additionally, for each $x \in \overline{\Gamma_N} \cap (\partial\Omega\backslash\Gamma_N)$ the mapping Ψ_x defined in Section 2 is a C^1-diffeomorphism from U_x onto V_x, $\mu_k \in BUC(\Omega)$*

A proof of the assertion of Proposition 4.2 can be found for i) in [28], for ii) in [39], for iii) in [13], for iv) and v) in [14], for vi) in [10] and for vii) in [15]. The localization principle is described in [24] and [15]. □

In the following we illustrate two admissable three-dimensional settings. In the figure on the left hand side one assumes Neumann conditions on the top of the upper cuboid, otherwise Dirichlet conditions. In the figure on the right hand side, the boundary of the cylinder is subject to Dirchlet conditions exept for the upper "hat", where Neumann conditions are prescribed.





Next we give two examples for the operators G_k:

4.3 Example. Let $g_k : \mathbb{R}^m \mapsto]0, \infty[$ be a twice continuously differentiable function and define $G_k(z)(x) = g_k(z(x))$ if $z \in \left(W^{1,2p}\right)^m$ and $x \in \Omega$.
In many applications g_k depends only on one variable and is a multiple of the exponential function.

As the second example we present a nonlocal operator arising in the diffusion of bacteria; see [6], [7] and references therein.

4.4 Example. Let η be a continuously differentiable function on $\mathbb{R}$ which is bounded from above and below by positive constants. Assume $\varphi \in L^2(\Omega)$ and define

$$G_k(z) := \eta(\int_\Omega z_k \varphi dx), \quad z = (z_1, ..., z_m) \in \left(W^{1,2p}\right)^m.$$

Now we give two examples for mappings R_k:

4.5 Example. Assume that $[T_0, T_1) = \cup_{l=1}^{j} [t_l, t_{l+1})$ is a (disjoint) decomposition of $[T_0, T_1)$ and let for $l \in \{1, ..., j\}$

$$S_l : \mathbb{R}^m \times \mathbb{R}^{nm} \mapsto \mathbb{R}$$

be a function which satisfies the following condition: For any compact set $K \subset \mathbb{R}^m$ there is a constant L_K such that for any $a, \tilde{a} \in K,\ b, \tilde{b} \in \mathbb{R}^{nm}$ the inequality

$$\begin{aligned} |S_l(a,b) - S_l(\tilde{a},\tilde{b})| \leq\ & L_K |a - \tilde{a}|_{\mathbb{R}^m} \left(|b|^2_{\mathbb{R}^{nm}} + |\tilde{b}|^2_{\mathbb{R}^{nm}}\right) \\ & + L_K |b - \tilde{b}|_{\mathbb{R}^{nm}} \left(|b|_{\mathbb{R}^{nm}} + |\tilde{b}|_{\mathbb{R}^{nm}}\right) \end{aligned}$$

holds. We define a mapping $S : [T_0, T_1[\times \mathbb{R}^m \times \mathbb{R}^{nm} \mapsto \mathbb{R}$ by setting

$$S(t,a,b) := S_l(a,b), \quad \text{if} \quad t \in [t_l, t_{l+1}).$$

The function S defines a mapping R in the following way: If z is the restriction of a $\mathbb{R}^m$-valued, continuously differentiable function on $\mathbb{R}^n$ to Ω, then we put

$$R(t, z, \nabla z)(x) = S(t, z(x), (\nabla z)(x)) \quad \text{for } x \in \Omega \tag{4.1}$$

and afterwards extend R by continuity to the whole set $[T_0, T_1) \times \left(W^{1,2p}(\Omega)\right)^m$.

4.6 Example. *Assume $\sigma : \mathbb{R} \mapsto (0, \infty)$ to be a continuously differentiable function. Further, let $\mathcal{S} : W^{1,2p} \mapsto W^{1,2p}$ be the mapping which assigns to $z \in W^{1,2p}$ the solution φ of the (inhomogeneous) Dirichlet problem*

$$-\nabla \cdot \sigma(z) \nabla \varphi = 0.$$

If one defines

$$R(z) = \sigma(z)|\nabla(\mathcal{S}(z))|^2$$

then, under a reasonable supposition on the boundary value of φ, the mapping R satisfies Assumption (Ra).

This second example comes from a model which describes electrical heat conduction; see [5] and the references therein.

5 Tools for the proof of Theorem 3.1

Let $1 < s < \infty$ and B be a densely defined sectorial operator in a Banach space X. Let again $J = (T_0, T_1)$ for some $T_0, T_1 > 0$. We say that the linear evolution equation

$$\begin{aligned} u' + Bu &= f, \\ u(T_0) &= 0, \end{aligned} \tag{5.1}$$

admits maximal L^s regularity on J if for any $f \in L^s(J;X)$ there exists a unique function $u \in W^{1,s}(J;X) \cap L^s(J;D(B))$ satisfying (5.1) in the L^s-sense. In that case, we write $B \in MR(s,X)$. Observe that

$$W^{1,s}(J;X) \cap L^s(J;D(B)) \hookrightarrow C(\overline{J};X_s), \tag{5.2}$$

where X_s is the real interpolation space $(X, D(B))_{1-\frac{1}{s},s}$. Consider now the quasilinear problem

$$\begin{aligned} u'(t) + \mathcal{B}(t,u(t))u(t) &= F(t,u(t)), \quad t \in J, \\ u(T_0) &= u_0. \end{aligned} \tag{5.3}$$

Here $u_0 \in X_s$, $B := \mathcal{B}(T_0, u_0)$ and $\mathcal{B} : J \times X_s \to \mathcal{L}(D(B);X)$ is continuous. $F : J \times X_s \to X$ is a Caratheodory map. We assume the following Lipschitz conditions on $\mathcal{B}$ and F:

(**B**): For each $R > 0$ there exists a constant $C_R > 0$, such that

$$\|\mathcal{B}(t,u)v - \mathcal{B}(t,\tilde{u})v\|_X \leq C_R\, ||u-\tilde{u}||_{X_s}||v||_{D(B)},\ t \in J, u, \tilde{u} \in X_s, ||u||_s,$$
$$||\tilde{u}||_s \leq R,\ v \in D(B). \tag{5.4}$$

(**F**): $F(\cdot,0) \in L^s(J;X)$ and for each $R > 0$ there is a function $\eta_R \in L^s(J)$ such that

$$\|F(t,u) - F(t,\tilde{u})\|_X \leq \eta_R(t)\ \|u-\tilde{u}\|_s,\ \text{a. a. } t \in J,\ u, \tilde{u} \in X_s,\ ||u||_s, \|\tilde{u}\|_s \leq R. \tag{5.5}$$

Then the following existence and uniqueness result due to Clément and Li [9] and Prüss [34] holds true.

5.1 Proposition. *Assume that (B) and (F) are satisfied and that* $B := \mathcal{B}(T_0, u_0)$ *has the property of maximal* L^s*-regularity. Then there exists* $T \in (T_0, T_1)$ *such that (5.3) admits a unique solution* u *on* $I := (T_0, T)$ *satisfying*

$$u \in W^{1,s}(I;X) \cap\ L^s(I;D(B)).$$

In order to verify the crucial condition that $B = \mathcal{B}(T_0, u_0)$ has maximal L^s-regularity in our situation we need the following results on traces, heat kernels, their multiplicative perturbations and maximal L^s-regularity. We start with the following result on traces.

5.2 Lemma. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lipschitz domain. Then the trace mapping* $tr : H^1(\Omega) \to L^2(\partial\Omega)$ *is order preserving.*

For a proof we refer to [33], Ch. 6.6.1.

5.3 Lemma. *Let* $\Omega \subset \mathbb{R}^n$ *be any domain. Assume that* $u_n \to u$ *in* $H^1(\Omega)$*. Then* $|u_n| \to |u|$, $u_n^+ \to u^+$ *and* $\inf(u_n, 1) \to \inf(u, 1)$ *in* $H^1(\Omega)$*.*

A proof is given in [3], see also [32] and references therein.

Consider a closed subspace V of $H^1(\Omega)$ which includes $H_0^1(\Omega)$. Let $\varrho \in L^\infty(\Omega, M_{n\times n})$ and assume it to be elliptic in the sense of (2.2). Define a bilinear form $a : V \times V \to \mathbb{R}$ on V by

$$a(u,v) = -\int_\Omega \varrho \nabla u \cdot \nabla u \,\mathrm{d}x,\ u, v \in V.$$

Let A be the operator associated to a in $L^2(\Omega)$ and $(e^{tA})_{t\geq 0}$ be the semigroup on $L^2(\Omega)$ generated by A. The following result gives sufficient conditions on the subspace V such that $(e^{tA})_{t\geq 0}$ satisfies an upper Gaussian bound. More precisely, the following holds, see [3].

5.4 Proposition. *Assume that V is a closed subspace of $H^1(\Omega)$ satisfying*

a) $H_0^1(\Omega) \subseteq V$,

b) V *has the* $L^1 - H^1$ *extension property,*

c) $u \in V$ *implies* $|u|$, $\inf(|u|, 1) \in V$,

d) $u \in V$, $v \in H^1(\Omega)$, $|v| \leq u$ *implies* $v \in V$.

Then e^{tA} *satisfies an upper Gaussian estimate, i.e.*

$$(e^{tA} f)(x) = \int_\Omega K_t(x, y) f(y) \mathrm{d}y, \ x \in \Omega, \ f \in L^2(\Omega)$$

for some measurable function $K_t : \Omega \times \Omega \to \mathbb{R}_+$ *and there exists constants* $\gamma, a > 0$ *and* $\omega \in \mathbb{R}$ *such that*

$$0 \leq K_t(x, y) \leq \frac{\gamma}{t^{\frac{n}{2}}} \, e^{\frac{-a|x-y|^2}{t}} \, e^{\omega t}, \quad t > 0, \ a.a. \ x, y \in \Omega. \tag{5.6}$$

5.5 Lemma. *Let $H^1_{\Gamma_N}(\Omega)$ be defined as above. Then $V := H^1_{\Gamma_N}(\Omega)$ satisfies the assumptions a) - d) of Proposition (5.4).*

Proof. Assertion a) is obvious. Concerning b) it seems that the required extension result for $H^1(\Omega)$ is known only for domains with Lipschitz boundary and not for Lipschitz domains. Hence, in the following we give a proof of the subsequent claim which implies the desired $L^1 - H^1$ extension property: Claim: *If Ω is a Lipschitz domain, then there exists a (linear, continuous) extension operator $\mathfrak{E} : L^1(\Omega) \to L^1(\mathbb{R}^n)$ whose restriction to $H^1(\Omega)$ maps this space continuously into $H^1(\mathbb{R}^n)$.*

By definition of Lipschitz domains (see [23]), for every $x \in \partial\Omega$ there is an open neighbourhood U_x of x and a bi-Lipschitz mapping $\Psi_x : U_x \mapsto \mathbb{R}^n$

such that $\Psi_x(x) = 0$ and $\Psi(U_x \cap \Omega)$ is the half cube $E_1 = \{x \in \mathbb{R}^n : \max\limits_{l=1...,n} |x_l| < 1, x_n < 0\}$. Since the image of U_x under Ψ_x is open, there is a number $\zeta_x \in (0,1)$ such that $\zeta_x E \subseteq \Psi_x(U_x \cap \Omega)$, where E is the cube $E = \{x \in \mathbb{R}^n : \max\limits_{l=1...,n} |x_l| < 1\}$. Define O_x as the image of $\zeta_x E$ under Ψ_x^{-1}. For $x \in \Omega$ let O_x be a ball around x whose closure is a subset of Ω. Clearly, the system $\{O_x\}_{x \in \bar{\Omega}}$ is an open covering of $\bar{\Omega}$. Let $O_{x_1}, ..., O_{x_j}, O_{x_{j+1}}, ..., O_{x_l}$ be a finite subcovering, where $x_1, ..., x_j \in \Omega$ and $x_{j+1}, ..., x_l \in \partial\Omega$. Let $\eta_1, ..., \eta_l$ be a partition of unity over $\bar{\Omega}$, subordinated to the covering $O_{x_1}, ..., O_{x_l}$. Obviously, then for any $\varphi \in L^1(\Omega)$ it holds $\varphi = \sum_{k=1}^l \eta_k \varphi$. Moreover, if $\varphi \in H^1(\Omega)$ then this equation holds also true as an equation in $H^1(\Omega)$. Further, one has $supp\, \eta_k \varphi \subseteq supp\, \eta_k \subseteq O_{x_k}$. Therefore, if $k \in \{1, ..., j\}$, the functions $\eta_k \varphi$ can be extended by zero (norm preserving) to whole $\mathbb{R}^n$ and one obtains again a function from $L^1(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$, respectively. For any $k \in \{j+1, ..., l\}$ the function $\eta_k \varphi$ may be transformed via Ψ_x to a function $\widetilde{\eta_k \varphi}$ on $\zeta_{x_k} E_1$, which is then from $L^1(\zeta_{x_k} E_1)$ or from $H^1(\zeta_{x_k} E_1)$, respectively. We define the function $\widehat{\eta_k \varphi}$ on $\zeta_{x_k} E$ as

$$\widehat{\eta_k \varphi}(y) := \begin{cases} \widetilde{\eta_k \varphi}(y) & \text{if} \quad y \in \zeta_{x_k} E_1 \\ \widetilde{\eta_k \varphi}(y_1, ..., y_{n-1}, -y_n) \text{ if} & (y_1, ..., y_{n-1}, -y_n) \in \zeta_{x_k} E_1. \end{cases}$$

Then $\widehat{\eta_k \varphi} \in L^1(\zeta_{x_k} E)$ and $\widehat{\eta_k \varphi} \in H^1(\zeta_{x_k} E)$ if $\varphi \in H^1(\Omega)$. Additionally, $\|\widehat{\eta_k \varphi}\|_{L^1(\zeta_{x_k} E)} = 2\|\widehat{\eta_k \varphi}\|_{L^1(\zeta_{x_k} E_1)}$ as well as $\|\widehat{\eta_k \varphi}\|_{H^1(\zeta_{x_k} E)} = 2\|\widehat{\eta_k \varphi}\|_{H^1(\zeta_{x_k} E_1)}$. Moreover, $supp\widehat{\eta_k \varphi} \subset \zeta_{x_k} E$. We transform $\widehat{\eta_k \varphi}$ back under Ψ_{x_k} and obtain a function which has its support within O_{x_k}, coincides with $\eta_{x_k} \varphi$ on $O_{x_k} \cap \Omega$ and belongs to $L^1(O_{x_k})$ or $H^1(O_{x_k})$, respectively. Trivially, by the support property, each of these functions may be extended by zero (hence norm preserving) to whole R^n. Clearly, this extension then also belongs to $L^1(\mathbb{R}^n)$ or $H^1(\mathbb{R}^n)$, respectively.

In order to prove the first assertion of c), notice first that it suffices to show that $u \in V$ implies $u^+ \in V$. Hence, let $u \in V$ and let $\{u_l\}_l \subset C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} u_l \cap (\partial\Omega \backslash \Gamma_N) = \emptyset$ and $u_l|_\Omega \to u$ in $H^1(\Omega)$. Clearly, then also $\operatorname{supp} u_l^+ \cap (\partial\Omega \backslash \Gamma_N) = \emptyset$, and by Lemma (5.3) we have $u_l^+|_\Omega \to u^+$ in $H^1(\Omega)$. A mollifier argument then yields the claim. The second assertion of c) follows similarly by Lemma 5.3.

In order to prove assertion d) note that Lemma (5.2) a) implies that $0 \leq$

$tr\ |v| \leq tr\ u$ a.e. on $\partial\Omega$. By (2.1), $tr\ u = 0$ a.e. on $\partial\Omega\backslash\Gamma_N$. Hence, $tr\ v = 0$ a.e. on $\partial\Omega\backslash\Gamma_N$, which yields, again by (2.1), that $v \in V = H^1_{\Gamma_N}(\Omega)$. $\square$

Consider the semigroup e^{tA_k} on $L^2(\Omega)$ generated by A_k associated to the form a_k defined in (2.3) with $V = H^1_{\Gamma_N}(\Omega)$. It follows from Proposition (5.4) and Lemma (5.5) that e^{tA_k} is a positive semigroup on $L^2(\Omega)$ satisfying an upper Gaussian bound. Hence, $(e^{tA_k})_{t\geq 0}$ extends to a positive C_0-semigroup of contractions on $L^q(\Omega)$ for all $1 \leq q < \infty$.

5.6 Theorem. *Let $b \in L^\infty(\Omega, \mathbb{R})$ such that $\inf\limits_{x\in\Omega} |b(x)| \geq \delta$ for some $\delta > 0$. Let $1 < s, q < \infty$. Then $bA_k \in MR(s, L^q(\Omega))$ for all $k \in \{1, \dots, m\}$.*

Proof. Let $k \in \{1, \dots, m\}$. By the above remark, e^{tA_k} is a positive contraction semigroup on $L^q(\Omega)$ satisfying an upper Gaussian bound. Hence, the kernel K_t of $e^{t(A_k - \alpha Id)})_{t\geq 0}$ satisfies (5.6) with $\omega = 0$ for suitable $\alpha \in \mathbb{R}$. Moreover, $A_k - \alpha Id$ is self-adjoint in $L^2(\Omega)$. By a result due to Duong and Ouhabaz [12], the semigroup on $L^2(\Omega)$ generated by $b(A_k - \alpha Id)$ satisfies an upper Gaussian bound with $\omega = 0$ as well. Thus $b(A_k - \alpha Id) \in MR(s, L^q(\Omega))$ by a result of Hieber and Prüss (see [26] or [11]). Finally, $bA_k \in MR(s, L^q(\Omega))$ due to the lower order perturbation result of maximal regularity; see [11]. $\square$

5.7 Proposition. *Let $p > \frac{n}{2}$ be the number from Assumption (Op) and assume $\theta \in (\frac{1}{2} + \frac{n}{4p}, 1]$. Then*

$$[L^p, D(A_k^p)]_\theta \hookrightarrow W^{1,2p}_{\Gamma_N}(\Omega)$$

A proof for the three dimensional case is given in [35]; the two dimensional case requires only obvious modifications. A complete, but technically more involved proof for the two dimensional case is contained in [29].

5.8 Corollary. *Let $r > \frac{4p}{2p-n}$. Then*

$$(L^p, D(A_k^p))_{1-\frac{1}{r},r} \hookrightarrow W^{1,2p}_{\Gamma_N}(\Omega)$$

Proof. Let θ be any number from the interval $]\frac{1}{2} + \frac{n}{4p}, 1 - \frac{1}{r}[$. By interpolation

$$(L^p, D(A_k^p))_{1-\frac{1}{r},r} \hookrightarrow (L^p, D(A_k^p))_{\theta,1} \hookrightarrow [L^p, D(A_k^p)]_\theta.$$

Then the assertion follows from the embedding property of the complex interpolation space into $W^{1,2p}_{\Gamma_N}(\Omega)$ established in Proposition 5.7. $\square$

6 Proof of the main result

We first set $X := \big(L^p(\Omega)\big)^m$, $\mathcal{D} := \times_{k=1}^m D(A_k^p)$ and $X_r := (X, \mathcal{D})_{1-\frac{1}{r},r}$ for r as above. By Assumption (IC), $w_0 \in X_r$. Further, for every pair $(t,z) \in [T_0, T_1) \times W^{1,2p}(\Omega)^m$ we define the mapping $H(t,z) : X \mapsto X$ via

$$\varphi := (\varphi_1, \dots, \varphi_m) \mapsto \big(H_1(t,z)\varphi_1, \dots, H_m(t,z)\varphi_m\big). \tag{6.1}$$

Since $H_k(t,z) \in L^\infty(\Omega)$ and since H_k possesses a strictly positive lower bound, it follows that

$$D(H_k(t,z)A_k^p) = D(A_k^p).$$

In particular, $D(H_k(T_0, w_0)A_k^p))$ is dense in $L^p(\Omega)$ (see [22] Thm. 4.5 and Thm. 4.7).

Consider the mapping $\mathcal{B} : J \times X_r \to \mathcal{L}(\mathcal{D}; X)$ given by

$$\mathcal{B}(t,z)\varphi := H(t,z)(A_1^p\varphi_1, \dots, A_m^p\varphi_m), \quad \varphi = (\varphi_1, ..., \varphi_m) \in \mathcal{D}.$$

By Corollary 5.8 and Morrey's theorem we have

$$X_r \hookrightarrow \big(W^{1,2p}_{\Gamma_N}(\Omega)\big)^m \hookrightarrow \big(C^\alpha(\Omega)\big)^m$$

for some $\alpha > 0$. Thus, the assumed properties on F_k, G_k and ϕ_k imply that

$$\mathcal{B} : J \times X_r \to \mathcal{L}(\mathcal{D}; X)$$

is continuous. Moreover, for $\beta > 0$ there exists $C_\beta > 0$ such that

$$\|H(t,z) - H(t,\tilde{z})\|_\infty \leq C_\beta \|z - \tilde{z}\|_{W^{1,2p}}$$

provided $t \in J$ and $\|z\|_{X_r}$ and $\|\tilde{z}\|_{X_r} \leq \beta$. Hence, (5.4) from Assertion (B) is fulfilled.

Furthermore, (5.5) from Assertion (F) holds due to the assumed properties of F_k, G_k, ϕ, R_k and Proposition 5.8. It remains to verify the key condition of Proposition 5.1, namely that $B := \mathcal{B}(T_0, w_0)$ has the property of maximal regularity. To this end, recall that $H(T_0, w_0) \in \big(L^\infty(\Omega)\big)^m$ with a strictly positive lower bound in each component. Thus, $B \in MR(r, X)$ by

Proposition 5.6. Finally, an application of Proposition 5.1 ends the proof of Theorem 3.1.

It remains to show that if w is a solution of (3.1) then $v := w+\phi$ provides a solution of (1.1). This will be done in the Appendix. □

We now give a proof of Corollary 3.2; in fact we prove the following sharper result:

6.1 Lemma. *There exists $\beta > 0$ such that each component w_k of the solution w of* (3.1) *belongs to the space $C^\beta((T_0,T);W^{1,2p}_\Gamma(\Omega)) \hookrightarrow C^\beta((T_0,T); C^\alpha(\Omega))$.*

Proof. We write for short $D_k = D(A_k)$ and $I = (T_0,T)$. Then

$$W^{1,r}(I;L^p) \cap L^r(I;D_k) \hookrightarrow C(\bar{I};(L^p,D_k)_{1-\frac{1}{r},r}) \hookrightarrow C(\bar{I};[L^p,D_k]_\theta),$$

if $\theta \in (0, 1-\frac{1}{r})$.
Moreover, we have the embedding

$$W^{1,r}(I;L^p) \hookrightarrow C^\delta(I;L^p) \quad \text{with} \quad \delta = 1-\frac{1}{r}.$$

Fix $\theta \in (\frac{1}{2}+\frac{n}{4p}, 1-\frac{1}{r})$ and let $\lambda \in (0,1)$ be given such that

$$\theta\lambda > \frac{1}{2}+\frac{n}{4p}.$$

In view of Proposition 5.7 and the reiteration theorem for complex interpolation (see [37]) we obtain

$$\begin{aligned}
&\frac{\|w_k(t)-w_k(s)\|_{W^{1,2p}}}{|t-s|^{\delta(1-\lambda)}} \le \\
&\le c\,\frac{\|w_k(t)-w_k(s)\|_{[L^p,D_k]_{\theta\lambda}}}{|t-s|^{\delta(1-\lambda)}} \sim \frac{\|w_k(t)-w_k(s)\|_{\left[L^p,[L^p,D_k]_\theta\right]_\lambda}}{|t-s|^{\delta(1-\lambda)}} \le \\
&\le \hat{c}\,\frac{\|w_k(t)-w_k(s)\|^{1-\lambda}_{L^p}}{|t-s|^{\delta(1-\lambda)}}\|w_k(t)-w_k(s)\|^{\lambda}_{[L^p,D_k]_\theta} = \\
&= \hat{c}\left(\frac{\|w_k(t)-w_k(s)\|_{L^p}}{|t-s|^{\delta}}\right)^{1-\lambda}\left(2\sup_{s\in\bar{I}}\|w_k(s)\|_{[L^p,D_k]_\theta}\right)^{\lambda}.
\end{aligned}$$

□

7 Appendix

It remains to show that if w is a solution of (3.1) then $v := w + \phi$ provides a solution of (1.1). One easily recognizes that all the manipulations which transfrom (1.1) into (3.1) are straight forward to justify within the distributional calculus - except one. Therefore, we will give a strict justification of this point in the following lemma. Throughout this appendix $f : \mathbb{R} \mapsto \mathbb{R}$ is always assumed to be twice continuously differentiable.

7.1 Lemma. *Assume $p, r \in]1, \infty[$ and $v \in W^{1,r}(]T_0, T[; L^p) \cap C([T_0, T]; C(\bar{\Omega}))$. Then the function $]T_0, T[\ni t \mapsto f(v(t))$ belongs to $W^{1,r}(]T_0, T[; L^p)$ and its distributional derivative is the function $]T_0, T[\ni t \mapsto f'(v(t))v'(t) \in L^r(]T_0, T[; L^p)$.*

7.2 Remark. We denote by $C^1(]T_0, T[; L^p)$ the space of all L^p-valued, continuously differentiable functions on $]T_0, T[$ with bounded derivatives on $]T_0, T[$.

In order to give a proof of Lemma 7.1 we use the following result.

7.3 Lemma. *Let $[T_0, T] \ni t \mapsto \psi(t, \cdot)$ be a mapping belonging to $C([T_0, T]; C(\bar{\Omega})) \cap C^1(]T_0, T[; L^p)$. Then the mapping*

$$]T_0, T[\ni t \mapsto f(\psi(t, \cdot)) \tag{7.1}$$

takes its values in $C(\bar{\Omega}) \hookrightarrow L^p$. It is continuously differentiable when regarded as L^p valued and its derivative in a point $s \in]T_0, T[$ is equal to the L^p-function $f'(\psi(s, \cdot))\psi'(s)$.

Proof. The first assertion is obvious. Concerning the second one, the set $\{\psi(t, x)/x \in \Omega, t \in [T_0, T]\}$ is bounded. Since f is twice continuously differentiable, for $s, t \in]T_0, T[$ and $x \in \Omega$ one may apply Taylor's formulae:

$$\frac{f(\psi(t, x)) - f(\psi(s, x))}{t - s} = f'(\psi(s, x))\frac{[\psi(t, x) - \psi(s, x)]}{t - s} + \tag{7.2}$$

$$+\int_0^1 (1-\tau) f''((1-\tau)\psi(t,x)+\tau\psi(s,x))\, d\tau \quad \frac{[\psi(t,x)-\psi(s,x)]^2}{t-s} \tag{7.3}$$

The family $\left\{f'(\psi(s,\cdot))\frac{[\psi(t,\cdot)-\psi(s,\cdot)]}{t-s}\right\}_t$ converges by the supposition on the differentiablity of the mapping $t \mapsto \psi(t,\cdot)$ in L^p to $f'(\psi(s,\cdot))\psi'(s)$ if t approaches s . It remains to show that the expression in (7.3) approaches zero in L^p. This follows easily from the uniform boundedness of the values $f''((1-\tau)\psi(t,x)+\tau\psi(s,x))$, the boundedness of $\left\{\frac{[\psi(t,\cdot)-\psi(s,\cdot)]}{t-s}\right\}_t$ in L^p and the convergence of $[\psi(t,\cdot)-\psi(s,\cdot)]$ to zero in $C(\bar{\Omega})$ for t approaching s. The continuity of the derivative follows from the continuity of ψ' and the continuity of the function $t \mapsto f'(\psi(t,\cdot))$ in $C(\bar{\Omega})$. □

7.4 Lemma. *Let $v \in W^{1,r}(]T_0,T[;L^p) \cap C([T_0,T];C(\bar{\Omega}))$. Then there is a sequence $\{\psi_l\}_l$ in $C([T_0,T];C(\bar{\Omega})) \cap C^1(]T_0,T[;L^p(\Omega))$ such that $\psi_l \mapsto v$ in $C([T_0,T];C(\bar{\Omega}))$ and $\psi_l' \mapsto v'$ in $L^r(]T_0,T[;L^p)$.*

Proof. Let us define a continuous extension $\tilde{v}$ to all of $\mathbb{R}$ which additionally has compact support as follows: we put

$$\hat{v}(t) := \begin{cases} v(T_0+(T_0-t)) \quad \text{if} \quad t \in]T_0-(T-T_0),T_0[\\ v(t) \text{ if} \quad t \in [T_0,T] \\ v(T-(t-T) \text{ if} \quad t \in]T, T+(T-T_0)[\end{cases} \tag{7.4}$$

(reflection at T_0, T, respectively). Afterwards we multiply $\hat{v}$ by a real valued, continuously differentiable function which is identical 1 on $[T_0,T]$ and which has its support in $]T_0-(T-T_0)/2, T+(T-T_0)/2[$. We define this product as $\tilde{v}$ and identify $\tilde{v}$ with its extension by zero to whole $\mathbb{R}$. Oviously, $\tilde{v}|_{[T_0,T]} = v$; further one verifies the property $\tilde{v} \in W^{1,r}(\mathbb{R};L^p) \cap C(\mathbb{R};C(\bar{\Omega}))$. Let ϑ be the usual mollifier function

$$\vartheta(s) = \begin{cases} \frac{1}{\int e^{-\frac{1}{1-s^2}}\,ds} e^{-\frac{1}{1-s^2}} \quad \text{if} \quad |s|<1 \\ 0 \text{ else on } \mathbb{R} \end{cases}$$

and $\vartheta_l(s) := l\vartheta(l\,s)$. Now we put

$$\psi_l(t) := \begin{cases} \int_{T_0}^t (\tilde{v}' * \vartheta_l)(s)\, ds + (\tilde{v} * \vartheta_l)(T_0), \quad \text{if} \quad t \geq T_0 \\ -\int_t^{T_0} (\tilde{v}' * \vartheta_l)(s)\, ds + (\tilde{v} * \vartheta_l)(T_0), \quad \text{if} \quad t < T_0. \end{cases} \tag{7.5}$$

Then ψ_l is nothing else but $\tilde{v} * \vartheta_l$. This yields $\psi_l \mapsto v$ in $C([T_0,T];C(\bar{\Omega}))$. On the other hand, (7.5) immediately gives $\psi_l' = \tilde{v}' * \vartheta_l$. This means that $\psi_l' \mapsto \tilde{v}'$ in $L^r(\mathbb{R};L^p)$, which implies $\psi_l'|_{]T_0,T[} \mapsto v'$ in $L^r(]T_0,T[;L^p)$. □

We now turn to the proof of Lemma 7.1: Let $\{\psi_l\}_l$ be the sequence from the previous lemma and $\varphi \in C_0^\infty(]T_0,T[)$. Then, considering the function $]T_0,T[\ni t \mapsto f(v(t))$ as a L^p-valued distribution, one gets by the definition of the weak derivative

$$\big(f(v)\big)'(\varphi) = -f(v)(\varphi') = -\int_{T_0}^T f(v(s))\varphi'(s)\,ds =$$

$$= -\int_{T_0}^T \lim_{l\mapsto\infty} f(\psi_l(s))\varphi'(s)\,ds = \lim_{l\mapsto\infty} -\int_{T_0}^T f(\psi_l(s))\varphi'(s)\,ds.$$

By Lemma 7.3, each $f(\psi_l)$ even has a strong (time) derivative which equals $f'(\psi_l)\psi_l'$. From this and integrating by parts one gets

$$-\int_{T_0}^T f(\psi_l(s))\varphi'(s)\,ds = \int_{T_0}^T f'(\psi_l(s))\psi_l'(s)\varphi(s)ds.$$

By construction, $\psi_l \mapsto v$ in $C([T_0,T];C(\bar{\Omega}))$, $\psi_l' \mapsto v'$ in $L^r(]T_0,T[;L^p)$, what implies $f'(\psi_l(\cdot))\psi_l'\varphi \mapsto f'(v(\cdot))v'\varphi$ in $L^r(]T_0,T[;L^p)$. But the integral is a continuous mapping from $L^r(]T_0,T[;L^p)$ into L^p; this finally gives

$$\int_{T_0}^T f'(v(s))v'(s)\varphi(s)\,ds = \int_{T_0}^T \lim_{l\mapsto\infty} f'(\psi_l(s))\psi_l'(s)\varphi(s)ds =$$

$$\lim_{l\mapsto\infty}\int_{T_0}^T f'(\psi_l(s))\psi_l'(s)\varphi(s)ds = \lim_{l\mapsto\infty} -\int_{T_0}^T f(\psi_l(s))\varphi'(s)\,ds = \big(f(v)\big)'(\varphi).$$

Thus, Lemma 7.1 is proved.

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