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# A shape calculus analysis for tracking type formulations in electrical impedance tomography


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Abstract. In the paper [17], the authors investigated the identification of an obstacle or void of perfectly conducting material in a two-dimensional domain by measurements of voltage and currents at the boundary. In particular, the reformulation of the given *nonlinear* identification problem was considered as a shape optimization problem using the Kohn and Vogelius criterion. The compactness of the *complete* shape Hessian at the optimal inclusion was proven, verifying strictly the ill-posedness of the identification problem. The aim of the paper is to present a similar analysis for the related *least square tracking* formulations. It turns out that the *two-norm-discrepancy* is of the same principal nature as for the Kohn and Vogelius objective. As a byproduct, the necessary first order optimality condition are shown to be satisfied *if and only if* the data are perfectly matching. Finally, we comment on possible consequences of the two-norm-discrepancy for the regularization issue.

## Introduction

Let $D \subset \mathbb{R}^2$ denote a bounded domain with boundary $\partial D = \Sigma$ and assume the existence of a simply connected subdomain $S \subset D$, consisting of perfectly conducting material, essentially different from the likewise constant conductivity of the material in the annular subregion $\Omega = D \setminus \overline{S}$. We consider the identification problem of this inclusion if the Cauchy data of the electrical potential $u$ are measured at the boundary $\Sigma$ , i.e., if a single pair $f = u|_\Sigma$ and $g = (\partial u/\partial \mathbf{n})|_\Sigma$ is known.

The problem under consideration is a special case of the general conductivity reconstruction problem and is severely ill-posed. It has been intensively investigated as an inverse problem. We refer for example to Akduman and Kress [1], Chapko and Kress [5] and Hettlich and Rundell [26] for numerical algorithms and to Friedmann and Isakov [21] as well as Alessandrini, Isakov and Powell [2] for particular results concerning uniqueness. Moreover, we refer to Brühl and Hanke [3, 4] for methods using the complete Dirichlet–to–Neumann operator at the outer boundary. We emphasize that we focus in the present paper on exact measurements and do not consider noisy data.

In [38], Roche and Sokolowski have been introduced a formulation as shape optimization problem using the Kohn and Vogelius criterion. The analysis and numerical results presented there for first order shape optimization algorithms are extended to second order methods in [17]. In particular,compactness of the shape Hessian is proven at the optimal domain $\Omega^\star = D \setminus \overline{S^\star}$, provided that the interface $\Gamma = \partial S$ is sufficiently regular. Note that the assumption on starshapeness of the inclusion with respect to a given pole $\mathbf{x}_0 \in D$ was only used to derive explicit expressions in terms of polar coordinates. This is not restrictive and can be bypassed by a generalization of the calculus, see for example Sokolowski and Zolesio [39] and Delfour and Zolesio [9]. However, since the related tracking formulations for either the Dirichlet- or Neumann-data at the outer boundary are quite often considered in the literature, the present paper aims at investigating these formulations by analogous methods.

Shape calculus techniques are also investigated and developed by e.g., Hettlich [25] and Rundell [26, 27], Hohage [30], Kirsch [31], Kress et. al. [23, 32, 33], Potthast [36, 37] (a rather incomplete list) for the study of various kinds of shape identification problems as nonlinear operator equations. That is, mainly the shape derivatives of solution of the state equation are considered and applied in Newton like iterative techniques. In view of these investigations, an aim of the present paper is, how higher order shape derivatives of least square objectives might provide a completion of the knowledge about the identification problem.

The numerical solution of the optimization problems on hand is not considered in the paper. Nevertheless, boundary integral equation methods could be exploited (see the appendix) by using efficient BEM implementations like wavelet based BEM or fast multipole methods. We refer to the likewise first or second order optimization methods explained in [17] (see also [13, 14] for more details about the principal setup). Of course, the extension to the numerical solution of problems in 3D is straightforward, see e.g., [18] for a principal outline. Nevertheless, it should be mentioned clearly that due to the ill-posedness of the problems, more appropriate regularization concepts have to be incorporated like those are already developed in the inverse problem community.

The present paper is organized as follows. In Section 1 we present the physical model and reformulate the identification problem as shape optimization problem(s) for either tracking the Dirichlet- or the Neumann data by a nonlinear least square. Some consequences of the unique continuation theorem for the Laplacian are stated. Moreover, we introduce the adjoint state equation for both formulations. Then, in Section 2, we compute first the gradient and the Hessian of the shape functionals. As a first consequence, we prove that a domain is stationary if and only if the data are perfectly matched with the (exact) measurements. Next, we analyze the shape Hessian in Section 3. By the particular structure of the second order form, the nature of the *two-norm-discrepancy* turns out to be completely analogous to the case of the Kohn and Vogelius criterion. We further prove degeneration of the shape Hessian at the optimal domain, hence the ill-posedness of the underlying identification problem. Some technicalities about boundary integral equation methods are postphoned to an appendix. Finally, we state some concluding remarks in Section 4.

## 1. Shape problem formulation

1.1. **The physical model and two alternatives for a least square formulation.** Let $D \in \mathbb{R}^2$ be a simply connected domain with boundary $\Sigma = \partial D$ and assume that an unknown simply connected inclusion $S$ with regular boundary $\Gamma = \partial S$ is located inside the domain $D$ satisfying $\operatorname{dist}(\Sigma, \Gamma) > 0$, cf. Figure 1.1. To determine the inclusion $S$ we measure for a given current distribution $g \in H^{-1/2}(\Sigma)$ the voltage distribution $f \in H^{1/2}(\Sigma)$ at the boundary $\Sigma$. Hence, we are seeking a domain $\Omega := D \setminus \overline{S}$ and an associated harmonic function $u$, satisfying the system of equations

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ u &= f && \text{on } \Sigma, \\ \frac{\partial u}{\partial \mathbf{n}} &= g && \text{on } \Sigma. \end{aligned}$$

This system denotes an overdetermined boundary value problem which should admit a solution only for the true inclusion $S$.

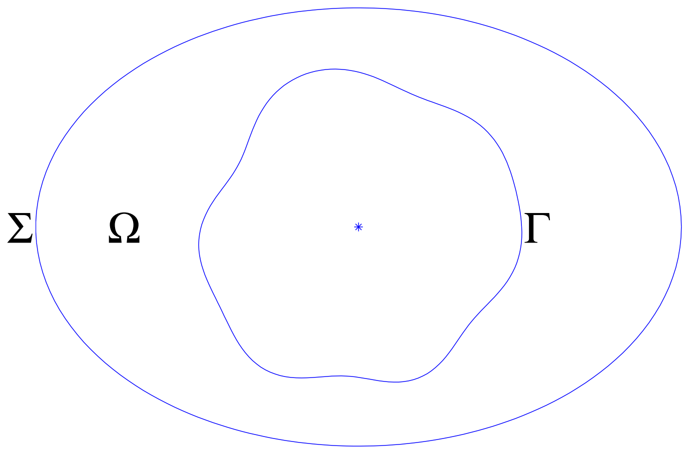


FIGURE 1.1. The domain $\Omega$ and its boundaries $\Gamma$ and $\Sigma$.

If the Neumann data (the current $g$) is assumed to be prescribed, the $L_2$-least square tracking of the Dirichlet data (the voltage distribution $f$) reads as follows

$$(P1) \qquad J(\Omega) = \frac{1}{2}\int_\Sigma (u-f)^2 \, d\sigma \to \inf, \tag{1.1}$$

subject to

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ \frac{\partial u}{\partial \mathbf{n}} &= g && \text{on } \Sigma. \end{aligned} \tag{1.2}$$

Likewise, the tracking of the Neumann data $g$ can be written as

$$(P2) \qquad J(\Omega) = \frac{1}{2}\int_\Sigma \left(g - \frac{\partial u}{\partial \mathbf{n}}\right)^2 d\sigma \to \inf, \tag{1.3}$$

where $u$ satisfies

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma, \\ u &= f && \text{on } \Sigma. \end{aligned} \tag{1.4}$$

Herein, the infimum has to be taken over all domains including a void with sufficiently regular boundary. We do not consider the interesting question of existence of optimal solutions in this paper. Instead, we will simply assume the existence of optimal domains, which is satisfied for example in case of perfectly matching data.

**Remark 1.1.** *Obviously, $L_2$-tracking is not completely compatible in both cases with the minimal requirements on $f$ and $g$ to provide a weak solution $u \in H^1(\Omega)$. However, it is more appropriate for considering noisy data later on. Moreover, assuming more regularity for $f$ and $g$ will simplify technicalities for regularity of the adjoint(s), and for the calculus. Nevertheless, we will briefly comment on possible relaxations with both respects in the concluding remarks (see Section 4).*

The following Lemma is an immediate consequence of the unique continuation theorem and will be of some importance for the investigations in the next sections.

**Lemma 1.2.** *In case of nonvanishing data $g \neq 0$, the solution $u$ of (1.2) has almost everwhere nonvanishing Neumann data along the inner boundary $\Gamma$, i.e.,*

$$meas\,\{x \in \Gamma \,|\, \frac{\partial u}{\partial \mathbf{n}}(x) = 0\} = 0.$$

*An analogous statement holds for the solution $u$ of (1.4).*

*Proof.* By the unique continuation theorem (cf. Hörmander [29]), we conclude from

$$\mathrm{meas}\,\{x \in \Gamma \,|\, \frac{\partial u}{\partial \mathbf{n}}(x) = 0\} > 0,$$

the consequence $u \equiv 0$ in $\Omega$. This contradicts $g \neq 0$ and proofs the assertion. □

1.2. **The adjoint equations.** According to the definition of the tracking type problems $(P1)$ and $(P2)$, the adjoint state for $(P1)$ have to satisfy the following equation

$$\begin{aligned} \Delta p &= 0 && \text{in } \Omega,\\ p &= 0 && \text{on } \Gamma,\\ \frac{\partial p}{\partial \mathbf{n}} &= (u - f) && \text{on } \Sigma. \end{aligned} \tag{1.5}$$

Similar, the adjoint equation for the second problem reads as

$$\begin{aligned} \Delta p &= 0 && \text{in } \Omega,\\ p &= 0 && \text{on } \Gamma,\\ p &= \left(\frac{\partial u}{\partial \mathbf{n}} - g\right) && \text{on } \Sigma. \end{aligned} \tag{1.6}$$

**Remark 1.3.** *To keep notations simple, we do not introduce subscripts for states $u$ and adjoints $p$ of $(P1)$ and $(P2)$, respectively. Whereas the equations (1.5) and (1.6) are quite similar to (1.2)and (1.4), respectively, the sources in the boundary condition imply different consequences for the regularity of the adjoint $p$ compared to the regularity of $u$. There is increasing regularity for the adjoint state in problem $(P1)$, but decreasing regularity* *in problem $(P2)$. Since we will not investigate this in more detail, we assume sufficient regularity of the boundary $\Sigma$ and the data to provide enough regularity for adjoints and their shape derivatives.*

Note the difference to the Kohn and Vogelius criterion: Since that objective is of the classical Dirichlet energy type (cf. [38]), no adjoint state have to be introduced there. More precisely, a formal calculus demonstrates that the related adjoint(s) coincide in principle with the original state(s). Obviously, the adjoint states $p$ are essentially different from $u$ in case of the least square formulations.

## 2. Shape calculus and the necessary first order optimality condition

2.1. **First and second order shape derivatives.** For sake of clearness in representation, we repeat the shape calculus concerning the problem under consideration by means of boundary variations. Since both objectives are defined on a fixed manifold far from the varying shape $\Gamma$, there exist two equivalent formulations for the shape gradient as well as for the shape Hessian either on $\Sigma$ or on $\Gamma$. But we emphasize that we mainly derive the boundary integral representation of the shape Gradient and the shape Hessian on $\Gamma$, which allows us to investigate in particular the natural two-norm discrepancy and the compactness at the stationary domain. Moreover, both expressions are more convenient for obtaining more efficiently descent directions for numerical algorithms, cf. [13, 14, 17]. For a survey on the shape calculus based on the material derivative concept, we refer the reader to Sokolowski and Zolesio [39] and Delfour and Zolesio [9] and the references therein. Concerning the Kohn and Vogelius criterion, the paper [38] contains the adaption of these general concepts to the particular case.

Let the underlying variational fields $\mathbf{V}$ be sufficiently smooth such that $C^{2,\alpha}$-regularity is preserved for all perturbed domains. Moreover, for sake of simplicity, we assume in addition the outer boundary and the measurements are sufficiently regular such that the state functions $u = u(\Omega)$ and the adjoints $p = p(\Omega)$ satisfy

$$u, p \in C^{2,\alpha}(\bar{\Omega}). \tag{2.7}$$

Then, the shape differentiability for both objectives (1.1) and (1.3) is provided up to second order *including* certain regularity for the shape Hessian representation. In particular, it provides Hölder-regularity of second spatial derivatives along $\Gamma$, arising from the shape differentiation of the state $u$ and adjoint $p$ (cf. the explicit representations below). Furthertmore, since the objective is defined on a fixed manifold far from the varying boundary, a formal differentiation of (1.1) in terms of local derivatives is possible and yields immediately

$$dJ(\Omega)[\mathbf{V}] = \int_{\Sigma} (u - f) du[V] d\sigma,$$

whereas the result for the objective (1.3) reads as follows

$$dJ(\Omega)[\mathbf{V}] = \int_{\Sigma} \left( \frac{\partial u}{\partial \mathbf{n}} - g \right) \frac{\partial du[V]}{\partial \mathbf{n}} d\sigma.$$

Here, the local shape derivatives $du = du[\mathbf{V}]$ for problem $(P1)$ and $(P2)$ satisfy

$$
\begin{array}{llcll}
 & \Delta du = 0 & & \Delta du = 0 & \text{in } \Omega, \\
(2.8) & du = -\langle \mathbf{V}, \mathbf{n}\rangle \dfrac{\partial u}{\partial \mathbf{n}} & \text{and} & du = -\langle \mathbf{V}, \mathbf{n}\rangle \dfrac{\partial u}{\partial \mathbf{n}} & \text{on } \Gamma, \\
 & \dfrac{\partial du}{\partial \mathbf{n}} = 0 & & du = 0 & \text{on } \Sigma,
\end{array}
$$

repectively. Note that the local shape derivatives for the two problems differ in *both* boundary conditions, since the state $u$ for problem $(P1)$ is different from those for $(P2)$ (see (1.2) and (1.4)).

It remains to compute the equivalent expressions for the shape gradients on the unknown boundary $\Gamma$, since they (have to) exist due to the Hadamard theorem.

**Lemma 2.1.** *The shape gradient representation on $\Gamma$ for both objectives reads as*

$$dJ(\Omega)[\mathbf{V}] = \int_\Gamma \langle \mathbf{V}, \mathbf{n}\rangle \frac{\partial u}{\partial \mathbf{n}} \frac{\partial p}{\partial \mathbf{n}} d\sigma. \tag{2.9}$$

*In case of $(P1)$, $u$ and $p$ solves (1.2) and (1.5), whereas the state $u$ and the adjoint $p$ satisfies (1.4) and (1.6) for $(P2)$, respectively.*

*Proof.* Using $\partial\Omega = \Gamma \cup \Sigma$ and the known boundary data from (1.5) and (2.8), the boundary integral representation of the shape gradient is obtained for the first objective via repeated integration by parts from the identity

$$
\begin{aligned}
0 &= \int_\Omega du \Delta p - p \Delta du d\mathbf{x}, \Rightarrow \\
0 &= \int_\Gamma du \frac{\partial p}{\partial \mathbf{n}} d\sigma + \int_\Sigma du (u - f) d\sigma, \quad \text{hence,} \\
dJ(\Omega)[\mathbf{V}] &= \int_\Sigma du (u - f) d\sigma = -\int_\Gamma du \frac{\partial p}{\partial \mathbf{n}} d\sigma,
\end{aligned}
$$

which is the desired result. Similarily, the same formula is derived in principle in the other case, but with different meaning for $u$ and $p$. □

If the hole $S$ is assumed to be starshaped with respect to some pole $\mathbf{x}_0 \in D$, the boundary $\Gamma = \partial S$ can be parametrized by a function $r = r(\varphi)$ of the polar angle $\varphi$ and the perturbation field $\mathbf{V}$ can be chosen as $\mathbf{V} = dr(\varphi)\mathbf{e}_r(\varphi)$. Herein, $\mathbf{e}_r(\varphi) := \mathbf{x}_0 + (\cos\varphi, \sin\varphi)^T$ denotes the radial direction with respect to the pole $\mathbf{x}_0$. The regulatity requirements imply $r, dr \in C^{2,\alpha}_{\text{per}}[0, 2\pi]$, where $r$ is a positive function such that $\operatorname{dist}(\Sigma, \Gamma) > 0$ and

$$C^{2,\alpha}_{\text{per}}[0, 2\pi] := \{r \in C^{2,\alpha}[0, 2\pi] : r^{(i)}(0) = r^{(i)}(2\pi),\ i = 0, 1, 2\}.$$

Then, the shape gradient $dJ[dr]$ becomes in both cases

$$dJ(\Omega)[dr] = -\int_0^{2\pi} dr(\varphi)\, r(\varphi) \left(\frac{\partial u}{\partial \mathbf{n}} \frac{\partial p}{\partial \mathbf{n}}\right)(\varphi)\, d\varphi, \tag{2.10}$$

where the minus sign issues from the fact that $\langle \mathbf{e}_r, \mathbf{n}\rangle = -r/\sqrt{r^2+r'^2}$. Similarily, we will keep the notation $du = du[dr]$ and $dp = dp[dr]$ for indicating the dependencies of related local shape derivatives.

**Remark 2.2.** *The class of bounded starshaped domains with regular $C^{k,\alpha}$-boundary possesses the* open set property *for $k \geq 1$, $\alpha \in [0,1]$: Within the general class of simply connected $C^{k,\alpha}$-domains, any such domain $\Omega$ has a neighbourhood $U_\eta$ in the $C^{k,\alpha}$-topology, containing only starshaped domains. Moreover, there is a* one-to-one *relation between (scalar) functions $dr \in U_\delta(\mathbf{0}) \subset C^{2,\alpha}_{per}[0,2\pi]$ and domains $\Omega_{dr} \in U_\eta(\Omega)$. Consequently, the calculus via polar coordinates provides the "complete information" like a general calculus for the class of domains under consideration. In arbitrary dimensions, the unit sphere might serve as an appropriate parameter manifold.*

To derive the shape Hessian, we proceed similar to [10, 11] by differentiating directly the shape gradient (2.10) while exploiting the relations

$$\nabla u|_\Gamma = \frac{\partial u}{\partial \mathbf{n}} \cdot \mathbf{n}, \quad \nabla p|_\Gamma = \frac{\partial p}{\partial \mathbf{n}} \cdot \mathbf{n} \;\Rightarrow\; \frac{\partial u}{\partial \mathbf{n}}\frac{\partial p}{\partial \mathbf{n}} = \langle \nabla u, \nabla p\rangle|_\Gamma.$$

**Lemma 2.3.** *The shape Hessian reads as*

$$\begin{aligned} d^2J(\Omega)[dr_1, dr_2] = -\int_0^{2\pi} & dr_1\, dr_2 \Big\{ \langle \nabla u, \nabla p\rangle + r\Big\langle \nabla\left(\langle \nabla u, \nabla p\rangle\right), \mathbf{e}_r \Big\rangle\Big\} \\ & + dr_1\, r \left\{ \frac{\partial u}{\partial \mathbf{n}} \frac{\partial dp[dr_2]}{\partial \mathbf{n}} + \frac{\partial p}{\partial \mathbf{n}} \frac{\partial du[dr_2]}{\partial \mathbf{n}} \right\} d\varphi, \end{aligned} \tag{2.11}$$

*where all data have to be understood as traces on the unknown boundary $\Gamma$.*

To give the expression (2.11) a meaning, it remains to compute the local shape derivatives of the adjoints $p$ for $(P1)$ and $(P2)$, i.e., the local shape derivatives of the solutions to (1.5) and (1.6), respectively. They are characterized for both problems as solutions of either

$$\begin{aligned} \Delta dp &= 0 & \qquad \Delta dp &= 0 & \quad &\text{in } \Omega, \\ dp &= -\langle \mathbf{V}, \mathbf{n}\rangle \frac{\partial p}{\partial \mathbf{n}} & \quad \text{or} \quad dp &= -\langle \mathbf{V}, \mathbf{n}\rangle \frac{\partial p}{\partial \mathbf{n}} & &\text{on } \Gamma, \\ \frac{\partial dp}{\partial \mathbf{n}} &= du[V] & dp &= \frac{\partial du[V]}{\partial \mathbf{n}} & &\text{on } \Sigma. \end{aligned} \tag{2.12}$$

**Remark 2.4.** *The formal equivalence for the shape Hessian* (2.11) *in both cases arises from formal similarity of formulae* (2.9) *or* (2.10)*, respectively. The differences can be seen more clearly when differentiating directly the expression for the shape Gradient on $\Sigma$, i.e.*

$$\begin{aligned} d^2J(\Omega)[V_1, V_2] &= \int_\Sigma du[V_1] du[V_2]\, d\sigma, & &\textit{for objective (1.1)}, \\ d^2J(\Omega)[V_1, V_2] &= \int_\Sigma \frac{\partial du[V_1]}{\partial \mathbf{n}} \frac{\partial du[V_2]}{\partial \mathbf{n}}\, d\sigma, & &\textit{for objective (1.3)}, \end{aligned}$$

*respectively, where $V_i = dr_i \cdot \mathbf{e}_r$, $i = 1, 2$, for example.*

**Remark 2.5.** *There is an important difference to "classical control problems" on fixed domains: The adjoints in shape optimization problems have nonvanishing derivatives on the primal optimization variable ("on the controls"), i.e., nonvanishing shape derivatives. That illustrates the stronger nonlinearity of the duality relation(s), since the "pde-constraint" in shape optimization problems cannot be directly considered as a "standard" equality constraint in a certain Banach space.*

2.2. **The necessary first order optimality condition.** For both problems, a first consequence can be derived from Lemma 1.2.

**Corollary 2.6.** *For any nontrivial variational field $V$, the local shape derivative $du[V]$ of problem $(P1)$ has almost everwhere nonvanishing Dirichlet data along the outer boundary $\Sigma$, i.e.,*

$$meas\,\{x \in \Sigma \,|\, du[V](x) = 0\} = 0 \;\Leftrightarrow\; \langle \mathbf{V}, \mathbf{n}\rangle|_\Gamma \not\equiv 0.$$

*Analogously, the local shape derivative $du$ of problem $(P2)$ has almost everwhere nonvanishing Neumann data along the outer boundary $\Sigma$, i.e.,*

$$meas\,\{x \in \Sigma \,|\, \frac{\partial du[V]}{\partial \mathbf{n}}(x) = 0\} = 0. \;\Leftrightarrow\; \langle \mathbf{V}, \mathbf{n}\rangle|_\Gamma \not\equiv 0.$$

*Proof.* Since we know $\frac{\partial u}{\partial \mathbf{n}}(x) \neq 0$ a.e. on $\Gamma$ from Lemma 1.2, the assumption

$$\text{meas}\,\{x \in \Sigma \,|\, du[V](x) = 0\} > 0$$

would lead to meas $\{x \in \Sigma \,|\, \langle \mathbf{V}, \mathbf{n}\rangle(x) \neq 0\} = 0$ in the first case. Contrary, from $\langle \mathbf{V}, \mathbf{n}\rangle|_\Gamma \equiv 0$ one easily derives $du[V] \equiv 0$ on $\bar{\Omega}$.

A similar reasoning remains valid in the second case. □

Since the shape gradient representation(s) (2.9) (or (2.10)) provide an easy structure, an important conclusion can be drawn from the first order necessary condition.

**Theorem 2.7.** *For both problems the validity of the necessary optimality condition on a certain domain $\Omega^\star$ is equivalent to a perfect matching of the data, i.e.*

$$\nabla J(\Omega^\star)[V] = 0 \;\; \textit{for all } V \;\Leftrightarrow\; u^\star|_\Sigma \equiv f, \tag{2.13}$$

*for problem $(P1)$, or similar for problem $(P2)$*

$$\nabla J(\Omega^\star)[V] = 0 \;\; \textit{for all } V \;\Leftrightarrow\; \frac{\partial u^\star}{\partial \mathbf{n}}|_\Sigma \equiv g. \tag{2.14}$$

*Proof.* Let us denote the state and the adjoint, associated with $\Omega^\star$ by $u^\star = u_{\Omega^\star}$ and $p^\star = p_{\Omega^\star}$. From (2.9) we immediately conclude for both problems

$$\nabla J(\Omega^\star)[V] = 0 \;\text{ for all } V \;\Leftrightarrow\; \left(\frac{\partial u^\star}{\partial \mathbf{n}}\frac{\partial p^\star}{\partial \mathbf{n}}\right)|_{\Gamma^\star} \equiv 0 \;\Leftrightarrow\; \frac{\partial p^\star}{\partial \mathbf{n}}|_{\Gamma^\star} \equiv 0,$$

where we have taken lemma 1.2 into account. Applying again the unique continuation theorem, we conclude in both cases $p^\star \equiv 0$ on $\bar{\Omega}^\star$. The theorem follows from the definition of $p^\star$ according to (1.5) or (1.6), respectively. □

**Remark 2.8.** *Consequently, no "spurios" stationary domains can appear for the EIT-problem in case of perfectly conducting inclusions. This is remarkable, since such a conclusion is challenging in case of arbitrary nonlinear least squares. Despite of corollary 2.6, the same conclusion cannot be obtained from the shape gradient representation on $\Sigma$, since it is a priori not clear, whether the traces $du[V]|_\Sigma$ covers a complete linear independent system for $L_2(\Sigma)$ or not (similar for problem (P2)).*

**Remark 2.9.** *Obviously, global optimality of a stationary domain $\Omega^\star$ is ensured by the particular structure of the objective(s) and theorem 2.7. In the next section we will discuss, whether $\Omega^\star$ is a strict local optimizer of second order or not.*

Finally, we want to mention that the considerations in this subsection are completely independent from the starshapeness of $\Omega^\star$.

## 3. The shape Hessian and sufficient optimality conditions

3.1. **The two-norm discrepancy and related remainder estimates.** According to remark 2.2, we will consider only starshaped domains for studying sufficient second order optimality conditions (SSOC) in shape optimization. This provides equivalence to sufficient conditions in related function spaces on the parameter manifold and avoids the nonuniqueness of more general domain or boundary variational approaches. Hence, it avoids to consider factorization procedures, preventing from e.g. the noninvertibility of related shape Hessians. Before investigating the shape Hessian at a stationary domain in more detail, we recall from [11] a general property of the shape Hessian at *arbitrary* domains.

**Lemma 3.1.** *The shape Hessian $\nabla^2 J(\Omega)$ defines a continuous bilinear form on $H^{1/2}[0,2\pi]\times H^{1/2}[0,2\pi]$, i.e.: It holds the estimate*

$$|\nabla^2 J(\Omega)[dr_1; dr_2]| \leq c_0 \, \|dr_1\|_{H^{1/2}} \cdot \|dr_2\|_{H^{1/2}}\,, \; c_0 = c_0(\Omega), \tag{3.15}$$

*but no similar estimate with respect to a weaker space is possible in general.*

We omit the proof, since (2.11) is a particular case of the shape Hessian structure, considered in [11]. To shorten notation, we use the identification $\Omega \Leftrightarrow r$, hence $\Omega_{dr} \Leftrightarrow r + dr$ in the next remark.

**Remark 3.2.** *Using Taylor expansion around $\Omega \Leftrightarrow r$, we have*

$$J(r+dr) = J(r) + \nabla J(r)[dr] + \frac{1}{2}\nabla^2 J(r)[dr, dr] + R_2(r, dr).$$

*where the second order remainder $R_2(r, dr)$ can be equivalently expressed as*

$$R_2(r, dr) = \frac{1}{2}\nabla^2 J(r+\rho dr)[dr, dr] - \frac{1}{2}\nabla^2 J(r)[dr, dr], \; \rho \in (0,1).$$

*In case of continuous depence on the argument $r$ of the second order bilinear form, this suggests together with* (3.15) *the validity of the* remainder estimate

$$
|R_2(r, dr)| \leq \eta(\|dr\|_{C^{2,\alpha}})\|dr\|^2_{H^{1/2}}, \quad \eta : \mathbb{R}_+ \to \mathbb{R}_+, \quad \lim_{s \to 0} \eta(s) = 0, \tag{3.16}
$$

*uniformly for all $r$ in a neighbourhood of $r^\star$. Of course, strict verification of* (3.16) *is highly appreciated, since it follows not immediately from the calculus. For a relatively large class of elliptic shape problems, such estimates are obtained by M. Dambrine* [7]*, see also* [6, 8]*. Note the difference to the estimates, ensured by the "standard calculus"*

$$
\begin{aligned}
|\nabla^2 J(r)[dr_1; dr_2]| &\leq c_0 \, \|dr_1\|_{C^{2,\alpha}} \cdot \|dr_2\|_{C^{2,\alpha}} \, , \ c_0 = c_0(r), \qquad \textit{and} \\
|R_2(r, dr)| &\leq \eta(\|dr\|_{C^{2,\alpha}})\|dr\|^2_{C^{2,\alpha}}.
\end{aligned}
$$

*Obviously, the estimate* (3.16) *provides a* sharper characterization *of the general behaviour of the remainder.*

**Remark 3.3.** *The estimate* (3.16) *has important consequences for a discussion of sufficient second order optimality conditions (SSOC) in shape optimization. Due to the particular structure of the shape Hessian, the validity of a uniform coercivity estimate cannot be expected in the "conventional" norm. Conversely, a coercivity estimate of the type*

$$
\nabla^2 J(\Omega^\star)[dr; dr]| \geq c_0 \, \|dr\|^2_{H^{1/2}} \, , \quad c_0 > 0, \tag{3.17}
$$

*would already provide strict local optimality of second order for a stationary domain $\Omega^\star$, if the estimates* (3.15) *and* (3.16) *are valid in a neighbourhood of $\Omega^\star$. Such a discrepancy between the (stronger) norm for differentiation and the (weaker) norm for the coercivity, already sufficient for optimality, is called a* two-norm-discrepancy*. Obviously, such a discrepancy can only occur in case of* nonquadratic *objectives, i.e., if a nontrivial second order remainder appears.*

**Remark 3.4.** *For shape functionals like the volume or the perimeter of a domain, different spaces arise for the two-norm-discrepancy. The shape Hessian for the volume is a continuous bilinear form in $L_2 \times L_2$, but the shape Hessian of perimeter defines naturally a bilinear form in $H^1 \times H^1$. For more details, including the discussion of additional functional constraints, see* [12].

3.2. **Compactness of the shape Hessian at the optimal domain.** Next, we will investigate the shape Hessian at the optimal domain $\Omega^\star$, that is, if the given inclusion is detected and the first order necessary condition (2.13) (or (2.14)) holds. Consequently, all quantities arising in the considerations are related to the optimal domain $\Omega^\star$ throughout this subsection. By theorem 2.7, the first two terms in (2.11) vanish and the shape Hessian simplifies according to

$$
d^2 J(\Omega)[dr_1, dr_2] = \int_0^{2\pi} dr_1 \, r \frac{\partial u}{\partial \mathbf{n}} \frac{\partial dp[dr_2]}{\partial \mathbf{n}} \, d\varphi. \tag{3.18}
$$

The next lemma is an immediate consequence of corollary 2.6 in combination with the shape Hessian representation on $\Sigma$ (cf. remark 2.4)

**Lemma 3.5.** *The second directional derivatives are strictly positive at a stationary domain $\Omega$ for both problems, i.e.,*

$$d^2 J(\Omega)[dr, dr] = \int_\Sigma (du[dr])^2 \, d\sigma > 0 \;\Leftrightarrow\; dr \neq 0,$$

*in case of $(P1)$, similar for problem $(P2)$.*

*Proof.* Since $\langle \mathbf{e}_r, \mathbf{n} \rangle|_\Gamma > 0$ for starshaped domains, the condition $dr \neq 0$ is equivalent to $\langle \mathbf{V}, \mathbf{n} \rangle|_\Gamma \not\equiv 0$. □

**Remark 3.6.** *The above lemma ensures the validity of the necessary second order conditions. Nevertheless, even strict positivity of any (nontrivial) second directional derivative is in general* not sufficient *for optimality in infinite dimensionsional optimization problems, cf. Maurer and Zowe* [34].

Whereas the subsequent analysis of the shape Hessian is quite similar to the investigations in [17], we repeat the main steps for convenience. We introduce first the multiplication operators

$$M_u dr := dr \cdot r \frac{\partial u}{\partial \mathbf{n}}\Big|_\Gamma \quad \text{and} \quad M_p dr := dr \cdot r \frac{\partial p}{\partial \mathbf{n}}\Big|_\Gamma \tag{3.19}$$

**Lemma 3.7.** *Let* (2.7) *hold, then the multiplication operators $M_u$, $M_p$ : $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ defined by* (3.19) *are continuous.*

*Proof.* Due to complete analogy, we consider only $M_u$. Abbreviating $v := r(\partial u / \partial \mathbf{n})|_\Gamma$ we may write $M_u dr = dr \cdot v$. Due to results of Triebel [40] or Mazja and Shaposhnikova [35], the multiplication operator $M_u$ is continuous from $H^{1/2}(\Gamma)$ to $H^{1/2}(\Gamma)$, provided that $v \in C^{0,\alpha}(\Gamma)$ for some $\alpha > 1/2$. From (2.7) we conclude $v \in C^{1,\alpha}(\Gamma)$ which implies the assertion. □

Again, the remaining steps will be outlined in detail only for $(P1)$. In general, we observe from (2.12) the dependence of the local shape derivative $dp = dp[dr]$ from the boundary variation $dr$ through both boundary conditions, where the dependence is explicit on $\Gamma$, but implicit on $\Sigma$, i.e.,

$$dp\big|_\Gamma = M_p dr, \qquad \frac{\partial dp}{\partial \mathbf{n}}\Big|_\Sigma = du[dr]\big|_\Sigma.$$

However, $\frac{\partial p}{\partial \mathbf{n}}\Big|_\Gamma$ vanishes at a stationary domain, see the proof of theorem 2.7. Moreover, the Dirichlet data $du[dr]\big|_\Sigma$ itself can be seen via (2.8) as the image of the shape variation $dr$ by a boundary integral operator $A$.

**Lemma 3.8.** *The linear operator $\mathbf{A} : dr \mapsto du[dr]\big|_\Sigma$, defined via* (2.8)*, is compact as a mapping from $H^{1/2}(\Gamma)$ to $H^{1/2}(\Sigma)$.*

The proof is given in the appendix.

Finally, the Neumann data $\frac{\partial dp[dr]}{\partial \mathbf{n}}\Big|_\Gamma$ depend on the direction of shape variation $dr$ by an additional Dirichlet-to-Neumann map

$$\Lambda(\mathbf{A}dr) := \frac{\partial dp[dr]}{\partial \mathbf{n}}\Big|_\Gamma. \tag{3.20}$$

With these operators at hand, we can rewrite (3.18) by

$$d^2J(\Omega^\star)[dr_1, dr_2] = \langle M_u dr_1, \Lambda(\mathbf{A}dr_2)\rangle, \tag{3.21}$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical $L^2(\Gamma)$-inner product. The proof of the next lemma is again postphoned to the appendix.

**Lemma 3.9.** *The operator* $\Lambda : H^{1/2}(\Sigma) \to H^{-1/2}(\Gamma)$ *defined by* (3.20) *is compact.*

Consequently, the composite mapping

$$\Lambda \circ \mathbf{A} : dr \mapsto \frac{\partial dp[dr]}{\partial \mathbf{n}}\Big|_\Gamma \quad (\text{as a map } H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma))$$

is compact at a stationary domain. Whereas the adjoint state $p$ vanishes identically at a stationary domain, it has nontrivial local shape derivatives in any direction $dr \neq 0$.

According to the Lemmas 3.7 and 3.9, the bilinear form $d^2J(\Omega^\star)$ defined in (3.21) is continuous on $H^{1/2}(\Gamma)\times H^{1/2}(\Gamma)$. Hence, it represents a continuous linear operator

$$H = M_u^\star\big(\Lambda\circ\mathbf{A}\big) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

As an immediate consequence of our considerations we conclude the following proposition.

**Proposition 3.10.** *The shape Hessian* $H : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is compact at the optimal domain* $\Omega^\star$.

**Remark 3.11.** *The situation changes essentially on nonstationary domains: In case of nonvanishing Neumann data* $\frac{\partial p}{\partial \mathbf{n}}\Big|_\Gamma$ *of the adjoint state, the map* $dr \mapsto \frac{\partial dp[dr]}{\partial \mathbf{n}}\Big|_\Gamma$ *defines a "conventional" pseudodifferential operator of order 1, i.e., a continuous operator from* $H^{1/2}(\Gamma)$ *to* $H^{-1/2}(\Gamma)$*, see remark 4.6 in the appendix. Whereas other parts are present in the shape Hessian representation on arbitrary domains, the main property of $H$ as a* pseudodiferential operator *of order 1 is governed by the expressions*

$$dr_1\, r\left\{\frac{\partial u}{\partial \mathbf{n}}\frac{\partial dp[dr_2]}{\partial \mathbf{n}} + \frac{\partial p}{\partial \mathbf{n}}\frac{\partial du[dr_2]}{\partial \mathbf{n}}\right\},$$

*from the variational formulation* (2.11) *of the operator $H$. Moreover, exactly these relations imply estimate* (3.15).

**Remark 3.12.** *Despite of the global optimality of $\Omega^\star$ (cf. remark 2.9), a regular strict minimizer of second order have to satisfy $H^{1/2}(\Gamma)$-coercivity of the shape Hessian*

$$d^2J(\Omega^\star)[dr, dr] \geq c\|dr\|^2_{H^{1/2}(\Gamma)},$$

*see subsection 3.1. The above proposition implies that this sufficient second order optimality condition cannot be valid, which characterizes the* ill-posedness *of the related* *identification problem. In particular, any nonregularized optimization algorithm cannot provide* stability *for a numerical solution of finite dimensional auxiliary problems as well as for the convergence of the solutions of these subproblems to the original domain.*

It is an easy task to illustrate the compactness of the maps $\Lambda$ and $\mathbf{A}$ as well as the compactness of the shape Hessian at a stationary domain by analyzing the situation of a ringshaped domain given by two concentric circles. While using analytical data for $g$ and $f$, this would in fact result in exponential decay of mapping coefficients in related Fourier series expansion, [17, 26], as well as exponential decay of the eigenvalues of $\nabla^2 J(\Omega^\star)$, [17]. Moreover, for arbitrary situations, one might exemplify this by computing the eigenvalues of the shape Hessian numerically like in [17]. For sake of breviety, we skip such illustrations.

## 4. Concluding remarks

We conclude the paper with a couple of remarks.

**Remark 4.1.** *The compactness proof in subsection 3.2 frequently uses smoothing properties of harmonic functions as solutions of the Laplace equation, either for the state and adjoint equation as well as for the governing equations of their local shape derivatives. Furthermore, we deal with objective(s) beeing defined on a compact manifold far from the varying shape. This gives rise for possibly providing enough regularity for $u$ and $p$ (and for $du$ and $dp$) around the unknown boundary $\Gamma$. Consequently, a similar shape calculus will be valid for lower regularity of the data $f, g$. Nevertheless, it might be challenging to point out the details, since this would ensure the same conclusions for the identification problem by essentially weaker assumptions.*

**Remark 4.2.** *As already discussed in remark 1.1, there is a degree of freedom in choosing the norm for the data tracking on $\Sigma$. At least, a $H^{-1/2}(\Sigma)$-tracking of the Neumann data $g$ would be compatible with considering noisy data thereafter. Tracking a Neumann condition in $H^{-1/2}$, but on the moving boundary $\Gamma$ was already investigated by Haslinger et. al. for a Bernoulli type free boundary problem, [24]. It would be interesting to study a shape calculus for such objectives.*

**Remark 4.3.** *To provide a (local) one-to-one correspondence to a scalar parametrization field in case of nonstarshaped domains, we can introduce a sufficiently regular $n$-dimensional reference manifold $\Gamma_0$ and consider a fixed boundary perturbation vector field. For example, the outer normal field $\mathbf{n}_0$ can be used. We suppose that the free boundary of each domain $\Omega \in \Upsilon$ can be parameterized via a sufficiently smooth function $r$ in terms of*

$$\boldsymbol{\gamma} : \Gamma_0 \to \Gamma, \quad \boldsymbol{\gamma}(\mathbf{x}) = \mathbf{x} + r(\mathbf{x})\mathbf{n}_0(\mathbf{x}).$$

*That is, we can identify a domain with the scalar function $r$. Defining the standard variation*

$$\boldsymbol{\gamma}_\varepsilon : \Gamma_0 \to \Gamma_\varepsilon, \quad \boldsymbol{\gamma}_\varepsilon(\mathbf{x}) := \boldsymbol{\gamma}(\mathbf{x}) + \varepsilon dr(\mathbf{x})\mathbf{n}_0(\mathbf{x}),$$

*where $dr$ is again a sufficiently smooth scalar function, we obtain the perturbed domain $\Omega_\varepsilon$. If an extension of the "vector support-field" to a neighbourhood of $\Gamma_0$ is required, one might use the oriented distance approach, see* [9].

**Remark 4.4.** *If an optimizer $\Omega^\star$ satisfies the strict coercivity assumption in the natural norm of the continuous extension of the second order form, the convergence $\Omega_N^\star \to \Omega^\star$ is shown in* [20] *for optimizers $\Omega_N^\star$ of finite dimensional auxiliary problems, if the ansatz spaces are properly chosen. Moreover, the numerical comptutation of optimizers in the auxiliary problems remain stable. Consequently, regularization of ill-posed problems can be already ensured in the continuity norm of the shape Hessian. Investigating regularization concepts in the $H^{1/2}$-norm for elliptic shape problems seems to be a challenging task. Nevertheless, noisy measurments or data might cause further influence on the regularization requirements. Furthermore, regularization in stronger norms can be considered as well.*

**Remark 4.5.** *The difficulties for the overall optimization process, resulting from ill-posedness does not directly concern the numerical calculation of the entries of the shape Hessian. As already confirmed in* [17]*, these computations turn out to be (relatively) stable by our approach up to the range of the considered entries near $\Omega^\star$ and is of the same accuracy even at the critical domain.*

## Appendix: Boundary integral equations

In the appendix we outline the remaining computations from subsection 3.2, where we focus mainly on $(P1)$. We introduce the single layer and the double layer operator with respect to the boundaries $\Phi, \Psi \in \{\Gamma, \Sigma\}$ by

$$(V_{\Phi\Psi}u)(\mathbf{x}) := -\frac{1}{2\pi}\int_\Phi \log\|\mathbf{x}-\mathbf{y}\| u(\mathbf{y}) d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Psi,$$

$$(K_{\Phi\Psi}u)(\mathbf{x}) := \frac{1}{2\pi}\int_\Phi \frac{\langle \mathbf{x}-\mathbf{y}, \mathbf{n}_{\mathbf{y}}\rangle}{\|\mathbf{x}-\mathbf{y}\|^2} u(\mathbf{y}) d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Psi.$$

Note that $V_{\Phi\Psi}$ denotes an operator of order $-1$ if $\Phi = \Psi$, i.e. $V_{\Phi\Phi} : H^{-1/2}(\Phi) \to H^{1/2}(\Phi)$, while it is an arbitrarily smoothing compact operator if $\Phi \neq \Psi$ since $\operatorname{dist}(\Gamma, \Sigma) > 0$. Likewise, if $\Sigma, \Gamma \in C^2$, the double layer operator $K_{\Phi\Phi} : H^{1/2}(\Phi) \to H^{1/2}(\Phi)$ is compact while it smoothes arbitrarily if $\Phi \neq \Psi$. We refer the reader to [22, 32] for more details concerning boundary integral equations.

For sake of simplicity we suppose that $\operatorname{diam}\Omega < 1$ to ensure that $V_{\Phi\Phi}$ is invertible, cf. [28]. Then, the normal derivative of $du = du[dr]$ is given by the Dirichlet-to-Neumann map

$$\begin{bmatrix} V_{\Gamma\Gamma} & V_{\Sigma\Gamma} \\ V_{\Gamma\Sigma} & V_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \frac{\partial du}{\partial \mathbf{n}}\big|_\Gamma \\ \frac{\partial du}{\partial \mathbf{n}}\big|_\Sigma \end{bmatrix} = \begin{bmatrix} 1/2 + K_{\Gamma\Gamma} & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & 1/2 + K_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} du\big|_\Gamma \\ du\big|_\Sigma \end{bmatrix}. \tag{4.22}$$

Likewise, the unknown boundary data of $dp$ are determined by

$$\begin{bmatrix} V_{\Gamma\Gamma} & V_{\Sigma\Gamma} \\ V_{\Gamma\Sigma} & V_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \frac{\partial dp}{\partial \mathbf{n}}\big|_\Gamma \\ \frac{\partial dp}{\partial \mathbf{n}}\big|_\Sigma \end{bmatrix} = \begin{bmatrix} 1/2 + K_{\Gamma\Gamma} & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & 1/2 + K_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} dp\big|_\Gamma \\ dp\big|_\Sigma \end{bmatrix}. \tag{4.23}$$

Note that here and in the sequel the operators $(1/2 + K_{\Phi\Phi})$, $\Phi \in \{\Gamma, \Sigma\}$, have to be understood as continuous and bijective operators in terms of $(1/2 + K_{\Phi\Phi}) : H^{1/2}(\Phi)/\mathbb{R} \to H^{1/2}(\Phi)/\mathbb{R}$.

**Proof of Lemma 3.8:**

*Proof.* We conclude from (4.24), (3.19) and $\frac{\partial du}{\partial \mathbf{n}}\big|_\Sigma = 0$ (see (2.8))

$$V_{\Gamma\Gamma}\frac{\partial du}{\partial \mathbf{n}}\Big|_\Gamma = [1/2 + K_{\Gamma\Gamma}]\,(M_u dr) + K_{\Sigma\Gamma} du\big|_\Sigma,$$
$$V_{\Gamma\Sigma}\frac{\partial du}{\partial \mathbf{n}}\Big|_\Gamma = [1/2 + K_{\Sigma\Sigma}]\,du\big|_\Sigma + K_{\Gamma\Sigma}(M_u dr).$$

Eliminating the unknown $\frac{\partial du}{\partial \mathbf{n}}\Big|_\Gamma$ and resolving for $du\big|_\Sigma$ yields

$$du\big|_\Sigma = \left[1/2 + K_{\Sigma\Sigma} - V_{\Gamma\Sigma}V_{\Gamma\Gamma}^{-1}K_{\Sigma\Gamma}\right]^{-1} \cdot \left\{V_{\Gamma\Sigma}V_{\Gamma\Gamma}^{-1}\left(1/2 + K_{\Gamma\Gamma}\right) - K_{\Gamma\Sigma}\right\}(M_u dr).$$

Compactness of the operator $\mathbf{A}$, defined by

$$\mathbf{A} = \left[1/2 + K_{\Sigma\Sigma} - V_{\Gamma\Sigma}V_{\Gamma\Gamma}^{-1}K_{\Sigma\Gamma}\right]^{-1} \cdot \left\{V_{\Gamma\Sigma}V_{\Gamma\Gamma}^{-1}\left(1/2 + K_{\Gamma\Gamma}\right) - K_{\Gamma\Sigma}\right\} \cdot M_u,$$

follows from compactness of $V_{\Gamma\Sigma}$ and of $K_{\Gamma\Sigma}$ as well as from the continuity of all other remaining operators. □

**Proof of Lemma 3.9:**

*Proof.* We observe first from (2.12) and from theorem 2.7

$$dp\big|_\Sigma = du\big|_\Sigma \quad \text{and} \quad \frac{\partial p}{\partial \mathbf{n}}\Big|_\Gamma = 0 \;\Rightarrow\; M_p dr = 0.$$

Hence, we conclude from (4.23) while eliminating $dp\big|_\Sigma$

$$dp\big|_\Sigma = (1/2 + K_{\Sigma\Sigma})^{-1}\left\{V_{\Gamma\Sigma}\frac{\partial dp}{\partial \mathbf{n}}\big|_\Gamma + V_{\Sigma\Sigma}du\big|_\Sigma\right\}, \Rightarrow$$

$$V_{\Gamma\Gamma}\frac{\partial dp}{\partial \mathbf{n}}\big|_\Gamma = K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}\left\{V_{\Gamma\Sigma}\frac{\partial dp}{\partial \mathbf{n}}\big|_\Gamma + V_{\Sigma\Sigma}du\big|_\Sigma\right\}, \Rightarrow$$

$$\frac{\partial dp}{\partial \mathbf{n}}\big|_\Gamma = \left[V_{\Gamma\Gamma} - K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Gamma\Sigma}\right]^{-1}\left\{K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Sigma\Sigma} - V_{\Sigma\Gamma}\right\} du\big|_\Sigma.$$

The compactness of the map $\Lambda$ now follows from compactness of the "transfer operators" $K_{\Sigma\Gamma}$ and $V_{\Sigma\Gamma}$. □

**Remark 4.6.** *In case of $M_p dr \neq 0$ (on nonstationary domains), the latter representation modifies to*

$$\begin{aligned}
\frac{\partial dp}{\partial \mathbf{n}}\big|_\Gamma &= \left[V_{\Gamma\Gamma} - K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Gamma\Sigma}\right]^{-1}\left\{K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Sigma\Sigma} - V_{\Sigma\Gamma}\right\} du\big|_\Sigma \\
&- \left[V_{\Gamma\Gamma} - K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Gamma\Sigma}\right]^{-1} K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}K_{\Gamma\Sigma}(M_p dr) \\
&+ \left[V_{\Gamma\Gamma} - K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Gamma\Sigma}\right]^{-1}\left(1/2 + K_{\Gamma\Gamma}\right)(M_p dr),
\end{aligned}$$

*where the operator $D$ in the last part*

$$D := \left[V_{\Gamma\Gamma} - K_{\Sigma\Gamma}(1/2 + K_{\Sigma\Sigma})^{-1}V_{\Gamma\Sigma}\right]^{-1}\left(1/2 + K_{\Gamma\Gamma}\right)M_p,$$

*is obviously* not compact*, but a "regular", i.e., nondegenerate pseudodifferential operator of order 1 in general.*

**Remark 4.7.** *The related considerations for* $(P2)$ *will be completely similar. Moreover, since the following BIE formulation is equivalent to* (1.2)*,*

$$\begin{bmatrix} V_{\Gamma\Gamma} & V_{\Sigma\Gamma} \\ V_{\Gamma\Sigma} & V_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \mathbf{n}}\big|_\Gamma \\ g \end{bmatrix} = \begin{bmatrix} 1/2 + K_{\Gamma\Gamma} & K_{\Sigma\Gamma} \\ K_{\Gamma\Sigma} & 1/2 + K_{\Sigma\Sigma} \end{bmatrix} \begin{bmatrix} 0 \\ du\big|_\Sigma \end{bmatrix}, \tag{4.24}$$

*it can be used for computing* $\frac{\partial u}{\partial \mathbf{n}}\big|_\Gamma$ *numerically by e.g., fast wavelet BEM. Analogously, we can compute* $\frac{\partial p}{\partial \mathbf{n}}\big|_\Gamma$ *as the other main ingredient for the shape gradient.*

## References


[1] I. Akduman and R. Kress. Electrostatic imaging via conformal mapping. *Inverse Problems*, 18:1659-1672, 2002.

[2] G. Alessandrini, V. Isakov, and J. Powell. Local uniqueness in the inverse problem with one measurement. *Trans. Am. Math. Soc.*, 347:3031–3041, 1995.

[3] M. Brühl. Explicit characterization of inclusions in electrical impedance tomography. SIAM J. Math. Anal. 32, no.6, 1327–1341, 2001.

[4] M. Brühl and M. Hanke. Numerical implementation of two noniterative methods for locating inclusions by impedance tomography. Inverse Problems 16, no.4, 1029–1042, 2000.

[5] R. Chapko and R. Kress. A hybrid method for inverse boundary value problems in potential theory. *J. Inverse Ill-Posed Probl.* 13:27–40, 2005.

[6] M. Dambrine and M. Pierre. About stability of equilibrium shapes. *M2AN* 34, No.4, 811–834, 2000.

[7] M. Dambrine. Hessiennes de forme et stabilite des formes critiques. PhD Thesis, Ecole Normale Superieure de Cachan, 2000.

[8] M. Dambrine. On variations of the shape Hessian and sufficient conditions for the stability of critical shapes. *RACSAM, Rev. R. Acad. Cien. Serie A. Mat.* 96, No.1, 95–121, 2002.

[9] M. Delfour and J.-P. Zolesio. *Shapes and Geometries.* SIAM, Philadelphia, 2001.

[10] K. Eppler. Optimal shape design for elliptic equations via BIE-methods. *J. of Applied Mathematics and Computer Science* 10:487–516, 2000.

[11] K. Eppler. Boundary integral representations of second derivatives in shape optimization. *Discussiones Mathematicae (Differential Inclusion Control and Optimization)*, 20:63–78, 2000.

[12] K. Eppler. Second derivatives and sufficient optimality conditions for shape functionals. *Control and Cybernetics*, 29:485–512, 2000.

[13] K. Eppler and H. Harbrecht. Numerical solution of elliptic shape optimization problems using wavelet-based BEM. *Optim. Methods Softw.*, 18:105–123, 2003.

[14] K. Eppler and H. Harbrecht. 2nd Order Shape Optimization using Wavelet BEM. *Optim. Methods Softw.*, 21:135–153, 2006.

[15] K. Eppler and H. Harbrecht. Exterior Electromagnetic Shaping using Wavelet BEM. *Math. Meth. Appl. Sci.* 28:387–405 (2005).

[16] K. Eppler and H. Harbrecht. Fast wavelet BEM for 3d electromagnetic shaping. *Appl. Numer. Math.* 54:537–554 (2005).

[17] K. Eppler and H. Harbrecht. A regularized Newton method in electrical impedance tomography using shape Hessian information. *Control & Cybernetics* 34:203–225, 2005.

[18] K. Eppler and H. Harbrecht. Shape optimization for 3D electrical impedance tomography. *Preprint 963, WIAS Berlin*, 2004. to appear in *Free and Moving Boundaries: Analysis, Simulation*, Lecture Notes in Pure and Applied Mathematics, Vol. 252, eds.: R. Glowinski and J.P. Zolesio, Dekker/CRC Press, 2006.

[19] K. Eppler and H. Harbrecht. Efficient Treatment of Stationary Free Boundary Problems. *Preprint 965, WIAS Berlin*, 2004. to appear in *Appl. Numer. Math.*

[20] K. Eppler, H. Harbrecht and R. Schneider. On Convergence in Elliptic Shape Optimization. WIAS-Preprint Nr. 1016, (2005), submitted.

[21] A. Friedman and V. Isakov. On the uniqueness in the inverse conductivity problem with one measurement. *Indiana Univ. Math. J.*, 38:563–579, 1989.

[22] W. Hackbusch. *Integralgleichungen.* B.G. Teubner, Stuttgart, 1989.

[23] H. Haddar and R. Kress. On the Fréchet derivative for obstacle scattering with an impedance boundary condition. *SIAM J. Appl. Math.*, 65:194–208, 2004.

[24] J. Haslinger, T. Kozubek, K. Kunisch, and G. Peichl. Shape optimization and fictitious domain approach for solving free boundary problems of Bernoulli type. *Comput. Optim. Appl.*, 26:231–251, 2003.

[25] F. Hettlich. Iterative regularization schemes in inverse scattering by periodic structures. *Inverse Probl.*, 18:701-714, 2002.

[26] F. Hettlich and W. Rundell. The determination of a discontinuity in a conductivity from a single boundary measurement. *Inverse Problems.* 14:67–82, 1998.

[27] F. Hettlich and W. Rundell. A Second Degree Method for Nonlinear Inverse Problems. *SIAM J. Numer. Anal.*, 37:587-620, 2000.

[28] G. Hsiao and W. Wendland. A finite element method for some equations of first kind. *J. Math. Anal. Appl.*, 58:449–481, 1977.

[29] L. Hörmander. *The analysis of Linear Partial Differential Operators.* Springer, New York, 1983-85.

[30] T. Hohage. Convergence rates of a regularized Newton method in sound-hard inverse scattering. *SIAM J. Numer. Anal.* 36:125–142, 1998.

[31] A. Kirsch. The domain derivative and two applications in inverse scattering theory. *Inverse Problems.*, 9:81–96, 1993.

[32] R. Kress. Newton's method for inverse obstacle scattering meets the method of least squares. *Inverse Probl.*, 19:S91–S104, 2003.

[33] R. Kress and S. Meyer. An inverse boundary value problem for the Oseen equation. *Math. Methods Appl. Sci.* 23:103–120, 2000.

[34] H. Maurer and J. Zowe. First and second-order necessary and sufficient optimality conditions for infinite-dimensional programming problems. *Math. Program.*, 16:98–110, 1979.

[35] V.G. Maz'ya and T.O. Shaposhnikova. *Theory of multipliers in spaces of differentiable functions.* Pitman, Boston, 1985.(Monographs and Studies in Mathematics, 23. Pitman Advanced Publishing Program. Boston - London - Melbourne: Pitman Publishing Inc. *XIII*, 344 p. (1985))

[36] R. Potthast. Domain derivatives in electromagnetic scattering. *Math. Methods Appl. Sci.* 19:1157-1175, 1996.

[37] R. Potthast. On the convergence of a new Newton-type method in inverse scattering. *Inverse Probl.* 17:1419–1434, 2001.

[38] J.-R. Roche and J. Sokolowski. Numerical methods for shape identification problems. *Control Cybern.* 25:867–894, 1996.

[39] J. Sokolowski and J.-P. Zolesio. *Introduction to Shape Optimization*. Springer, Berlin, 1992.

[40] H. Triebel. *Theory of function spaces*. Birkhäuser, Basel-Boston-Stuttgart, 1983.