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A jump-diffusion Libor model and its robust calibration

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Abstract

In this paper we propose a jump-diffusion Libor model with jumps in a high-dimensional space ($\mathbb{R}^m$) and test a stable non-parametric calibration algorithm which takes into account a given local covariance structure. The algorithm returns smooth and simply structured Lévy densities, and penalizes the deviation from the Libor market model. In practice, the procedure is FFT based, thus fast, easy to implement, and yields good results, particularly in view of the severe ill-posedness of the underlying inverse problem.

1 Introduction

The calibration of financial models has become an important topic in financial engineering because of the need to price increasingly complex options consistent with prices of standard instruments liquidly traded in the market. The choice of an underlying model is crucial with respect to its statistical relevance on the one hand, and the possibility of calibrating it with ease on the other. In order to cover stylized facts in financial data such as implied volatility smiles, more complex models, i.e. models beyond Black-Scholes, are called for.

During the last decade Lévy-based models have drawn much attention, as these models are capable to describe complex but realistic behavior of financial time series. In particular, these models may cover jumps, heavy tails, and are principally able to match implied volatility surfaces observed in stock and interest rate markets. For modelling stock prices, pure jump Lévy processes were already proposed in Eberlein, Keller and Prause (1998). In Cont & Tankov (2003) regularized approaches for calibrating jump-diffusion stock price models were considered.

In the interest rate world the Libor market model developed by Brace, Gatarek, Musiela (1997), Jamshidian (1997), and Miltersen, Sandmann, Sondermann (1997), has become one of the most popular and advanced tools for modelling interest rates and interest rate derivatives. This in spite of a main drawback; the Libor market model cannot explain implied volatility surfaces typically observed in the cap markets. In order to handel this issue different extensions of the Libor market model using processes with jumps have been proposed. Glasserman and Kou (2003) developed a jump diffusion Libor model and gave some useful explicit specifications. The most general framework for Libor models driven by jump measures is provided in Jamshidian (2001).

The central theme in this paper is a well structured jump-diffusion Libor model which allows for robust and efficient calibration. Our starting point will be a given

Libor market model with known deterministic volatility structure. For instance, this market model might be obtained from a calibration procedure involving at the money (ATM) caps, ATM swaptions, and/or a historically identified forward rate correlation structure. Meanwhile, calibration procedures for Libor market models are well studied in the literature (e.g. Schoenmakers (2005), or Brigo & Mercurio (2001)). Yet, our main goal is the development of a specific jump-diffusion Libor model which can be calibrated to the cap-strike matrix in a robust way and which is, in a sense, as near as possible to the given market model. In particular, this model will be furnished in such a way that *the (local) covariance structure of the jump-diffusion model coincides with the (local) covariance structure of the market model.* We have three main reasons for doing so: (1) The price of a cap in a Libor market model does not depend on the (local) correlation structure of the forward Libors. However, this correlation structure may contain important information such as, for instance, prices of ATM swaptions. We therefore do not want to destroy this correlation structure as given by the input market model when calibrating the extended model to the cap(let)-strike volatility matrix. (2) The lack of smile behavior of the input market model, which is regarded as a rough intermediate approximation of a smile explaining jump-diffusion model, is considered to be a consequence of Gaussianity of the driving random forces (Wiener processes). So, loosely speaking, we want to perturb these forces to non-Gaussian ones by using jumps, while maintaining the (local) covariance structure of the given market model, hence the correlation structure implicitly. (3) Last but not least, by preserving the covariance structure we obtain a very robust calibration procedure.

The literature on calibration methods for asset models based on Lévy processes has mainly focused on certain parametrization of the underlying Lévy process. Since the characteristic triplet of a Lévy process is a priori an infinite-dimensional object, the parametric approach is always exposed to the problem of misspecification, in particular when there is no inherent economic foundation of the parameters and they are only used to generate different shapes of possible jump distributions. Therefore, we employ a nonparametric approach of Belomestny & Reiss (2004) which utilizes explicit inversion of a Fourier based pricing formula and a regularization in the spectral domain.

The outline of the paper is as follows. We recall in Section 2 the general arbitrage-free Libor framework developed in Jamshidian (2001). It will serve as the baseplate of this article. The covariance preserving jump-diffusion extension of the Libor market model is constructed in Section 3. In Section 4 we recap Fourier-based representations for Caplet prices in the spirit of Car & Madan (1999), see also Glasserman & Merener (2003), Eberlein & Özkan (2005). The algorithm for calibrating to a full cap-strike matrix is developed in Section 5, and a real life calibration is carried out in Section 6. Technical details and derivations are given in the Appendix-section.

2 General framework for Libor models with jumps

Consider a fixed sequence of tenor dates $0 =: T_0 < T_1 < T_2 < \ldots T_n$, called a tenor structure, together with a sequence of so called day-count fractions $\delta_i := T_{i+1} - T_i,\ i = 1, \ldots, n-1$. With respect to this tenor structure we consider zero bond processes $B_i,\ i = 1, \ldots, n$, where each B_i lives on the interval $[0, T_i]$ and ends up with its face value $B_i(T_i) = 1$. With respect to this bond system we deduce a system of forward rates, called Libor rates, which are defined by

$$L_i(t) := \frac{1}{\delta_i}\left(\frac{B_i(t)}{B_{i+1}(t)} - 1\right), \quad 0 \le T_i,\ 1 \le i \le n-1.$$

Note that L_i is the annualized effective forward rate to be contracted for at the date t, for a loan over a forward period $[T_i, T_{i+1}]$. Based on this rate one has to pay at T_{i+1} an interest amount of $\$\delta_i L_i(T_i)$ on a \$1 notional.

2.1 Arbitrage free dynamics

On a filtered measurable space $(\Omega, \mathcal{F}, \mathcal{F}_t)$ we consider a Libor model under the terminal measure P_n within the following framework (Jamshidian (2001)),

$$\begin{aligned}\frac{dL_i}{L_{i-}} = &-\sum_{j=i+1}^{n-1} \frac{\delta_j L_{j-}}{1+\delta_j L_{j-}} \eta_i^\top \eta_j dt + \eta_i^\top dW^{(n)} \\ &- \int_E \nu^{(n)}(dt, du)\psi_i(t,u)\left(\prod_{j=i+1}^{n-1}\left(1 + \frac{\delta_j L_{j-}\psi_j(t,u)}{1+\delta_j L_{j-}}\right) - 1\right) \\ &+ \int_E \psi_i(t,u)(\mu - \nu^{(n)})(dt, du), \qquad i = 1, \ldots, n-1, \end{aligned} \tag{1}$$

with $\omega \to \mu(dt, du, \omega)$, being a random point measures on $\mathbb{R}_+ \times E$, where E is an abstract Lusin space, and $\nu^{(n)}(dt, du, \omega)$ is the $(P_n, \mathcal{F})$-compensator on $\mathbb{R}_+ \times E$ of μ. In (1), $W^{(n)}$ is a d-dimensional standard Brownian motion under P_n, and the filtration $(\mathcal{F}_t)_{t\ge 0}$ is assumed to contain the natural filtrations generated by $W^{(n)}$ and μ, respectively. Further, $(\omega, t) \to \psi_i(t, \cdot, \omega)$ are predictable processes of functions on E and η_i are d-dimensional predictable column vector processes. The random measure μ is assumed to be of the form

$$\mu = \sum_{n\ge 1} 1_{T_n(\omega)=t}\delta_{(t,\beta_t(\omega))}(dt, du), \tag{2}$$

where β is in general an optional process and $T_n,\ n = 1, 2, ..$ is a sequence of stopping times with disjoint graphs, i.e. $T_n(\omega) \ne T_m(\omega)$ for $n \ne m$.
The framework (1) may be casted into a somewhat different form. Let us consider a partition $E := \bigcup_{k=1}^{m} E_k$, where $E_1, \ldots, E_m$ are Lusin spaces with $E_k \cap E_l = \varnothing$ for

$k \neq l$, and define $\mu_k := \mu|_{E_k}$, $\psi_{ik} := \psi_i|_{E_k}$, $\nu_k^{(n)} := \nu^{(n)}|_{E_k}$, for $k = 1, ..., m$. Then (1) becomes

$$\frac{dL_i}{L_{i-}} = -\sum_{j=i+1}^{n-1} \frac{\delta_j L_{j-}}{1+\delta_j L_{j-}} \eta_i^\top \eta_j dt + \eta_i^\top dW^{(n)}$$
$$-\sum_{k=1}^{m} \int_{E_k} \nu_k^{(n)}(dt, du_k) \psi_{ik}(t, u_k) \left(\prod_{j=i+1}^{n-1} \left(1 + \frac{\delta_j L_{j-} \psi_{jk}(t, u_k)}{1+\delta_j L_{j-}} \right) - 1 \right)$$
$$+\sum_{k=1}^{m} \int_{E_k} \psi_{ik}(t, u_k)(\mu_k - \nu_k^{(n)})(dt, du_k), \qquad i = 1, ..., n-1. \tag{3}$$

In particular, it easily follows that $\nu_k^{(n)}$ is the P_n-compensator of μ_k with respect to $\mathcal{F}$. Note that in general $E^{\mathcal{F}_t^{(k)}} \nu_k^{(n)}(\omega, dt, du)$ is the compensator of μ_k with respect to the restricted filtration $\mathcal{F}_t^{(k)} := \mathcal{F}_t \cap \sigma\{\mu([0, s] \times C) : s \leq t, C \in \mathcal{B}(E_k)\}$, $t \geq 0$ (thus not $\nu_k^{(n)}$). As shown in Appendix 7.1, the representation (3) is in fact equivalent to (1), but somewhat more natural as it suggest the use of a system of m point processes with phase space $\mathbb{R}_+ \times \mathbb{R}$ as in the papers of Glasserman & Kou (2001) and Glasserman & Merener (2003).

Henceforth we consider in (1) only random point measures with finite activity, i.e., μ is of the form (2) and for each $t > 0$, $\mu([0, t] \times E) < \infty$. In order to guarantee that the Libor processes L_i are nonnegative we further require that $\psi_i > -1$ in (1), and then set $\varphi_i := \ln(\psi_i + 1)$. Let (s_l, u_l), $l = 1, ..., N_t$, denote the jumps of μ up to time t for an $\omega \in \Omega$. Using the fact that at a jump time s_l, $\Delta L_i(s_l, \omega) = L_i(s_l-, \omega)\psi_i(s_l, u_l, \omega) = L_i(s_l-, \omega)(e^{\varphi_i(s_l, u_l, \omega)} - 1)$, and hence $L_i(s_l, \omega) = L_i(s_l-, \omega)e^{\varphi_i(s_l, u_l, \omega)}$, we obtain by the Ito-substitution rule for jump processes (with ω suppressed),

$$d \ln L_i = \frac{1}{L_{i-}} dL_i - \frac{1}{2}|\eta_i|^2 dt + d\sum_{l=1}^{N_t} (\varphi_i(s_l, u_l) - \psi_i(s_l, u_l))$$
$$= -\frac{1}{2}|\eta_i|^2 dt - \sum_{j=i+1}^{n-1} \frac{\delta_j L_{j-}}{1+\delta_j L_{j-}} \eta_i^\top \eta_j dt + \eta_i^\top dW^{(n)}$$
$$- \int_E \nu^{(n)}(dt, du)(e^{\varphi_i(s,u)} - 1) \prod_{j=i+1}^{n-1} \frac{1 + \delta_j L_{j-} e^{\varphi_j(s,u)}}{1+\delta_j L_{j-}} + d\sum_{l=1}^{N_t} \varphi_i(s_l, u_l). \tag{4}$$

The logarithmic analogue of (3) directly follows from (4),

$$
\begin{aligned}
d\ln L_i = {} & -\frac{1}{2}|\eta_i|^2 dt - \sum_{j=i+1}^{n-1} \frac{\delta_j L_{j-}}{1+\delta_j L_{j-}} \eta_i^\top \eta_j dt + \eta_i^\top dW^{(n)} \qquad (5)\\
& - \sum_{k=1}^{m} \int_{E_k} \nu_k^{(n)}(dt, du_k)(e^{\varphi_{ik}(s,u_k)} - 1) \prod_{j=i+1}^{n-1} \frac{1+\delta_j L_{j-} e^{\varphi_{jk}(s,u_k)}}{1+\delta_j L_{j-}}\\
& + d\sum_{k=1}^{m} \sum_{l=1}^{N_t^{(k)}} \varphi_{ik}(s_l^{(k)}, u_l^{(k)}),
\end{aligned}
$$

with $\varphi_{ik} := \ln(\psi_{ik} + 1)$ and $(s_l^{(k)}, u_l^{(k)})$, $l = 1, ..., N_t^{(k)}$, denoting the jumps of μ_k up to time t. The logarithmic representation (4) (or equivalently (5)) will be the basic framework for our purposes.

3 Jump diffusion extension of a Libor market model

We first specialize to a jump-diffusion Libor model which is driven by a Poisson random measure with marks in some multi-dimensional space.

3.1 Poisson driven multi-dimensional jumps

Consider the Lusin product space $E := E_1 \times \cdots \times E_m$, with E_k Lusin for $k = 1, ..., m$ (e.g. $E_k = \mathbb{R}$). Suppose that on a common probability space, equipped with some probability measure P_n, we are given random measures μ_k on $\mathbb{R}_+ \times E_k$. We then consider the product Lusin space $E := E_1 \times \ldots \times E_m$ (e.g. $E = \mathbb{R}^m$), and on $\mathbb{R}_+ \times E$ the random measure $\mu(dt, du, \omega)$ such that for any $t \geq 0$, $\mu(\{t\}, \cdot, \omega) := \mu_1(\{t\}, \cdot, \omega) \otimes \ldots \otimes \mu_m(\{t\}, \cdot, \omega)$. We assume that the random measures μ_k are such that almost surely for each $t \geq 0$ either $\mu_k(\{t\}, E_k, \omega) = 1$ for all k, or $\mu_k(\{t\}, E_k, \omega) = 0$ for all k. Thus, all random measures μ_k throw a point in E_k at the same time. Then each $\mu_k(\{t\}, \cdot, \omega)$ can be seen as the image of $\mu(\{t\}, \cdot, \omega)$ under the projection of E onto E_k. In addition, we assume that given $\mu_k(\{t\}, E_k, \omega) = 1$ for all k, the Dirac measures $\mu_k =: \delta_{(t,u_k)}$ are mutually independent for $k = 1, ..., m$, independent of t, and u_k is distributed on E_k with probability $p_k(du_k)$. The (simultaneous) jump-times, i.e. times t at which $\mu_k(\{t\}, E_k, \omega) = 1$ for all k, are assumed to be Poisson distributed with locally finite intensity measure $\lambda(t)dt$. We then consider (4) (or (1)) for the thus constructed jump measure μ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$ which is generated by μ and $W^{(n)}$, where the P_n standard Brownian motion $W^{(n)}$ is independent of μ. Under these assumptions it follows that the $(P_n, \mathcal{F})$-compensator of μ is deterministic and is given by

$$
\nu^{(n)}(dt, du_1, ..., du_m) := \lambda(t)p_1(du_1) \cdots p_m(du_m)dt =: \lambda(t)p(du)dt.
$$

3.2 Extending the Libor market model

Within the particular framework constructed above we now introduce a jump-diffusion Libor model which in a sense can be seen as an extension or perturbation of a (given) Libor market model. Let $\gamma_i(t) \in \mathbb{R}^d$ be the (given) deterministic volatility structure of the market model, resulting for instance from some standard calibration procedure to ATM caps and ATM swaptions or historical data. To exclude local redundancies we assume that the matrix $(\gamma_{i,l}(t))_{1\leq i<n, 1\leq l\leq d}$ has full rank d for all t. Let $E := \mathbb{R}^m$ for some integer m and consider deterministic vector functions $\beta_i(t) \in \mathbb{R}^m$, $i = 1, ..., n-1$. We then take a sequence of constants r_i with $-1 < r_i < 1$, and set

$$\eta_i := \sqrt{1 - r_i^2}\gamma_i, \quad \varphi_i(t, u) := r_i\, u^\top \beta_i(t) \tag{6}$$

in (4) to yield,

$$\begin{aligned} d\ln L_i = &-\frac{1}{2}(1 - r_i^2)|\gamma_i|^2 dt - \sum_{j=i+1}^{n-1} \frac{\delta_j L_{j-}}{1 + \delta_j L_{j-}} \sqrt{(1 - r_i^2)(1 - r_j^2)}\gamma_i^\top \gamma_j dt \\ &+ \sqrt{1 - r_i^2}\gamma_i^\top dW^{(n)} + r_i d \sum_{l=1}^{N_t} u_l^\top \beta_i(s_l) \\ &- \lambda(t) dt \int_{\mathbb{R}^m} \left(\exp(r_i\, u^\top \beta_i) - 1\right) p(du) \prod_{j=i+1}^{n-1} \frac{1 + \delta_j L_{j-} \exp(r_j\, u^\top \beta_j)}{1 + \delta_j L_{j-}}. \end{aligned} \tag{7}$$

Note that in (7) the market model is retrieved by taking $r_i \equiv 0$, and so, for small r_i, (7) may be seen as a jump diffusion perturbation of the Libor market model.

3.3 The jump drift of $\ln L_i$ under P_n

Let us consider the third term in (7), i.e. the "log jump drift" of $\ln L_i$ under the terminal measure P_n. The computation of this term is of particular importance, for example, in a Monte Carlo simulation of the model. For a fixed time $t > 0$ we consider the expression

$$(*) := \int_{\mathbb{R}^m} p(du) \left(\exp(r_i u^\top \beta_i(t) - 1\right) \prod_{j=i+1}^{n-1} \left[1 + \delta_j L_{j-}(t) \exp(r_j u^\top \beta_j(t))\right]. \tag{8}$$

Using the abbreviation $x_j := \delta_j L_{j-}(t) \exp(r_j u^\top \beta_j(t))$, the product in (8) my be expanded as

$$\begin{aligned} \prod_{j=i+1}^{n-1} (1 + x_j) = 1 &+ \sum_{i<j<n} x_j + \sum_{i<j_1<j_2<n} x_{j_1} x_{j_2} \\ &+ \sum_{i<j_1<j_2<j_3<n} x_{j_1} x_{j_2} x_{j_3} + \ldots + x_{i+1} \cdots x_{n-1}. \end{aligned}$$

Let us take a generic term of degree $1 \leq d < n-i$ (with t suppressed),

$$x_{j_1} \cdots x_{j_d} = \delta_{j_1} L_{j_1-} \cdots \delta_{j_d} L_{j_d-} \exp(r_{j_1} u^\top \beta_{j_1}) \cdots \exp(r_{j_d} u^\top \beta_{j_d}),$$

for $i < j_1 < j_2 < \cdots < j_d < n$, and observe that

$$\begin{aligned}
&\int_{\mathbb{R}^m} p(du) e^{r_i u^\top \beta_i} \exp(r_{j_1} u^\top \beta_{j_1}) \cdots \exp(r_{j_d} u^\top \beta_{j_d}) \\
&= \int_{\mathbb{R}^m} p(du) \exp\left[u^\top (r_i \beta_i + r_{j_1} \beta_{j_1} + \cdots + r_{j_d} \beta_{j_d})\right] \\
&= \prod_{l=1}^m \int_{\mathbb{R}} p_l(du_l) \exp\left[u_l (r_i \beta_{il} + r_{j_1} \beta_{j_1 l} + \cdots + r_{j_d} \beta_{j_d l})\right] \\
&= \prod_{l=1}^m \phi_{p_l}(-\mathrm{i} r_i \beta_{il} - \mathrm{i} r_{j_1} \beta_{j_1 l} \cdots - \mathrm{i} r_{j_d} \beta_{j_d l}),
\end{aligned}$$

with ϕ_{p_l} being the characteristic function of p_l. Note that the existence of $\phi_{p_l}(z)$ in some ball $\{z \in \mathbb{C} : |z| < A\}$ has to be assumed. By analogue computations and collecting terms we thus obtain

$$\begin{aligned}
(*) = -1 + \prod_{l=1}^m \phi_{p_l}(-\mathrm{i} r_i \beta_{il}) + \\
\sum_{d=1}^{n-1-i} \sum_{i < j_1 < j_2 < \cdots < j_d < n} \delta_{j_1} L_{j_1-} \cdots \delta_{j_d} L_{j_d-} \times \\
\times \left[\prod_{l=1}^m \phi_{p_l}(-\mathrm{i} r_i \beta_{il} - \mathrm{i} r_{j_1} \beta_{j_1 l} \cdots - \mathrm{i} r_{j_d} \beta_{j_d l}) - \prod_{l=1}^m \phi_{p_l}(-\mathrm{i} r_{j_1} \beta_{j_1 l} \cdots - \mathrm{i} r_{j_d} \beta_{j_d l}) \right] \\
=: \varrho_i^{p,r,\beta} + \sum_{d=1}^{n-1-i} \sum_{i < j_1 < j_2 < \cdots < j_d < n} \delta_{j_1} L_{j_1-} \cdots \delta_{j_d} L_{j_d-} \varrho_{i;j_1,\ldots,j_d}^{p,r,\beta}.
\end{aligned}$$

Once the model inputs r_i, jump loadings $t \to \beta_i(t)$ for $1 \leq i < n$, and jump component measures p_l with characteristic functions ϕ_{p_l} for $1 \leq l \leq m$, are calibrated or simply given, the real valued functions $t \to \varrho_i^{p,r,\beta}(t)$, $t \to \varrho_{i;j_1,\ldots,j_d}^{p,r,\beta}(t)$, $1 \leq i < n$, $i < j_1 < j_2 < \cdots < j_d < n$, can be computed in closed form and, in principle, even be stored outside the Monte Carlo simulator. Thus considering these functions as given, the simulation of $\ln L_i$ in the terminal measure may be carried out straightforwardly via the formula

$$
\begin{aligned}
d\ln L_i = -\frac{1}{2}(1-r_i^2)|\gamma_i|^2 dt - \sum_{j=i+1}^{n-1} \frac{\delta_j L_{j-}}{1+\delta_j L_{j-}} \sqrt{(1-r_i^2)(1-r_j^2)}\gamma_i^\top \gamma_j dt \\
+\sqrt{1-r_i^2}\gamma_i^\top dW^{(n)} + r_i d\sum_{l=1}^{N_t} u_l^\top \beta_i(s_l) \qquad (9) \\
-\prod_{j=i+1}^{n-1} (1+\delta_j L_{j-})^{-1} \lambda(t) dt \Big[\varrho_i^{p,r,\beta}(t) + \\
+\sum_{d=1}^{n-1-i} \sum_{i<j_1<j_2<\cdots<j_d<n} \delta_{j_1} L_{j_1 -} \cdots \delta_{j_d} L_{j_d -} \varrho^{p,r,\beta}_{i;j_1,\ldots,j_d}(t) \Big].
\end{aligned}
$$

We underline that the structure of the dynamics (9), hence the feasibility of standard Monte Carlo simulation of every forward Libor in the terminal measure, is a consequence of our model design in Sections 3.1 and 3.2. In particular it is due to the special product structure of the principally high dimensional jump measure p and the linear structure of the log-Libor factor loadings (6).

Remark 1 Based on (9) we may consider different Libor model approximations. For example we may freeze L_{j-} at zero (see Glasserman & Merener (2003)), hence replace L_{j-} with $L_j(0)$ in (9). As an alternative, if the r_i are small enough and the magnitudes of $\delta_j L_j$ are small enough as well, one could drop in (9) the terms of order $(\delta_j L_j)^2$ and higher. Of course, any such attempt needs careful investigation which is considered beyond the scope of this article. For related approximations in the context of the standard Libor market model, see for instance Kurbanmuradov, Sabelfeld and Schoenmakers (2002).

3.4 Dynamics of L_i under P_{i+1}

We now consider for $i = 1, ..., n-1$ the dynamics of L_i under P_{i+1}. From (7) we see that the logarithm of the last Libor rate L_{n-1} has the following simple dynamics in the P_n measure,

$$
\begin{aligned}
d\ln L_{n-1} = -\frac{1}{2}(1-r_{n-1}^2)|\gamma_{n-1}|^2 dt + \sqrt{1-r_{n-1}^2}\gamma_{n-1}^\top dW^{(n)} + \\
r_{n-1} d\sum_{l=1}^{N_t} u_l^\top \beta_{n-1}(s_l) - \lambda(t)dt \int_{\mathbb{R}^m} \big(\exp(r_{n-1}\, u^\top \beta_{n-1}) - 1\big)\; p(du) \qquad (10)
\end{aligned}
$$

and thus belongs to the class of additive models, i.e., the process $X_{n-1}(t) := \ln L_{n-1}(t) - \ln L_{n-1}(0)$ has independent increments. By using Lemma 2 below for instance, we can derive straightforwardly the characteristic function of $X_{n-1}(t)$,

$$\Phi_n(z;t) := E_{P_n} \exp[\mathrm{i}zX_{n-1}(t)] = \exp\left[\psi_n(z;t)\right] \quad \text{with} \tag{11}$$

$$\psi_n(z;t) := -\frac{z^2}{2}(1-r_{n-1}^2)\int_0^t |\gamma_{n-1}(s)|^2 ds - \mathrm{i}z\int_0^t \Big[\frac{1}{2}(1-r_{n-1}^2)|\gamma_{n-1}(s)|^2 ds +$$
$$\lambda(s)ds\int_{\mathbb{R}^m}\big(\exp(r_{n-1}\,u^\top\beta_{n-1}(s))-1\big)\,p(du)\Big]$$
$$+\int_0^t \lambda(s)ds\int_{\mathbb{R}^m}(e^{\mathrm{i}z\,r_{n-1}u^\top\beta_{n-1}(s)}-1)p(du). \tag{12}$$

For $1 \le i < n-1$ the dynamics of L_i under P_{i+1} is more complicated. By the fact that L_i is a martingale under P_{i+1} we observe from the general framework (1) that

$$\frac{dL_i}{L_{i-}} =: \eta_i^\top dW^{(i+1)} + \int_E \psi_i(t,u)\left(\mu-\nu^{(i+1)}\right)(dt,du), \tag{13}$$

where

$$dW^{(i+1)} = -\sum_{j=i+1}^{n-1}\frac{\delta_j L_{j-}}{1+\delta_j L_{j-}}\eta_j dt + dW^{(n)}$$

is a standard Brownian motion under P_{i+1}, and

$$\nu^{(i+1)}(dt,du) = \nu^{(n)}(dt,du)\prod_{j=i+1}^{n-1}\left(1+\frac{\delta_j L_{j-}\psi_j(t,u)}{1+\delta_j L_{j-}}\right) \tag{14}$$

is the compensator process of μ under the measure P_{i+1}. For the more specialized setup introduced in this section, which is based on (6), (14) reads

$$\nu^{(i+1)}(dt,du) = \lambda(t)p(du)dt\prod_{j=i+1}^{n-1}\frac{1+\delta_j L_{j-}\exp(r_j\,u^\top\beta_j)}{1+\delta_j L_{j-}}, \tag{15}$$

and (13) reads

$$\frac{dL_i}{L_{i-}} = \sqrt{1-r_i^2}\gamma_i^\top dW^{(i+1)} + \int_{\mathbb{R}^m}\left(e^{r_i\,u^\top\beta_i(t)}-1\right)\left(\mu-\nu^{(i+1)}\right)(dt,du), \tag{16}$$

$i = 1,\dots,n-1$. The logarithmic version of (16) is seen from (7) to be

$$d\ln L_i = -\frac{1}{2}(1-r_i^2)|\gamma_i|^2 dt + \sqrt{1-r_i^2}\gamma_i^\top dW^{(i+1)} \tag{17}$$
$$+r_i d\sum_{l=1}^{N_t} u_l^\top\beta_i(s_l) - \int_{\mathbb{R}^m}\left(\exp(r_i\,u^\top\beta_i)-1\right)\nu^{(i+1)}(dt,du).$$

In particular, for $i < n-1$ the compensator (15) is non-deterministic in the present setup and, as a consequence, $\ln L_i$ is generally not additive under P_{i+1} for $i < n-1$. However, by freezing in (15) the Libor terms, i.e. replacing L_{i-} by $L_{i-}(0)$, we may get a deterministic approximative compensator and so an additive approximation of $\ln L_i$ under P_{i+1}.

3.5 Preserving the local covariance structure

We recall the following standard lemma which is proved in Appendix 7.2.

Lemma 2 *If $J(t) = \sum_{l=1}^{N_t} \varphi(s_l, u_l)$ is a compound Poisson process in $\mathbb{R}^q$ with jump intensity $\lambda(t)dt$, independent jumps in a measurable space E with probability measure $p(du)$, and $\varphi : \mathbb{R}_+ \times E \to \mathbb{R}^q$ is deterministic, then (i) the characteristic function of $J(t)$ is given by*

$$Ee^{\mathrm{i}z^\top J(t)} = \exp\left[\int_0^t \lambda(s)ds \int_E (e^{\mathrm{i}z^\top \varphi(s,u)} - 1)p(du)\right], \qquad z \in \mathbb{R}^q.$$

and (ii) for the expectation and covariance structure of $J(t)$ we have

$$\begin{aligned} EJ_l(t) &= \int_0^t \lambda(s)ds \int_E \varphi_l(s,u)p(du), \\ \mathrm{Cov}(J_l(t), J_{l'}(t)) &= \int_0^t \lambda(s)ds \int_E \varphi_l(s,u)\varphi_{l'}(s,u)p(du), \qquad 1 \le l, l' \le q. \end{aligned}$$

Let us now write the integrated random term in (7) as

$$\begin{aligned} \xi_i(t) &:= \sqrt{1 - r_i^2} \int_0^t \gamma_i^\top dW^{(n)} + r_i \sum_{l=1}^{N_t} u_l^\top \beta_i(s_l) \\ &=: \sqrt{1 - r_i^2}\, \xi_i^D(t) + r_i \xi_i^J(t). \end{aligned} \tag{18}$$

By Lemma 2 the characteristic function of the jump process ξ^J is then given by

$$Ee^{\mathrm{i}z^\top \xi^J(t)} = \exp\left[\int_0^t \lambda(s)ds \left(\phi_p\left(\sum_{j=1}^{n-1} z_j \beta_j(s)\right) - 1\right)\right],$$

with $\phi_p(y) := \int p(du) \exp\left[\mathrm{i}u^\top y\right]$, $y \in \mathbb{R}^m$ being the characteristic function of p. For the covariance matrix Lemma 2 yields

$$\begin{aligned} \mathrm{Cov}(\xi_i^J(t), \xi_j^J(t)) &= \int_0^t \lambda(s)ds \int_{\mathbb{R}^m} \beta_i^\top(s) u u^\top \beta_j(s) p(du) \\ &=: \int_0^t \lambda(s)ds \beta_i^\top(s) \Sigma \beta_j(s) \end{aligned}$$

with $\Sigma_{kl} := \int u_k u_l p(du)$ being the cross moments of jump components u_k and u_l. Since the Brownian motion and the jumps are assumed to be independent, we have for the local covariance of the random term in (7),

$$\mathrm{Cov}(d\xi_i(t), d\xi_j(t))/dt = \sqrt{(1 - r_i^2)(1 - r_j^2)}\gamma_i^\top(t)\gamma_j(t) + r_i r_j \lambda(t) \beta_i^\top(t) \Sigma \beta_j(t). \tag{19}$$

Our main idea is to consider jump diffusion extensions of a (given) pure Libor market model which preserve the (given) local covariance structure of the market model. To this aim we consider in (7) the case where $r :\equiv r_i$ for all i. Then (19) yields

$$\mathrm{Cov}(d\xi_i, d\xi_j)/dt = (1 - r^2)\gamma_i^\top \gamma_j + r^2 \lambda \beta_i^\top \Sigma \beta_j.$$

We then assume $\beta_j = A\gamma_j$ for some $m \times d$ matrix A which gives

$$\mathrm{Cov}(d\xi_i, d\xi_j)/dt = \gamma_i^\top (I - r^2 I + r^2 \lambda A^\top \Sigma A)\gamma_j.$$

Now the requirement that the local covariances (19) coincide with the local covariances of the market model leads to the condition

$$\lambda A^\top \Sigma A = I_d,$$

and in particular $m \geq d$. Since Σ is (time independent) positive definite there is a unique positive symmetric $m \times m$ matrix C such that $\Sigma = C^2$. Then for any column-orthogonal $m \times d$ matrix Q we have a solution

$$A = \lambda^{-1/2} C^{-1} Q.$$

Note that in general Q and λ may depend on t. Without loss of generality (i.e. without affecting the input Libor market model) we may assume that the $(n-1) \times d$ matrix $(\gamma_{j,r})$ is an upper triangular matrix in the sense

$$\gamma_{n-j,l} = 0 \quad \text{for} \quad 1 \leq l < d - j + 1, \quad j = 1, ..., d.$$

We assume (for technical reasons in fact) that the $(n-1) \times m$ matrix $(\beta_{j,r})$ is also an upper triangular matrix,

$$\beta_{n-j,l} = \sum_{r=1}^{d} A_{l,r} \gamma_{n-j,r} = 0, \quad \text{for} \quad 1 \leq l < m - j + 1, \quad j = 1, ..., m. \tag{20}$$

In particular this entails that the jumps of L_{n-1} are driven by a single jump measure. We will achieve (20) by the additional requirement $m = d$ (dimension of the jump space equal to the number of Brownian motions) and by taking the orthogonal matrix Q such that $C^{-1}Q$, hence A, is a lower triangular (square) matrix with positive diagonal elements. Thus, A is uniquely determined by

$$AA^\top = \lambda^{-1}\Sigma^{-1}, \qquad A \text{ is lower triangular with positive diagonal.} \tag{21}$$

As a further specialization we take λ to be time independent. Note that $u^\top \beta_i = (Du)^\top D^{-1} \beta_i$ for any regular diagonal matrix D. So, multiplication of all jump random variables with an arbitrary factor and respective components of β_i with this factors inverse yields the same model. We thus need to standardize the jump components in a suitable way. Without any restriction we may fix the jump variances α_k defined as

$$\begin{aligned} \alpha_k &:= \int u_k^2 p_k(du_k) - \kappa_k^2 \quad \text{where} \\ \kappa_k &:= \int u_k p_k(du_k) \end{aligned}$$

is the mean of the kth jump component, as we like. As a convenient choice we take them all equal, i.e. we set $\alpha_k \equiv: \alpha$, $k = 1, \ldots, m$. We will choose α such that $||A||_F := \sqrt{\sum_{k,l=1}^{m} |A_{kl}|^2} = \sqrt{m} = ||I_m||_F$, which is equivalent to

$$||C^{-1}||_F^2 = \sum_{k=1}^{m} \frac{1}{\lambda_k^{\Sigma}} = \lambda m, \tag{22}$$

where λ_k^{Σ}, $k = 1, ..., m$, denote the eigenvalues of Σ. Then by the result of Appendix 7.3 it follows that (22) is equivalent to

$$\alpha\lambda = \frac{\alpha + \frac{m-1}{m}\sum_{p=1}^{m}\kappa_p^2}{\alpha + \sum_{p=1}^{m}\kappa_p^2}.$$

It is easy to show that this quadratic equation in α has one positive and one negative solution, and that for large m the positive solution $\alpha_+ \approx 1/\lambda$. We therefore set

$$\alpha := \frac{1}{\lambda} \equiv \alpha_k, \quad k = 1, \ldots, m.$$

For all $k, l = 1, ..., m$, $c_k^\top c_l = e_k^\top C^2 e_l = \Sigma_{kl} = \alpha_k \delta_{kl} + \kappa_k \kappa_l$. We so have in particular $\beta_{n-1,l}(s) \equiv 0$ for $1 \leq l < m$, and

$$\begin{aligned}
\beta_{n-1,m}(s) &= A_{m,m}\gamma_{n-1,m}(s) = \lambda^{-1/2}(e_m^\top C^2 e_m)^{-1}\gamma_{n-1,m}(s) \\
&= \frac{\gamma_{n-1,m}(s)}{\sqrt{\lambda(\alpha + \kappa_m^2)}} = \frac{\gamma_{n-1,m}(s)}{\sqrt{1 + \lambda\kappa_m^2}}.
\end{aligned} \tag{23}$$

Hence the dynamics of $\ln L_{n-1}$ is driven by a single jump variable u_m under a jump distribution with density p_m with mean κ_m and variance λ^{-1}.

4 Pricing caplets

A caplet for the period $[T_j, T_{j+1}]$ with strike K is an option which pays $(L_j(T_j) - K)^+\delta_j$ at time T_{j+1}, where $1 \leq j < n$. It is well-known that under the T_{j+1} - forward measure the caplet price has the following simple representation. Writing E_{j+1} for the expectation under this measure, we have

$$C_j(K) = B_{j+1}(0)E_{j+1}[(L_j(T_j) - K)^+\delta_j]$$

for price of the j-th caplet at time zero. Thus the j-th caplet price is determined by the dynamics of L_j under P_{j+1} only. We now recall the FFT pricing method of

Carr & Madan, which basically goes as follows. It turns out natural to transform for a fixed j the strike variable into a log-forward moneyness variable defined by

$$v := \ln \frac{K}{L_j(0)}.$$

In terms of log-forward moneyness the j-th caplet price is then given by

$$\mathcal{C}_j(v) := \delta_j B_{j+1}(0) L_j(0) E_{j+1}[(e^{X_j(T_j)} - e^v)^+],$$

where $X_j(t) := \ln L_j(t) - \ln L_j(0)$. We further introduce an auxiliary function

$$\begin{aligned}\mathcal{O}_j(v) &:= \delta_j^{-1} B_{j+1}^{-1}(0) L_j^{-1}(0)\mathcal{C}_j(v) - (1-e^v)^+ \\ &= E_{j+1}(e^{X_j(T_j)} - e^v)^+ - (1-e^v)^+ \\ &= 1_{v\geq 0} E_{j+1}(e^{X_j(T_j)} - e^v)^+ + 1_{v\leq 0} E_{j+1}(e^v - e^{X_j(T_j)})^+,\end{aligned}$$

where the third expression is basically due to the put-call parity and follows from the identity $(a-b)^+ = a - b + (b-a)^+$ and the fact $E_{j+1}e^{X_j(T_j)} = 1$. In Appendix we derive further characteristic properties of the function $\mathcal{O}_j$. In particular, it holds (for a proof see Appendix 7.4)

$$\mathfrak{F}\{\mathcal{O}_j\}(z) = \int_{-\infty}^{\infty} \mathcal{O}_j(v) e^{\mathrm{i}vz} dv = \frac{1 - \Phi_{j+1}(z - \mathrm{i}; T_j)}{z(z-\mathrm{i})}. \tag{24}$$

Most importantly, if the characteristic function of $X_j(T_j)$ is explicitly given, for example by (11), and (12) in the case $j = n-1$, we obtain an analytical caplet pricing formula via Fourier inversion,

$$\begin{aligned}C_j(K) &= \delta_j B_{j+1}(0)(L_j(0) - K)^+ + \\ &\quad \frac{\delta_j B_{j+1}(0) L_j(0)}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \Phi_{j+1}(z-\mathrm{i}; T_j)}{z(z-\mathrm{i})} e^{-\mathrm{i}z \ln \frac{K}{L_j(0)}} dz.\end{aligned} \tag{25}$$

For a fixed j, $j < n-1$, let now $\ln L_j$ be given by (17). As noted at the end of Section 3, we may then obtain an additive approximation $\widetilde{X}_j(T_j)$ of $X_j(T_j)$ via (17) by replacing $\nu^{(j+1)}$ with the approximative compensator

$$\widetilde{\nu}^{(j+1)}(dt, du) := \lambda(t) dt\, p(du) \prod_{l=j+1}^{n-1} \frac{1 + \delta_l L_l(0) \exp(r_l u^\top \beta_l)}{1 + \delta_l L_l(0)}. \tag{26}$$

Hence, approximative caplet prices $\widetilde{C}_j(K)$ are obtained from (25), using an approximation $\widetilde{\Phi}_{j+1}$ of the characteristic function Φ_{j+1}, which in turn is obtained by replacing in (11)-(12), $n-1, n$, and $\nu^{(n)}(dt, du) = \lambda(dt)p(du)$, respectively with $j, j+1$, and $\widetilde{\nu}^{(j+1)}(dt, du)$ from (26).

5 Calibration

Let us first consider the calibration to a panel of caplets corresponding to maturity T_{n-1} and different strikes $K_{-N} < \cdots < K_{-1} < K_0 := L_{n-1}(0) < K_1 < \cdots < K_N$. So, suppose that caplet prices $C_{n-1,j}$ corresponding to K_j, $-N \le j \le N$, are available. We first transform the observations $C_{n-1,j}$ and strikes K_j to

$$O_{n-1,j} := \delta_{n-1,}^{-1} B_n^{-1}(0) L_{n-1}^{-1}(0) C_{n-1,j} - (1 - e^{v_j})^+, \tag{27}$$

$$v_j := \ln \frac{K_j}{L_j(0)}, \quad -N \le j \le N. \tag{28}$$

Our calibration procedure relies essentially upon the next formula which follows from (11), (12), (24), and taking the assumptions of Section 3.5 into account.

$$\begin{aligned} \psi_n(z; T_{n-1}) &= \mathrm{Ln}(\Phi_n(z; T_{n-1})) = \mathrm{Ln}\Big(1 - z(z+\mathrm{i})\mathfrak{F}\{\mathcal{O}_{n-1}\}(z+\mathrm{i})\Big) \\ &= -\frac{\theta_{n-1}^2 z^2}{2} - \mathrm{i}\varkappa_{n-1} z - \zeta_{n-1} + \zeta_{n-1}\mathfrak{F}\{\mu_{n-1}\}(z), \end{aligned} \tag{29}$$

with abbreviations

$$\begin{aligned} \theta_{n-1}^2 &:= (1 - r_{n-1}^2) \int_0^{T_{n-1}} |\gamma_{n-1}(s)|^2 ds, \\ \varkappa_{n-1} &:= \lambda T_{n-1} \int_{\mathbb{R}} \Big(\exp(r_{n-1}\, u \beta_{n-1,m}(s)) - 1 \Big)\, \mathfrak{p}_m(u)\, du \\ &+ \frac{1}{2} \int_0^{T_{n-1}} (1 - r_{n-1}^2) |\gamma_{n-1}(s)|^2 ds \end{aligned} \tag{30}$$

$$\zeta_{n-1} := \lambda T_{n-1}, \tag{31}$$

$$\mu_{n-1}(\cdot) := T_{n-1}^{-1} \int_0^{T_{n-1}} r_{n-1}^{-1} \beta_{n-1,m}^{-1}(s)\, \mathfrak{p}_m(r_{n-1}^{-1} \beta_{n-1,m}^{-1}(s)\, \cdot)\, ds, \tag{32}$$

with $\mathrm{Ln}(w) := \ln|w| + \mathrm{i}\operatorname{Arg} w$, $-\pi < \operatorname{Arg} w \le \pi$ denoting the main branch of the logarithm, and $\mathfrak{p}_m$ being the density of p_m which we now assume to exist.

In principle, the constants θ_{n-1}^2, $\varkappa_{n-1}$, ζ_{n-1}, and the mixed density μ_{n-1} can be recovered via (29) from complete knowledge of function $\mathcal{O}_{n-1}$, hence a complete system of model consistent caplet prices $C_{n-1}(K)$, $0 < K < \infty$. Indeed, since $\mathfrak{F}\{\mu_{n-1}\}(z)$ tends to zero as $|z| \to \infty$ due to the Riemann-Lebesgue lemma, we have

$$\begin{aligned} \theta_{n-1}^2 &= -2 \lim_{z \to +\infty} z^{-2} \psi_n(z; T_{n-1}) \\ \varkappa_{n-1} &= - \lim_{z \to +\infty} z^{-1} \operatorname{Im} \psi_n(z; T_{n-1}), \quad \text{and next,} \\ \zeta_{n-1} &= \lim_{z \to +\infty} \left(-\psi_n(z; T_{n-1}) - \frac{\theta_{n-1}^2 z^2}{2} - \mathrm{i}\varkappa_{n-1} z \right), \end{aligned}$$

and then the function $\mathfrak{F}\{\mu_{n-1}\}(z)$ can be found from (29). In practice this approach breaks down due to incomplete knowledge of $\mathcal{O}_{n-1}$ and lack of numerical stability however.

In Belomestny and Reiss (2004) a more stable procedure is developed which estimates all spot characteristics $\theta_{n-1}^2, \varkappa_{n-1}$, ζ_{n-1}, and $\mu_{n-1}(\cdot)$, for a given set of noisy observations (27) due to a discrete set of strikes (28). This procedure consists basically of four steps: *(i)* first, a continuous piece-wise linear approximation $\widetilde{\mathcal{O}}_{n-1}$ of $\mathcal{O}_{n-1}$ is built from the data; *(ii)* from $\widetilde{\mathcal{O}}_{n-1}$ an approximation $\widetilde{\psi}_n$ of ψ_n is obtained; *(iii)* next the coefficients of the quadratic polynomial on the right-hand side in (29) are estimated from $\widetilde{\psi}_n$, under the presence of the nonparametric nuisance part $\mathfrak{F}\{\mu_{n-1}\}$ (which vanishes at infinity) using appropriate weighting schemes; *(iv)* finally an estimator for μ_{n-1} is obtained via FFT inversion of the remainder. The steps *(i)*–*(iv)* are spelled out in detail below.

(i) In view of Appendix 7.5, we construct a continuous piece-wise linear function $v \to \widetilde{\mathcal{O}}_{n-1}(v)$ on a grid v_j, $-N-1 \le j \le N+1$, with $v_{-N-1} \ll v_{-N} < \cdots < v_{-1} < v_0 := 0 < v_1 < \cdots < v_N \ll v_{N+1}$, , such that $\widetilde{\mathcal{O}}_{n-1}(v)$ fits the data at $v_j, j \neq 0$, $\widetilde{\mathcal{O}}_{n-1}(v_{-N-1}) := \widetilde{\mathcal{O}}_{n-1}(v_{N+1}) := 0$, and $\widetilde{\mathcal{O}}'_{n-1}(0-) - \widetilde{\mathcal{O}}'_{n-1}(0+) = 1$. The boundary strikes v_{-N-1}, v_{N+1} are included to reflect the fact that $\lim_{v\to\pm\infty} \mathcal{O}_{n-1}(v) = 0$.

(ii) By straightforward FFT we compute $\mathfrak{F}\{\widetilde{\mathcal{O}}_{n-1}\}(z+\mathrm{i})$ and so obtain

$$\widetilde{\psi}_n(z) := \mathrm{Ln}\Big(1 - z(z+\mathrm{i})\mathfrak{F}\{\widetilde{\mathcal{O}}_{n-1}\}(z+\mathrm{i})\Big), \quad z \in \mathbb{R}. \tag{33}$$

(iii) With an estimate $\widetilde{\psi}_n$ of ψ_n at hand, we obtain estimators for the parametric part $(\theta_{n-1}^2, \varkappa_{n-1}, \zeta_{n-1})$ by an averaging procedure using the polynomial structure in (29) and the decay property of $\mathfrak{F}\{\mu_{n-1}\}$. For suitable weight functions w_θ, $w_\varkappa$, and w_ζ constructed in Section 5.1, which have bounded support $\mathcal{U} := [-U, U]$ with $U > 0$, and satisfy

$$\begin{gathered}\int w_\theta du = 0, \int u^2 w_\theta(u) du = -2, \quad \int u w_\varkappa(u) du = 1, \\ \int u^2 w_\zeta(u) du = 0, \int w_\zeta(u) du = -1,\end{gathered} \tag{34}$$

we compute the estimates

$$\begin{aligned}\widetilde{\theta}_{n-1}^2 &:= \int \mathrm{Re}(\widetilde{\psi}_n(u)) w_\theta(u) du, \\ \widetilde{\varkappa}_{n-1} &:= \int \mathrm{Im}(\widetilde{\psi}_n(u)) w_\varkappa(u) du, \\ \widetilde{\zeta}_{n-1} &:= \int \mathrm{Re}(\widetilde{\psi}_n(u)) w_\zeta(u) du,\end{aligned} \tag{35}$$

for the parameters $\theta_{n-1}^2, \varkappa_{n-1}$, and ζ_{n-1}, respectively.

(iv) The estimate for μ_{n-1} is obtained via the inverse Fourier transform,

$$\widetilde{\mu}_{n-1} := \widetilde{\zeta}_{n-1}^{-1}\mathfrak{F}^{-1}\left\{\left(\widetilde{\psi}_n(\cdot) + \frac{\widetilde{\theta}_{n-1}^2}{2}(\cdot)^2 - \mathrm{i}\widetilde{\varkappa}_{n-1}(\cdot) + \widetilde{\zeta}_{n-1}\right)\mathbf{1}_{\mathcal{U}}\right\}, \tag{36}$$

where $u \in \mathbb{R}$ and $\mathbf{1}_{\mathcal{U}}$ is the indicator function of the set $\mathcal{U}$.

The computational complexity of this estimation procedure is very low. The only time consuming steps are the three integrations in step *(iii)* and the inverse Fourier transform (inverse FFT) in step *(iv)*.

5.1 Determination of the weights w_θ, $w_\varkappa$, and w_ζ

Let us assume that for some natural number p and $C > 0$,

$$\max_{0\le q\le p} \|\mu_{n-1}^{(q)}\|_{L^2(\mathbb{R})} \le C \tag{37}$$

and consider for some $U > 0$ the following weight functions,

$$\begin{aligned}
w_\theta^{U,p}(u) &:= \frac{p+3}{(1-2^{-2/(p+1)})\,U^{p+3}}|u|^p(1_{|u/U|\le 1} - 2\cdot 1_{2^{-1/(p+1)}\le|u/U|\le 1}), \qquad (38)\\
w_\varkappa^{U,p}(u) &:= \frac{p+2}{2U^{p+2}}|u|^p\mathrm{sign}(u)1_{|u/U|\le 1},\\
w_\zeta^{U,p}(u) &:= \frac{p+1}{2\,(2^{2/(p+3)}-1)\,U^{p+1}}|u|^p(2\cdot 1_{2^{-1/(p+3)}\le|u/U|\le 1} - 1_{|u/U|\le 1}),
\end{aligned}$$

which satisfy the conditions (34) by straightforwardly checking.
Following Belomestny and Reiss (2005), we can estimate

$$\begin{aligned}
|\widetilde{\theta}_{n-1}^2 - \theta_{n-1}^2| &\le \left|\int \mathrm{Re}(\widetilde{\psi}_n(u) - \psi_n(u))w_\theta^{U,p}(u)du\right| + \left|\int \mathrm{Re}(\mathfrak{F}\{\mu_{n-1}\}(u))w_\theta^{U,p}(u)du\right|\\
&= (1) + (2). \qquad (39)
\end{aligned}$$

The second term can be estimated using the identity $(\mathrm{i}u)^p\mathfrak{F}\{\mu_{n-1}\}(u) = \mathfrak{F}\{\mu_{n-1}^{(p)}\}(u)$, two times Parseval's isometry, and (38),

$$\begin{aligned}
(2) &\le \left|\int \mathfrak{F}\{\mu_{n-1}\}(u)w_\theta^{U,p}(u)du\right| = \left|\int (\mathrm{i}u)^p\mathfrak{F}\{\mu_{n-1}\}(u)\overline{\left(\frac{w_\theta^{U,p}(u)}{(\mathrm{i}u)^p}\right)}du\right|\\
&= \left|\int \mathfrak{F}\{\mu_{n-1}^{(p)}\}(u)\overline{\left(\frac{w_\theta^{U,p}(u)}{(\mathrm{i}u)^p}\right)}du\right| = \frac{1}{2\pi}\left|\int \mu_{n-1}^{(p)}(s)\overline{\mathfrak{F}^{-1}\left\{\frac{w_\theta^{U,p}(\cdot)}{(\mathrm{i}\cdot)^p}\right\}(s)}ds\right|\\
&\le \frac{C}{\sqrt{2\pi}}\left\|\frac{w_\theta^{U,p}(\cdot)}{(\cdot)^p}\right\|_{L^2(\mathbb{R})} = \frac{C(p+3)}{\sqrt{\pi}\,(1-2^{-2/(p+1)})\,U^{p+5/2}} \le C_1\frac{(p+1)(p+3)}{U^{p+5/2}},
\end{aligned}$$

for some $C_1 > 0$, which explains the construction of $w_\theta^{U,p}$: for fixed p and U large, (2) falls with $O(U^{-(p+5/2)})$. The first term (1) is due to the noise and lack of data. It can be estimated by

$$
\begin{aligned}
(1) \quad &\leq \quad ||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}||w_\theta^{U,p}||_{L^1(\mathcal{U})} = ||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}\frac{2(p+3)}{(p+1)\left(1-2^{-2/(p+1)}\right)U^2} \\
&\leq \quad C_2||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}\frac{p+3}{U^2},
\end{aligned}
$$

for some $C_2 > 0$. So we have,

$$
|\widetilde{\theta}_{n-1}^2 - \theta_{n-1}^2| \leq C_2||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}\frac{p+3}{U^2} + C_1\frac{(p+1)(p+3)}{U^{p+5/2}}. \tag{40}
$$

In a similar way we obtain for $\varkappa_{n-1}$, and ζ_{n-1},

$$
|\widetilde{\varkappa}_{n-1} - \varkappa_{n-1}| \quad \leq \quad C_3||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}\frac{p+2}{U(p+1)} + C_4\frac{(p+2)}{U^{p+3/2}}, \tag{41}
$$

$$
|\widetilde{\zeta}_{n-1} - \zeta_{n-1}| \quad \leq \quad C_5||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}(p+3) + C_6\frac{(p+1)(p+3)}{U^{p+1/2}}, \tag{42}
$$

for some $C_3, C_4, C_5, C_6 > 0$. Note that even when $||\mu_{n-1}^{(q)}||_{L_2(\mathbb{R})}$ is finite for very large q it is not wise in view of (42) to take p too large. In practice one needs to accomplish that $||\widetilde{\psi}_n - \psi_n||_{L^\infty(\mathcal{U})}$ is small for a large enough U and then $p = 1$ or 2 turns out to be a proper choice.

Correction of $\widetilde{\mu}_{n-1}$

Due to numerical as well as statistical errors the estimated $\widetilde{\mu}_{n-1}$ may not be a probability density and thus needs to be corrected. Besides that we also want the variance of X_{n-1} to be equal to the Black variance $T_{n-1}(\gamma_{n-1}^B)^2$, where

$$
\gamma_{n-1}^B := \sqrt{\frac{1}{T_{n-1}}\int_0^{T_{n-1}} |\gamma_{n-1}|^2(s)ds}.
$$

In order to accomplish all these requirements we construct a new estimate $\widetilde{\mu}_{n-1}^+$ as a solution of the following optimization problem,

$$
||\widetilde{\mu}_{n-1}^+ - \widetilde{\mu}_{n-1}||_{L^2(\mathbb{R})}^2 \to \min, \quad \inf_{x\in\mathbb{R}} \widetilde{\mu}_{n-1}^+(x) \geq 0 \tag{43}
$$

subjected to

$$
\int \widetilde{\mu}_{n-1}^+(v)dv = 1, \quad \int v^2\widetilde{\mu}_{n-1}^+(v)dv = \frac{T_{n-1}\left(\gamma_{n-1}^B\right)^2 - \widetilde{\theta}_{n-1}^2}{\widetilde{\zeta}_{n-1}}. \tag{44}
$$

The solution has a rather simple form and is given by

$$
\widetilde{\mu}_{n-1}^+(x;\xi,\eta) := \max\{0, \widetilde{\mu}_{n-1}(x) - \xi - \eta x^2\}, \quad x \in \mathbb{R},
$$

where ξ and η need to be determined such that (44) is satisfied. Note that by representing $\widetilde{\mu}^+$ as a mixture of given densities, (43)-(44) boils down to a finite dimensional quadratic optimization problem.

5.2 Procedure for calibration against terminal caplets

For $U > 0$ we denote the estimates (35) obtained using the weight functions (38) by $\theta_{n-1}(U)$, $\varkappa_{n-1}(U)$, $\zeta_{n-1}(U)$, and the corrected Lévy density is denoted by $\mu^+_{n-1}(\cdot;U)$. From (30) and (31) we can directly infer estimates $r_{n-1}(U)$ and $\lambda(U)$, respectively. We further have to identify a jump density $\mathfrak{p}_m$ from $\mu^+_{n-1}(\cdot;U)$ via (32), while taking into account (23). Since the function β is usually not constant this might be not easy in general. We therefore go the following pragmatic way. Let us define in the spirit of (23) $\beta^B_{n-1} := \gamma^B_{n-1}/\sqrt{1+\lambda\kappa_m^2}$. We then consider as candidate jump density

$$\begin{aligned}\widehat{\mathfrak{p}}_m(u;U) &:= r_{n-1}(U)\beta^B_{n-1}\mu^+_{n-1}\big(r_{n-1}(U)\beta^B_{n-1}u;U\big) \\ &= \frac{r_{n-1}(U)\gamma^B_{n-1}}{\sqrt{1+\lambda(U)\kappa_m^2}}\mu^+_{n-1}\left(\frac{r_{n-1}(U)\gamma^B_{n-1}}{\sqrt{1+\lambda(U)\kappa_m^2}}u;U\right). \end{aligned} \tag{45}$$

Due to the very construction

$$-\mathfrak{F}\{\mu^+_{n-1}(\cdot,U)\}''(0) = \int v^2\mu^+_{n-1}(v;U)dv = \frac{r^2_{n-1}(U)\big(\gamma^B_{n-1}\big)^2}{\lambda(U)}, \tag{46}$$

and so by (45) it holds $\int u^2\widehat{\mathfrak{p}}_m(u;U)\,du = \lambda^{-1}(U)+\kappa_m^2$. By next requiring that the first moment of the r.h.s. in (45) is equal to κ_m, we simply obtain

$$\kappa_m(U) := \frac{\kappa_{\mu^+}}{\sqrt{\lambda(U)\alpha_{\mu^+}}}, \tag{47}$$

with κ_{μ^+} and α_{μ^+} denoting the expectation and the variance, respectively, of a random variable with density $\mu^+_{n-1}(\cdot;U)$. Substituting (47) in (45) then yields

$$\widehat{\mathfrak{p}}_m(u;U) = \frac{r_{n-1}(U)\gamma^B_{n-1}}{\sqrt{1+\kappa^2_{\mu^+}/\alpha_{\mu^+}}}\mu^+_{n-1}\left(\frac{r_{n-1}(U)\gamma^B_{n-1}}{\sqrt{1+\kappa^2_{\mu^+}/\alpha_{\mu^+}}}u;U\right). \tag{48}$$

Finally we consider in view of (32)

$$\begin{aligned}\widehat{\mu}^+_{n-1}(\cdot;U) &:= \frac{1}{T_{n-1}}\int_0^{T_{n-1}}\frac{\sqrt{1+\kappa^2_{\mu^+}/\alpha_{\mu^+}}}{r_{n-1}(U)\gamma_{n-1,m}(s)}\times \\ &\quad\times\widehat{\mathfrak{p}}_m\left(\frac{\sqrt{1+\kappa^2_{\mu^+}/\alpha_{\mu^+}}}{r_{n-1}(U)\gamma_{n-1,m}(s)}\cdot;U\right)ds. \end{aligned} \tag{49}$$

Note that the second moments of $\widehat{\mu}^+_{n-1}$ and μ^+_{n-1} coincide and are given by the r.h.s. of (46) (the first moments coincide approximately).

Choice of U

We find U^* as a solution of the following *minimization problem*

$$U^* = \operatorname{arginf}_U \sum_{i=-N}^{N} |\widehat{C}_{n-1}(K_i; U) - C_{n-1,i}|^2, \tag{50}$$

where $\widehat{C}_{n-1}(\cdot; U)$ are prices computed from the model due to $\theta_{n-1}(U)$, $\varkappa_{n-1}(U)$, $\zeta_{n-1}(U)$, and $\widehat{\mu}^+_{n-1}(\cdot; U)$.

5.3 Calibration to other caplets

With U^* is determined via (50) and $\mathfrak{p}_m := \mathfrak{p}_m(U^*)$, we introduce the shifted densities

$$\mathfrak{p}_j(u) := \mathfrak{p}_m(u - \kappa_j + \kappa_m),$$

hence

$$\kappa_j = \int_{\mathbb{R}} u\,\widehat{\mathfrak{p}}_j(u)du, \qquad j = 1, \ldots, m. \tag{51}$$

Because we want to preserve the input local covariance structure we set $r_j = r_m(U^*)$, $j = 1, \ldots, m-1$. Let $\mathcal{U}$ be the upper triangular $m \times m$ matrix with positive diagonal elements such that $\Sigma = \mathcal{U}\mathcal{U}^\top$. This decomposition exists because Σ is invertible. From (21) we then have $A = \lambda^{-1/2}\mathcal{U}^{-\top}$. Let us define $\Sigma^{(k)}_{rr'}$, $k \le r, r' \le m$, $k = 1, ..., m$. Since $\mathcal{U}$ is an upper triangular we have $\Sigma^{(k)} = \mathcal{U}^{(k)}(\mathcal{U}^{(k)})^\top$ and $A^{(k)} = \lambda^{-1/2}(\mathcal{U}^{(k)})^{-\top}$ with $A^{(k)}$ and $\mathcal{U}^{(k)}$ defined analogously to $\Sigma^{(k)}$. Thus, for knowing $A^{(k)}$ it is sufficient to know $\Sigma^{(k)}$.

Now let us suppose that $m = n-1$. We determine κ_j, $j = 1, \ldots, n-1$, recursively in the following way. For $j = n-1$, κ_{n-1} is determined from (47), then $\beta_{n-1,n-1}$ from (23), and $\Sigma^{(n-1)}_{n-1,n-1} = \alpha + \kappa^2_{n-1}$. Suppose $\beta_{l,k}$ is determined for $l = j, ..., n-1$, $k = l, ..., n-1$, where $j > 1$. For $j = m = n-1$ we are in the situation of Section 5.2. We then consider the matrix

$$\Sigma^{(j-1)}(\kappa_{j-1}) := \left[\begin{array}{c|c} \alpha + \kappa^2_{j-1} & \kappa_{j-1}a^\top \\ \kappa_{j-1}a & \Sigma^{(j)} \end{array} \right], \tag{52}$$

with $a := [\kappa_j, \cdots, \kappa_{n-1}]^\top$, and where the $(n-j) \times (n-j)$ matrix $\Sigma^{(j)}_{rr'}$ is assumed to be already determined. Note that $\alpha = \lambda^{-1}(U^*)$ is the common jump variance. In fact the only unknown parameter to be determined in (52) is κ_{j-1}. Further, it easily follows that,

$$\mathcal{U}^{(j-1)}(\kappa_{j-1}) = \left[\begin{array}{c|c} \left(\alpha + \kappa^2_{j-1}\left(1 - a^\top(\Sigma^{(j)})^{-1}a\right)\right)^{1/2} & \kappa_{j-1}a^\top(\mathcal{U}^{(j)})^{-\top} \\ & \mathcal{U}^{(j)} \end{array} \right]$$

and so

$$
F^{(j-1)}(\kappa_{j-1}) := \left(\mathcal{U}^{(j-1)}\right)^{-\top}(\kappa_{j-1}) =
$$
$$
\left[\begin{array}{c|c} \left(\alpha + \kappa_{j-1}^2\left(1 - a^\top(\Sigma^{(j)})^{-1}a\right)\right)^{-1/2} & \\ -\left(\alpha + \kappa_{j-1}^2\left(1 - a^\top(\Sigma^{(j)})^{-1}a\right)\right)^{-1/2}\kappa_{j-1}a & \left(\mathcal{U}^{(j)}\right)^{-\top} \end{array}\right]
$$

Next, set according to (20)

$$
\beta_{j-1,k}(\kappa_{j-1}) = \lambda^{-1/2}\sum_{r=j-1}^{k} F_{k,r}^{(j-1)}(\kappa_{j-1})\gamma_{j-1,r}, \quad k = j-1, ..., n-1,
$$
$$
\beta_{j-1,k}(\kappa_{j-1}) = 0, \quad 1 \le k < j-1.
$$

By a simple trial and error search we then determine κ_{j-1} such that the least squares fit error of the T_{j-1} caplet panel is as small as possible. For each guess of κ_{j-1} the model caplet prices may be computed by Monte Carlo simulation of the model, or as an alternative by approximating caplet prices as proposed at the end of Section 4.

6 Calibration to real data

In this section we calibrate the model (7) to market data given on 11.01.2004. The caplet-strike volatility matrix is partially shown in Table 1. The corresponding implied volatility surface is shown in Figure 1.

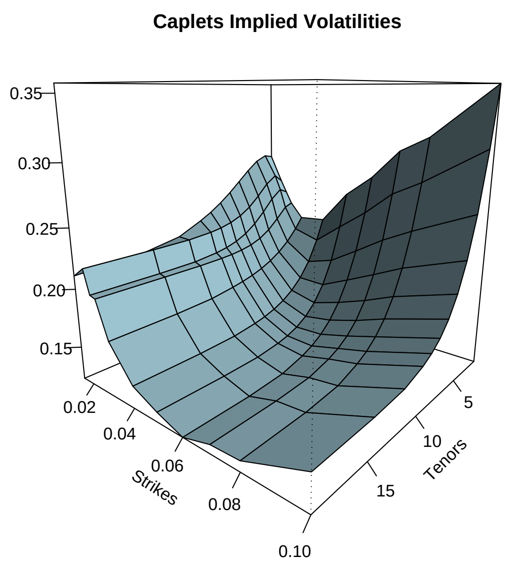

Figure 1: Smoothed caplet implied volatility surface σ_T^K.

K/T	0.150	0.200	0.225	0.250	0.300	0.400	0.500	0.600
0.50	0.2604	0.1735	0.1819	0.1969	0.2453	0.2708	0.3197	0.3407
0.75	0.2678	0.2036	0.2052	0.2136	0.2401	0.2598	0.3052	0.3258
1.75	0.2832	0.2587	0.2475	0.2365	0.2227	0.2246	0.2539	0.2733
2.50	0.2850	0.2651	0.2513	0.2334	0.2125	0.2051	0.2234	0.2412
3.50	0.2804	0.2581	0.2432	0.2233	0.2016	0.1856	0.1924	0.2071
4.50	0.2720	0.2474	0.2319	0.2142	0.1934	0.1720	0.1711	0.1821
5.50	0.2625	0.2381	0.2219	0.2079	0.1872	0.1625	0.1566	0.1640
6.50	0.2531	0.2314	0.2144	0.2039	0.1824	0.1557	0.1470	0.1510
7.50	0.2447	0.2270	0.2092	0.2016	0.1788	0.1510	0.1407	0.1418
8.50	0.2375	0.2241	0.2058	0.2002	0.1761	0.1477	0.1367	0.1355
9.50	0.2315	0.2224	0.2036	0.1995	0.1740	0.1454	0.1342	0.1311
11.50	0.2212	0.2206	0.2011	0.1988	0.1707	0.1424	0.1312	0.1253
14.50	0.2149	0.2201	0.2003	0.1987	0.1689	0.1410	0.1302	0.1228
19.50	0.2111	0.2200	0.2001	0.1987	0.1678	0.1404	0.1300	0.1219

Table 1: Caplet volatilities σ_T^K for different strikes and different tenor dates (in years).

Pronounced smiles are clearly observable. Due to the structure of the given data we are going to calibrate the jump diffusion model based on semi-annual tenors, i.e. $\delta_j \equiv 0.5$, with $n = 41$, and where the initial calibration date 01.11.04 is identified with $T_0 = 0$.

In a pre-calibration a standard market model is calibrated to ATM caps and ATM swaptions using Schoenmakers (2005). However, we emphasize that the method by which this input market model is obtained is not essential nor a discussion point for this paper. For the pre-calibration we have used a volatility structure of the form

$$\gamma_i(t) = c_i g(T_i - t) e_i, \quad 0 \le t \le \min(T_i, T_j), \quad 1 \le i, j < n,$$

where g is a simple parametric function and e_i are unit vectors. The calibration routine returned $e_i \in \mathbb{R}^{40}$ with

$$e_i^\top e_j = \rho_{ij} = \exp[-0.005|i-j|] \quad 1 \le i, j < 41,$$

such that the matrix $(e_{i,k})$ is upper triangular, and

$$g(s) = 0.8 + 0.2e^{-2.0s}.$$

The c_i can be readily computed from

$$(\sigma_{T_i}^{ATM})^2 T_i = c_i^2 \int_0^{T_i} g^2(s)\, ds, \quad i = 1, \ldots, n-1,$$

using the initial Libor curve, which is obtained by a standard stripping procedure from the yield curve at 11.01.04, and is given in Table 2.

T_i	$L_i(0)$	T_i	$L_i(0)$	T_i	$L_i(0)$	T_i	$L_i(0)$
0.5	0.0238176	5.5	0.0451931	10.5	0.0509249	15.5	0.0539696
1	0.0264201	6	0.0465074	11	0.0512114	16	0.0540521
1.5	0.0292798	6.5	0.0475881	11.5	0.0515804	16.5	0.0540931
2	0.0320656	7	0.0484201	12	0.0520317	17	0.0540933
2.5	0.0345508	7.5	0.0490942	12.5	0.0524639	17.5	0.054053
3	0.0366693	8	0.0496402	13	0.0528456	18	0.0539728
3.5	0.0385821	8.5	0.0500331	13.5	0.0531757	18.5	0.0538533
4	0.040381	9	0.0502848	14	0.0534529	19	0.053695
4.5	0.0420863	9.5	0.0504889	14.5	0.0536757	19.5	0.0534984
5	0.0437079	10	0.0506932	15	0.0538451	20	0.053268

Table 2: Initial Libor curve.

The further steps are as follows

1. The model for L_{n-1} is calibrated as described in Section 5.2 and the calibrated parameters are shown in Table 3. The calibrated density $p_m(x)$ is plotted in

r	λ	κ_m
0.7	0.1	-0.005

Table 3: Parameters calibrated using terminal caplet volas $\sigma^K_{T_{n-1}}$.

Figure 2. Note that the variance of the distribution corresponding to p_m is equal to $1/\lambda = 10.0$ in order to ensure (22).

2. Remaining parameters κ_j, $j = 1, \ldots, 39$, are calibrated sequentially as described in Section 5.3 with approximation formula (26) being used for pricing caplets. It turned out experimentally that κ_j can be taken on the line

$$\kappa_j = \kappa_{40} - 0.0751 * (40 - j), \quad j = 40, \ldots, 1.$$

The quality of the calibration can be seen in Figure 3, where calibrated volatility curves are shown for several caplet maturities together with original caplet volas and ATM caplet volas. The overall root-mean-square fit we have reached shows to be 0.5%-5%, when the number of caplet panels ranges from 2 to 20. Fitting all the 40 caplet panels with an acceptable accuracy (e.g. $\leq$5%), would require a more flexible structure for $\mathfrak{p}_j$, $j < m$, however.

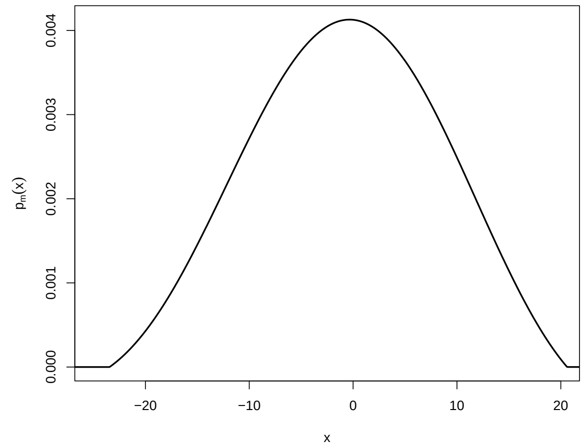

Figure 2: Density $p_m(x)$ calibrated using terminal caplet volas $\sigma^K_{T_{n-1}}$

7 Appendix

7.1 Equivalence of (1) and (3)

Suppose on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_n)$ we are given η and $W^{(n)}$ as in (3), and for $k = 1, ..., m$ we are given a random measure μ_k on $\mathbb{R}_+ \times E_k$, with E_k Lusin, of the form (2)

$$\mu_k = \sum_{n\geq 1} 1_{T_n^{(k)}(\omega)=t}\delta_{(t,\beta_t^{(k)}(\omega))}(dt, du),$$

where the stopping times $(T_n^{(k)})_{k=1,...,m,n\geq 1}$ satisfy $T_n^{(k)}(\omega) \neq T_m^{(l)}(\omega)$ for $n \neq m$ or $k \neq l$. Further let for $i = 1, ..., n-1$, $k = 1, ..., m$, the E_k-valued function processes ψ_{ik} be predictable. By treating E_k and E_l for $k \neq l$ as completely different spaces, i.e.$E_k \cap E_l = \varnothing$ (which may be achieved by giving them different colors if need be), we may construct straightforwardly the Lusin space $E := \bigcup_{k=1}^{m} E_k$ and define a random measure $\mu := \sum_{k=1}^m \mu_k$ on $\mathbb{R}_+ \times E$. Let now $\nu_k^{(n)}$ be the $(P_n, \mathcal{F})$-compensator of μ_k (which is concentrated on E_k), then it easily follows that $\nu^{(n)} := \sum_{k=1}^m \nu_k^{(n)}$ is the $(P_n, \mathcal{F})$-compensator of μ, and by defining $\psi_i(t, u, \omega) := \psi_{ik}(t, u, \omega)$ if $u \in E_k$, (3) may be written as (1).

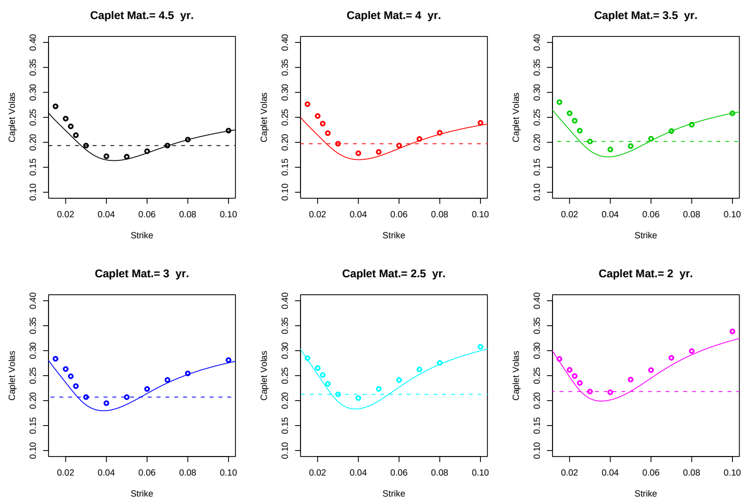

Figure 3: Caplet volas from the calibrated model (solid lines), original caplets volas σ_T^K (points) and ATM caplet volas σ_T^{ATM} (dashed lines) for different caplet maturities T.

7.2 Proof of Lemma 2

Proof of (i):

$$
\begin{aligned}
Ee^{\mathrm{i}z^\top J(t)} &= E\ E\left[e^{\mathrm{i}z^\top \sum_{l=1}^{N_t}\varphi(s_l,u_l)}|N_t\right] = E\left[\prod_{l=1}^{N_t} e^{\mathrm{i}z^\top \varphi(s_l,u_l)}|N_t\right] \\
&= E\left(\int_0^t \frac{\lambda(s)ds}{\int_0^t \lambda(\tau)d\tau}\int_E e^{\mathrm{i}z^\top\varphi(s,u)}p(du)\right)^{N_t} \\
&= \sum_{k=0}^{\infty}\frac{\left(\int_0^t \lambda(\tau)d\tau\right)^k}{k!}e^{-\int_0^t\lambda(\tau)d\tau}\left(\int_0^t \frac{\lambda(s)ds}{\int_0^t \lambda(\tau)d\tau}\int_E e^{\mathrm{i}z^\top\varphi(s,u)}p(du)\right)^k \\
&= \exp\int_0^t \lambda(s)ds\int_E (e^{\mathrm{i}z^\top\varphi(s,u)}-1)p(du).
\end{aligned}
$$

Proof of (ii): By differentiating the characteristic function with respect to z_l and z_l' we obtain

$$
\begin{aligned}
\frac{\partial}{\partial z_l}Ee^{\mathrm{i}z^\top J(t)} = \mathrm{i}\int_0^t \lambda(s)ds\int_E e^{\mathrm{i}z^\top\varphi(s,u)}\varphi_l(s,u)p(du)\cdot \\
\cdot\exp\int_0^t \lambda(s)ds\int_E (e^{\mathrm{i}z^\top\varphi(s,u)}-1)p(du),
\end{aligned}
$$

$$\frac{\partial^2}{\partial z_l \partial z_{l'}} E e^{\mathrm{i} z^\top J(t)} = -\int_0^t \lambda(s) ds \int_E e^{\mathrm{i} z^\top \varphi(s,u)} \varphi_{l'}(s,u) p(du) \cdot$$
$$\cdot \int_0^t \lambda(s) ds \int_E e^{\mathrm{i} z^\top \varphi(s,u)} \varphi_l(s,u) p(du) \ \exp \int_0^t \lambda(s) ds \int_E (e^{\mathrm{i} z^\top \varphi(s,u)} - 1) p(du)$$
$$- \int_0^t \lambda(s) ds \int_E e^{\mathrm{i} z^\top \varphi(s,u)} \varphi_l(s,u) \varphi_{l'}(s,u) p(du)$$
$$\cdot \exp \int_0^t \lambda(s) ds \int_E (e^{\mathrm{i} z^\top \varphi(s,u)} - 1) p(du).$$

Hence

$$E J_l(t) = \int_0^t \lambda(s) ds \int_E \varphi_l(s,u) p(du),$$

and

$$E J_l(t) J_{l'}(t) = \int_0^t \lambda(s) ds \int_E \varphi_{l'}(s,u) p(du) \cdot \int_0^t \lambda(s) ds \int_E \varphi_l(s,u) p(du)$$
$$+ \int_0^t \lambda(s) ds \int_E \varphi_l(s,u) \varphi_{l'}(s,u) p(du),$$

and then note that $\mathrm{Cov}(J_l(t), J_{l'}(t)) = E J_l(t) J_{l'}(t) - E J_l(t) E J_{l'}(t)$.

7.3 Summed reciprocal eigenvalues of Σ

Consider the determinant

$$D_m := \begin{vmatrix} \alpha_1 + \kappa_1^2 & \kappa_1\kappa_2 & \kappa_1\kappa_3 & & \kappa_1\kappa_{m-1} & \kappa_1\kappa_m \\ \kappa_2\kappa_1 & \alpha_2 + \kappa_2^2 & \kappa_2\kappa_3 & & & \kappa_2\kappa_m \\ \kappa_3\kappa_1 & \kappa_3\kappa_2 & \alpha_3 + \kappa_3^2 & & & \\ & & & & & \\ \kappa_{m-1}\kappa_1 & \kappa_{m-1}\kappa_2 & & & \alpha_{m-1} + \kappa_{m-1}^2 & \kappa_{m-1}\kappa_m \\ \kappa_m\kappa_1 & \kappa_m\kappa_2 & & & \kappa_m\kappa_{m-1} & \alpha_m + \kappa_m^2 \end{vmatrix}$$

$$= \begin{vmatrix} \alpha_1 + \kappa_1^2 & \kappa_1\kappa_2 & \kappa_1\kappa_3 & & \kappa_1\kappa_{m-1} & \kappa_1\kappa_m \\ \kappa_2\kappa_1 & \alpha_2 + \kappa_2^2 & \kappa_2\kappa_3 & & & \kappa_2\kappa_m \\ \kappa_3\kappa_1 & \kappa_3\kappa_2 & \alpha_3 + \kappa_3^2 & & & \\ & & & & & \\ \kappa_{m-1}\kappa_1 & \kappa_{m-1}\kappa_2 & & & \alpha_{m-1} + \kappa_{m-1}^2 & \kappa_{m-1}\kappa_m \\ -\frac{\kappa_m}{\kappa_1}\alpha_1 & 0 & & & 0 & \alpha_m \end{vmatrix}$$

$$\alpha_m D_{m-1} - \frac{\kappa_m}{\kappa_1}\alpha_1 (-1)^{m-1} \begin{vmatrix} \kappa_1\kappa_2 & \kappa_1\kappa_3 & & \kappa_1\kappa_{m-1} & \kappa_1\kappa_m \\ \alpha_2 + \kappa_2^2 & \kappa_2\kappa_3 & & & \kappa_2\kappa_m \\ \kappa_3\kappa_2 & \alpha_3 + \kappa_3^2 & & & \\ & & & & \\ \kappa_{m-1}\kappa_2 & \kappa_{m-1}\kappa_3 & & \alpha_{m-1} + \kappa_{m-1}^2 & \kappa_{m-1}\kappa_m \end{vmatrix}.$$

Since

$$
\begin{vmatrix}
\kappa_1\kappa_2 & \kappa_1\kappa_3 & & \kappa_1\kappa_{m-1} & \kappa_1\kappa_m \\
\alpha_2+\kappa_2^2 & \kappa_2\kappa_3 & & & \kappa_2\kappa_m \\
\kappa_3\kappa_2 & \alpha_3+\kappa_3^2 & & & \kappa_3\kappa_m \\
 & & & & \\
\kappa_{m-1}\kappa_2 & & & \alpha_{m-1}+\kappa_{m-1}^2 & \kappa_{m-1}\kappa_m
\end{vmatrix}
$$

$$
= \ldots = \kappa_1\kappa_m(-1)^{m-2}
\begin{vmatrix}
\alpha_2 & 0 & & 0 \\
0 & \alpha_3 & & \\
 & 0 & & \\
 & & & 0 \\
0 & 0 & & \alpha_{m-1}
\end{vmatrix}
= \kappa_1\kappa_m(-1)^{m-2}\alpha_2\cdots\alpha_{m-1},
$$

we obtain

$$
\begin{aligned}
D_m &= \alpha_m D_{m-1} - \frac{\kappa_m}{\kappa_1}\alpha_1(-1)^{m-1}\kappa_1\kappa_m(-1)^{m-2}\alpha_2\cdots\alpha_{m-1} \\
&= \alpha_m D_{m-1} + \kappa_m^2\alpha_1\alpha_2\cdots\alpha_{m-1} = \ldots \\
&= \left(1+\sum_{p=1}^{m}\frac{\kappa_p^2}{\alpha_p}\right)\prod_{q=1}^{m}\alpha_p.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
D_m(\lambda) = |\Sigma-\lambda I_m| &= \left(1+\sum_{p=1}^{m}\frac{\kappa_p^2}{\alpha_p-\lambda}\right)\prod_{q=1}^{m}(\alpha_q-\lambda) \\
&= \prod_{q=1}^{m}(\alpha_q-\lambda)+\sum_{p=1}^{m}\kappa_p^2\prod_{\substack{q=1,\\ q\neq p}}^{m}(\alpha_q-\lambda) =: \cdots + K\lambda + |\Sigma|,
\end{aligned}
$$

where the coefficient of λ is given by

$$
K := -\sum_{p=1}^{m}\prod_{\substack{q=1,\\ q\neq p}}^{m}\alpha_q - \sum_{p=1}^{m}\sum_{\substack{r=1,\\ r\neq p}}^{m}\kappa_p^2\prod_{\substack{q=1,\\ q\neq p, q\neq r}}^{m}\alpha_q.
$$

We finally obtain

$$\sum_{p=1}^{m}\frac{1}{\lambda_i} = -\frac{K}{|\Sigma|} = \frac{\displaystyle\sum_{p=1}^{m}\prod_{\substack{q=1,\\ q\neq p}}^{m}\alpha_q + \sum_{p=1}^{m}\sum_{\substack{r=1,\\ r\neq p}}^{m}\kappa_p^2\prod_{\substack{q=1,\\ q\neq p, q\neq r}}^{m}\alpha_q}{\displaystyle\prod_{q=1}^{m}\alpha_q + \sum_{p=1}^{m}\kappa_p^2\prod_{\substack{q=1,\\ q\neq p}}^{m}\alpha_q}$$

$$= \frac{\displaystyle\sum_{p=1}^{m}\frac{1}{\alpha_p} + \sum_{p=1}^{m}\sum_{\substack{r=1,\\ r\neq p}}^{m}\frac{\kappa_p^2}{\alpha_p\alpha_r}}{\displaystyle 1+\sum_{p=1}^{m}\frac{\kappa_p^2}{\alpha_p}}.$$

7.4 Derivation of (24)

On the one hand we have,

$$\begin{aligned}\int_0^{\infty}\mathcal{O}_j(v)e^{\mathrm{i}vz}dv &= \int_0^{\infty} e^{\mathrm{i}vz}E_{j+1}(e^{X_j(T_j)}-e^v)^+dv \qquad (53)\\ &= \int_0^{\infty} e^{\mathrm{i}vz}\int_v^{\infty} P_{j+1}(X_j(T_j)\in dx)(e^x-e^v)dv\\ &= \int_0^{\infty} P_{j+1}(X_j(T_j)\in dx)\int_0^{x} e^{\mathrm{i}vz}(e^x-e^v)dv\\ &= \int_0^{\infty} P_{j+1}(X_j(T_j)\in dx)\left[e^{(\mathrm{i}z+1)x}\left(\frac{1}{\mathrm{i}z}-\frac{1}{\mathrm{i}z+1}\right)+\frac{1}{\mathrm{i}z+1}-\frac{e^{\mathrm{i}xz}}{\mathrm{i}z}\right]\end{aligned}$$

and on the other hand,

$$\begin{aligned}\int_{-\infty}^{0}\mathcal{O}_j(v)e^{\mathrm{i}vz}dv &= \int_{-\infty}^{0} e^{\mathrm{i}vz}E_{j+1}(e^v-e^{X_{T_j}})^+dv \qquad (54)\\ &= \int_{-\infty}^{0} e^{\mathrm{i}vz}dv\int_{-\infty}^{v} P_{j+1}(X_j(T_j)\in dx)(e^v-e^x)\\ &= \int_{-\infty}^{0} P_{j+1}(X_j(T_j)\in dx)\int_x^{0}(e^v-e^x)e^{\mathrm{i}vz}dz\\ &= \int_{-\infty}^{0} P_{j+1}(X_j(T_j)\in dx)\left(e^{(\mathrm{i}z+1)x}\left(\frac{1}{\mathrm{i}z}-\frac{1}{\mathrm{i}z+1}\right)+\frac{1}{\mathrm{i}z+1}-\frac{e^x}{\mathrm{i}z}\right)\end{aligned}$$

Note that the characteristic function $\Phi_{j+1}(z;T_j)$ of $X_j(T_j)$ exist in the strip $\{z = x+\mathrm{i}y\in\mathbb{C}: -\infty<x<\infty, -1\le y\le 0\}$ since $E_{j+1}L_j(T_j)=L_j(0)$ exists. Hence, by combining (53), (54), and using the martingale property of $X_j(T_j)$ again, we obtain (24).

7.5 Characteristic properties of $\mathcal{O}_j$

By denoting the density of $L_j(T_j)$ with $\rho_{L_j(T_j)}$ we may write

$$\begin{aligned}
C_j(K) &= B_{j+1}(0)E_{j+1}[(L_j(T_j)-K)^+\delta_j] \\
&= B_{j+1}(0)\delta_j\int_K^\infty (y-K)\rho_{L_j(T_j)}(y)dy,
\end{aligned}$$

and then by differentiating two times with respect to K we obtain

$$C_j''(K) = B_{j+1}(0)\delta_j\rho_{L_j(T_j)}(K).$$

The density of $X_j := \ln L_j(T_j) - \ln L_j(0)$ is obviously given by $\rho_{X_j}(v) := \rho_{L_j(T_j)}(L_j(0)e^v)L_j(0)e^v$, so

$$\begin{aligned}
\rho_{X_j}(v) &= B_{j+1}^{-1}(0)\delta_j^{-1}C_j''(L_j(0)e^v)L_j(0)e^v \\
&= B_{j+1}^{-1}(0)\delta_j^{-1}L_j^{-1}(0)\left(\mathcal{C}_j''(v)-\mathcal{C}_j'(v)\right)e^{-v} \\
&= \left(\mathcal{O}_j''(v)-\mathcal{O}_j'(v)\right)e^{-v},\ v\neq 0,
\end{aligned}$$

where $\mathcal{O}_j''-\mathcal{O}_j'$ extends continuously at $v=0$. In particular, $\mathcal{O}_j$ satisfies

$$\mathcal{O}_j''(v)-\mathcal{O}_j'(v)>0 \quad\text{and}\quad \mathcal{O}'(0-)-\mathcal{O}'(0+)=1. \tag{55}$$

On the grid v_j, $-N-1\leq j\leq N+1$ we consider a continuous piecewise linear approximation $\widetilde{\mathcal{O}}_{n-1}$ of $\mathcal{O}_{n-1}$,

$$\widetilde{\mathcal{O}}_{n-1}(v) :=$$
$$\sum_{j=-N}^{N+1}\frac{1}{v_j-v_{j-1}}(O_{n-1,j-1}v_j - v_{j-1}O_{n-1,j} + v(O_{n-1,j}-O_{n-1,j-1}))1_{[v_{j-1},v_j)}(v)$$

with v_j and $O_{n-1,j-1}$ given by (27) and (28), extended with $O_{n-1,-N-1}=O_{N+1,n-1}=0$ (note that $v_0:=0$). Then it follows that (with suppressed subscript $n-1$)

$$\frac{d}{dv}^{distr}\widetilde{\mathcal{O}}(v) = \sum_{j=-N}^{N+1}\frac{O_j-O_{j-1}}{v_j-v_{j-1}}1_{[v_{j-1},v_j)}(v) \tag{56}$$

in (Schwartz) distribution sense. Differentiating in distribution again yields

$$\begin{aligned}
\frac{d^2}{dv^2}^{distr}\widetilde{\mathcal{O}}(v) &= \frac{O_{-N}}{v_{-N}-v_{-N-1}}\delta_{v_{-N-1}} + \frac{O_N}{v_{N+1}-v_N}\delta_{v_{N+1}} \\
&+ \sum_{j=-N}^{N}\left(\frac{O_{j+1}-O_j}{v_{j+1}-v_j} - \frac{O_j-O_{j-1}}{v_j-v_{j-1}}\right)\delta_{v_j}.
\end{aligned} \tag{57}$$

Because $\mathcal{O}$ satisfies

$$\mathcal{O}''(v) - \mathcal{O}'(v) = \frac{d^2}{dv^2}^{distr} \mathcal{O} - \frac{d}{dv}^{distr} \mathcal{O}, \quad v \neq 0,$$

we consider for $v \neq 0$,

$$\begin{aligned}\left(\frac{d^2}{dv^2}^{distr} \widetilde{\mathcal{O}} - \frac{d}{dv}^{distr} \widetilde{\mathcal{O}} \right) e^{-v} = &-\frac{O_1 - O_0}{v_1} 1_{[0,v_1)}(v) e^{-v} + \frac{O_N}{v_{N+1} - v_N} \delta_{v_{N+1}} e^{-v_{N+1}} \\ &\frac{O_{-N}}{v_{-N} - v_{-N-1}} \delta_{v_{-N-1}} e^{-v_{-N-1}} - \frac{O_{-N}}{v_{-N} - v_{-N-1}} 1_{[v_{-N-1}, v_{-N})}(v) e^{-v} \\ &+ \sum_{\substack{j=-N \\ j \neq 0}}^{N} \left[\left(\frac{O_{j+1} - O_j}{v_{j+1} - v_j} - \frac{O_j - O_{j-1}}{v_j - v_{j-1}} \right) \delta_{v_j} e^{-v_j} - \frac{O_{j+1} - O_j}{v_{j+1} - v_j} 1_{[v_j, v_{j+1})}(v) e^{-v} \right],\end{aligned} \tag{58}$$

which follows from (56) and (57) and some rearranging of terms. Since the generalized function (58) should be an approximation of the density $\rho_{X_{n-1}}$, integrals over each interval $[v_{j-1}, v_j)$, $j = -N, ..N + 1$, should be non-negative. This leads to

$$\begin{aligned} 0 \leq &\left(\frac{O_{j+1} - O_j}{v_{j+1} - v_j} - \frac{O_j - O_{j-1}}{v_j - v_{j-1}} \right) e^{-v_j} - \frac{O_{j+1} - O_j}{v_{j+1} - v_j} \int 1_{[v_j, v_{j+1})}(v) e^{-v} dv \\ = &\frac{O_{j+1} - O_j}{v_{j+1} - v_j} e^{-v_{j+1}} - \frac{O_j - O_{j-1}}{v_j - v_{j-1}} e^{-v_j}, \quad j = -N, ..., -N, \quad j \neq 0. \end{aligned} \tag{59}$$

Note that (59) holds if the input data are consistent with a function $\mathcal{O}$ which is convex on both $v < 0$ and $v > 0$, and if the grid v_j is fine enough. Further, the total mass of (58) should be one. This leads straightforwardly to the requirement,

$$\frac{O_0 - O_{-1}}{-v_{-1}} - \frac{O_1 - O_0}{v_1} = 1,$$

which is a discretisation of the boundary condition (55) in fact.

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