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No-arbitrage pricing beyond semimartingales

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Abstract

We show how no-arbitrage pricing can be extended to some non-semimartingale models by restricting the class of admissible strategies. However, this restricted class is big enough to cover hedges for relevant options. Moreover, we show that the hedging prices depend essentially only on a path property of the stock price process, viz. on the quadratic variation. As a consequence, we can incorporate many stylized facts to a pricing model without changing the option prices.

1 Introduction

The fundamental theorem of asset pricing states that a notion of absence of arbitrage, namely the property of 'no free lunch with vanishing risk', is equivalent to the existence of an equivalent local martingale measure. See Delbaen and Schachermayer (1994) for the general version of this theorem for markets with continuous trajectories. Since semimartingales are stable under equivalent change of measure, non-semimartingale models have been ruled out for use in mathematical finance by means of this theorem. While the fundamental theorem connects an important economic concept (the absence of arbitrage) with an important mathematical one (the martingale property) in an impressive way, the validity of the fundamental theorem crucially depends on the appropriate choice of 'admissible' strategies. Indeed, the usual class of admissible strategies is – in some sense – as big as possible from a mathematical point of view. Apart from being self-financing and a condition excluding doubling strategies, any predictable and integrable (w.r.t. the stock process S) process is an admissible strategy.

In this paper we are concerned with a smaller class of strategies, but, at the same time, go beyond semimartingale models. As strategies we consider functions that depend on time, the spot of the stock, and a finite number of factors which include the running maximum, the running minimum, and the running average of the stock in a deterministic and (piecewise) smooth way. We hence allow path-depending strategies, but restrict the kind of path dependence. We believe that, apart from the idealization of continuous readjustment of the portfolio, this class of portfolios covers many economically relevant strategies. Concerning the models we group them according to their local quadratic variation. Given a function $\sigma(x)$ we consider continuous processes S_t as discounted models with pathwise quadratic variation given by $\mathrm{d}\langle S\rangle_t = \sigma(S_t)\mathrm{d}t$ and satisfying a certain small ball condition (in dependence of σ). We shall show under rather weak conditions on σ that such models are free of arbitrage with the 'smooth' strategies described above. In particular this no-arbitrage result covers the mixed fractional Black-Scholes model, our prime example throughout the paper (see Mishura and Valkeila (2002), Androshchuk and Mishura (2005), and Zähle (2002) for related results). We hence contribute to the arbitrage discussion related to fractional Brownian motion which has gained considerable interest in recent years (see Guasoni (2006) for some discussion on arbitrage results for pure fractional models).

Moreover, we discuss the robustness of hedges for models with the same local quadratic variation as functionals of the discounted stock price. In this respect we extend the results of Schoenmakers and Kloeden (1999) for European options in constant volatility models in several ways: Our results cover stochastic volatility models and the construction of robust hedges via PDEs is extended to exotic options such as Asian options and lookback options. Indeed robustness of hedges for general continuous payoff functionals is shown, even if no PDE is related to the hedge.

As a consequence of the no-arbitrage result and the robustness of hedges, no-arbitrage pricing can be extended beyond semimartingales. In our framework prices are model independent given the local quadratic variation function σ. We emphasize that the quadratic variation can be viewed as a path property and, hence, prices are basically independent of probabilistic properties. As an example we demonstrate how stylized facts such as dependent increments and heavy tails can be incorporated into a standard Black-Scholes model without changing the prices. For a survey on stylized facts we refer the reader to Cont (2001).

The paper is organized as follows: In Section 2 we review some facts on forward integration and pathwise quadratic variation. Model classes in dependence of the local quadratic variation are introduced in Section 3. In Section 4 we present the no-arbitrage result while robust replication is studied in Section 5. In Section 6 we discuss the possibility of an approximative arbitrage in the context of the mixed fractional Black-Scholes model. Section 7 concludes with a discussion on how to incorporate stylized facts and on different interpretations of volatility.

2 Simple review of forward integration

We consider processes that are not semimartingales. So, the classical stochastic integration theory is not at our disposal. However, there is an economically meaningful notion of integral, viz. the *forward integral*, that can be applied for non-semimartingales. Actually, there are many slightly different versions of the forward integral. In this paper we use a simplistic approach introduced by Föllmer (1981). For different kinds of forward integrals we refer to Lin (1995), Russo and Vallois (1993) and Zähle (2002).

Let $T > 0$ be fixed throughout the paper and let $\pi_n = \{0 = t_{n,0} < \cdots < t_{n,K(n)} = T\}$ be such partitions of $[0, T]$ that

$$\mathrm{mesh}(\pi_n) := \max_{t_{n,k}\in\pi_n} |t_{n,k} - t_{n,k-1}| \quad \to \quad 0$$

as $n \to \infty$. Suppose $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ is a filtered probability space satisfying the usual conditions of completeness and right continuity of the filtration $(\mathcal{F}_t)_{t\in[0,T]}$.

Later we cannot assume that our processes are properly integrable over the entire interval $[0, T]$. Thus we define the integrals over the subintervals $[0, t]$, $t < T$. The integral over the interval $[0, T]$ will be then interpreted as an improper forward integral.

Definition 2.1 *Let $t < T$ and let $X = (X_s)_{s\in[0,T]}$ be a continuous process. The* forward integral *of a process $Y = (Y_s)_{s\in[0,T]}$ with respect to X* along the sequence *$(\pi_n)_{n=1}^{\infty}$ is*

$$\int_0^t Y_s \,\mathrm{d}X_s \quad := \quad \lim_{n\to\infty} \sum_{\substack{t_{n,k}\in\pi_n \\ t_{n,k}\le t}} Y_{t_{n,k-1}} \left(X_{t_{n,k}} - X_{t_{n,k-1}}\right),$$

where the limit is assumed to exist $\mathbf{P}$-a.s. The forward integral over the whole interval $[0, T]$ is the improper forward integral

$$\int_0^T Y_s \,\mathrm{d}X_s \quad := \quad \lim_{t\uparrow T} \int_0^t Y_s \,\mathrm{d}X_s,$$

where the limit is again understood in the $\mathbf{P}$-a.s. sense.

Definition 2.2 *A process $X = (X_t)_{t\in[0,T]}$ is a* quadratic variation process along the sequence *$(\pi_n)_{n=1}^{\infty}$ if for all $t \le T$ the limit*

$$\langle X\rangle_t \quad := \quad \sum_{\substack{t_{n,k}\in\pi_n \\ t_{n,k}\le t}} \left(X_{t_{n,k}} - X_{t_{n,k-1}}\right)^2$$

exists $\mathbf{P}$-a.s., and is continuous in t.

Example 2.3 *(i) For standard Brownian motion W we have $\mathrm{d}\langle W\rangle_t = \mathrm{d}t$ if the sequence (π_n) is refining. This follows from the Borel-Cantelli lemma.*

(ii) If Z is a continuous process with zero quadratic variation along (π_n) and X is a continuous quadratic variation process along (π_n) then $\mathrm{d}\langle X+Z\rangle_t = \mathrm{d}\langle X\rangle_t$. This follows from the Cauchy-Schwartz inequality.

(iii) If X is a quadratic variation process along (π_n) and $f \in \mathcal{C}^1(\mathbb{R})$ then $Y = f \circ X$ is also a quadratic variation process along (π_n). Indeed,

$$\mathrm{d}\langle Y\rangle_t = f'(X_t)\,\mathrm{d}\langle X\rangle_t$$

(cf. Föllmer (1981) p. 148).

In what follows the sequence (π_n) will be fixed and omitted in the text.

The following Itô formula for the forward integral is a simple generalization of the theorem that can be found in Föllmer (1981) p. 144. The proof is based on a second order multidimensional Taylor expansion. Actually, it is basically the same as in the semimartingale case.

Lemma 2.4 (Itô formula) *Let X be a continuous quadratic variation process, $Y^1, \ldots, Y^m$ continuous bounded variation processes and suppose $f \in \mathcal{C}^{1,2,1}([0,T) \times \mathbb{R} \times \mathbb{R}^m)$. Let $0 \le s \le t < T$. Then*

$$\begin{aligned}
f(t, X_t, Y_t^1, \ldots, Y_t^m) \;=\;& f(s, X_s, Y_s^1, \ldots, Y_s^m) \\
&+ \int_s^t \frac{\partial}{\partial t} f(u, X_u, Y_u^1, \ldots, Y_u^m)\,\mathrm{d}u \\
&+ \int_s^t \frac{\partial}{\partial x} f(u, X_u, Y_u^1, \ldots, Y_u^m)\,\mathrm{d}X_u \\
&+ \frac{1}{2}\int_s^t \frac{\partial^2}{\partial x^2} f(u, X_u, Y_u^1, \ldots, Y_u^m)\,\mathrm{d}\langle X\rangle_u \\
&+ \sum_{n=1}^m \int_s^t \frac{\partial}{\partial y_n} f(u, X_u, Y_u^1, \ldots, Y_u^m)\,\mathrm{d}Y_u^n.
\end{aligned}$$

In particular, this formula implies the forward integral on the right hand side exists and has a continuous modification.

Remark 2.5 *In the remainder of the paper we choose continuous modifications of forward integrals, whenever possible.*

Remark 2.6 *The forward integral with non-semimartingale integrator does not satisfy a dominated convergence theorem. Therefore, we have to impose some continuity assumptions on the integrands (cf. Definition 4.1 of hindsight factors in Section 4). The lack of dominated convergence theorem may cause some sort of approximative arbitrage, see Section 6.*

3 Model classes

We now introduce model classes in dependence of the quadratic variation. A discounted *market model* is a five-tuple $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$ such that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ is a filtered probability space satisfying the usual conditions and $S = (S_t)_{t\in[0,T]}$ is an $(\mathcal{F}_t)$-progressively measurable quadratic variation process with continuous paths starting at $s_0 > 0$.

Suppose a continuously differentiable function $\sigma : \mathbb{R} \to \mathbb{R}$ with linear growth is given. The corresponding model class $\mathcal{M}_\sigma$ will be defined via the quadratic variation property

$$\mathrm{d}\langle S\rangle_t \;=\; \sigma^2(S_t)\,\mathrm{d}t \qquad \mathbf{P}-\text{a.s.} \tag{1}$$

and a non-degeneracy property. In order to formulate the latter property let f_σ denote the unique solution of the ordinary differential equation

$$f'(x) \;=\; \sigma(f(x)), \quad f(0) = s_0.$$

Since σ is continuously differentiable, f_σ belongs to $\mathcal{C}^2(\mathbb{R})$ and

$$f''_\sigma(x) \;=\; f'_\sigma(x)\sigma'(f_\sigma(x)). \tag{2}$$

Define the space

$$\mathcal{C}_{\sigma,s_0} \;:=\; \{f_\sigma \circ \eta \;;\; \eta \in \mathcal{C}([0,T]),\ \eta(0) = 0\}\,.$$

We assume that the following small ball condition is satisfied: Given $\eta \in \mathcal{C}_{\sigma,s_0}$ and $\varepsilon > 0$

$$\mathbf{P}\Big(\|S-\eta\|_\infty < \varepsilon\Big) \;>\; 0, \tag{3}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on the interval $[0,T]$. Summarizing the foregoing, the model class $\mathcal{M}_\sigma$ is defined to contain those discounted market models which satisfy (1) and (3).

We illustrate this definition by an example.

Example 3.1 *Suppose $\sigma(x) = \sigma x$ for some constant $\sigma > 0$. Obviously,*

$$f_\sigma(x) \;=\; s_0 e^{\sigma x}.$$

Hence, condition (3) means that the support of the stochastic process S is the space of nonnegative continuous functions starting in s_0. In particular, the standard Black-Scholes model belongs to this class $\mathcal{M}_\sigma$. We will consider its risk-neutral version (i.e. with zero drift) as reference model in $\mathcal{M}_\sigma$, because of its martingale property. A prominent non-semimartingale model in the class $\mathcal{M}_\sigma$ is the mixed fractional Black-Scholes model, where the Brownian motion of the standard Black-Scholes model is replaced by a sum of a Brownian motion and an independent fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, \frac{3}{4}]$. Recall that a fractional Brownian motion Z is a centred stationary increment Gaussian process with variance $\mathbf{E}(Z_t^2) = t^{2H}$ for some $H \in (0,1)$, and for $H \in (\frac{1}{2}, 1)$ it has zero quadratic variation. Another way of characterizing the fractional Brownian motion is to say that it is the unique (up to a multiplicative constant) centred H-self-similar Gaussian process with stationary increments. It is known from Cheridito (2001) that the sum of independent Brownian and fractional Brownian motion is a semimartingale if and only if $H \in (\frac{3}{4}, 1)$.

Next we construct a reference model which plays the role of the risk-neutral Black-Scholes model for general $\mathcal{M}_\sigma$. Suppose $(\tilde{\Omega}, \tilde{\mathcal{F}}, \bar{\mathbf{P}})$ is the canonical Wiener space on the time interval $[0,T]$, $W_t(\omega) = \omega(t)$ the coordinate process Brownian motion and $\tilde{\mathcal{F}}_t$ the augmented filtration generated by W. We impose the following standing assumption:

(H) The process

$$M_t = \exp\left\{-\frac{1}{2}\int_0^t \sigma'(f_\sigma(W_r))\mathrm{d}W_r - \frac{1}{8}\int_0^t \left(\sigma'(f_\sigma(W_r))\right)^2 \mathrm{d}r\right\}$$

is well defined (i.e. the integrals exist) and is a martingale under $\bar{\mathbf{P}}$.

Under hypothesis **(H)** we can define a probability measure $\tilde{\mathbf{P}}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}_T)$ by

$$\tilde{\mathbf{P}}(A) \;=\; \int_A M_t \,\mathrm{d}\bar{\mathbf{P}}, \quad A \in \tilde{\mathcal{F}}_T.$$

Then $(\tilde{W}_t)_{t\in[0,T]}$ given by

$$\tilde{W}_t \;=\; W_t + \frac{1}{2}\int_0^t \sigma'(f_\sigma(W_r))\,\mathrm{d}r$$

is a Brownian motion under $\tilde{\mathbf{P}}$ by the Girsanov theorem.

Define a discounted stock price by $\tilde{S}_t = f_\sigma(W_t)$. We obtain:

Lemma 3.2 *$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ and $\tilde{S}$ is a local martingale. We call $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}})$ the* reference model.

Proof. The small ball property is trivially satisfied under $\bar{\mathbf{P}}$ and hence under $\tilde{\mathbf{P}}$. Moreover, Itô's formula and (2) yield,

$$\begin{aligned} \tilde{S}_t &= s_0 + \int_0^t f'_\sigma(W_r)\,\mathrm{d}W_r + \frac{1}{2}\int_0^t f'_\sigma(W_r)\sigma'(f_\sigma(W_r))\,\mathrm{d}r \\ &= s_0 + \int_0^t f'_\sigma(W_r)\,\mathrm{d}\tilde{W}_r \\ &= s_0 + \int_0^t \sigma(\tilde{S}_r)\,\mathrm{d}\tilde{W}_r. \end{aligned}$$

Thus, (1) is satisfied and $\tilde{S}$ is a local martingale under $\tilde{\mathbf{P}}$. ∎

We now give an important example where condition **(H)** is satisfied.

Example 3.3 *Suppose $\sigma(x) = x\tilde{\sigma}(x)$ with $\tilde{\sigma} \in \mathcal{C}^1(\mathbb{R})$ bounded and $x\tilde{\sigma}'(x)$ bounded. Both boundedness conditions are met, when $\tilde{\sigma}$ is constant for $|x|$ sufficiently large. Then $\sigma'(x) = \tilde{\sigma}(x) + x\tilde{\sigma}'(x)$ is bounded and consequently* **(H)** *follows from Novikov's condition. In this situation the reference model is a risk neutral generalized Black-Scholes model with stochastic volatility depending on the spot,*

$$\tilde{S}_t = s_0 + \int_0^t \tilde{\sigma}(\tilde{S}_r)\tilde{S}_r\,\mathrm{d}\tilde{W}_r.$$

These so-called local volatility models were suggested by Dupire (1994) in order to capture the implied volatility smile.

We conclude this section with an example on how to construct further models in $\mathcal{M}_\sigma$.

Example 3.4 *Suppose W is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ and Y is a continuous process with zero quadratic variation independent of W which satisfies the small ball condition*

$$\mathbf{P}\Big(\|Y\|_\infty < \varepsilon\Big) > 0.$$

For instance, Y could be a fractional Brownian motion with Hurst parameter bigger than a half. Define $S_t = f_\sigma(W_t + Y_t)$. Then $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$ belongs to $\mathcal{M}_\sigma$. Indeed, the quadratic variation of S is easily calculated by Example 2.3 and from the small ball property of Y around zero and the independence one obtains (3). Observe that by Itô's formula

$$S_t = s_0 + \int_0^t \sigma(S_r)\,\mathrm{d}(W+Y)_r + \frac{1}{2}\int_0^t f'_\sigma(W_r + Y_r)\sigma'(f_\sigma(W_r + Y_r))\,\mathrm{d}r.$$

More general drifts can be introduced by performing a Girsanov change of measure on the Brownian motion only. (Note, the law of Y remains unchanged by the Girsanov transformation due to independence.)

Remark 3.5 *The introduction of time dependent local quadratic variation functions $\sigma(t,x)$ does not cause any difficulties, but makes the presentation more cumbersome.*

4 A no-arbitrage result

We shall now derive a no-arbitrage result for strategies that depend in a deterministic and smooth way on time, the spot price S_t and some additional economically relevant factors such as e.g. the running maximum, minimum, and average of the stock.

We first specify some assumptions on the additional economic factors on which the strategy may depend.

Definition 4.1 *A mapping $g : [0,T] \times \mathcal{C}_{\sigma,s_0} \to \mathbb{R}$ is a* hindsight factor*, if*

(i) $g(t,\eta) = g(t,\tilde{\eta})$ whenever $\eta(s) = \tilde{\eta}(s)$ for all $0 \le s \le t$,

(ii) $g(\cdot;\eta)$ is of bounded variation and continuous for every $\eta \in \mathcal{C}_{\sigma,s_0}$,

(iii) there is a constant K such that for every continuous function f

$$\left| \int_0^t f(s)\mathrm{d}g(s,\eta) - \int_0^t f(s)\mathrm{d}g(s,\tilde{\eta}) \right| \le K \max_{0\le r\le t} |f(r)| \cdot \|\eta - \tilde{\eta}\|_\infty. \qquad (4)$$

Property (i) is the natural assumption that the factors must not contain information about the future stock prices. Properties (ii)–(iii) are technical assumptions which we need since the forward integral is not continuous in terms of the integrands.

The running maximum, minimum, and average are denoted, respectively,

$$\begin{aligned} \eta^*(t) &:= \max_{s\in[0,t]} \eta(s), \\ \eta_*(t) &:= \min_{s\in[0,t]} \eta(s), \\ \bar{\eta}(t) &:= \int_0^t \eta(s)\,\mathrm{d}s. \end{aligned}$$

(We do not include the factor $1/t$ in the running average. This is just a matter of convenience since one can always include the factor $1/t$ in the 'strategy function' φ in (7).)

Proposition 4.2 *The running maximum, minimum, and average are hindsight factors.*

Proof. Properties (i) and (ii) of Definition 4.1 are obviously satisfied for the running maximum, minimum, and average. Moreover, property (iii) is trivial for the running average. We now prove a somewhat stronger assertion than (iii) for the running maximum. Suppose f, g, $\tilde{g}$ are continuous functions on $[0,t]$ and define

$$\mathcal{I}(t; f, g) = \int_0^t f(s)\, \mathrm{d}g^*(s),$$

where $g^*(s) = \max_{u\in[0,s]} g(u)$. We shall show that

$$|\mathcal{I}(t; f, g) - \mathcal{I}(t; f, \tilde{g})| \;\leq\; 4 \max_{r\in[0,t]} |f(r)| \max_{r\in[0,t]} |g(r) - \tilde{g}(r)|. \tag{5}$$

We first consider the case of non-negative f. Since

$$\mathrm{d}(g^* + \tilde{g}^*) \;\geq\; \mathrm{d}(g + \tilde{g})^*,$$

we obtain for non-negative f the sub-additivity of $\mathcal{I}(t; f, \cdot)$:

$$\mathcal{I}(t; f, g + \tilde{g}) \;\leq\; \mathcal{I}(t; f, g) + \mathcal{I}(t; f, \tilde{g}).$$

Hence

$$\mathcal{I}(t; f, g) - \mathcal{I}(t; f, \tilde{g}) \;\leq\; \mathcal{I}(t; f, g - \tilde{g}). \tag{6}$$

Using the Love-Young inequality (with sup-norm and total variation norm) and the fact that the total variation of the running maximum is dominated by two times the running maximum, we have for general f

$$\begin{aligned} |\mathcal{I}(t; f, g)| &\leq \max_{r\in[0,t]} |f(r)|\, \mathsf{TV}_{[0,t]}(g^*) \\ &\leq 2 \max_{r\in[0,t]} |f(r)| \max_{r\in[0,t]} |g(r)|. \end{aligned}$$

Combining this with (6) the inequality (5) follows for non-negative f with constant 2 instead of 4. From (6) and noting $\mathcal{I}(t; f, g) = -\mathcal{I}(t; -f, g)$ we get for non-positive f

$$\mathcal{I}(t; f, g) - \mathcal{I}(t; f, \tilde{g}) \;\leq\; \mathcal{I}(t; -f, \tilde{g} - g),$$

which yields (5) for non-positive f with constant 2 instead of 4. The general case with constant 4 follows now from the linearity of $\mathcal{I}(t; \cdot, g)$ and the triangle inequality.

The analogous inequality of (5) for the running minimum can be straightforwardly reduced to the case of the running maximum, since

$$\int_0^t f(s)\, \mathrm{d}g_*(s) \;=\; -\int_0^t f(s)\, \mathrm{d}(-g)^*(s).$$

■

Suppose hindsight factors $g_1, \dots, g_m$ and a function $\varphi : [0,T] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is given. We consider strategies of the form

$$\Phi_t \quad = \quad \varphi\Big(t, S_t, g_1(t,S), \dots, g_m(t,S)\Big). \tag{7}$$

Here Φ_t denotes the number of stocks held at time t by an investor. Hence, the wealth process corresponding to the strategy Φ is

$$V_t(\Phi, v_0; S) \quad = \quad v_0 + \int_0^t \Phi_u \,\mathrm{d}S_u, \tag{8}$$

where $v_0 \in \mathbb{R}$ denotes the investor's initial capital. (Recall, the stochastic integral is defined as a limit of forward sums. Thus, this definition reflects the classical and economically meaningful condition for a self-financing portfolio.)

We now prove a result on absence of arbitrage under the smoothness condition $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R} \times \mathbb{R}^m)$[1]. Recall that a strategy Φ is an *arbitrage* in the market model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$, if

$$V_T(\Phi, 0; S) \;\geq\; 0 \quad \mathbf{P}-\text{a.s.} \qquad \text{and} \qquad \mathbf{P}(V_T(\Phi,0;S) > 0) \;>\; 0.$$

Theorem 4.3 *Suppose the standing assumption* **(H)** *holds. Let* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *and suppose* Φ *is of form (7) with* $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R} \times \mathbb{R}^m)$. *Moreover, assume that* Φ *is nds-admissible*[2] *in the classical sense, i.e. there is a constant* $a > 0$ *such that for all* $t \in [0,T]$

$$\int_0^t \Phi_u \,\mathrm{d}S_u \;\geq\; -a \quad \mathbf{P}-\text{a.s.}$$

Then Φ *cannot be an arbitrage in the model* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$.

To prepare the proof we define an auxiliary wealth functional for $\tau \in [0,T]$ by

$$v : [0,\tau] \times \mathcal{C}_{\sigma,s_0} \times \mathcal{C}^1([0,\tau] \times \mathbb{R} \times \mathbb{R}^m) \to \mathbb{R}$$

as the Itô formula suggests:

$$\begin{aligned} v(t,\eta;\varphi) \quad := \quad & u(t, \eta(t), g_1(t;\eta), \dots, g_m(t;\eta)) \\ & - \sum_{n=1}^m \int_0^t \frac{\partial}{\partial y_n} u(r, \eta(r), g_1(r;\eta), \dots, g_m(r;\eta)) \,\mathrm{d}g_n(r;\eta) \\ & - \int_0^t \frac{\partial}{\partial t} u(r, \eta(r), g_1(r;\eta), \dots, g_m(r;\eta)) \,\mathrm{d}r \\ & - \frac{1}{2} \int_0^t \frac{\partial}{\partial x} \varphi(r, \eta(r), g_1(r;\eta), \dots, g_m(r;\eta)) \sigma^2(\eta(r)) \,\mathrm{d}r, \end{aligned} \tag{9}$$

[1]Of course, smoothness conditions can be relaxed to, say, $[0,T] \times \mathbb{R}_+ \times \mathbb{R}_+^m$ when one knows a priori, that the stock and the hindsight factors are positive.

[2]nds stands for no-doubling-strategies

where

$$u(t, x, y_1, \ldots, y_m) \quad = \quad \int_{s_0}^{x} \varphi(t, \xi, y_1, \ldots, y_m)\, \mathrm{d}\xi. \tag{10}$$

The next crucial lemma shows that v is a continuous wealth functional.

Lemma 4.4 *Let $0 \leq t \leq \tau \leq T$ and $\varphi \in \mathcal{C}^1([0, \tau] \times \mathbb{R}_+ \times \mathbb{R}^m)$. Suppose on $0 \leq t \leq \tau$ a strategy for $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ is given by $\Phi_t = \varphi(t, S_t, g_1(t, S), \ldots, g_m(t, S))$. Then we have, for $0 \leq t \leq \tau$,*

$$V_t(\Phi, v_0; S) = v_0 + v(t, S; \varphi) \quad \mathbf{P} - \text{a.s.}$$

Moreover, for all $0 \leq t \leq \tau$ the functional $v(t, \cdot; \varphi)$ is continuous in the supremum norm.

Proof. Applying Itô's formula to u given by (10) we obtain for $0 \leq t \leq \tau$ that

$$\int_0^t \Phi_u \mathrm{d}S_u = v(t, S; \varphi) \quad \mathbf{P} - \text{a.s.}$$

Hence,

$$V_t(\Phi, v_0; S) = v_0 + v(t, S; \varphi) \quad \mathbf{P} - \text{a.s.}$$

To prove continuity of $v(t, \cdot; \varphi)$ let $(\eta_n) \subset \mathcal{C}_{\sigma, s_0}$ be a sequence which converges to $\eta \in \mathcal{C}_{\sigma, s_0}$ in the sup-norm. Then η_n and η take values in a compact set $A_0 \subset \mathbb{R}$. Hence, due to (4) applied to $f = 1$, there is a compact set $A \subset \mathbb{R}$ such that for all $0 \leq j \leq m$, $n \in \mathbb{N}$ and $0 \leq t \leq T$ $g_j(t, \eta_n)$ and $g_j(t, \eta)$ take values in A. Moreover, applying (4) again to $f = 1$ we see that

$$|g_j(t, \eta_n) - g_j(t, \eta)| \leq K \|\eta_n - \eta\|_\infty. \tag{11}$$

Thus, by the continuity of φ, $(\partial/\partial x)\varphi$, and $(\partial/\partial t)u$, the dominated convergence theorem yields

$$\begin{aligned} & u(\Sigma(t, \eta_n)) - \int_0^t \frac{\partial}{\partial t} u(\Sigma(r, \eta_n))\, \mathrm{d}r - \frac{1}{2} \int_0^t \frac{\partial}{\partial x} \varphi(\Sigma(r, \eta_n)) \sigma^2(\eta_n(r))\, \mathrm{d}r \\ \to \quad & u(\Sigma(t, \eta)) - \int_0^t \frac{\partial}{\partial t} u(\Sigma(r, \eta))\, \mathrm{d}r - \frac{1}{2} \int_0^t \frac{\partial}{\partial x} \varphi(\Sigma(r, \eta)) \sigma^2(\eta(r))\, \mathrm{d}r \end{aligned}$$

where, for notational convenience,

$$\Sigma(t, \eta) = t, \eta(t), g_1(t; \eta), \ldots, g_m(t; \eta).$$

To prove convergence of the integrals with respect to the hindsight factors we decompose

$$\begin{aligned}
&\left|\int_0^t \frac{\partial}{\partial y_j} u(\Sigma(r,\eta_n))\,\mathrm{d}g_j(r;\eta_n) - \int_0^t \frac{\partial}{\partial y_j} u(\Sigma(r,\eta))\,\mathrm{d}g_j(r;\eta)\right| \\
\leq\ & \left|\int_0^t \frac{\partial}{\partial y_j} u(\Sigma(r,\eta))\,\mathrm{d}(g_j(r;\eta) - g_j(r;\eta_n))\right| \\
&+ \left|\int_0^t \left\{\frac{\partial}{\partial y_j} u(\Sigma(r,\eta_n)) - \frac{\partial}{\partial y_j} u(\Sigma(r,\eta))\right\}\mathrm{d}g_j(r;\eta_n)\right| = (I) + (II).
\end{aligned}$$

By (4),

$$(I) \leq K \max_{0\leq r\leq t}\left|\frac{\partial}{\partial y_j} u(\Sigma(r,\eta))\right| \|\eta_n - \eta\|_\infty \to 0.$$

Analogously,

$$(II) \leq K \max_{0\leq r\leq t}\left|\frac{\partial}{\partial y_j} u(\Sigma(r,\eta)) - \frac{\partial}{\partial y_j} u(\Sigma(r,\eta_n))\right| \|\eta_n\|_\infty.$$

By (11), given $\delta > 0$, there is an n_0 such that for all $n \geq n_0$, $0 \leq j \leq m$ and $0 \leq r \leq T$

$$|g_j(r,\eta_n) - g_j(r,\eta)| < \delta.$$

Exploiting the uniform continuity of $(\partial/\partial y_j)u$ on the compact set $[0,t] \times A_0 \times A^m$ we deduce

$$\max_{0\leq r\leq t}\left|\frac{\partial}{\partial y_j} u(\Sigma(r,\eta)) - \frac{\partial}{\partial y_j} u(\Sigma(r,\eta_n))\right| \to 0.$$

Since $\|\eta_n\|_\infty$ is bounded, $(II) \to 0$. ■

We now proceed with the proof of Theorem 4.3.

Proof of Theorem 4.3. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ be the reference model and $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ be some model. Consider $\Phi_t = \varphi(t, S_t, g_1(t,S), \ldots, g_m(t,S))$, with continuously differentiable φ as a strategy for the model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P})$. Further suppose the investor has initial capital zero and

$$V_T(\Phi, 0; S) \geq 0 \quad \mathbf{P} - \text{a.s.} \tag{12}$$

By Lemma 4.4

$$v(T, S; \varphi) = V_T(\Phi, 0; S) \quad \mathbf{P} - \text{a.s.}$$

By the small ball condition (3) and the continuity of $v(T, \cdot, \varphi)$ we see that the inequality (12) holds in the functional sense:

$$v(T, \eta; \varphi) \geq 0$$

for all $\eta \in \mathcal{C}_{\sigma,s_0}$. Indeed, otherwise $v(T,\cdot;\varphi)$ would be negative in some ball in $\mathcal{C}_{\sigma,s_0}$ by continuity. Since all balls have positive $\mathbf{P}$-measure by the small ball condition (3), the assumption (12) would be violated.

We hence obtain

$$v(T,\tilde{S};\varphi) \geq 0 \quad \tilde{\mathbf{P}} - \text{a.s.}$$

Analogously, the nds-admissibilty of Φ implies for all $t \geq 0$

$$v(t,\tilde{S};\varphi) \geq -a \quad \tilde{\mathbf{P}} - \text{a.s.}$$

Since $\tilde{P}$ itself is an equivalent local martingale measure for $\tilde{S}$, we may conclude from the classical no-arbitrage theory that

$$v(T,\tilde{S};\varphi) = 0 \quad \tilde{\mathbf{P}} - \text{a.s.}$$

Interchanging the roles of $\tilde{S}$ and S and applying the same argument as above yields

$$V_T(\Phi,0;S) = 0 \quad \mathbf{P} - \text{a.s.}$$

Hence, Φ is not an arbitrage. ■

Remark 4.5 *The most important ingredient for the proof of Theorem 4.3 is the existence of a continuous wealth functional $v(t,\cdot;\varphi)$. This property remains unchanged when φ is only piecewise smooth. Precisely suppose $0 = s_0 < s_1 < \cdots < s_J = T$ and $\varphi_j : [s_{j-1}, s_j] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ are continuously differentiable in the first $m+2$ variables and continuous in the last one. Then the no-arbitrage results holds true for strategies of the form*

$$\Phi_t = \sum_{j=1}^{J} \mathbf{1}_{(s_{j-1},s_j]}(t)\varphi_j\Big(t, S_t, g_1(t,S), \ldots, g_m(t,S), \xi_j(S)\Big),$$

where $\xi_j : \mathcal{C}_{\sigma,s_0} \to \mathbb{R}$ is continuous and $\xi_j(\eta)$ depends on the segment $\{\eta(r);\ 0 \leq r \leq s_{j-1}\}$ only. Note the introduction of the functionals ξ_j allows dependence of the strategy on the discretely sampled maximum, minimum, or average.

We also note that the nds-admissibilty can be relaxed to

$$\int_0^t \Phi_u \, \mathrm{d}S_u \geq -a(t,S_t) \quad \mathbf{P} - \text{a.s.}$$

where $a : [0,T] \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ is continuous and $\int_{\tilde{\Omega}} \sup_{t\in[0,T]} |a(t,\tilde{S}_t)| \mathrm{d}\tilde{\mathbf{P}} < \infty$.

5 Robust replication

In this section we discuss the robustness of hedges within a model class $\mathcal{M}_\sigma$. Our aim is to show that hedges for a large class of claims do not depend of the specific model in $\mathcal{M}_\sigma$ as a functional of the stock. In particular we obtain that the initial capital for such a hedge does not depend on the chosen model. In combination with the no-arbitrage result (Theorem 4.3) this means that the fair price of those contingent claims coincides for all models from $\mathcal{M}_\sigma$.

We first motivate a slight enhancement of the class of allowed strategies. In the standard Black-Scholes model the Black-Scholes PDE yields the hedge for a call option with strike K and maturity T as a function $\varphi(t,x)$ of time and spot. This function fails to be continuous at $t=T, x=K$. More generally, hedges which are obtained via PDEs often do not satisfy the smoothness condition at $t=T$. To overcome this difficulty we suggest to enlarge the class of allowed strategies in the following way: We relax the smoothness condition of φ to $\varphi \in \mathcal{C}^1([0,T)\times\mathbb{R}\times\mathbb{R}^m,\mathbb{R})$. The next lemma explains what happens at the terminal date $t=T$.

Lemma 5.1 *Suppose Φ is of the form (7) with $\varphi \in \mathcal{C}^1([0,T)\times\mathbb{R}_+\times\mathbb{R}^m)$. Then the following assertions are equivalent:*

(i) *For all $(\Omega,\mathcal{F},S,(\mathcal{F}_t),\mathbf{P}) \in \mathcal{M}_\sigma$ and all $0 \le t \le T$ the (improper) integral*

$$\int_0^t \Phi_u \,\mathrm{d}S_u$$

exists.

(ii) *There is a dense subset $D \subset \mathcal{C}_{\sigma,s_0}$ and a limiting wealth functional $F : D \to \mathbb{R}$ such that for all $(\Omega,\mathcal{F},S,(\mathcal{F}_t),\mathbf{P}) \in \mathcal{M}_\sigma$ we have $\mathbf{P}(S \in D) = 1$ and for all $\eta \in D$ we have*

$$\lim_{t\uparrow T} v(t,\eta;\varphi) = F(\eta).$$

Proof. Let $\varphi \in \mathcal{C}^1([0,T)\times\mathbb{R}_+\times\mathbb{R}^m)$. By Lemma 4.4 we have for every model $(\Omega,\mathcal{F},S,(\mathcal{F}_t),\mathbf{P}) \in \mathcal{M}_\sigma$ and all $0 \le t < T$ that

$$\int_0^t \varphi(r,S_r,g_1(r,S),\ldots,g_m(r,S))\,\mathrm{d}S_r = v(t,S;\varphi). \tag{13}$$

As we always choose continuous modifications of the forward integrals the above identity holds up to $\mathbf{P}$-indistinguishability. Hence, for every model $(\Omega,\mathcal{F},S,(\mathcal{F}_t),\mathbf{P}) \in \mathcal{M}_\sigma$ we may choose a set Ω^1 such that (13) holds on Ω^1 and $\mathbf{P}(\Omega^1)=1$. Note also, that by (13) assertion (i) is equivalent to the existence of the corresponding improper forward integrals at $t = T$.

Suppose now existence of a limiting functional, i.e. (ii). For a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ set $\Omega^2 = \Omega^1 \cap S^{-1}(D)$. Then Ω^2 has full $\mathbf{P}$-measure and for each $\omega \in \Omega^2$ we have

$$\lim_{t \uparrow T} \left(\int_0^t \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S)) \, \mathrm{d}S_r \right)(\omega) = \lim_{t \uparrow T} v(t, S(\omega); \varphi) = F(\omega).$$

This means that the integral $\int_0^T \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S)) \, \mathrm{d}S_r$ exists in the improper forward sense in the model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$, The claim (i) follows.

We now suppose existence of the forward integrals, i.e. (i), and construct a limiting functional. In view of (13), given a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ we find a set Ω^3 of full $\mathbf{P}$-measure such that on Ω^3

$$\int_0^T \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S)) \, \mathrm{d}S_r$$

exists and (13) holds. Define

$$D = \bigcup_{(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma} S(\Omega^3).$$

For $\eta \in D$ choose a model $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ such that $\eta = S(\omega)$ with $\omega \in \Omega^3$ and define

$$F(\eta) = \left(\int_0^T \varphi(r, S_r, g_1(r,S), \ldots, g_m(r,S)) \, \mathrm{d}S_r \right)(\omega).$$

Note, F is well-defined due to (13) and D is a dense set due to the small ball condition (3). ■

The previous lemma characterizes the minimal assumption of existence of $\int_0^\cdot \Phi_u \, \mathrm{d}S_u$ in terms of existence of a limiting functional of $v(t, \cdot; \varphi)$ as $t \uparrow T$. To define allowed strategies we will strengthen this minimal requirement by imposing a continuity assumption on the limiting functional.

Definition 5.2 *A strategy Φ is* allowed *for the model class $\mathcal{M}_\sigma$ if*

(A1) *there is a finite number of hindsight variables $g_1, \ldots, g_m$ and a function $\varphi \in \mathcal{C}^1([0,T) \times \mathbb{R} \times \mathbb{R}^m)$ such that*

$$\Phi_t = \varphi(t, S_t, g_1(t,S), \ldots, g_m(t,S)).$$

(A2) *There is a dense subset* $D \subset \mathcal{C}_{\sigma,s_0}$ *and a functional* $F : D \to \mathbb{R}$ *such that for all models* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *we have* $\mathbf{P}(S \in D) = 1$ *and for all* $\eta \in D$ *we have*

$$\lim_{t \uparrow T} v(t, \eta; \varphi) = F(\eta).$$

Moreover, we assume that F *is continuous in* D.

(A3) *There is a constant* $a > 0$ *such that for all* $0 \le t \le T$

$$\int_0^t \Phi_u \, \mathrm{d}S_u \ge -a \quad \mathbf{P} - \text{a.s.}$$

Recall that **(A3)** is the classical concept of nds-admissibility which is typically imposed to exclude doubling strategies. Also, if **(A3)** holds for one model then it holds for all models (given (**A1)** and (**A2)**), as was shown in the proof of Theorem 4.3. Also note that in **(A2)** we do not assume that F can be continuously extended to the whole space $\mathcal{C}_{\sigma,s_0}$.

Obviously (**A2)** holds, if $\varphi \in \mathcal{C}^1([0,T] \times \mathbb{R} \times \mathbb{R}^m)$. Moreover, Theorem 4.3 carries over to allowed strategies without any additional difficulties:

Theorem 5.3 *Suppose condition* **(H)**. *Then every model in* $\mathcal{M}_\sigma$ *is free of arbitrage with allowed strategies.*

The next theorem states that hedges are robust as functionals of the stock within the class $\mathcal{M}_\sigma$.

Theorem 5.4 *Suppose condition* **(H)** *holds and* G *is a continuous functional on* $\mathcal{C}_{\sigma,s_0}$ *such that* $G(\tilde{S})$ *is replicable* $\tilde{\mathbf{P}}$*-a.s. in the reference model* $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}}) \in \mathcal{M}_\sigma$ *with an allowed strategy*

$$\tilde{\Phi}^*_t = \varphi^*(t, \tilde{S}_t, g_1(t, \tilde{S}), \ldots, g_m(t, \tilde{S}))$$

and initial capital v_0. *Then* $G(S)$ *is replicable* $\mathbf{P}$*-a.s. in every model* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *with the same initial capital* v_0 *and the replicating allowed strategy is given by*

$$\Phi^*_t = \varphi^*(t, S_t, g_1(t, S), \ldots, g_m(t, S)),$$

i.e. replicating allowed strategies are, as functionals of the stock prices, independent of the model.

The converse also holds, i.e. any 'functional' hedge φ^* *in some model* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$ *is also a 'functional' hedge for the reference model.*

Proof. Given φ^* let F be the limiting wealth functional on a dense set D as in assumption **(A2)**. Note that by Lemma 4.4

$$v_0 + F(\tilde{S}) = V_T(\tilde{\Phi}^*, v_0; \tilde{S}) = G(\tilde{S}), \quad \mathbf{P}\text{-a.s.}$$

Hence,

$$v_0 + F(\tilde{S}(\tilde{\omega})) = G(\tilde{S}(\tilde{\omega}))$$

for all $\tilde{\omega}$ from a set $\tilde{\Omega}^1$ of full $\tilde{\mathbf{P}}$-measure such that $\tilde{S}(\tilde{\Omega}^1) \subset D$. Hence, $F = G - v_0$ on D by the continuity of G and **(A2)**. Again by Lemma 4.4, $V_T(\Phi^*, v_0; S) = v_0 + F(S) = G(S)$ $\mathbf{P}$-a.s.

For the converse one simply interchanges the roles of $\tilde{S}$ and S. ■

Remark 5.5 *From the previous theorem and the classical no-arbitrage theory we may derive the following result: With the notation from the previous theorem the initial capital v_0 satisfies the inequality*

$$v_0 \geq \mathbf{E}^{\tilde{\mathbf{P}}}[G(\tilde{S})].$$

Moreover identity holds, if and only if $v(t, \tilde{S}; \varphi^*)$ is a martingale under $\tilde{\mathbf{P}}$. In the latter case $\mathbf{E}^{\tilde{\mathbf{P}}}[G(\tilde{S})]$ is the fair price (by no-arbitrage arguments) relative to the class of allowed strategies of the contingent claim $G(S)$ for all models $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$.

We now give some sufficient conditions for hedgeability of some relevant options via PDEs. This PDE approach to robust replication was first considered in Schoenmakers and Kloeden (1999) for European options.

Example 5.6 *Suppose $G \in \mathcal{C}(\mathbb{R}^4)$. We define an option by plugging the time-T-values of the spot, the running average, maximum, and minimum into the arguments of G. To construct robust hedges for this type of option let $\Gamma_1(t, y_2, y_3) \cup \Gamma_2(t, y_2, y_3) = \{(x, y_1);\ x \leq y_2,\ y_1 \leq ty_2\} \cup \{(x, y_1);\ x \geq y_3,\ y_1 \geq ty_3\}$. Suppose for $0 \leq t < T$, $0 \leq y_3 \leq s_0 \leq y_2$, and $(x, y_1) \in \Gamma_1(t, y_2, y_3) \cup \Gamma_2(t, y_2, y_3)$ the PDE*

$$\begin{aligned}
\frac{\partial}{\partial t} U(t, x, y_1, y_2, y_3) &= -\frac{\sigma(x)}{2} \frac{\partial^2}{\partial x^2} U(t, x, y_1, y_2, y_3) \\
&\quad - x \frac{\partial}{\partial y_1} U(t, x, y_1, y_2, y_3) \\
U(T, x, y_1, y_2, y_3) &= G(x, y_1, y_2, y_3) \\
\frac{\partial}{\partial y_2} U(t, \cdot, \cdot, y_2, y_3)_{|\partial \Gamma_1(t, y_1, y_2)} &= 0 \\
\frac{\partial}{\partial y_3} U(t, \cdot, \cdot, y_2, y_3)_{|\partial \Gamma_2(t, y_1, y_2)} &= 0
\end{aligned}$$

has a solution $U \in \mathcal{C}^{1,2,1}([0,T) \times \mathbb{R}^4) \cap \mathcal{C}([0,T] \times \mathbb{R}^4)$ *which is bounded from below, i.e.* $U \geq -a$ *for some* $a \geq 0$*. Let* $(\Omega, \mathcal{F}, S, (\mathcal{F}_t), \mathbf{P}) \in \mathcal{M}_\sigma$*. By Itô's formula, for* $0 \leq t_0 \leq t \leq T$*,* $\mathbf{P}$*-almost surely,*

$$\begin{aligned} U(t, S_t, \bar{S}_t, S_t^*, S_{*,t}) &= U(t, S_{t_0}, \bar{S}_{t_0}, S_{t_0}^*, S_{*,t_0}) \\ &\quad + \int_{t_0}^{t} \frac{\partial}{\partial x} U(r, S_r, \bar{S}_r, S_r^*, S_{*,r}) dS_r. \end{aligned} \tag{14}$$

(Here, we used that

$$\int_{t_0}^{t} \frac{\partial}{\partial y_2} U(r, S_r, \bar{S}_r, S_r^*, S_{*,r}) dS_r^* = 0$$

by the boundary condition and similarly for the integral with respect to the minimum.) Define $\Phi_t = \frac{\partial}{\partial x} U(t, S_t, \bar{S}_t, S_t^*, S_{*,t})$*. In particular we obtain by the usual continuity argument for* $\eta \in \mathcal{C}_{\sigma,s_0}$

$$U(t, \eta(t), \bar{\eta}(t), \eta^*(t), \eta_*(t)) - U(0, s_0, 0, s_0, s_0) = v(t, \eta; \frac{\partial}{\partial x} U).$$

This shows, Φ_t *is an allowed strategy. Moreover formula (14) with* $t = T$ *shows that* Φ_t *is a hedge for* $G(S_t, \bar{S}_t, S_t^*, S_{*,t})$*.*

We note, the above PDE has two difficulties: (i) it is degenerate parabolic, since the second derivative in y_1*-direction does not appear; (ii) the boundary conditions in terms of the derivatives in direction of the parameters* y_2 *and* y_3 *are a rather unusual. Nonetheless there are some well known existence results for practically important exotic options such as lookback options (which depend on the maximum or minimum only) and Asian options (which depend on the average only). Of course European options with continuous payoff functions are also covered by this PDE approach. For details we refer to Willmott (1998).*

Remark 5.7 *Theorem 5.4 requires that the option is continuous as a function of the paths. The most prominent option which fails to satisfy this assumption is the digital option* $G(\eta) = \mathbf{1}_{[K,\infty)}(\eta(T))$ *with strike* K*. A straightforward modification of the argument in Theorem 5.4 shows that hedges for the digital option are robust in any subclass of* $\mathcal{M}_\sigma$ *which contains only models that satisfy* $\mathbf{P}(S_T = K) = 0$*.*

6 Approximative arbitrage exemplified

The basic idea of the no-arbitrage result in Theorem 4.3 was to extend absence of arbitrage in the reference model by means of the continuous wealth functional. We will now show that there can exist an approximative arbitrage in some models in

$\mathcal{M}_\sigma$, which fail to be an approximative arbitrage in the reference model. Our notion of approximative arbitrage is different from the notion of a free lunch with vanishing risk. However it admits to construct a very intuitive example in the context of mixed fractional Black-Scholes model.

Example 6.1 *Suppose that $\sigma(x) = \sigma x$. Hence, the reference model $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{S}, (\tilde{\mathcal{F}}_t), \tilde{\mathbf{P}})$ is the risk neutral Black-Scholes model. Now, the mixed fractional Black-Scholes model*

$$\mathrm{d}S_t \;=\; \sigma S_t \,\mathrm{d}X_t$$

where $X_t = W_t + Z_t$ is a sum of an independent Brownian motion and a fractional Brownian motion belongs to same class $\mathcal{M}_\sigma$ if the Hurst parameter H of the fractional Brownian motion satisfies $H \in (1/2, 3/4)$.

We define a sequence of strategies via the functionals

$$\varphi^n(t,\eta) = n^{2H-1} \sum_{k=1}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \frac{\tilde{\eta}(T\frac{k-1}{n}) - \tilde{\eta}(T\frac{k-2}{n})}{\eta(t)},$$

with $\tilde{\eta}(t) = \log(\eta(t))/\sigma - 1/2\sigma^2 t$ which fits into the context of Remark 4.5.

Write

$$\begin{aligned} S_t\varphi^n(t,S) \;&=\; n^{2H-1} \sum_{k=1}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \left(W_{T\frac{k}{n}} - W_{T\frac{k-1}{n}}\right) \\ &\quad + n^{2H-1} \sum_{k=1}^{n} \mathbf{1}_{\left(T\frac{k-1}{n}, T\frac{k}{n}\right]}(t) \left(Z_{T\frac{k}{n}} - Z_{T\frac{k-1}{n}}\right) \\ &=: \; K_t^n + L_t^n. \end{aligned}$$

Since $H \in (\frac{1}{2}, \frac{3}{4})$, $S_t\varphi^n(t,S)$ converges uniformly to zero in probability with respect to $\mathbf{P}$. (Indeed, this convergence holds for K_t^n and L_t^n). Hence the risk of the strategies $\Phi_t^n = \varphi^n(t,S)$ becomes smaller and smaller in the sense that the number of risky assets held by the investor tends to zero.

We now decompose

$$\int_0^T \Phi_t^n \,\mathrm{d}S_t \;=\; \int_0^T K_t^n \,\mathrm{d}W_t + \int_0^T L_t^n \,\mathrm{d}W_t + \int_0^T K_t^n \,\mathrm{d}Z_t + \int_0^T L_t^n \,\mathrm{d}Z_t.$$

The first and the second term go to zero in probability by Theorem II.11 in Protter (2004). The third terms converges to zero in probability, since, by the independence of W and Z,

$$\mathit{Law}\left(\int_0^T K_t^n \,\mathrm{d}Z_t \,\Big|\, \mathbf{P}\right) = \mathit{Law}\left(\int_0^T L_t^n \,\mathrm{d}W_t \,\Big|\, \mathbf{P}\right).$$

However, for the fourth term, we obtain,

$$\begin{aligned}\int_0^T L_s^n \, \mathrm{d}Z_s &= n^{2H-1}\sum_{k=2}^{n-1}\left(Z_{T\frac{k+1}{n}} - Z_{T\frac{k}{n}}\right)\left(Z_{T\frac{k}{n}} - Z_{T\frac{k-1}{n}}\right) \\ &\to T^{2H}(2^{2H-1}-1)\end{aligned}$$

in $L^1(\mathbf{P})$. *Hence, the limiting wealth of the strategies* Φ_t^n *is strictly positive, namely,*

$$\lim_{n\to\infty}\int_0^T \Phi_t^n \, \mathrm{d}S_t = T^{2H}(2^{2H-1}-1)$$

in probability. We consider such sequence an approximative arbitrage.

Of course, the corresponding sequence of strategies does not constitute an approximative arbitrage in the reference model. As above, we obtain that

$$\tilde{S}_t\varphi^n(t,\tilde{S}) = n^{2H-1}\sum_{k=1}^{n}\mathbf{1}_{\left(T\frac{k-1}{n},T\frac{k}{n}\right]}(t)\left(\tilde{W}_{T\frac{k}{n}} - \tilde{W}_{T\frac{k-1}{n}}\right)$$

converges to zero uniformly in probability with respect to $\tilde{\mathbf{P}}$. *Hence, with* $\tilde{\Phi}_t^n = \varphi^n(t,\tilde{S})$,

$$\lim_{n\to\infty}\int_0^t \tilde{\Phi}_t^n \, \mathrm{d}\tilde{S}_t = \lim_{n\to\infty}\int_0^t \tilde{S}_t\varphi^n(t,\tilde{S}) \, \mathrm{d}\tilde{W}_t = 0$$

as the Itô integral is continuous in terms of the integrand.

Remark 6.2 *The construction of the approximative arbitrage in the mixed fractional Black-Scholes model above follows an easy intuition. Due to the memory of the fractional Brownian motion the stock tends to increase, if it already increased in the previous time period. How to exploit this intuition is made precise above. The example also shows that integrals with respect to the mixed fractional Brownian motion with* $H \in (1/2, 3/4)$ *are not continuous in terms of the integrands. Hence, it may be considered a simple proof that mixed fractional Brownian motion is not a semimartingale for this range of the Hurst parameter. The reader is invited to compare our argument with the proof by Cheridito (2001).*

7 Concluding discussion

We have seen that the no arbitrage replication prices of options depend essentially only on the quadratic variation. Now, the quadratic variation is a path property that does not tell much about the probabilistic structure. Because of this we can incorporate many stylized facts to the model without changing the prices of the options.

Next we illustrate how to incorporate some stylized facts to the classical Black-Scholes model by using mixed models

$$\mathrm{d}S_t \;=\; S_t\,\mathrm{d}X_t, \quad t\in[0,T].$$

Consider first a generalized mixed fractional Black-Scholes model: $X_t = \sigma W_t + \delta Z_t$, where σ, $\delta>0$ and Z is a fractional Brownian motion with Hurst index $H\in(1/2,1)$. If we assume that (W,Z) is jointly Gaussian then the small ball property (3) is satisfied. Moreover, since Z has zero quadratic variation this model belongs to the same class $\mathcal{M}_\sigma$ as the classical Black-Scholes model. If we now take W and Z to be independent then

$$\mathbf{E}\left[X_1(X_{n+1}-X_n)\right] \;\sim\; H(2H-1)\delta^2\cdot n^{2H-2}.$$

So, the log-returns are *long-range dependent*. We can go even further: We can choose *any correlation structure* for the jointly Gaussian pair (W,Z) and we are still in the same model class $\mathcal{M}_\sigma$ as the classical Black-Scholes model. So, it seems that the second order structure of the stock price process is quite irrelevant in option pricing. (It was the great result of Black and Scholes that the first order structure is irrelevant in option pricing.) Next we show how to incorporate *heavy tails*. Let

$$C_t \;=\; \sum_{k=1}^{\infty} U_k \mathbf{1}_{\{\tau_k\le t\}}$$

be a compound Poisson process with unit density and positive jumps that are heavy tailed:

$$\mathbf{P}(U_k\ge x) \;\sim\; x^{-\alpha}$$

for some $\alpha>0$. Let Y then be the integrated compound Poisson process:

$$Y_t \;=\; \int_0^t C_s\,\mathrm{d}s$$

and consider the model $X_t=\sigma W_t - Y_t$, where Y is independent of W. Since Y is continuous with zero quadratic variation and $\mathbf{P}(Y_T=0)>0$ this model belongs to the same class $\mathcal{M}_\sigma$ as the classical Black-Scholes model. But this is a model with heavy tailed log-returns. Indeed, since Y is independent of W it is enough to show that Y has heavy tailed increments:

$$\begin{aligned}
\mathbf{P}(Y_{t+\Delta t}-Y_t\ge x) &= \mathbf{P}(Y_{\Delta t}\ge x)\\
&\ge \mathbf{P}(Y_{\Delta t}\ge x, \tau_1<\Delta t/2, \tau_2>\Delta t)\\
&\ge \mathbf{P}(U_1\cdot t/2\ge x, \tau_1<\Delta t/2, \tau_2>\Delta t)\\
&= c(\Delta t)\mathbf{P}(U_1\ge x\cdot 2/\Delta t)\\
&\sim c(\Delta t)(\Delta t/2)^{\alpha}\cdot x^{-\alpha}.
\end{aligned}$$

So, the increments of the left tail follow the power law. Finally, one can combine the long-range dependence and heavy tails of the log-returns by putting $X_t = \sigma W_t + \delta Z_t - Y_t$ with independent W, Z and Y.

We end this discussion by noting that the pathwise quadratic variation of the log-returns need not be their standard deviation. Indeed, this is obvious from the mixed fractional Black-Scholes model. So, one should not be surprised if the historical and implied volatility do not agree: The former is an estimate of the variance and the latter is an estimate of the quadratic variation.

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