Weierstraß–Institut
für Angewandte Analysis und Stochastik
im Forschungsverbund Berlin e.V.

A note on pathwise approximation of stationary Ornstein–Uhlenbeck processes with diagonalizable drift

Henri Schurz

submitted: 9th August 1994

Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstraße 39
D - 10117 Berlin
Germany

Preprint No. 112
Berlin 1994

1991 Mathematics Subject Classification. Primary 65U05; Secondary 60H10, 58G32, 34F05.
Key words and phrases. Stochastic differential equations, additive noise, implicit Euler methods.

Edited by
Weierstraß–Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e–mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e–mail (Internet): preprint@iaas-berlin.d400.de

A NOTE ON PATHWISE APPROXIMATION OF STATIONARY ORNSTEIN–UHLENBECK PROCESSES WITH DIAGONALIZABLE DRIFT

HENRI SCHURZ

Department of Stochastic Algorithms and Nonparametric Statistics

Weierstrass Institute for Applied Analysis and Stochastics

Mohrenstr. 39, Berlin 10117, Germany

There is a lack of appropriate replication of the asymptotical behaviour of stationary stochastic differential equations solved by numerical methods. The paper illustrates this fact with the stationary Ornstein-Uhlenbeck process with real–diagonalizable drift and the familiy of implicit Euler methods. For the description of the occuring bias the notions of asymptotical p–th mean, mean, variance and equilibrium preservation are introduced. The main result can be useful for the implementation of numerical algorithms requiring more precise long–term runs, such as in discrete parametric estimation or in numerical computation of top Lyapunov exponents.

KEY WORDS: Stochastic differential equations, additive noise, implicit Euler methods

1. INTRODUCTION

In numerous fields models with additive noise are used to express uncertainty, environmental fluctuations or parameter excitations. They also serve as a possible base for further investigation of the qualitative behaviour of dynamical systems, e.g. how the systems behave under random perturbations which are state–independent. Then it turns out that the stationary Ornstein–Uhlenbeck Process is often met in statistical modelling and seems to be very useful for the purposes mentioned above. The dynamical behaviour of this object $\{X_t, t \geq 0\} \in \mathbb{R}^d$ can be described by the stochastic differential equation (SDE)

$$dX_t \;=\; A\,X_t\,dt \;+\; \sum_{j=1}^{m} b^j\,dW_t^j \tag{1}$$

with an initial value $X_t = X_0 \in \mathbb{R}^d$ (deterministic or Gaussian distributed). The system (1) is driven by the Brownian motion $W_t = (W_t^1, ..., W_t^m)$ which represents m independent, identically distributed Gaussian random variables ($\in \mathcal{N}(0,t)$). Details about this stochastic object and corresponding calculus can be found, e.g. in Karatzas and Shreve [7]. We suppose that throughout this paper $\mathbb{E}\,\|X_0\|^2 < +\infty$ and X_0 is independent of $\mathcal{F}_t^j = \sigma\{W_s^j, 0 \leq s \leq t\}$ $(j = 1, 2, ..., m)$, the σ–algebra generated by the underlying Wiener process. Matrix A as a real–valued $d \times d$ matrix in (1) may or may not depend on time t, however its eigenvalues have only negative real parts. The diffusion vectors $b^j \in \mathbb{R}^d$ are assumed to be bounded as well, in case of dependence on time t. The solution expression of this stochastic process is well–known with

$$X_t = \exp(At) \left(X_0 + \sum_{j=1}^{m} \int_0^t \exp(-As)\, b^j(s) dW_s^j \right). \tag{2}$$

One is even capable of stating its corresponding probability distribution. More precisely,

$$\exp(-At)X_t - X_0 =: Q_t \in \mathcal{N}\Big(0, \sum_{j=1}^{m} \int_0^t \exp(-As)\, b^j(s) b^{j\,T}(s) \exp(-A^T s)\, ds\Big).$$

$\mathcal{N}(\mu, \sigma^2)$ denotes the law of Gaussian distribution with mean μ and covariance σ^2. Assume that drift matrix A is diagonalizable. Then it exists an invertible matrix L such that

$$A = L^{-1}\, D\, L \tag{3}$$

where D is a $d \times d$ diagonal matrix with complex–valued elements. Exploiting this fact we can transform $X_t \to Z_t = L\, X_t$ and obtain the new SDE

$$d\, Z_t = D\, Z_t\, dt + \sum_{j=1}^{m} L b^j\, dW_t^j \tag{4}$$

starting in $Z_0 = LX_0$. Obviously system (4) consists of d separated components, hence for the analytical solution of this system we can separately consider its single components and find

$$d\, Z_t^i = d_i\, Z_t^i\, dt + \sum_{j=1}^{m} [Lb^j]_i\, dW_t^j \tag{5}$$

with $d_i \in \mathbb{C}$ (space of complex numbers). The solution expression of (5) for autonomous systems (i.e. systems with the time–independent drift d_i and diffusion components $[Lb^j]_i$) is very simple and found to be

$$Z_t^i = \exp(d_i t) \left(Z_0^i + \sum_{j=1}^{m} [Lb^j]_i \int_0^t \exp(-d_i s)\, dW_s^j \right). \tag{6}$$

Thus we know explicit solutions of (1) and (4) as well. However, in expressions both (2) and (6) we have to calculate the value of stochastic integrals for pathwise evolution of the processes X_t and Z_t along a Wiener path. The probability distribution of these stochastic integrals is known under complete information of the underlying Wiener process.

An objective of this paper is to provide a concept and some results for assessment of asymptotical behaviour of discrete time approximations for SDEs with additive noise. Therein Ornstein–Uhlenbeck processes (1) and (5) serve as a test system for

approximations of more general SDEs with additive noise to some extent, e.g. by linearization of the drift part around equilibria. There are several ways to approximate SDEs and stochastic integrals over functionals of solutions of SDEs on finite time intervals (in fact a large variety!).
Instead of proceeding on with the description of different generation possibilities, we want to examine the following task. Given the information about the underlying Wiener path at discrete time points $(t_n)_{n\in\mathbb{N}}$, i.e. $\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n)$ is known and fixed. Now we are especially interested in an adequate replication of the long–term behaviour of the Ornstein–Uhlenbeck process (1) by corresponding approximations along that fixed Wiener path. This interest is naturally given. If one computes Lyapunov exponents (see [23]) or estimates parameters in drift and diffusion part of (1) (see [6], [9]) then accurate and stable long–term integration is required. Only then one receives reasonable and reliable results. The computation along one and same Wiener path is especially important when one compares stochastic integration techniques with respect to one and the same Wiener path, and one is aiming at crystallizing out an appropriate technique. In particular, for parametric estimation under discretely observed diffusions while approximating continuous time models one needs some guarantee for correct replication of asymptotical behaviour of the exact solution of SDEs. There this problem basicly arises during stochastic integration which is necessary for the computation of likelihood estimators under discrete observation, cf. [6], [9], [11] or [19]. One uses substitutions of continuous time estimators by corresponding discrete versions and supposes that these discretizations correctly provide the behaviour of the continuous time estimates as integration time t tends to infinity. A general justification and proof of this approach seems to be very complicated, due to the nonlinear structure of likelihood quotients. A similar effect can be observed in estimation of Lyapunov exponents. It should be clarified whether one estimates the top Lyapunov exponent of the discrete or continuous time solution. Clearly, as integration time tends to zero one would obtain the correct Lyapunov exponent (of continuous time system) under sufficient smoothness conditions. However, the usage of 'almost vanishing' (very small) step sizes contradicts to the requirement of 'finiteness and efficiency' on practical algorithms.
Although one exactly knows the probability distribution of (2) and (6), one already arrives into troubles in order to replicate the asymptotical behaviour of the exact stochastic process under discrete time observation of the underlying Wiener path. This fact will be verified by section 3. Before, in section 2 we clarify which numerical solutions are to be under further investigation.

2. NUMERICAL SOLUTIONS FOR SDES (1) AND (4)

It would be a natural way to make use of numerical techniques for solving of SDEs (1) and (4). They allow to get a straight forward, pathwise link between the current Wiener process increments and the stochastic integration. Let Y_n be the value of approximation using equidistant step size $\Delta > 0$ at time point $t_n \in [0, \infty)$. For simplicity we only consider equidistant approximations. Then from [9] we know the familiy of *implicit Euler schemes* with

$$\begin{aligned} Y_{n+1} &= Y_n + \left(\alpha\, A\, Y_{n+1} + (1-\alpha)\, A\, Y_n\right)\Delta + \sum_{j=1}^{m} b^j\, \Delta W_n^j \\ Y_0 &= X_0 \in \mathbb{R}^d \qquad (n = 0,1,2,\dots) \end{aligned} \tag{7}$$

for system (1). $\alpha \in [0,1]$ represents an implicitness parameter to be choosen appropriately. In fact, these schemes provide us with the simplest class of numerical methods solving system (1) at discrete time points t_n. On finite time intervals $[0,T]$ ($T < +\infty$) one is entitled to use them as strong approximations of SDE (1), i.e. the criterion of *strong convergence*

$$\forall\ \xi = (t_n)_{n\in\mathbb{N}} \in [0,T]: \quad \sup_{t_n \le T} \mathbb{E}\, ||X_{t_n} - Y_n|| \ \le\ K_1(T)\, \Delta^{\gamma_1} \tag{8}$$

is satisfied with $\gamma_1 = 1.0$. ξ denotes a discretization of the time axis as collection of time points t_n. Moreover one could also show the validity of *mean–square convergence* towards (2). This criterion has the form

$$\forall\ \xi = (t_n)_{n\in\mathbb{N}} \in [0,T]: \quad \sup_{t_n \le T} \mathbb{E}\, ||X_{t_n} - Y_n||^2 \ \le\ K_2^2(T)\, \Delta^{2\gamma_2} \tag{9}$$

with order $\gamma_2 = 1.0$. Note that the schemes (7) are identical with the family of *implicit Mil'shtein schemes* for system (1), cf. [9]. There is a large variety of further numerical schemes. For references and some aspects, e.g. see [2], [9], [14], [15], [16], [18], [23] or [10]. In particular, Shkurko [21] and Törok [25] have already dealt with linear numerical schemes. An alternative to these references is given by Kushner and Dupuis [12] via constructing Markov chain approximations for solving problems in stochastic control (Time and space are discretized for computation of control functionals). Here we follow the direct approach of references above. However, most of the suggested schemes require more smoothness on drift and diffusion functions or more information on the σ–algebra generated by the underlying Wiener process in order to achieve higher order of strong or mean square convergence. Clark and Cameron [3] showed that the highest possible order of mean square convergence is one, provided that only the Wiener increments are used. Thus, we naturally restrict the main attention to 'lower order methods'.

3. THE PRESERVATION OF ASYMPTOTICAL PROPERTIES

In the following we will show that there can be a distance between the asymptotical behaviour of the exact and numerical solution generated by (7). For this purpose we introduce the notions of *asymptotical p–th mean, mean, variance and equilibrium preservation.* Each of these notions reflects an asymptotical property of numerical solutions compared with the asymptotical behaviour of the exact solution. It also gives some information on the replication of possible equilibria of the considered stochastic systems.

DEFINITION: *Let $\{X_t, t \geq 0\} \subseteq \mathbb{R}^d$ be a stationary, ergodic stochastic process governed by SDE (1). Then the numerical solution $(Y_n)_{n\in\mathbb{N}}$ is said to be* ***(asymptotical) p–th mean preserving*** *for SDE (1) if*

$$\lim_{n\to+\infty} \mathbb{E}||Y_n||^p = \mathbb{E}||X_\infty^p||^p := \lim_{t\to+\infty} \mathbb{E}||X_t||^p$$

for an exponent $p \in \mathbb{R}^1$. Furthermore, it is called ***(asymptotical) mean preserving*** *for SDE (1) if*

$$\lim_{n\to+\infty} \mathbb{E}Y_n = \mathbb{E}X_\infty := \lim_{t\to+\infty} \mathbb{E}X_t\,,$$

(asymptotical) variance preserving *for SDE (1) if*

$$\lim_{n\to+\infty} \mathbb{E}Y_nY_n^T = \mathbb{E}X_\infty X_\infty^T := \lim_{t\to+\infty} \mathbb{E}X_tX_t^T$$

and ***(asymptotical) equilibrium preserving*** *for SDE (1) if*

$$\mathcal{L}\left(\lim_{n\to+\infty} Y_n\right) = \mathcal{L}\left(X_\infty\right) := \mathcal{L}\left(\lim_{t\to+\infty} X_t\right)$$

where $\mathcal{L}(.)$ denotes the probability law of the corresponding random variable.

$(.)^T$ denotes the transpose of the inscribed vectors or matrices throughout this paper. The involved norm can be any choosen vector norm. For the sake of simplicity, we take the Euclidean vector norm, i.e. $||x||^2 = \sum_{i=1}^d x_i^2$ for all $x \in \mathbb{R}^d$. If $p = 2$ we also use the notation of mean square preservation.

Remarks. The conditions of the definition above ensure the convergence of the process X_t towards the stationary solution (equilibrium) of SDE (1) as t tends to infinity. In contrast to deterministic analysis and to stochastic bilinear models with purely multiplicative noise, the corresponding stationary solution for the differential system (1) is a random variable which has Gaussian distribution with mean zero, hence not a simple, deterministic number. In the case of linear, multiplicative noise (i.e. state–dependent diffusion functions $b^j(x) = B^jx$ with $d \times d$ matrices B^j) the herein introduced notion of asymptotical p–th mean preservation would be identical with the notion of asymptotical p–th mean stability of numerical solutions, cf. Khas'minskij [8] or Kozin [13]. The notion of mean preservation represents the weakest notion among the presented ones. Moreover, in case of linear systems the concept of mean preservation reduces to the stability problem as known in deterministic numerical analysis. Thereby we may consider the concept of asymptotical preservation as an extension of stability concepts being common so far.
For the sake of simplicity, from now on we only consider autonomous systems (1). That is, systems with time–independent drift and diffusion components.

THEOREM: *There is only one numerical method (7) which exactly replicates the asymptotical behaviour of the stationary Ornstein–Uhlenbeck processes governed by SDE (1) with arbitrary in $\mathbb{R}^d$ diagonalizable drift matrix A for all step sizes $\Delta > 0$. More precisely, $(Y_n)_{n\in\mathbb{N}}$ generated by (7) with equidistant step size Δ and implicitness degree $\alpha = 0.5$ is asymptotical mean, p–th mean, variance and equilibrium preserving for the model class of stationary SDEs (1) with in $\mathbb{R}^d$ diagonalizable drift matrices A.*

Proof: In analogous manner to deterministic analysis, for $\alpha \geq 0.5$ we easily verify the property of asymptotical mean preservation by (7) for all possible step sizes $\Delta > 0$. Now we continue with investigating the variance evolution of implicit Euler schemes. Consider $V_n = L\, Y_n$ where $A = L^{-1}DL$ with real $d \times d$ matrices L and D ($D = I(d_i)$ is the diagonal Jordan form of A, $d_i \in \mathbb{C}$, I unit matrix of $\mathbb{R}^{d\times d}$). Then the transformed Euler scheme has the form

$$V_{n+1} = V_n + (\alpha\, D\, V_{n+1} + (1-\alpha)\, D\, V_n)\Delta + \sum_{j=1}^{m} Lb^j\, \Delta W_n^j\,. \tag{10}$$

where $A = L^{-1}DL$. Because of the stationarity of SDE (1) the drift matrix A must have only eigenvalues with negative real parts, hence matrix D too. Thus A and D are invertible. Now we can rewrite (10) to the explicit form

$$V_{n+1} = (I - \alpha\, D\, \Delta)^{-1} \left(V_n(I + (1-\alpha) D\, \Delta) + \sum_{j=1}^{m} Lb^j\, \Delta W_n^j \right)\,.$$

This system has completely separated components, hence we are able to treat it componentwisely. Let V_n^i denote the $i-th$ component of the approximation $V_n (i = 1, 2, ..., d)$. Then one encounters with

$$V_{n+1}^i = \frac{V_n^i\,(1 + (1-\alpha) d_i \Delta) + \sum_{j=1}^{m} \sigma_i^j\, \Delta W_n^j}{1 - \alpha d_i \Delta}$$

where $V_0^i = [LX_0]_i$ and $\sigma_i^j = [L\, b^j]_i$. Assume that $d_i \in \mathbb{R}$. After introducing the abbreviation

$$U_{n+1}^{i,k} \;:=\; \mathbb{E}\, V_{n+1}^i V_{n+1}^k$$

for all $i, k = 1, 2, ..., d$, a computation leads to the iterative relation

$$U_{n+1}^{i,k} = \nu_{i,k}\, U_n^{i,k} + \beta_{i,k} = \beta_{i,k} \sum_{l=0}^{n} (\nu_{i,k})^l + (\nu_{i,k})^{n+1}\, U_0^{i,k}$$

where

$$\nu_{i,k} = \frac{\Big(1 + (1-\alpha) d_i \Delta\Big)\Big(1 + (1-\alpha) d_k \Delta\Big)}{\Big(1 - \alpha d_i \Delta\Big)\Big(1 - \alpha d_k \Delta\Big)} \quad \text{and} \quad \beta_{i,k} = \frac{\sum_{j=1}^{m} \sigma_i^j \sigma_k^j\, \Delta}{\Big(1 - \alpha d_i \Delta\Big)\Big(1 - \alpha d_k \Delta\Big)}\,.$$

Because the real part of $d_i \in \mathbb{C}$ is negative, we find that $(\nu_{i,k})^{n+1} \underset{n\to+\infty}{\longrightarrow} 0$ for all step sizes $\Delta > 0$ under the assumption of $\alpha \geq 0.5$. Just as well the series $\sum_{l=0}^{n} (\nu_{i,k})^l$ must converge to the limit $1/(1 - \nu_{i,k})$. Consequently, it holds

$$U_{n+1}^{i,k} \underset{n\to+\infty}{\longrightarrow} \frac{\beta_{i,k}}{1 - \nu_{i,k}} =: U_\infty^{i,k}.$$

Now, we analyze $U_\infty^{i,k} = U_\infty^{i,k}(\Delta)$ by

$$\begin{aligned} U_\infty^{i,k} &= \frac{\beta_{i,k}}{1-\nu_{i,k}} = \frac{\sum_{j=1}^m \sigma_i^j \sigma_k^j \Delta}{(1-\alpha d_i \Delta)(1-\alpha d_k\Delta) - (1+(1-\alpha)d_i\Delta)(1+(1-\alpha)d_k\Delta)} \\ &= -\sum_{j=1}^m \frac{\sigma_i^j \sigma_k^j}{d_i + d_k + (1-2\alpha)\Delta d_i d_k}. \end{aligned}$$

Calculating the second moment evolution of the exact solution one encounters with

$$\mathbb{E}\, Z_\infty^i Z_\infty^k = -\sum_{j=1}^m \frac{\sigma_i^j \sigma_k^j}{d_i + d_k}.$$

Note that $d_i, d_k < 0$. That is

$$U_\infty^{i,k}(\Delta) = \mathbb{E}\, Z_\infty^i Z_\infty^k \quad \text{iff} \quad \alpha = 0.5$$

for all step sizes $\Delta > 0$. Thus asymptotical variance preservation is only observed for $\alpha = 0.5$. After those steps above one transforms numerical solution $(V_n)_{n\in\mathbb{N}}$ back to $(Y_n)_{n\in\mathbb{N}}$ via relation $Y_n = L^{-1}V_n$. Besides one uses the connections

$$\mathbb{E}\, Y_n Y_n^T = L^{-1}\left(\mathbb{E}\, V_n V_n^T\right) L^{-1^T} \quad \text{and} \quad \mathbb{E}\, X_t X_t^T = L^{-1}\left(\mathbb{E}\, Z_t Z_t^T\right) L^{-1^T}$$

and obtains the validity of

$$\lim_{n\to+\infty} \mathbb{E}\, Y_n Y_n^T = \lim_{t\to+\infty} \mathbb{E}\, X_t X_t^T$$

for $\alpha = 0.5$ under real diagonalizability of matrix A (i.e. $d_i \in \mathbb{R}$). Thus asymptotical variance preservation can be verified for the original system (1). Furthermore, we know that the limit distribution of (7) is Gaussian. Consequently, the limit distributions of the exact and numerical solution are identical (preservation of the equilibrium law), i.e. the distance between the asymptotical behaviour of the numerical (7) and exact solution of (1) only vanishes for $\alpha = 0.5$ and arbitrary step sizes $\Delta > 0$, as claimed in the theorem. Asymptotical p–th mean preservation is obvious from the equality of limit distributions. Thereby the proof has been completed. ◊

4. REMARKS AND CONCLUSIONS

In this paper we established a remarkable asymptotical bias between the behaviour of exact and simplest numerical solutions with additive noise. This distance significantly depends on the step size of numerical integration. Only the half drift–implicit Euler scheme (i.e. implicitness $\alpha = 0.5$) could exactly replicate the asymptotical behaviour of the stationary Ornstein–Uhlenbeck process. With the introduced notions of asymptotical p–th mean, mean, variance and equilibrium preservation one can assess to some extent the goodness of stochastic approximations with respect to their replication of the stationary behaviour of exact solutions of dynamical systems, at least in the sense of the mean, variance and absolute moments. Moreover, because the stationary numerical behaviour for the Ornstein–Uhlenbeck

Process is given as a Gaussian distributed random variable with mean zero and corresponding covariance matrix, we know numerical solutions providing the same stationary Gaussian probability distribution as that of the corresponding stationary, exact solution. Note that the Gaussian distribution is uniquely described by the behaviour of first and second moments. Consequently, with the asymptotical mean and variance preservation by the half drift–implicit Euler scheme one only receives the correct limit distribution within the class of numerical methods with lower smoothness requirements. This is clear for Ornstein–Uhlenbeck Processes with in $\mathbb{R}^d$ diagonalizable drift.
A corresponding approach to systems with multiplicative noise (i.e. with state–dependent diffusion part) is presented in [20]. There some stability analysis of the implicit Euler schemes leads to their mean square stability (hence to a preservation of deterministic equilibria) under appropriate conditions on the corresponding continuous time systems and with implicitness degree $\alpha \geq 0.5$. Therefore, summarizing the main result of this paper and [20], one obtains the superiority of half drift–implicit Euler methods ($\alpha = 0.5$), at least for linear systems of Itô SDEs.

Acknowledgements

Special thanks are given to **Prof. Michael Sørensen** (University of Aarhus, Denmark) and **Prof. Marianne Hübner** (Tufts University, USA) who inspired the author on writing down the gist of this paper.

References

[1] L. Arnold, *Stochastic Differential Equations*, Wiley, New York, 1974.
[2] A.T. Averina and S.S. Artemiev, Numerical solution of systems of stochastic differential equations, *Soviet J. Num. Anal. Math. Modelling* **3** (1988), No. 4, 267-285.
[3] J.M.C. Clark and R.J. Cameron, The maximum rate of convergence of discrete approximations for stochastic differential equations, in: B. Grigelionis (Ed.), *Stochastic Differential Systems*, Lecture Notes in Control and Information Systems **25**, Springer, Berlin, 1980, pp. 162-171.
[4] T.C. Gard, *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York, 1988.
[5] C.W. Gardiner, *Handbook of Stochastic Methods*, Springer, New York, 1985.
[6] D. Florenz–Zmirou, Approximate discrete–time schemes for statistics of diffusion processes, *Statistics* **20** (1989), No. 4, 547–557.
[7] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer, New York, 1988.
[8] R.Z. Khas'minskij, *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.
[9] P.E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Appl. of Math. **23**, Springer, New York, 1992.
[10] P.E. Kloeden, E. Platen and H. Schurz, *Numerical Solution of Stochastic Differential Equations Through Computer Experiments*, Universitext, Springer, New York, 1994.
[11] P.E. Kloeden, E. Platen, H. Schurz and M. Sørensen, On effects of discretization of estimators of diffusions, *Theory Prob. Appl.* (submitted), Report No. **249**, University of Aarhus, Aarhus, 1992.

[12] H. Kushner and G. Dupuis, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Appl. of Math. **24**, Springer, New York, 1992.

[13] F. Kozin, A survey of stability of stochastic systems, *Automatica* **5** (1969), 95-112.

[14] G.N. Mil'shtein, *Numerical Integration of Stochastic Differential Equations* (in Russian), Uralski Univ. Press, Sverdlovsk, 1988.

[15] N.J. Newton, An asymptotically efficient difference formula for solving stochastic differential equations, *Stochastics* **19** (1986), 175-206.

[16] N.J. Newton, Asymtotically efficient Runge–Kutta methods for a class of Itô and Stratonovich equations, *SIAM J. Appl. Math.* **51** (1991), 542–567.

[17] B. Øksendal, *Stochastic Differential Equations* (3rd edition), Universitext, Springer, 1992.

[18] E. Pardoux and D. Talay, Discretization and simulation of stochastic differential equations, *Acta Appl. Math.* **3** (1985), 23-47.

[19] A.R. Pedersen, *A new approach to maximum likelihood estimation for stochastic differential equations based on discrete observations*, Report No. **264**, University of Aarhus, Aarhus, 1993.

[20] H. Schurz, Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise, *Stoch. Anal. Appl.* (submitted), Preprint No. **108**, WIAS, Berlin, 1994.

[21] I.O. Shkurko, Numerical solution of linear systems of stochastic differential equations (in Russian), in *Numerical Methods for Statistics and Modelling. Collected Scientific Works*, Novosibirsk, 1987, pp. 101–109.

[22] K. Sobczyk, *Stochastic Differential Equations*, Kluwer, Dordrecht, 1991.

[23] D. Talay, *Simulation and numerical analysis of stochastic differential systems: A review*, Rapports de Recherche, No. **1313**, INRIA, Valbonne, 1990.

[24] D. Talay, Approximation of upper Lyapunov exponents of bilinear stochastic differential equations, *SIAM J. Num. Anal.* **28** (1991), 1141–1164.

[25] C. Tõrok, Numerical solution of linear stochastic differential equations, *Comp. Math. Appl.* **27** (1994), No. 4, 1–10.

[26] E. Wong and M. Zakai, On the relation between ordinary and stochastic differential equations, *Int. J. Eng. Sci.* **3** (1965), 213–229.

Recent publications of the Weierstraß–Institut für Angewandte Analysis und Stochastik

Preprints 1994

83. Klaus Matthes, Rainer Siegmund–Schultze, Anton Wakolbinger: Recurrence of ancestral lines and offspring trees in time stationary branching populations.

84. Karmeshu, Henri Schurz: Moment evolution of the outflow–rate from nonlinear conceptual reservoirs.

85. Wolfdietrich Müller, Klaus R. Schneider: Feedback stabilization of nonlinear discrete–time systems.

86. Gennadii A. Leonov: A method of constructing of dynamical systems with bounded nonperiodic trajectories.

87. Gennadii A. Leonov: Pendulum with positive and negative dry friction. Continuum of homoclinic orbits.

88. Reiner Lauterbach, Jan A. Sanders: Bifurcation analysis for spherically symmetric systems using invariant theory.

89. Milan Kučera: Stability of bifurcating periodic solutions of differential inequalities in $\mathbb{R}^3$.

90. Peter Knabner, Cornelius J. van Duijn, Sabine Hengst: An analysis of crystal dissolution fronts in flows through porous media Part I: Homogeneous charge distribution.

91. Werner Horn, Philippe Laurençot, Jürgen Sprekels: Global solutions to a Penrose–Fife phase–field model under flux boundary conditions for the inverse temperature.

92. Oleg V. Lepskii, Vladimir G. Spokoiny: Local adaptivity to inhomogeneous smoothness. 1. Resolution level.

93. Wolfgang Wagner: A functional law of large numbers for Boltzmann type stochastic particle systems.

94. Hermann Haaf: Existence of periodic travelling waves to reaction–diffusion equations with excitable–oscillatory kinetics.

95. Anton Bovier, Véronique Gayrard, Pierre Picco: Large deviation principles for the Hopfield model and the Kac–Hopfield model.

96. Wolfgang Wagner: Approximation of the Boltzmann equation by discrete velocity models.

97. Anton Bovier, Véronique Gayrard, Pierre Picco: Gibbs states of the Hopfield model with extensively many patterns.

98. Lev D. Pustyl'nikov, Jörg Schmeling: On some estimations of Weyl sums.

99. Michael H. Neumann: Spectral density estimation via nonlinear wavelet methods for stationary non–Gaussian time series.

100. Karmeshu, Henri Schurz: Effects of distributed delays on the stability of structures under seismic excitation and multiplicative noise.

101. Jörg Schmeling: Estimates of Weyl sums over subsequences of natural numbers.

102. Grigori N. Milstein, Michael V. Tret'yakov: Mean–square approximation for stochastic differential equations with small noises.

103. Valentin Konakov: On convergence rates of suprema in the presence of non–negligible trends.

104. Pierluigi Colli, Jürgen Sprekels: On a Penrose–Fife model with zero interfacial energy leading to a phase–field system of relaxed Stefan type.

105. Anton Bovier: Self–averaging in a class of generalized Hopfield models.

106. Andreas Rathsfeld: A wavelet algorithm for the solution of the double layer potential equation over polygonal boundaries.

107. Michael H. Neumann: Bootstrap confidence bands in nonparametric regression.

108. Henri Schurz: Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise.

109. Gottfried Bruckner: On the stabilization of trigonometric collocation methods for a class of ill–posed first kind equations.

110. Wolfdietrich Müller: Asymptotische Input–Output–Linearisierung und Störgrößenkompensation in nichtlinearen Reaktionssystemen.

111. Vladimir Maz'ya, Gunther Schmidt: On approximate approximations using Gaussian kernels.