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A nonlocal phase-field model with nonconstant specific heat

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Abstract

We prove the existence, uniqueness, thermodynamic consistency, global boundedness from both above and below, and continuous data dependence for a strong solution to an integrodifferential model for nonisothermal phase transitions under nonhomogeneous mixed boundary conditions. The specific heat is allowed to depend on the order parameter, and the convex component of the free energy may or may not be singular.

1 Introduction

Phase-field models have been designed to describe the evolution of the state variables $\theta > 0$ and χ, representing the absolute temperature and a scalar order parameter, respectively, during temperature-induced phase transitions in a body $\Omega \subset \mathbb{R}^N$ ($N = 3$, for instance) if no mechanical motion takes place. For example, in a simple melting-solidification process, χ attains its values in the interval $[0,1]$, where $\{\chi = 0\}$ characterizes the solid phase, $\{\chi = 1\}$ the liquid, and $\{0 < \chi < 1\}$ is the liquid fraction in a mixture of both phases. Solid-solid phase transitions between two crystallographic variants with different mechanical properties (martensite and austenite, say) may also exhibit a similar behavior provided the experiment is uniaxial and is carried out under constant strain. Then the stress may play the role of an order parameter characterizing the phase, but no natural restriction on the admissible order parameter range is necessary.

We deal here with an integrodifferential model for volume preserving nonisothermal phase transitions that takes into account long-range interactions between particles. The physical relevance of nonlocal interaction phenomena in phase separation and phase transition models was already described in the pioneering papers [28] and [8]; however, only recently both isothermal and nonisothermal models containing nonlocal terms have been analyzed in a more systematic way (cf., e. g., [2]–[3], [9]–[21], [26]). The difference between local and nonlocal models consists in a different choice of the particle interaction potential in the free energy functional. The nonlocal contribution to the free energy has typically the form $\int_\Omega k(x,y)\,|\chi(x) - \chi(y)|^2\,dy$ with a given symmetric kernel $k(x,y)$; its classical local Ginzburg-Landau counterpart has the form $(\nu/2)|\nabla\chi(x)|^2$ as, e. g., in [7], with a positive parameter ν, and can be obtained as a formal limit as $m \to \infty$ from the nonlocal one with the choice $k(x,y) = m^{N+2}K(|m(x-y)|^2)$, where K is a nonnegative function with support in

$[0,1]$. This follows from the formula

$$\int_\Omega m^{N+2} K(|m(x-y)|^2)\,|\chi(x)-\chi(y)|^2\,dy = \int_{\Omega_m(x)} K(|z|^2)\left|\frac{\chi\left(x+\frac{z}{m}\right)-\chi(x)}{\frac{1}{m}}\right|^2 dz$$

$$\stackrel{m\to\infty}{\longrightarrow} \int_{\mathbb{R}^N} K(|z|^2)\,\langle\nabla\chi(x),z\rangle^2\,dz = \frac{\nu}{2}|\nabla\chi(x)|^2$$

for a sufficiently regular χ, where we denote $\nu = 2\int_{\mathbb{R}^N} K(|z|^2)|z|^2\,dz$ and $\Omega_m(x) = m(\Omega - x)$. Let us also mention the "Penrose-Fife" potential $(\nu/2)\,\theta\,|\nabla\chi(x)|^2$, see [6, 25]. Its nonlocal version might also deserve appropriate attention (cf. [21]), but we do not consider this issue here.

The passage from a nonlocal to a local potential changes dramatically the properties of the model. For example, the maximum principle is lost in the limit, and in general it is not possible to guarantee without additional hypotheses that the absolute temperature remains positive during the process.

We pursue here the investigations begun in [19] and consider a local free energy of the form

$$F[\theta,\chi] \;=\; c_V(\chi)\theta(1-\log\theta) + \theta\sigma(\chi) + \lambda(\chi) + (\beta+\theta)\varphi(\chi) + B[\chi]\,, \tag{1.1}$$

where σ and λ are smooth functions describing the local dependence on χ of the entropy and of the latent heat, respectively; $\beta > 0$ is a constant parameter, $B[\chi]$ is a nonlocal operator of the form

$$B[\chi](x,t) := \int_\Omega k(x,y)\,G(\chi(x,t)-\chi(y,t))\,dy \tag{1.2}$$

with a bounded, symmetric kernel $k : \Omega\times\Omega \to \mathbb{R}$ and an even smooth function G; φ is a general proper, convex, and lower semicontinuous function. Its domain $\mathcal{D}(\varphi)$ may be bounded or unbounded, depending on the specific model situation. The main novelty here is that the specific heat c_V may depend on the order parameter χ. In the solid-liquid system mentioned above, for example, we may have different values c_V^0 in the solid and c_V^1 in the liquid. Assuming that their dependence on temperature can be neglected in each phase, we may define $c_V(\chi) = c_V^0 + \chi(c_V^1 - c_V^0)$, cf. [27, Section IV.4]. The value of χ can be kept between 0 and 1 by setting in this case $\varphi = I_{[0,1]}$ (the indicator function of $[0,1]$).

With the above free energy, we associate the local internal energy E and entropy S according to the formulas

$$S = -\frac{\partial F}{\partial\theta}\,, \quad E = F + \theta S\,, \tag{1.3}$$

that is,

$$\begin{cases} S[\theta,\chi] \;=\; c_V(\chi)\log\theta - \sigma(\chi) - \varphi(\chi)\,, \\ E[\theta,\chi] \;=\; c_V(\chi)\theta + \lambda(\chi) + \beta\varphi(\chi) + B[\chi]\,. \end{cases} \tag{1.4}$$

The temperature dynamics is governed by the internal energy balance over an arbitrary control volume $\Omega' \subset \Omega$,

$$\frac{d}{dt}\int_{\Omega'} E[\theta,\chi]\,dx + \int_{\partial\Omega'} \langle \mathbf{q},\mathbf{n}\rangle\,ds = \Psi(\Omega')\,, \tag{1.5}$$

where $\mathbf{q}$ is the heat flux vector, $\mathbf{n}$ is the unit outward normal to $\partial\Omega'$, and $\Psi(\Omega')$ is the energy exchange through the boundary of Ω' due to the nonlocal interactions. The order parameter dynamics is assumed in the form

$$\mu(\theta)\chi_t \in -\delta_\chi \mathcal{F}[\theta,\chi] \tag{1.6}$$

with a factor $\mu(\theta) > 0$, where we denote

$$\mathcal{F}[\theta,\chi] \;=\; \int_\Omega F[\theta,\chi]\,dx\,,$$

and where $\delta_\chi\mathcal{F}$ stands for the variational derivative of $\mathcal{F}$ with respect to the variable χ. The inclusion sign in (1.6) accounts for the fact that $\mathcal{F}$ may contain components that are not Fréchet differentiable, but are convex, and the derivative can be interpreted as the subdifferential, which may be multivalued. Condition (1.6) is based on the assumption that the system tends to move towards local minima of the free energy with a speed proportional to $1/\mu(\theta)$. Using (1.1), we can rewrite (1.6) as

$$\mu(\theta)\chi_t + c_V'(\chi)\,\theta\,(1-\log\theta) + \lambda'(\chi) + \theta\sigma'(\chi) + (\beta+\theta)\partial\varphi(\chi) + b[\chi] \;\ni\; 0, \tag{1.7}$$

with the notation

$$b[\chi](x,t) \;:=\; 2\int_\Omega k(x,y)\,G'\left(\chi(x,t)-\chi(y,t)\right)\,dy. \tag{1.8}$$

The interaction term $\Psi(\Omega')$ in (1.5) and the constitutive law for the heat flux have to comply with the Clausius-Duhem inequality

$$S_t + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \;\geq\; 0\,, \tag{1.9}$$

which is understood here almost everywhere in the regularity context of Theorem 2.2. Assuming $\theta > 0$ (this will have to be justified in the next sections), and using (1.1) with (1.7) and (1.3), we obtain the identities

$$\begin{aligned} \theta\left(S_t + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right)\right) &= E_t + \operatorname{div}\mathbf{q} - F_t - \theta_t S - \frac{\langle \mathbf{q},\nabla\theta\rangle}{\theta} \\ &= E_t + \operatorname{div}\mathbf{q} + \mu(\theta)\chi_t^2 - \frac{\langle \mathbf{q},\nabla\theta\rangle}{\theta} + b[\chi]\chi_t - B[\chi]_t\,. \end{aligned} \tag{1.10}$$

We assume the Fourier heat flux law

$$\mathbf{q} = -\kappa\nabla\theta\,,$$

with a constant positive heat conductivity κ. Then the right-hand side of (1.10) stays nonnegative without prescribing any relationship between $\mu(\theta)$ and $B[\chi]$, provided that we choose $\Psi(\Omega')$ in (1.5) as

$$\Psi(\Omega') = \int_{\Omega'} (-b[\chi]\chi_t + B[\chi]_t)\, dx\,. \tag{1.11}$$

In agreement with natural expectations, we have $\Psi(\Omega) = 0$. The differential form of the energy balance (1.5) then reads

$$E_t + \operatorname{div} \mathbf{q} \;=\; -b[\chi]\chi_t + B[\chi]_t\,, \tag{1.12}$$

that is,

$$(c_V(\chi)\,\theta + \lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t - \kappa\Delta\theta \;=\; 0\,. \tag{1.13}$$

In real materials, the dependence of c_V on the phase may be very strong (the specific heat in water is considerably higher than both in ice and in vapor, for instance). Introducing the term $c_V(\chi)$ into the above system may however create substantial difficulties from both the physical and mathematical viewpoints, which can again be illustrated on the two-phase system mentioned above. More specifically, consider the thermodynamically insulated (i. e., with homogeneous Neumann boundary conditions) relaxed Stefan problem corresponding to the choice $\varphi = I_{[0,1]}$, $\lambda'(\chi) = L$, $\sigma'(\chi) = -L/\theta_c$, $B[\chi] \equiv 0$, where L and θ_c are positive constants (the latent heat and phase transition temperature, respectively), $c_V'(\chi) = \bar{c} := c_V^1 - c_V^0$. Thermodynamic equilibria are located on the curve

$$\partial I_{[0,1]}(\chi) \;\ni\; \bar{c}\,\theta(\log\theta - 1) + \frac{L}{\theta_c}(\theta - \theta_c)\,,$$

or, equivalently,

$$\chi \;\in\; \mathfrak{H}\left(\bar{c}\,\theta(\log\theta - 1) + \frac{L}{\theta_c}(\theta - \theta_c)\right),$$

where $\mathfrak{H}$ is the maximal monotone extension of the Heaviside function. We see that if $c_V^1 < c_V^0$, like in the water-vapor system, then the only (stable!) equilibrium for both very high and very low temperatures is $\chi = 0$, which is an obvious physical paradox. Between water and ice, this contradiction does not occur.

We focus here on mathematical problems arising in connection with this model. On the boundary of Ω, we prescribe nonhomogeneous mixed boundary conditions. Our main results include the proof of existence and uniqueness of a global strong solution (θ, χ) to (1.7) and (1.13) on the whole time axis $(0, +\infty)$. We also prove that θ is uniformly bounded from above and below on $(0, +\infty)$, with the intention to study the asymptotic behavior $t \to +\infty$ in the future. Note that there are only few works in the literature dealing with the convergence of trajectories towards equilibrium for nonlocal phase-field systems. The case of analytic potentials φ has been solved first in [11] and then in [17] for a time-relaxed model and in [13] for a time-discrete scheme. The nonsmooth case is not straightforward even if the nonlocal term is absent, see [23], and deserves special attention.

The paper is organized as follows. The main results are stated in Section 2. Section 3 is devoted to some auxiliary results on a class of differential inclusions, on maximum principles for parabolic equations with nonconstant coefficients and non-homogeneous mixed boundary conditions, and on L^∞-estimates based on Moser-type iterations. Uniqueness is proved in Section 4, existence and global boundedness in time in Section 5.

2 Main results

Consider a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and the time interval $[0, \infty)$. For $T \in (0, \infty]$ (∞ included) we denote by $Q_T = \Omega \times (0, T)$ the open space-time cylinder, and by Σ_T its lateral boundary $\partial\Omega \times (0, T)$. We use, for the sake of simplicity, the same symbol H for both $L^2(\Omega)$ and $L^2(\Omega\,;\mathbb{R}^N)$, and H^1 for $H^1(\Omega)$.

We rewrite the system (1.13), (1.7), putting, for simplicity and without loss of generality, $\kappa = 1$, in the form

$$(2.1) \qquad \left(c_V(\chi)\theta + \lambda(\chi) + \beta\varphi(\chi)\right)_t + b[\chi]\chi_t - \Delta\theta = 0\,,$$

$$(2.2) \qquad \mu(\theta)\chi_t + c_V'(\chi)\theta\,(1 - \log\theta) + \lambda'(\chi) + \theta\sigma'(\chi) + (\beta + \theta)\partial\varphi(\chi) + b[\chi] \ni 0\,,$$

to be satisfied a. e. in Q_∞, with $b[\chi]$ defined in (1.8), and prescribe the boundary and initial conditions

$$(2.3) \qquad \partial_{\mathbf{n}}\theta + \gamma\,(\theta - \theta_\Gamma) = 0 \quad \text{a. e. on } \Sigma_\infty\,,$$

$$(2.4) \qquad \theta(0) = \theta_0, \quad \chi(0) = \chi_0 \quad \text{a. e. in } \Omega,$$

where $\partial_{\mathbf{n}}$ denotes the outward normal derivative, and the data fulfil the following hypothesis.

Hypothesis 2.1. *We fix positive constants β, c_0, $\bar\theta_\Gamma$, μ_*, C_0, θ_*, and assume that:*

(i) $\gamma \in L^\infty(\partial\Omega)$ *is a nonnegative function.*

(ii) *There exist constants $\psi^* > \psi_* > 0$ such that $\psi^* \geq \psi_1(x) \geq \psi_*$ a. e., where $\psi_1 \in H^1$ is the eigenfunction with unit H-norm corresponding to the smallest eigenvalue $\lambda_1 \geq 0$ of the elliptic problem*

$$(2.5) \qquad -\Delta\psi_1 \;=\; \lambda_1\psi_1 \quad \text{in } \Omega, \quad \partial_{\mathbf{n}}\psi_1 + \gamma\psi_1 \;=\; 0 \quad \text{on } \partial\Omega\,.$$

(iii) *$\varphi : \mathbb{R} \to [0, +\infty]$ is a proper, convex, and lower semicontinuous function, $\mathcal{D}(\varphi)$ is its domain, and $0 \in \partial\varphi(0)$.*

(iv) $\sigma, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi))$.

(v) *$G \in W^{2,\infty}(\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, $G(z) = G(-z)$ for all $z \in (\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, $k \in L^\infty(\Omega \times \Omega)$, $\quad k(x, y) = k(y, x)$ a. e. in $\Omega \times \Omega$.*

(vi) $c_V \in W^{2,\infty}(\mathcal{D}(\varphi))$, $c_V(z) \geq c_0 > 0$ *for all* $z \in \mathcal{D}(\varphi)$.

(vii) $\theta_\Gamma \in L^\infty(\Sigma_\infty)$, $\quad (\theta_\Gamma)_t \in L^2_{\text{loc}}(\Sigma_\infty)$, $\quad \theta_\Gamma \geq \bar{\theta}_\Gamma > 0$ *a. e. in* Σ_∞.

(viii) μ *is locally Lipschitz in* $\mathbb{R}^+$, $\mu(\tau) \geq \mu_*(1+\tau)$ *for all* $\tau \in \mathbb{R}^+$.

(ix) *For any* $C > 0$ *set* $\mathcal{D}_C(\varphi) = \{\chi \in \mathcal{D}(\varphi)\,;\, \partial\varphi(\chi) \cap [-C, C] \neq \emptyset\}$, *and assume that* $\chi_0 \in L^\infty(\Omega)$, $\chi_0(x) \in \mathcal{D}_{C_0}(\varphi)$ *a. e. in* Ω.

(x) $\theta_0 \in H^1 \cap L^\infty(\Omega)$, $\theta_0(x) \geq \theta_*$ *a. e. in* Ω.

If γ vanishes on some part of $\partial\Omega$, then (2.3) is a mixed Neumann-Robin boundary condition. Below in Remark 2.3, we will show some sufficient conditions for Hypothesis 2.1 (ii) to hold. Note also that by [5, Example 2.3.4], $\partial\varphi$ is maximal monotone, hence $\mathcal{D}_C(\varphi)$ is a closed (possibly unbounded or degenerate) interval for every $C > 0$.

We are in the position of stating the existence theorem.

Theorem 2.2. *Let Hypothesis 2.1 hold. Then there exists at least one pair* (θ, χ) *which solves the system (2.1–2.4), and such that*

$$\theta \in L^\infty(Q_\infty)\,,\ \theta_t, \Delta\theta \in L^2_{\text{loc}}(0,\infty\,;\, L^2(\Omega))\,, \tag{2.6}$$

$$\theta(x,t) > 0 \quad \text{a. e. in } Q_\infty\,, \tag{2.7}$$

$$\chi \in L^\infty_{\text{loc}}(Q_\infty), \quad \chi_t \in L^\infty(Q_\infty); \quad \exists\, C > 0 \,:\, \chi(x,t) \in \mathcal{D}_C(\varphi) \quad \text{a. e. in } Q_\infty\,. \tag{2.8}$$

Moreover, there exist two positive constants $\overline{\theta}$ *and* $\underline{\theta}$ *(independent of* t*) such that the following uniform upper and lower bounds hold:*

$$\underline{\theta} < \theta(x,t) < \overline{\theta} \quad \text{for a. e. } (x,t) \in Q_\infty. \tag{2.9}$$

Remark 2.3. Hypothesis 2.1 (ii) is fulfilled for example if $\gamma \equiv 0$. Then $\lambda_1 = 0$ and ψ_1 is a constant function. Another easy case is when $\Omega = (a_1, b_1) \times \cdots \times (a_N, b_N)$ is an orthogonal parallelepiped with γ constant on each side. As a last example, let us mention the case that both $\partial\Omega$ and γ are of class C^∞. The first eigenfunction ψ_1 is defined as a minimizer of the Rayleigh functional

$$\mathcal{R}(u) \;=\; \int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} \gamma u^2\,ds$$

on the set $S^1_1 := \{u \in H^1; |u|_H = 1\}$, and λ_1 is given as $\lambda_1 = \min\{\mathcal{R}(u)\,;\, u \in S^1_1\}$. The function ψ_1 thus satisfies the variational equation

$$\int_\Omega \langle \nabla\psi_1, \nabla w\rangle\; dx + \int_{\partial\Omega} \gamma\psi_1\, w\, ds \;=\; \lambda_1 \int_\Omega \psi_1\, w\, dx \tag{2.10}$$

for every $w \in H^1$. Choosing $w = \psi_1^+$, $w = \psi_1^-$ (the positive and negative parts of ψ_1, respectively), we see that both ψ_1^+, ψ_1^-, as well as $|\psi_1| = \psi_1^+ + \psi_1^-$, satisfy the variational equation (2.10). Then $|\psi_1|$ is a weak solution of the problem

$$-\Delta|\psi_1| = \lambda_1|\psi_1| \text{ in } \Omega\,, \quad \partial_{\mathbf{n}}|\psi_1| + \gamma|\psi_1| = 0 \text{ on } \partial\Omega\,. \tag{2.11}$$

By [24, Chap. 2, Thm. 5.1], we have, denoting $H^r = W^{r,2}(\Omega)$, $H^0 = H$, that

$$||\psi_1||_{H^{2+r}} \leq C_r \left(|\lambda_1|\psi_1||_{H^r} + ||\psi_1||_{H^{1+r}}\right) \tag{2.12}$$

for every integer $r \geq 0$ with some constants $C_r > 0$ provided the right-hand side is well defined. This is the case for $r = 0$, hence we may iterate in (2.12) for $r = 1, 2, \ldots$ until we obtain $|\psi_1| \in C^2(\bar{\Omega})$ taking r sufficiently large. The function $|\psi_1|$ does not vanish in Ω, by virtue of Maximum Principle I in [4, Part II, Chap. 2]. Assuming that there exists $x \in \partial\Omega$ such that $|\psi_1(x)| = 0$, also leads to a contradiction. Indeed, we find a ball lying entirely in Ω and touching $\partial\Omega$ at the point x. Maximum Principle III in [4, Part II, Chap. 2] then yields that $\partial_{\mathbf{n}}(|\psi_1(x)|) < 0$, which contradicts (2.11). Hence ψ_1 does not vanish in $\bar{\Omega}$. This argument also shows that this is the unique eigenfunction of the problem (2.5) up to a constant multiple.

To conclude this section, we state a result on uniqueness and continuous data dependence for (2.1–2.4).

Theorem 2.4. *Let Hypothesis 2.1 hold, and let (θ_1, χ_1), (θ_2, χ_2) be solutions to (2.1–2.4) in the sense of Theorem 2.2 associated with respective boundary and initial data $\theta_{\Gamma 1}, \theta_{01}, \chi_{01}$ and $\theta_{\Gamma 2}, \theta_{02}, \chi_{02}$. Put $\hat{\theta} = \theta_1 - \theta_2$, $\hat{\chi} = \chi_1 - \chi_2$, $\hat{\theta}_\Gamma = \theta_{\Gamma 1} - \theta_{\Gamma 2}$, $\hat{\chi}_0 = \chi_{01} - \chi_{02}$, $\hat{\theta}_0 = \theta_{01} - \theta_{02}$. Then for every $T > 0$ there exists a constant $C_T > 0$ such that*

$$\begin{aligned} &\int_0^T \int_\Omega |\hat{\theta}(x,t)|^2 \, dx \, dt + \max_{t\in[0,T]} \int_\Omega |\hat{\chi}(x,t)|^2 \, dx \\ &\qquad \leq C_T \left(|\hat{\theta}_0|_H^2 + |\hat{\chi}_0|_H^2 + \int_0^T \int_{\partial\Omega} \gamma \hat{\theta}_\Gamma^2(x,t) ds \, dt \right) . \end{aligned} \tag{2.13}$$

3 Auxiliary results

In this section we provide some auxiliary results that are used in the remainder of the paper. The first part of this section deals with the continuity of solution operators to general differential inclusions, while the second one recalls some parabolic maximum principle results and a variant of the Moser iteration scheme.

3.1 Solution operators to differential inclusions

Consider a functional φ as in Hypothesis 2.1 (iii). For a given initial condition χ_0, a fixed final time $T > 0$, and a given function $\theta \in L^1(Q_T)$, we solve the following differential inclusion:

$$\alpha(\theta)\,\chi_t + \partial\varphi(\chi) \ \ni \ f[\chi, \theta] \quad \text{a. e. in } Q_T, \quad \chi(x,0) = \chi_0(x) \quad \text{a. e. in } \Omega, \tag{3.1}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a given function, and $f : L^1(Q_T) \times L^1(Q_T) \to L^1(Q_T)$ is a given continuous operator satisfying the following hypothesis.

Hypothesis 3.1. *There exist positive constants α_0, L, C such that:*

(i) $\alpha_0 \le \alpha(\theta)$ *for all* $\theta \in \mathbb{R}$.

(ii) $|\alpha(\theta_1) - \alpha(\theta_2)| \ \le \ L|\theta_1 - \theta_2|$ *for all* $\theta_1,\ \theta_2 \in \mathbb{R}$.

(iii) $|f[\mathcal{X},\theta](x,t)| \le C$ *a. e. in* Q_T *for all* $\mathcal{X},\theta \in L^1(Q_T)$ *such that* $\mathcal{X}(x,t) \in \mathcal{D}(\varphi)$ *a. e. in* Q_T .

(iv) $|f[\mathcal{X}_1,\theta] - f[\mathcal{X}_2,\theta]|_{L^1(Q_t)} \ \le \ L|\mathcal{X}_1 - \mathcal{X}_2|_{L^1(Q_t)}$ *for all* $\mathcal{X}_1,\mathcal{X}_2,\theta \in L^1(Q_T)$

and $t \in [0,T]$.

This is slightly different from [19, Subsection 3.1], where f is assumed to be Lipschitz continuous also with respect to θ. Here, f is only continuous, and we therefore only get the following weaker result.

Proposition 3.2. *Let Hypothesis 3.1 hold, and let* $\mathcal{D}_C(\varphi)$ *be as in Hypothesis 2.1. Then, for every* $\theta \in L^1(Q_T)$, *and for every* $\mathcal{X}_0 \in L^\infty(\Omega)$, $\mathcal{X}_0(x) \in \mathcal{D}_C(\varphi)$ *a. e. in* Ω, *there exists a unique solution* $\mathcal{X} \in L^\infty(Q_T)$ *to Eq. (3.1) such that* $\mathcal{X}_t \in L^\infty(Q_T)$, *and we have*

$$\mathcal{X}(x,t) \in \mathcal{D}_C(\varphi)\,, \quad |f[\mathcal{X},\theta](x,t) - \alpha(\theta(x,t))\,\mathcal{X}_t(x,t)| \le C \quad \text{a. e. in } Q_T\,. \tag{3.2}$$

Moreover, let $\{\theta^{(n)}\}$ *be a sequence that converges strongly in* $L^1(Q_T)$ *to* θ, *let* $\mathcal{X}_0^{(n)} \in L^\infty(\Omega)$ *be initial conditions such that* $\mathcal{X}_0^{(n)}(x) \in \mathcal{D}_C(\varphi)$ *a. e. in* Ω *and* $\mathcal{X}_0^{(n)} \to \mathcal{X}_0$ *in* $L^1(\Omega)$, *and let* $\mathcal{X}^{(n)}, \mathcal{X}$ *be the respective solutions to (3.1). Then* $\mathcal{X}^{(n)} \to \mathcal{X}$, $\mathcal{X}_t^{(n)} \to \mathcal{X}_t$ *strongly in* $L^1(Q_T)$.

Remark 3.3. As a complement to the above Proposition, notice that the strong continuity $L^1(Q_T) \to L^p(Q_T)$ of the solution mapping for $1 \le p < \infty$ follows from the uniform L^∞-bound (3.2). Indeed, testing (3.1) by $\mathcal{X}_t$, we obtain the identity

$$\varphi(\mathcal{X})_t \ = \ -\alpha(\theta)\,\mathcal{X}_t^2 + f[\mathcal{X},\theta]\,\mathcal{X}_t \quad \text{a. e. in } Q_T\,. \tag{3.3}$$

If $\theta^{(n)}, \theta, \mathcal{X}^{(n)}, \mathcal{X}$ are as in Proposition 3.2, the L^∞-bounds (3.2) yield that $\mathcal{X}^{(n)} \to \mathcal{X}$, $\mathcal{X}_t^{(n)} \to \mathcal{X}_t$, $\varphi(\mathcal{X}^{(n)})_t \to \varphi(\mathcal{X})_t$, strongly in any $L^p(Q_T)$ for $1 \le p < \infty$.

The proof of Proposition 3.2 is based on properties of the corresponding space-independent problem. For a given initial condition $\mathcal{X}_0 \in \mathcal{D}(\varphi)$ and a given function $\theta \in L^1(0,T)$, we consider the differential inclusion

$$\alpha(\theta(t))\,\dot{\mathcal{X}}(t) + \partial\varphi(\mathcal{X}(t)) \ \ni \ g(t) \quad \text{a. e. in } (0,T), \quad \mathcal{X}(0) = \mathcal{X}_0, \tag{3.4}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is as in Hypothesis 3.1 and $g \in L^\infty(0,T)$ is such that

$$|g(t)| \le C \quad \text{a. e. in } (0,T). \tag{3.5}$$

We recall from [19, Proposition 3.4] the following result.

Proposition 3.4. *Let Hypotheses 3.1 (i–ii) and (3.5) hold. Then, for every* $\theta \in L^1(0,T)$ *and every* $\mathcal{X}_0 \in \mathcal{D}_C(\varphi)$*, there exists a unique solution* $\mathcal{X} \in W^{1,\infty}(0,T)$ *to Eq. (3.4), and we have*

$$(3.6) \qquad \mathcal{X}(t) \in \mathcal{D}_C(\varphi) \ \ \forall t \in [0,T]\,, \qquad \left|g(t) - \alpha(\theta(t))\,\dot{\mathcal{X}}(t)\right| \ \le \ C \quad \text{a. e. in } (0,T)\,.$$

Moreover, there exists a positive constant R *depending only on* C*,* α_0*, and* L*, such that the solutions* $\mathcal{X}_1, \mathcal{X}_2 \in W^{1,\infty}(0,T)$ *associated with* $\mathcal{X}_{01}, \mathcal{X}_{02} \in \mathcal{D}_C(\varphi)$*,* $\theta_1, \theta_2 \in L^1(0,T)$*, and* $g_1, g_2 \in L^\infty(0,T)$ *with the constraint (3.5), satisfy the inequality*

$$(3.7) \quad |\dot{\mathcal{X}}_1 - \dot{\mathcal{X}}_2|(t) + \frac{d}{dt}|\mathcal{X}_1 - \mathcal{X}_2|(t) \le R\Big(|\theta_1 - \theta_2|(t) + |g_1 - g_2|(t)\Big) \quad \text{a. e. in } (0,T).$$

We now use (3.7) to prove the convergence statement in Proposition 3.2.

Proof of Proposition 3.2. For given $\theta \in L^1(Q_T)$ and $\mathcal{X}_0 \in L^\infty(\Omega)$, $\mathcal{X}_0(x) \in \mathcal{D}_C(\varphi)$ a. e., we obtain the existence of a unique solution to (3.1) by the Banach contraction argument in the same way as in the proof of [19, Proposition 3.2]. To prove the continuity of the solution mapping, consider the sequences $\mathcal{X}_0^{(n)}, \theta^{(n)}, \mathcal{X}^{(n)}$ as above. For almost all $x \in \Omega$, we use (3.7) with $\theta_1(t) = \theta(x,t)$, $\theta_2(t) = \theta^{(n)}(x,t)$, $\mathcal{X}_1(t) = \mathcal{X}(x,t)$, $\mathcal{X}_2(t) = \mathcal{X}^{(n)}(x,t)$, $g_1(t) = f[\mathcal{X},\theta](x,t)$, $g_2(t) = f[\mathcal{X}^{(n)},\theta^{(n)}](x,t)$. Integrating over $\Omega \times (0,t)$ for $t \in (0,T]$, and using Hypothesis 3.1, we obtain that

$$(3.8) \qquad \begin{aligned} &\int_0^t\!\!\int_\Omega |\mathcal{X}_t - \mathcal{X}_t^{(n)}|(x,s)\,dx\,ds + \int_\Omega |\mathcal{X} - \mathcal{X}^{(n)}|(x,t)\,dx - |\mathcal{X}_0 - \mathcal{X}_0^{(n)}|_{L^1(\Omega)} \\ &\quad \le \ R\int_0^t\!\!\int_\Omega \left(|\theta - \theta^{(n)}| + L|\mathcal{X} - \mathcal{X}^{(n)}|\right)(x,s)dx\,ds \\ &\qquad + R\int_0^t\!\!\int_\Omega |f[\mathcal{X},\theta] - f[\mathcal{X},\theta^{(n)}]|(x,s)\,dx\,ds\,, \end{aligned}$$

and Gronwall's argument yields

$$(3.9) \quad \begin{aligned} &\int_0^t\!\!\int_\Omega |\mathcal{X}_t - \mathcal{X}_t^{(n)}|(x,s)\,dx\,ds + \int_\Omega |\mathcal{X} - \mathcal{X}^{(n)}|(x,t)\,dx \ \le \ \mathrm{e}^{RLt}\Big(|\mathcal{X}_0 - \mathcal{X}_0^{(n)}|_{L^1(\Omega)} \\ &\qquad + R\int_0^t\!\!\int_\Omega \left(|\theta - \theta^{(n)}| + |f[\mathcal{X},\theta] - f[\mathcal{X},\theta^{(n)}]|\right)(x,s)\,dx\,ds\Big)\,, \end{aligned}$$

which concludes the proof. ∎

3.2 The maximum principle and Moser iteration

We modify here an elementary maximum principle result from [19, Prop. 3.5, 3.6] to the case of more general boundary conditions. For a fixed final time $T > 0$, we consider in Q_T, for given functions $a : Q_T \to \mathbb{R}$, $u_\Gamma : \Sigma_T \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$, $R : Q_T \times \mathbb{R} \to \mathbb{R}$, the evolution problem

(3.10)

$$\left\{ \begin{aligned} &\int_\Omega a u_t w\,dx + \int_\Omega \langle \nabla u, \nabla w\rangle\,dx + \int_{\partial\Omega} \gamma(u - u_\Gamma) w\,ds \ = \ \int_\Omega R(x,t,u) w\,dx \quad \forall w \in H^1 \\ &u(x,0) = u_0(x) \quad \text{a. e.} \end{aligned} \right.$$

under the following hypothesis.

Hypothesis 3.5. *The data in (3.10) have the following properties.*

(i) $a \in L^\infty(Q_T)$, $a(x,t) \geq a_* > 0$ a. e.

(ii) $\gamma \in L^\infty(\partial\Omega)$, $\gamma \geq 0$ a. e.

(iii) $\exists h \in L^\infty(0,T)$: $|R(x,t,u_1) - R(x,t,u_2)| \leq h(t)|u_1 - u_2|$ a. e. $\forall u_1, u_2 \in \mathbb{R}$.

(iv) $R(\cdot,\cdot,0) \in L^2(Q_T)$, $R(x,t,0) \leq 0$ a. e.

(v) $u_\Gamma \in L^2(\Sigma_T)$, $(u_\Gamma)_t \in L^2(\Sigma_T)$, $u_\Gamma \leq 0$ a. e.

(vi) $u_0 \in H^1$, $u_0 \leq 0$ a. e.

Proposition 3.6. *Let Hypothesis 3.5 hold. Then Problem (3.10) admits a unique solution $u \in L^2(Q_T)$ such that $u_t \in L^2(Q_T)$, $\Delta u \in L^2(Q_T)$, $\nabla u \in C([0,T];H)$. Moreover, we have $u(x,t) \leq 0$ a. e. in Q_T.*

Sketch of the proof. For a sufficiently large discretization parameter $n \in \mathbb{N}$, we consider the time-discrete problem with time step $\delta = T/n$,

$$(3.11)\qquad \frac{1}{\delta}\int_\Omega a_k(u_k - u_{k-1})\, w\, dx + \int_\Omega \langle \nabla u_k, \nabla w\rangle\, dx + \int_{\partial\Omega} \gamma(u_k - u_{\Gamma k}) w\, ds = \int_\Omega g_k(u_k, x) w\, dx \quad \forall w \in H^1 \quad k = 1, \dots, n,$$

where u_0 is defined as in (3.10), and where $a_k(x)$, $g_k(\cdot, x)$, for $x \in \Omega$, $u_{\Gamma k}(x)$, for $x \in \partial\Omega$, are the integral means of the corresponding functions in (3.10) over the time interval $[(k-1)\delta, k\delta]$. The existence of $u_k \in H^1$ with $\Delta u_k \in H$ is obtained e. g. from the Lax-Milgram Lemma, recursively for $k = 1, \dots, n$, whenever $n > T|h|_\infty/a_*$. Choosing $w = u_k^+$ (the positive part of u_k) in (3.11), and assuming that $u_{k-1} \leq 0$ a. e., we obtain that

$$(3.12)\qquad \int_\Omega \frac{a_k}{\delta}|u_k^+|^2\, dx + \int_\Omega |\nabla u_k^+|^2 dx + \int_{\partial\Omega} \gamma |u_k^+|^2\, ds \;\leq\; |h|_\infty \int_\Omega |u_k^+|^2\, dx\,.$$

We have $a_k(x) \geq a_*$, hence $u_k \leq 0$ a. e. for all $k = 1, \dots, n$. Choosing now $w = u_k - u_{k-1}$ in (3.11), we derive in a standard way a priori estimates that enable us to pass to the limit as $n \to \infty$ and prove the existence of a nonpositive solution to (3.10). To check that this solution is unique, consider two solutions u^1, u^2 and set $\bar{u} = u^1 - u^2$. We now test the difference of Eqs. (3.10), written for u^1 and u^2, by a suitable regularization of $\bar{u}_t$, Galerkin for instance, choosing basis functions from the complete orthonormal system $\{\psi_k\,;\, k \in \mathbb{N}\}$ in H of eigenfunctions of the problem

$$(3.13)\qquad -\Delta\psi_k \;=\; \lambda_k \psi_k \quad \text{in } \Omega, \quad \partial_{\mathbf{n}}\psi_k + \gamma\psi_k \;=\; 0 \quad \text{on } \partial\Omega\,.$$

Passing to the limit in the Galerkin approximations, we obtain that

$$(3.14)\qquad a_* \int_0^t \int_\Omega |\bar{u}_t|^2 dx\, d\tau \;\leq\; |h|_\infty \int_0^t \int_\Omega |\bar{u}\bar{u}_t| dx\, d\tau\,,$$

hence $\bar{u} = 0$ by Gronwall's Lemma. ■

Corollary 3.7. *Let Hypotheses 3.5 (i)–(iii) hold, and let (iv)–(vi) be replaced by*

(iv)' $R(\cdot,\cdot,0) \in L^2(Q_T)$, $R(x,t,0) \le R^*$ *a. e.*

(v)' $u_\Gamma \in L^2(\Sigma_T)$, $(u_\Gamma)_t \in L^2(\Sigma_T)$, $u_\Gamma \le u_\Gamma^*$ *a. e.*

(vi)' $u_0 \in H^1$, $u_0 \le u^*$ *a. e.*

with some positive constants R^*, u_Γ^*, u^*. *Set* $K = \max\{u^*, u_\Gamma^*\}$, *consider some constant* $B_1 \ge R^*/K$, *and for* $t \in [0,T]$ *put*

$$H(t) \;=\; \frac{1}{a_*}\int_0^t (B_1 + h(\tau))\, d\tau\,.$$

Then Problem (3.10) has a unique solution with the regularity from Proposition 3.6 such that

$$u(x,t) \;\le K\, \mathrm{e}^{H(t)} \quad \text{a. e.} \tag{3.15}$$

Proof. For $(x,t) \in Q_T$ and $v \in \mathbb{R}$ set $v_\Gamma(x,t) = u_\Gamma(x,t) - K\mathrm{e}^{H(t)}$, $\tilde{R}(x,t,v) = R(x,t,v + K\mathrm{e}^{H(t)}) - K\dot{H}(t)a(x,t)\mathrm{e}^{H(t)}$. Then $\tilde{R}$ fulfils the conditions 3.5 (iii)–(iv), since $\tilde{R}(\cdot,\cdot,0) \in L^2(Q_T)$ and

$$\begin{aligned}
\tilde{R}(x,t,0) &= R(x,t,K\mathrm{e}^{H(t)}) - K\dot{H}(t)a(x,t)\mathrm{e}^{H(t)}\\
&\le R(x,t,0) + K\mathrm{e}^{H(t)}\left(h(t) - a(x,t)\dot{H}(t)\right)\\
&\le R^* + K\mathrm{e}^{H(t)}\left(h(t) - a(x,t)\dot{H}(t)\right)\\
&\le \left(1 - \frac{a(x,t)}{a_*}\right)\left(R^* + Kh(t)\mathrm{e}^{H(t)}\right) \;\le\; 0\,.
\end{aligned}$$

A function v on Q_T with appropriate regularity is a solution to the equation

$$\int_\Omega a v_t w\, dx + \int_\Omega \langle \nabla v, \nabla w\rangle\, dx + \int_{\partial\Omega} \gamma(v - v_\Gamma) w\, ds \;=\; \int_\Omega \tilde{R}(x,t,v) w\, dx \quad \forall w \in H^1 \tag{3.16}$$

with initial condition $v(x,0) = v_0 := u_0(x) - K$ if and only if $u(x,t) := v(x,t) + K\mathrm{e}^{H(t)}$ is a solution to Problem (3.10). Since $\tilde{R}$, a, γ, v_Γ, and v_0 fulfil Hypothesis 3.5, from Proposition 3.6 it follows that v is uniquely determined and $v(x,t) \le 0$ a. e. Hence u is uniquely determined and satisfies the desired growth condition (3.15). ∎

Corollary 3.8. *Let Hypotheses 3.5 (i)–(iii) hold, and let (iv)–(vi) be replaced by*

(iv)" $R(\cdot,\cdot,0) \in L^2(Q_T)$, $R(x,t,0) \ge 0$ *a. e.*

(v)" $u_\Gamma \in L^2(\Sigma_T)$, $(u_\Gamma)_t \in L^2(\Sigma_T)$, $u_\Gamma \ge 0$ *a. e.*

(vi)" $u_0 \in H^1$, $u_0(x) \ge u_*\psi_1(x)$ *a. e.*

with some positive constant u_*, where ψ_1 is the positive eigenfunction corresponding to the smallest eigenvalue $\lambda_1 \geq 0$ of (3.13) for $k = 1$. Consider any constant $B_2 \geq \lambda_1$, and for $t \in [0, T]$ put

$$H(t) \;=\; \frac{1}{a_*}\int_0^t (B_2 + h(\tau))\,d\tau\,.$$

Then Problem (3.10) has a unique solution with the regularity from Proposition 3.6 and such that

$$u(x,t) \;\geq u_*\psi_1(x)\mathrm{e}^{-H(t)} \quad \text{a. e.}$$

Proof. For $(x,t) \in Q_T$ and $v \in \mathbb{R}$ set $\tilde{R}(x,t,v) = -R(x,t,-v + u_*\psi_1(x)\mathrm{e}^{-H(t)}) + u_*\mathrm{e}^{-H(t)}(\lambda_1 - a\dot{H}(t))\psi_1(x)$, $v_\Gamma(x,t) = -u_\Gamma(x,t)$. Then $\tilde{R}$ fulfils again the conditions 3.5 (iii)–(iv), since $\tilde{R}(\cdot,\cdot,0) \in L^2(Q_T)$ and

$$\begin{aligned}\tilde{R}(x,t,0) &= -R\big(x,t,u_*\psi_1(x)\mathrm{e}^{-H(t)}\big) + u_*\mathrm{e}^{-H(t)}(\lambda_1 - a\dot{H}(t))\psi_1(x)\\ &\leq -R(x,t,0) + u_*\psi_1(x)\mathrm{e}^{-H(t)}\left(\lambda_1 + h(t) - a(x,t)\dot{H}(t)\right)\\ &\leq u_*\psi_1(x)\mathrm{e}^{-H(t)}\left(1 - \frac{a(x,t)}{a_*}\right)(\lambda_1 + h(t)) \;\leq\; 0\,.\end{aligned}$$

As in the proof of Corollary 3.7, we have a one-to-one correspondence between the solution v to (3.16) with initial condition $v(x,0) = u_*\psi_1(x) - u_0(x)$ and the solution $u(x,t) = u_*\psi_1(x)\mathrm{e}^{-H(t)} - v(x,t)$ to Problem (3.10). By Proposition 3.6 we have again $v \leq 0$ a. e., and the assertion immediately follows. ■

Consider now in Q_T the problem

(3.17)
$$\begin{cases} \displaystyle\int_\Omega a u_t w\,dx + \int_\Omega \langle\nabla u, \nabla w\rangle\,dx + \int_{\partial\Omega}\gamma(u - u_\Gamma)w\,ds \\ \displaystyle\qquad = \int_\Omega \big(r(x,t) + h_1(x,t)\,u\; + h_2(x,t)\,u\,|\log|u||\big)\,w\,dx \quad \forall w \in H^1\,, \\ u(x,0) = u_0(x) \quad \text{a. e.} \end{cases}$$

under Hypotheses 3.5 (i)–(ii), where u_Γ has the regularity as in (v), and with given functions $r, h_1, h_2 \in L^\infty(Q_T)$, assuming that

(3.18)
$$0 \leq r(x,t) \leq r^*\,,\; 0 \leq u_\Gamma(x,t) \leq u_\Gamma^*\,,\; |h_i(x,t)| \leq h^* \text{ for } i = 1,2\,,\; u_*\psi_1(x) \leq u_0(x) \leq u^*$$

a. e. in the respective domains, where $r^*, h^*, u_\Gamma^*, u_*, u^*$ are fixed positive constants. Set

$$\begin{aligned} K &= \max\{u^*, u_\Gamma^*\}\,,\\ A &= \max\{0, \log K, -\log(u_*\psi_*)\}\,,\\ B &= \max\left\{\lambda_1, \frac{r^*}{K}\right\} + (2 + A)h^*\,,\\ C &= \frac{1}{a_*}(B + h^*)\,, \end{aligned}$$

where $\psi_* > 0$ is a uniform lower bound for $\psi_1(x)$.

Proposition 3.9. *Problem (3.17) has a unique solution, and it holds*

$$u_*\psi_1(x)\mathrm{e}^{-H(t)} \le u(x,t) \le K\mathrm{e}^{H(t)} \text{ a.e. }, \tag{3.19}$$

where

$$H(t) \;=\; \frac{1}{a_*}\int_0^t \left(B + h^*\mathrm{e}^{C\tau}\right) d\tau\,.$$

Proof. For all admissible values of the arguments set

$$R(x,t,u) = r(x,t) + h_1(x,t)u + h_2(x,t)u\min\{|\log|u||, A + \mathrm{e}^{Ct}\}\,. \tag{3.20}$$

Then $R(x,t,0) = r(x,t) \in [0,r^*]$, and

$$|\partial_u R(x,t,u)| \le h^*(2 + A + \mathrm{e}^{Ct}) \text{ a.e.}$$

We thus may apply Corollaries 3.7-3.8 and conclude that the solution to (3.10) satisfies the estimates (3.19). It remains to check that u is a solution to (3.17). From (3.19) it follows that

$$\log(u_*\psi_*) - H(t) \le \log u(x,t) \le \log K + H(t) \text{ a.e. }, \tag{3.21}$$

hence the constraint $A + \mathrm{e}^{Ct}$ in (3.20) is never active, and the assertion follows. ∎

Finally, we derive here a global in time Moser-type estimate (cf. [1]) for non-homogeneous mixed boundary conditions. We follow in principle the scheme of [22], showing in addition the explicit dependence upon some parameters of the problem, which is needed in the proof of Theorem 2.2. We state the result in the space

$$L^\infty_{\mathrm{loc}}(Q_\infty) \;:=\; \{u : \Omega\times(0,\infty) \to \mathbb{R}\,;\, u\big|_{Q_T} \in L^\infty(Q_T) \text{ for all } T > 0\}\,.$$

We will also make repeated use of the following well-known interpolation inequality

$$|v|_H \le A\left(\eta|\nabla v|_H + \eta^{-N/2}|v|_{L^1(\Omega)}\right), \tag{3.22}$$

which holds for every $v \in H^1$ and every $\eta \in (0,1)$, with a positive constant A that depends on Ω, but neither on v nor on η.

Proposition 3.10. *Given a nonnegative function $\gamma \in L^1(\partial\Omega)$, consider the problem*

$$a(x,t)u_t - \Delta u = \mathcal{H}[u] \quad \text{a.e. on } Q_\infty\,, \tag{3.23}$$

$$\partial_{\mathbf{n}} u(x,t) + \gamma(x)\,\left(h(x,t,u(x,t)) - u_\Gamma(x,t)\right) = 0 \quad \text{a.e. on } \Sigma_\infty\,, \tag{3.24}$$

$$u(x,0) = u^0 \quad \text{a.e. in } \Omega\,, \tag{3.25}$$

under the assumption that there exist positive constants $H_0, H_1, C_h, a_0, a_1, A_0, U, U_\Gamma, E_0$ such that the following holds:

(i) *The mapping $\mathcal{H} : L^\infty_{\mathrm{loc}}(Q_\infty) \to L^\infty_{\mathrm{loc}}(Q_\infty)$ has the property that*

$$u(x,t)\,\mathcal{H}[u](x,t) \le H_1|u(x,t)| + H_0|u(x,t)|^2 \quad \text{a.e. in } Q_\infty\,, \quad \forall u \in L^\infty_{\mathrm{loc}}(Q_\infty)\,.$$

(ii) *h is a Carathéodory function on $Q_\infty \times \mathbb{R}$ such that $h(x,t,u)\,u \geq C_h\, u^2$ a. e. for all $u \in \mathbb{R}$.*

(iii) *$a, a_t \in L^\infty(Q_\infty)$ are such that $a_0 \leq a(x,t) \leq A_0$ and $|a_t(x,t)| \leq a_1$ a. e. in Q_∞.*

(iv) *$u^0 \in L^\infty(\Omega)$, $|u^0(x)| \leq U$ a. e. in Ω.*

(v) *$u_\Gamma \in L^\infty(\Sigma_\infty)$, $|u_\Gamma(x,t)| \leq U_\Gamma$ a. e. on Σ_∞.*

(vi) *There exists a solution $u \in L^\infty_{\text{loc}}(Q_\infty) \cap L^2_{\text{loc}}(0,\infty; H^1)$ to (3.23–3.25) satisfying the a priori estimate*

$$\int_\Omega |u(x,t)|\, dx \leq E_0 \quad \text{a. e. in } (0,\infty)\,.$$

Then there exists a positive constant C^* depending only on A (cf. (3.22)), $|\Omega|$, $|\gamma|_{L^1(\partial\Omega)}$, C_h, U, U_Γ, a_0, and A_0 such that

$$|u(t)|_{L^\infty(\Omega)} \leq C^* \left(1 + a_1 + H_0\right)^{1+N/2} \left(1 + H_1 + E_0\right) \quad \text{for a. e. } t > 0. \tag{3.26}$$

Proof. We prove Proposition 3.10 under the additional hypothesis

$$U \;=\; U_\Gamma \;=\; C_h \;=\; H_1 \;=\; E_0 \;=\; 1\,. \tag{3.27}$$

The general result is then easily obtained via the transformation

$$\tilde{\gamma} = C_h \gamma\,, \quad \tilde{u} = \frac{1}{K} u\,, \quad K = \max\left\{ E_0, H_1, U, \frac{U_\Gamma}{C_h} \right\}\,. \tag{3.28}$$

During the proof we will denote by the symbol C_i, $i = 1, 2, \ldots$, some positive constants depending only on A, $|\Omega|$, $|\gamma|_{L^1(\partial\Omega)}$, a_0, and A_0.

For $k \in \mathbb{N}$, test equation (3.23) by $u|u|^{2^k-2}$. Using the boundary conditions in (3.24), we obtain that

$$a(x,t)\, u_t\, u\, |u|^{2^k-2} = 2^{-k} \frac{\partial}{\partial t}\left(a(x,t)|u|^{2^k}\right) - 2^{-k} a_t |u|^{2^k}$$

$$-\int_\Omega \Delta u\, u\, |u|^{2^k-2}\, dx = \int_\Omega \left\langle \nabla u, \nabla\left(u\,|u|^{2^k-2}\right)\right\rangle dx + \int_{\partial\Omega} \gamma\left(h(x,t,u) - u_\Gamma\right) u\,|u|^{2^k-2}\, ds$$

$$\geq \frac{2^k-1}{2^{2k-2}} \int_\Omega \left|\nabla\left(u\,|u|^{2^{k-1}-1}\right)\right|^2 dx + \int_{\partial\Omega} \gamma\,\left(|u|^{2^k} - |u|^{2^k-1}\right) ds\,.$$

Set $\Phi_k = u\,|u|^{2^{k-1}-1}$. Then we get, using Hölder's and Young's inequalities

$$
\begin{aligned}
2^{-k}\frac{d}{dt}\int_\Omega a(x,t)\,|\Phi_k|^2\,dx &+ \frac{2^k-1}{2^{2k-2}}\int_\Omega |\nabla\Phi_k|^2\,dx + \int_{\partial\Omega}\gamma\,|\Phi_k|^2\,ds \\
&\le \left(\int_{\partial\Omega}\gamma\,|\Phi_k|^2\,ds\right)^{1-2^{-k}}\left(\int_{\partial\Omega}\gamma\,ds\right)^{2^{-k}} + a_1 2^{-k}\int_\Omega |\Phi_k|^2\,dx \\
&\quad + \int_\Omega \left(H_0|\Phi_k|^2 + |\Phi_k|^{2(1-2^{-k})}\right)\,dx \\
&\le \left(1-2^{-k}\right)\int_{\partial\Omega}\gamma\,|\Phi_k|^2\,ds + 2^{-k}\int_{\partial\Omega}\gamma\,ds \\
&\quad + a_1\,2^{-k}\int_\Omega |\Phi_k|^2\,dx + (H_0+1)\int_\Omega |\Phi_k|^2\,dx + 2^{-k}|\Omega|\,.
\end{aligned}
$$

Setting $H_2 := 1 + a_1 + H_0$ and multiplying the above inequality by 2^k, we find that

$$
\frac{d}{dt}\int_\Omega a(x,t)\,|\Phi_k|^2\,dx + 2\int_\Omega |\nabla\Phi_k|^2\,dx \;\le\; 2^k H_2\int_\Omega |\Phi_k|^2\,dx + |\Omega| + \int_{\partial\Omega}\gamma\,ds. \tag{3.29}
$$

We now use the interpolation inequality (3.22) to derive the inequalities

$$
\int_\Omega |\Phi_1|^2\,dx \le\; 2A^2\left(\eta^2|\nabla\Phi_1|_H^2 + \eta^{-N}E_0^2\right) \tag{3.30}
$$

$$
\int_\Omega |\Phi_k|^2\,dx \le\; 2A^2\left(\eta^2|\nabla\Phi_k|_H^2 + \eta^{-N}|\Phi_{k-1}|_H^4\right) \quad \text{for } k>1\,. \tag{3.31}
$$

For $k=1$, we infer from (3.29) and (3.30) that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega a(x,t)\,|\Phi_1|^2\,dx + 2\int_\Omega |\nabla\Phi_1|^2\,dx \\
&\quad\le\; 4H_2\,A^2\left(\eta^2\int_\Omega |\nabla\Phi_1|^2\,dx + \eta^{-N}E_0^2\right) + |\Omega| + \int_{\partial\Omega}\gamma\,ds.
\end{aligned}
$$

Choosing $\eta = 1/(2A\sqrt{H_2})$, we find that

$$
\frac{d}{dt}\int_\Omega a(x,t)\,|\Phi_1|^2\,dx + \int_\Omega |\nabla\Phi_1|^2\,dx \le C_1\,H_2^{1+N/2}\,. \tag{3.32}
$$

For $k>1$, we choose $\eta = 1/(A\sqrt{2^{k+1}H_2})$, we conclude from (3.29) and (3.31) that

$$
\frac{d}{dt}\int_\Omega a(x,t)|\Phi_k|^2\,dx + |\nabla\Phi_k|_H^2 \le C_2\left(1 + (2^kH_2)^{1+N/2}|\Phi_{k-1}|_H^4\right)\,. \tag{3.33}
$$

Using again (3.31) with $\eta = 1/(\sqrt{2}A)$, it follows for a.e. $t>0$ that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega a(x,t)|\Phi_1|^2\,dx + |\Phi_1(t)|_H^2 \le C_3\,H_2^{1+N/2}\,, \\
&\frac{d}{dt}\int_\Omega a(x,t)|\Phi_k|^2\,dx + |\Phi_k(t)|_H^2 \le C_4\left(1 + (2^kH_2)^{1+N/2}|\Phi_{k-1}(t)|_H^4\right)\,.
\end{aligned}
$$

By assumption, we have $a_0|\Phi_k(t)|_H^2 \le \int_\Omega a(x,t)|\Phi_k|^2 dx \le A_0|\Phi_k(t)|_H^2$, and $|\Phi_k(0)|_H^2 \le |\Omega|$. Hence,

$$|\Phi_1(t)|_H^2 \le C_5\, H_2^{1+N/2}\,,$$
$$|\Phi_k(t)|_H^2 \le C_6\Big(1 + (2^k H_2)^{1+N/2} \max_{0\le\tau\le t} |\Phi_{k-1}(\tau)|_H^4\Big).$$

Define now

$$z_k(t) = \max_{0\le\tau\le t} |u(\tau)|_{L^{2^k}(\Omega)} = \max_{0\le\tau\le t} |\Phi_k(\tau)|_H^{2^{-k}}\,.$$

Then we have

$$z_1(t) \le C_7\, H_2^{(1+N/2)/2}\,,$$
$$z_k(t) \le C_8^{2^{-k}}\, \big(2^k H_2\big)^{(1+N/2)2^{-k}} \max\{1, z_{k-1}(t)\}\,.$$

In particular, putting $y_k(t) = \max\{1, z_k(t)\}$, we get

$$y_1(t) \le C_9\, H_2^{(1+N/2)/2},$$
$$y_k(t) \le \Big(C_{10}\, H_2^{(1+N/2)}\Big)^{2^{-k}} 2^{k\,(1+N/2)\,2^{-k}}\, y_{k-1}(t)\,, \quad \text{for } k \ge 2.$$

Hence, we conclude that

$$\begin{aligned}\log y_k(t) \le\ & \sum_{j=1}^{k} 2^{-j}\Big(\log\Big(C_{11}\, H_2^{1+N/2}\Big) + j\,(1+N/2)\,\log 2\Big)\\ &\le (1+N/2)\,\big(C_{12} + \log H_2\big)\,,\end{aligned}$$

independently of k and $t > 0$. Thus, it suffices to choose $C^* = \exp(C_{12}\,(1+N/2))$, and conclude that

$$\sup_{t\ge 0,\, k\in\mathbb{N}} |u(t)|_{L^{2^k}(\Omega)} \ \le\ C^* H_2^{1+N/2} \ =\ C^*(1 + a_1 + H_0)^{1+N/2}\,.$$

Formula (3.26) now follows from (3.27–3.28). ■

4 Proof of the uniqueness result

We start with the proof of Theorem 2.4. Equation (2.2) is for (almost) all $x \in \Omega$ of the form (3.4), with

$$\alpha(\theta) = \frac{\mu(\theta)}{\beta + \theta}\,,$$
$$g = f[\theta, \chi] = -\frac{1}{\beta+\theta}\,\left(c_V'(\chi)\theta\,(1 - \log\theta) + \lambda'(\chi) + \theta\sigma'(\chi) + b[\chi]\right)\,.$$

Within the range $\underline{\theta} < \theta < \overline{\theta}$ and $\chi \in \mathcal{D}_C(\varphi)$, $\chi_t \le C$ of admissible values for the solutions (taking indeed $\underline{\theta} = \min\{\underline{\theta}_1, \underline{\theta}_2\}$ etc.), all nonlinearities in (2.1–2.2) are Lipschitz continuous. Using the notation from Theorem 2.4, we obtain, as a consequence of (3.7), for a. e. $(x,t) \in Q_\infty$ the estimate

$$|\hat{\chi}_t(x,t)| + \frac{\partial}{\partial t}|\hat{\chi}(x,t)| \le R_0 \left(|\hat{\theta}(x,t)| + |\hat{\chi}(x,t)| + \int_\Omega |\hat{\chi}(y,t)|\, dy \right), \tag{4.1}$$

with some constant R_0. Let us fix some $T > 0$. In what follows, we denote by $R_1, R_2, \dots$ suitable constants depending possibly on T, but independent of the solutions. Integrating (4.1) over Ω, we obtain by Gronwall's argument that

$$\int_\Omega |\hat{\chi}(y,t)|\, dy \le R_1 \left(\int_\Omega |\hat{\chi}_0(y)|\, dy + \int_0^t \int_\Omega |\hat{\theta}(y,\tau)|\, dy\, d\tau \right). \tag{4.2}$$

Hence, testing (4.1) by $\mathrm{e}^{-R_0 t}$, and using (4.2),

$$\int_0^t |\hat{\chi}_t(x,\tau)|\, d\tau + |\hat{\chi}(x,t)| \le R_2 \left(|\hat{\chi}_0(x)| + \int_0^t |\hat{\theta}(x,\tau)|\, d\tau + \int_0^t \int_\Omega |\hat{\theta}(y,\tau)|\, dy\, d\tau \right) \tag{4.3}$$

for a. e. $x \in \Omega$ and every $t \in [0,T]$. In particular,

$$\int_0^t \int_\Omega |\hat{\chi}_t(x,\tau)|\, dx\, d\tau \le R_3 \left(\int_\Omega |\hat{\chi}_0(x)|\, dx + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|\, dx\, d\tau \right). \tag{4.4}$$

We now multiply (4.3) by $|\hat{\chi}(x,t)|$ and integrate over Ω to get for all $t \in [0,T]$ that

$$\int_\Omega |\hat{\chi}(x,t)|^2\, dx \le R_4 \left(|\hat{\chi}_0|_H^2 + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2\, dx\, d\tau \right). \tag{4.5}$$

The crucial point is to exploit Eq. (2.1) properly. Notice first that we have

$$b[\chi]\chi_t(x,t) = 2B[\chi]_t(x,t) + 2\int_\Omega k(x,y)\, G'(\chi(x,t) - \chi(y,t))\, \chi_t(y,t)\, dy. \tag{4.6}$$

We integrate the difference of the two equations (2.1), written for (θ_1, χ_1) and (θ_2, χ_2), from 0 to t, rewriting the terms $b[\chi_i](\chi_i)_t$ according to (4.6). We test the result by $\hat{\theta}(x,t)$. Using the Lipschitz continuity of all nonlinearities (φ is Lipschitz continuous on $\mathcal{D}_C(\varphi)$ with constant C), and denoting $\hat{\Theta}(x,t) = \int_0^t \hat{\theta}(x,\tau) d\tau$, $\hat{\Theta}_\Gamma(x,t) = \int_0^t \hat{\theta}_\Gamma(x,\tau) d\tau$, we obtain for each $t > 0$ that

$$\begin{aligned} c_0 \int_\Omega |\hat{\theta}(x,t)|^2\, dx &+ \frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla\hat{\Theta}|^2 dx + \frac{1}{2}\int_{\partial\Omega} \gamma\hat{\Theta}^2 ds - \int_{\partial\Omega} \gamma\hat{\Theta}\hat{\Theta}_\Gamma ds \right) + \int_{\partial\Omega} \gamma\hat{\Theta}\hat{\theta}_\Gamma ds \\ &\le R_5 \left(|\hat{\theta}_0|_H^2 + |\hat{\chi}_0|_H^2 + \int_\Omega |\hat{\chi}(x,t)|^2\, dx + \int_0^t \int_\Omega \int_\Omega k(x,y) |\hat{\chi}_t(y,\tau)| |\hat{\theta}(x,t)| dx\, dy\, d\tau \right). \end{aligned}$$

The last term on the right-hand side of the above inequality can be estimated, using (4.4), as

$$\begin{aligned} \int_0^t \int_\Omega \int_\Omega k(x,y) |\hat{\chi}_t(y,\tau)| |\hat{\theta}(x,t)| dx\, dy\, d\tau &\le R_6 \int_\Omega |\hat{\theta}(x,t)| dx \int_0^t \int_\Omega |\hat{\chi}_t(y,\tau)| dy\, d\tau \\ &\le R_7 \left(\int_\Omega |\hat{\theta}(x,t)|^2 dx \right)^{1/2} \left(\int_\Omega |\hat{\chi}_0(x)|^2\, dx + \int_0^t \int_\Omega |\hat{\theta}(x,\tau)|^2\, dx\, d\tau \right)^{1/2}. \end{aligned}$$

Combining the last two inequalities again with the Gronwall lemma, we obtain for each $t \in [0,T]$ the estimate

$$
\begin{aligned}
(4.7)\quad & \int_0^t \int_\Omega |\hat\theta(x,\tau)|^2 dx\, d\tau + \int_\Omega |\nabla \hat\Theta(x,t)|^2 dx + \int_{\partial\Omega} \gamma \hat\Theta^2(s,t) ds \\
& \le R_8 \left(|\hat\theta_0|_H^2 + |\hat\chi_0|_H^2 + \int_0^t \int_{\partial\Omega} \gamma \hat\theta_\Gamma^2(s,\tau) ds\, d\tau + \int_0^t \int_\Omega |\hat\chi(x,\tau)|^2\, dx\, d\tau \right) .
\end{aligned}
$$

We now multiply (4.7) by $2R_4$, add the result to (4.5), and see that Gronwall's argument can be applied again to arrive at the final estimate

(4.8)

$$
\int_\Omega |\hat\chi(x,t)|^2\, dx + \int_0^t \int_\Omega |\hat\theta(x,\tau)|^2 dx\, d\tau \le R_8 \left(|\hat\theta_0|_H^2 + |\hat\chi_0|_H^2 + \int_0^t \int_{\partial\Omega} \gamma \hat\theta_\Gamma^2(s,\tau) ds\, d\tau \right) .
$$

With this, Theorem 2.4 is proved. ■

5 Proof of the existence result

This section is devoted to the proof of Theorem 2.2 (i. e. the existence of solutions to the system (2.1–2.4), (2.6–2.8)). We use a standard technique: we first truncate the system (2.1–2.2), prove existence of solutions to the truncated problem, and finally show that the solution of this system is also a solution of original system, removing the truncation.

5.1 Approximation

Assuming Hypothesis 2.1 to hold, we proceed as follows: first solve the problem corresponding to (2.1–2.4), in which we regularize the coefficient μ and the logarithmic contribution in (2.2), replace θ by $|\theta|$ at suitable places, and then derive upper and lower bounds for θ that will allow us to conclude that the solution of the modified problem satisfies also (2.1–2.4), (2.6–2.8). For some sufficiently large cut-off parameter $\varrho > 0$, which will be specified later, we introduce for $\theta \in \mathbb{R}$ the functions

$$
(5.1)\qquad \mu_\varrho(\theta) = \begin{cases} \mu(|\theta|) & \text{for } |\theta| \le \varrho\,, \\ \mu(\varrho) + \mu_*(|\theta| - \varrho) & \text{for } |\theta| > \varrho\,, \end{cases}
$$

$$
(5.2)\qquad L_\varrho(\theta) = \begin{cases} 0 & \text{for } \theta \le 0\,, \\ \log\theta & \text{for } 0 < \theta < \varrho\,, \\ \log\varrho & \text{for } \theta \ge \varrho\,, \end{cases}
$$

and consider the following problem:

Problem 5.1. For $T > 0$ find a pair (θ, χ) with the regularity (2.6) and (2.8) restricted to the time interval $[0,T]$, solving a. e. in Q_T the system of equations

$$
(5.3)\qquad (c_V(\chi)\theta + \lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t - \Delta\theta = 0\,,
$$

$$
(5.4)\qquad \mu_\varrho(\theta)\chi_t + c_V'(\chi)\theta\,(1 - L_\varrho(\theta)) + \lambda'(\chi) + \theta\sigma'(\chi) + (\beta + |\theta|)\partial\varphi(\chi) + b[\chi] \ni 0\,,
$$

with boundary and initial conditions (2.3–2.4).

Lemma 5.2. *For each fixed $\varrho > 0$, there exists at least one solution to Problem 5.1. Moreover, there exist positive numbers $c_{\varrho,T} < C_{\varrho,T}$ such that $c_{\varrho,T} \le \theta(x,t) \le C_{\varrho,T}$ a. e.*

Proof. Consider the Faedo-Galerkin approximations

$$\theta^m(x,t) \; = \; \sum_{k=1}^{m} \theta_k(t)\, \psi_k(x)\,,$$

where $\{\psi_k\,;\, k \in \mathbb{N}\}$ is the complete orthonormal system in H of eigenfunctions of the problem (3.13), and where $\theta_k(t)$ satisfy the system of equations

(5.5)
$$\int_\Omega (c_V(\chi)\theta^m)_t \psi_k\, dx + \int_\Omega \langle \nabla\theta^m, \nabla\psi_k\rangle\, dx + \int_{\partial\Omega} \gamma(\theta^m - \theta_\Gamma)\psi_k\, ds$$
$$= - \int_\Omega \big((\lambda(\chi) + \beta\varphi(\chi))_t + b[\chi]\chi_t \big) \psi_k\, dx\,, \quad k = 1, \dots, m\,,$$

(5.6)
$$\mu_\varrho(\theta^m)\chi_t + c_V'(\chi)\theta^m(1 - L_\varrho(\theta^m)) \; + \lambda'(\chi) + \theta\sigma'(\chi) + (\beta + |\theta^m|)\partial\varphi(\chi) + b[\chi] \ni 0\,,$$

with the initial conditions

$$\theta_k(0) = \int_\Omega \theta_0\psi_k\, dx\,, \tag{5.7}$$

$$\chi(x,0) = \chi_0(x)\,. \tag{5.8}$$

It follows from Proposition 3.2 and Remark 3.3 that Eq. (5.6) defines a mapping that with each $\theta^m \in L^1(Q_T)$ associates continuously χ, χ_t and $\varphi(\chi)_t$ in any $L^p(Q_T)$. Equation (5.5) therefore has the form

$$\int_\Omega F[\theta^m]\theta_t^m \psi_k\, dx + \lambda_k\, \theta_k \; = \; \int_{\partial\Omega} \gamma\theta_\Gamma\psi_k ds + \int_\Omega H[\theta^m]\, \psi_k\, dx\,, \tag{5.9}$$

with continuous operators $F, H : L^1(Q_T) \to L^p(Q_T)$ for suitably chosen $p > 1$, and such that $\tilde{C}_1 \ge F[\theta](x,t) \ge c_0$, $|H[\theta](x,t)| \le \tilde{C}_1$ a. e. for all $\theta \in L^1(Q_T)$ with a constant $\tilde{C}_1 > 0$ independent of θ. The matrix $A_{jk}[\theta](t) = \int_\Omega F[\theta](x,t)\psi_j(x)\psi_k(x)\, dx$ is symmetric and positive definite, hence Lipschitz continuous solutions $\theta_1(t), \dots, \theta_m(t)$ to (5.9), (5.7) are well defined on $[0,T]$. Testing (5.9) by $\dot\theta_k$, and summing over k, yields

$$\int_\Omega F[\theta^m]|\theta_t^m|^2\, dx + \frac{d}{dt}\left(\frac{1}{2}\int_\Omega |\nabla\theta^m|^2 dx + \frac{1}{2}\int_{\partial\Omega} \gamma|\theta^m|^2 ds - \int_{\partial\Omega} \gamma\,\theta_\Gamma\,\theta^m\, ds \right) \tag{5.10}$$
$$= - \int_{\partial\Omega} \gamma\,(\theta_\Gamma)_t\,\theta^m\, ds + \int_\Omega H[\theta^m]\,\theta_t^m\, dx$$
$$\le \left(\int_{\partial\Omega} \gamma\,|(\theta_\Gamma)_t|^2\, ds \right)^{1/2} \left(\int_{\partial\Omega} \gamma\,|\theta^m|^2\, ds \right)^{1/2} + \tilde{C}_2 \left(\int_\Omega |\theta_t^m|^2\, dx \right)^{1/2}.$$

Integrating from 0 to $t \in (0,T]$, we obtain from Gronwall's argument the estimate

$$|\theta_t^m|_{L^2(Q_T)} + |\nabla\theta^m|_{L^\infty(0,T;H)} + \int_{\partial\Omega} \gamma\,|\theta^m|^2\,ds + |\Delta\theta^m|_{L^2(Q_T)} \;\le\; \tilde C_3\,, \tag{5.11}$$

with constants $\tilde C_2, \tilde C_3 > 0$ independent of m (depending possibly on T, but T is kept fixed here). Selecting a subsequence, if necessary, we may pass to the limit in (5.5–5.6) as $m \to \infty$ to obtain the existence result.

To derive the bounds for θ, we first estimate χ_t using Proposition 3.2 and equation (5.4), taking into account (3.3). Note that the term $|\theta(1-L_\varrho(\theta))|$ is bounded from above independently of t by a constant multiple of $1+|\theta|(1+\log\varrho)$. We thus deduce the following bound on the χ-component of the solution (θ,χ) to Problem 5.1:

$$|\chi_t|_{L^\infty(Q_T)} + |\partial\varphi(\chi)_t|_{L^\infty(Q_T)} \le c_1\,(1+\log\varrho)\,. \tag{5.12}$$

Here, and in the sequel, $c_1, c_2, \ldots$ denote constants independent of ϱ and T. We rewrite Eq. (5.3) in the form

$$c_V(\chi)\theta_t - \Delta\theta \;=\; \mu_\varrho(\theta)\chi_t^2 - c_V'(\chi)\chi_t\theta\,L_\varrho(\theta) + \theta\sigma(\chi)_t + |\theta|\,\varphi(\chi)_t\,. \tag{5.13}$$

We are thus in the situation of (3.17–3.18) with the choice $u=\theta$, and

$$\begin{aligned}
a(x,t) &= c_V(\chi),\\
r(x,t) &= \frac{\mu_\varrho(\theta)}{1+|\theta|}\chi_t^2,\\
h_1(x,t) &= \sigma(\chi)_t + \operatorname{sign}(\theta)\Big(\varphi(\chi)_t + \frac{\mu_\varrho(\theta)}{1+|\theta|}\chi_t^2\Big),\\
h_2(x,t) &= -c_V'(\chi)\chi_t\frac{L_\varrho(\theta)}{|\log|\theta||}\,,\\
u_\Gamma &= \theta_\Gamma, \quad u^0 = \theta_0, \quad u_* = \theta_*/\psi^*,
\end{aligned}$$

and the upper and lower bounds for θ follow from Proposition 3.9. ∎

Remark 5.3. If we examine the proof more closely, we see that the hypothesis on $(\theta_\Gamma)_t$ can be relaxed using the trace theorem for functions from H^1 for $N \ge 2$. The argument still works for $(\theta_\Gamma)_t \in L^2(0,T\,;\,L^p(\partial\Omega))$ with $p \ge 2(N-1)/N$, or $(\theta_\Gamma)_t \in L^2(0,T\,;\,H^{-1/2}(\partial\Omega))$, if $\partial\Omega$ is smooth.

The uniqueness and continuous dependence result in Theorem 2.4 holds indeed for the problem (5.3–5.4), (2.3–2.4) as well. We therefore can extend the solution to (5.3–5.4), (2.3–2.4) to the whole time interval $[0,\infty)$ and obtain the following result.

Corollary 5.4. *There exists a solution to (5.3), (5.4), (2.3–2.4) on Q_∞ with the properties (2.7–2.8), and functions $\theta_\varrho^l, \theta_\varrho^u : (0,\infty) \to (0,\infty)$ such that θ_ϱ^l is nonincreasing, $\theta_\varrho^u(T)$ is nondecreasing, and $\theta_\varrho^l(T) \le \theta(x,t) \le \theta_\varrho^u(T)$ for a. e. $(x,t) \in Q_T$ and for all $T > 0$.*

We see in particular that we can remove the absolute values in (5.3–5.4). Our aim is now to prove that the solution to Problem 5.1 satisfies also (2.1–2.4), (2.6–2.8) for suitably chosen ϱ. To this end, we derive a uniform upper bound for θ. Then, choosing ϱ above this bound, we will check that the solution to (5.3–5.4) is the desired solution to (2.1–2.2).

Equation (5.3) is of the form as in Proposition 3.10, with

$$u = \theta\,, \quad a(x,t) = c_V(\chi)\,, \quad h(x,t,u) = u\,, \tag{5.14}$$

$$\mathcal{H}[u] = -\big(\lambda(\chi) + \beta\varphi(\chi)\big)_t - b[\chi]\chi_t - c_V'(\chi)\chi_t\, u\,. \tag{5.15}$$

Referring to Proposition 3.10, we have $a_0 = c_0$, $C_h = 1$, and $U + U_\Gamma + A_0 \le c_2$. The other parameters, however, namely a_1, H_0, H_1, and E_0, do depend on ϱ by virtue of (5.12). Hypothesis 2.1 and (5.12) yield that

$$a_1 + H_0 + H_1 \;\le\; c_3\,(1 + \log\varrho)\;. \tag{5.16}$$

It remains to determine the dependence of E_0 on ϱ. To do so, we test (5.3) by ψ_1 from (2.5). This yields

$$\begin{aligned} \frac{d}{dt}\int_\Omega c_V(\chi)\theta\,\psi_1\,dx + \int_\Omega \langle\nabla\theta, \nabla\psi_1\rangle\;dx + \int_{\partial\Omega}\gamma(\theta-\theta_\Gamma)\psi_1\,ds \\ = -\int_\Omega \left((\lambda(\chi)+\beta\varphi(\chi))_t + b[\chi]\chi_t\right)\psi_1\,dx\,. \end{aligned} \tag{5.17}$$

If $\gamma \equiv 0$, we may take $\psi_1 \equiv 1$, and using the symmetry of $B[\chi]$, we obtain from (5.17) that

$$\begin{aligned} \int_\Omega c_V(\chi)\theta(x,t)\,dx &= \int_\Omega c_V(\chi_0)\theta_0\,dx + \int_\Omega \left(\lambda(\chi_0) + \beta\varphi(\chi_0) + B[\chi_0]\right)\,dx \\ &\quad - \int_\Omega \left(\lambda(\chi) + \beta\varphi(\chi) + B[\chi]\right)(x,t)\,dx \\ &\le c_4\,. \end{aligned} \tag{5.18}$$

Assume now that $\int_{\partial\Omega}\gamma\,ds > 0$. Then $\lambda_1 > 0$, and we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega c_V(\chi)\theta\,\psi_1\,dx + \lambda_1\int_\Omega \theta\psi_1\,dx \\ = \int_{\partial\Omega}\gamma\theta_\Gamma\psi_1\,ds - \int_\Omega \left((\lambda(\chi)+\beta\varphi(\chi))_t + b[\chi]\chi_t\right)\psi_1\,dx\,. \end{aligned}$$

From Hypothesis 2.1 and estimate (5.12), we infer that

$$\frac{d}{dt}\int_\Omega c_V(\chi)\theta\,\psi_1\,dx + c_5\int_\Omega c_V(\chi)\theta\psi_1\,dx \le c_6(1+\log\varrho)\,. \tag{5.19}$$

Hence, $\int_\Omega \theta\,\psi_1\,dx \;\le\; c_7(1+\log\varrho)$, and using Hypothesis 2.1 (ii), we obtain the final estimate

$$E_0 \;\le\; c_8\,(1+\log\varrho)\;. \tag{5.20}$$

Referring to (3.26) in Proposition 3.10, we find that

$$\theta(x,t) \;\le\; c_9\,(1+\log\varrho)^{2+N/2} \tag{5.21}$$

for a. e. $(x,t)\in Q_\infty$. Taking now any ϱ such that

$$\varrho \;>\; c_9\,(1+\log\varrho)^{2+N/2}\,,$$

we see that the solution to Problem 5.1 is also a solution to (2.1–2.4), (2.6–2.8), and the upper bound in (2.9) is satisfied.

It remains to derive the uniform (in time) lower bound in (2.9). To this end, let us consider the function

$$w = \log\overline{\theta} - \log\theta > 0 \quad \text{a. e. in } Q_\infty\,,$$

with $\overline{\theta}$ defined in (2.9). Using (2.2), we rewrite (2.1) as

$$c_V(\chi)\frac{\theta_t}{\theta} - \frac{\Delta\theta}{\theta} \;=\; \frac{\mu(\theta)}{\theta}\chi_t^2 + (\varphi(\chi)+\sigma(\chi))_t - (c_V(\chi))_t\log\theta\,.$$

Then

$$(c_V(\chi)w)_t - \Delta w \;=\; -(\varphi(\chi)+\sigma(\chi))_t - \frac{\mu(\theta)}{\theta}\chi_t^2 - \frac{|\nabla\theta|^2}{\theta^2}\,, \tag{5.22}$$

with boundary condition

$$\partial_{\mathbf{n}} w + \gamma\left(\frac{\theta_\Gamma}{\overline{\theta}}\,\mathrm{e}^{w} - 1\right) = 0 \quad \text{a. e. on } \Sigma_\infty\,. \tag{5.23}$$

We are thus again in the situation of Proposition 3.10, with $h(x,t,u) = (\theta_\Gamma/\bar{\theta})(\mathrm{e}^u - 1)$ suitably extended for $u<0$, and it only remains to find a uniform L^1-bound for w as in Proposition 3.10 (vi). We proceed as above and test (5.22) by ψ_1. This yields

$$\frac{d}{dt}\int_\Omega (c_V(\chi)w + \varphi(\chi)+\sigma(\chi))\;\psi_1\,dx \;+\lambda_1\int_\Omega w\,\psi_1\,dx \\ +\int_{\partial\Omega}\gamma\left(\frac{\theta_\Gamma}{\overline{\theta}}\,\mathrm{e}^{w} - 1 - w\right)\psi_1\,ds \le 0.$$

The case $\gamma\equiv 0$, $\lambda_1 = 0$ is again straightforward. For $\lambda_1>0$ we notice that $(\theta_\Gamma/\overline{\theta})\mathrm{e}^w - 1 - w \ge -c_{10}$, hence a uniform bound for w in $L^1(\Omega)$ follows again from the uniform Gronwall lemma. From Proposition 3.10 we conclude that

$$w(x,t)\le c_{11} \quad \text{a. e. in } Q_\infty. \tag{5.24}$$

Hence, $\theta(x,t)\ge\bar{\theta}\mathrm{e}^{-c_{11}}$ a. e., which completes the proof of Theorem 2.2. ∎

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