On approximate approximations using Gaussian kernels

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This paper discusses quasi–interpolation and interpolation with Gaussians from a new point of view concerning accuracy in numerical computations. Estimates are obtained showing a high order approximation up to some saturation error negligible in numerical applications. The construction of local high order quasi–interpolation formulas is given.

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1. Introduction

In Maz'ya (1991), (1994) a new approximation method was proposed mainly directed to the numerical solution of operator equations. This method is characterised by a very accurate approximation in a certain range relevant for numerical computations, but in general the approximations do not converge. For that reason such processes were called *approximate approximations* (see also Maz'ya and Schmidt (1994)).

The present paper is devoted to an application of this method to the approximation of multivariate functions using Gaussian kernels. We study some examples to approximate functions u in $\mathbf{R}^n$ by sums of the form

$$u_h(\mathbf{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} u_{\mathbf{m}} \exp\Big(-\frac{|\mathbf{x}-h\mathbf{m}|^2}{\mathcal{D}h^2}\Big) \tag{1.1}$$

with two positive parameters, "small" h and fixed "large" $\mathcal{D}$, and certain scalars $u_{\mathbf{m}}$ depending on u. One of the main results can be described as follows.

Let $N = 2M+2$ be an even natural number. For any given $\varepsilon > 0$ there exist $\mathcal{D} > 1$ and a mask $\{c_{\mathbf{k}}\,,\ \mathbf{k} = (k_1, \dots, k_n) \in \mathbf{Z}^n\,,\ \sum_{j=1}^{n} |k_j| \le M\}$ such that the quasi–interpolation formula (1.1) with $u_{\mathbf{m}} = \sum c_{\mathbf{k}}\, u(h(\mathbf{m}-\mathbf{k}))$ provides for any function $u \in C^N(\mathbf{R}^n) \cap W^N_\infty(\mathbf{R}^n)$ the estimate

$$\|u - u_h\|_{L_\infty(\mathbf{R}^n)} \le c_N\, h^N\, |u|_{W^N_\infty(\mathbf{R}^n)} + \varepsilon \|u\|_{W^{N-1}_\infty(\mathbf{R}^n)} \tag{1.2}$$

with some constant c_N not depending on u. Roughly spoken, linear combinations of the $h\mathbf{Z}^n$-shifts of the Gaussian kernel $\exp(-h^{-2}\mathcal{D}^{-1}|\mathbf{x}|^2)$ approximate with arbitrary order $\mathcal{O}(h^N)$, but only for all h greater than some lower bound h_0. If $h_0 > h \to 0$ then

$$u_h(\mathbf{x}) - u(\mathbf{x}) = u(\mathbf{x}) \sum_{\boldsymbol{\nu} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} \exp(-\pi^2 \mathcal{D} |\boldsymbol{\nu}|^2) \, e^{\frac{2\pi i}{h} \langle \mathbf{x}, \boldsymbol{\nu} \rangle} + \mathcal{O}(h) \, ,$$

i.e. under the assumption that $\mathcal{D}$ is fixed the quasi–interpolation does not converge.

The idea to consider such non-converging approximation processes comes from numerical applications providing always some inaccuracies. For example, if $\mathcal{D}$ is chosen such that ε is smaller than the machine precision then (1.1) performs like a usual high order approximation process. But additionally and in contrast to other approximations, formulas of the from (1.1) are easy to implement and give the possibility to determine analytically the action of various important differential and integral operators, which is very significant especially in the multivariate case (see Maz'ya (1994)).

The main purpose of our paper is to construct quasi–interpolation formulas providing a prescribed approximation order and to analyse the saturation error ε. In Sect. 2 we obtain explicit error expansions of quasi–interpolation with rapidly decaying kernels for functions from the Sobolev space $W_\infty^N(\mathbf{R}^n)$.

Based on a result on multivariate polynomial interpolation in Sect. 3 we construct the indicated quasi–interpolants with Gaussians, show that these formulas are optimal with respect to the number of mask elements and obtain estimates of the approximation error.

In Sect. 4 we derive error estimates for the interpolation with formula (1.1) to classes of functions which are characterised by conditions on the Fourier transform. In particular, the interpolating sum (1.1) approximates a given function u satisfying $\mathcal{F}u(\boldsymbol{\lambda}) \exp(a|\boldsymbol{\lambda}|) \in L_1(\mathbf{R}^n)$ with the order $\mathcal{O}(\exp(-a/2h))$ up to the saturation error $\approx 4n \exp(-\pi^2 \mathcal{D})$.

The approximation with Gaussian kernels $\exp(-|\mathbf{x}|^2)$ is often mentioned in the literature and has been studied recently in connection with radial-basis functions and principal shift-invariant spaces. It is well known that no linear combination of translates of this generating function reproduces polynomial. This in fact causes the saturation errors. The underlying idea of our research is, that these errors can be made arbitrarily small since the kernels $\exp(-\mathcal{D}^{-1}|\mathbf{x}|^2)$ reproduce polynomials very accurate.

Let us mention some results concerning the approximation with Gaussians. In Buhmann (1990) the interpolation problem with sums of the form (1.1) is considered. Since the polynomial reproduction is absent the author concludes that the interpolating functions cannot yield good convergence results. However, if one looks from the point of view of practical applications then the Gaussian kernels show remarkable approximation results. For instance, already the simple formula

$$\frac{1}{\sqrt{27\pi^3}} \sum_{\mathbf{m} \in \mathbf{Z}^3} u(h\mathbf{m}) \exp\Big(-\frac{|\mathbf{x} - h\mathbf{m}|^2}{3h^2}\Big) \tag{1.3}$$

provides in $\mathbf{R}^3$ the approximation order $\mathcal{O}(h^2)$ modulo a saturation error which is bounded pointwise by

$$8.303 \cdot 10^{-13} \cdot |u(\mathbf{x})| + 2.609 \cdot 10^{-12} \cdot h \cdot \sum_{i=1}^{3} \Big| \frac{\partial u(\mathbf{x})}{\partial x_i} \Big| \, .$$

Higher order approximations with negligible saturation errors and their application in numerical methods were considered in Maz'ya (1991), (1994). In particular it was shown that the quasi–interpolant

$$(\mathcal{D}\pi)^{-n/2} \sum_{\mathbf{m} \in \mathbf{Z}^n} u(h\mathbf{m}) \, L_M^{(n/2)}\Big(-\frac{|\mathbf{x} - h\mathbf{m}|^2}{\mathcal{D}h^2}\Big) \exp\Big(-\frac{|\mathbf{x} - h\mathbf{m}|^2}{\mathcal{D}h^2}\Big)$$

approximates a smooth function u with the order $\mathcal{O}(h^{2M+2})$ plus some small saturation error, where $L_M^{(\alpha)}$ denotes the generalized Laguerre polynomial. A detailed analysis concerning the errors of approximate approximations with respect to different norms for the quasi–interpolation with rapidly decreasing kernels and construction methods for such kernels can be found in Maz'ya and Schmidt (1994).

The estimate (1.2) indicates that it is possible to obtain approximations with Gaussian kernels converging for all $h \to 0$ if one choose the parameter $\mathcal{D}$ depending on h. This was studied in some papers recently. In Wu and Schaback (1993) interpolants of the form

$$\sum_{\mathbf{m}} u_{\mathbf{m}} \exp\Big(-\frac{|\mathbf{x}-\mathbf{x_m}|^2}{\mathcal{D}}\Big)$$

are considered, where the points $\mathbf{x_m}$ are allowed to be irregularly distributed over a domain Ω, and de Boor and Ron (1992) deals with best approximants of the forms

$$\sum_{\mathbf{m}\in\mathbf{Z}^n} u_{\mathbf{m}} \exp\Big(-\frac{|\mathbf{x}-h\mathbf{m}|^2}{\mathcal{D}h}\Big) .$$

It is shown that the corresponding expressions approximate smooth functions with arbitrarily high order. In the book of Stenger (1993), Section 5.8, a one-dimensional approximation formula similar to (1.3) is considered, where $\mathcal{D} = h^{2(\beta-1)}$, $0<\beta<1$. It is proved that this quasi–interpolant converges for continuous functions. In Beatson and Light (1992) quasi–interpolants of the form (1.1) with variable $\mathcal{D}$ are analysed providing high order approximation. In particular a tensor product construction of a quasi–interpolation formula is given converging to $u \in C^N(\mathbf{R}^n) \cap W_\infty^N(\mathbf{R}^n)$ with the order $\mathcal{O}(h^N |\ln h|^N)$ if $\mathcal{D}(h) = N\,|\ln h|/\pi^2$.

In all of these approaches the use of finer grids enlarge the number of summands necessary to compute the approximate value at a fixed point $\mathbf{x}$ within a given tolerance. This is in contrast to the case of fixed $\mathcal{D}$, which we prefer, because this is advantageous in numerical applications and reflects the local character of the quasi–interpolants constructed in Sect. 3.

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2. Preliminaries

Here we prove some results for use in later analysis. First we consider for fixed $\mathcal{D} > 0$ the behaviour of the quasi–interpolation formula

$$u_h(\mathbf{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\, \eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) , \tag{2.1}$$

where u is a bounded function and the continuous generating function η satisfies the decay condition

$$|\eta(\mathbf{t})| \le A\,(1+|\mathbf{t}|)^{-N-n-\delta} , \ \mathbf{t} \in \mathbf{R}^n , \tag{2.2}$$

for some natural number N and positive constants A and δ.

Additionally we suppose that η is subjected to the moment condition

$$\int_{\mathbf{R}^n} \eta(\mathbf{t})\, d\mathbf{t} = 1 , \ \int_{\mathbf{R}^n} \mathbf{t}^{\boldsymbol{\alpha}} \eta(\mathbf{t})\, d\mathbf{t} = 0, \quad \forall \boldsymbol{\alpha} , \ 1 \le |\boldsymbol{\alpha}| < N . \tag{2.3}$$

Here and henceforth we use the notations:
Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbf{Z}^n_{\geq 0}$ a multiindex. We denote $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$, $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_n!$,

$$\partial^{\boldsymbol{\alpha}} u(\mathbf{x}) = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u(\mathbf{x}) \, .$$

For $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbf{Z}^n$ we define the multiindex $\xi(\mathbf{k}) := (|k_1|, \ldots, |k_n|) \in \mathbf{Z}^n_{\geq 0}$. The usual scalar product in $\mathbf{R}^n$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$ and $|\mathbf{x}| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$. With the abbreviation

$$e_{\boldsymbol{\lambda}}(\mathbf{x}) := e^{2\pi i \langle \mathbf{x}, \boldsymbol{\lambda} \rangle}$$

the Fourier transform of an L_1-function is defined by

$$\mathcal{F}\varphi(\boldsymbol{\lambda}) = \int_{\mathbf{R}^n} \varphi(\mathbf{x}) \, e_{\boldsymbol{\lambda}}(-\mathbf{x}) \, d\mathbf{x} \, .$$

Lemma 2.1 *Suppose that η satisfies* (2.2), (2.3) *and for given $\mathcal{D} > 0$ the Fourier transforms of $\mathbf{t}^{\boldsymbol{\alpha}} \eta(\mathbf{t})$ are such that*

$$\{\mathcal{F}(\mathbf{t}^{\boldsymbol{\alpha}} \eta(\mathbf{t}))(\sqrt{\mathcal{D}} \, \cdot)\} \in l_1(\mathbf{Z}^n) \, , \; 0 \leq |\boldsymbol{\alpha}| < N \, . \tag{2.4}$$

Then for any $u \in C^N(\mathbf{R}^n) \cap W^N_\infty(\mathbf{R}^n)$ it holds

$$u_h(\mathbf{x}) - u(\mathbf{x}) = R_h(\mathbf{x}) + \sum_{|\boldsymbol{\alpha}|=0}^{N-1} (-\sqrt{\mathcal{D}} h)^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x})}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{\nu} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} \mathcal{F}(\mathbf{t}^{\boldsymbol{\alpha}} \eta(\mathbf{t}))(\sqrt{\mathcal{D}} \boldsymbol{\nu}) \, e_{\boldsymbol{\nu}}(\frac{\mathbf{x}}{h}) \, ,$$

where

$$|R_h(\mathbf{x})| \leq (\sqrt{\mathcal{D}} h)^N \sum_{|\boldsymbol{\alpha}|=N} \rho_{\boldsymbol{\alpha}} \, \|\partial^{\boldsymbol{\alpha}} u\|_{L_\infty(\mathbf{R}^n)} \tag{2.5}$$

and

$$\rho_{\boldsymbol{\alpha}} = \frac{1}{\boldsymbol{\alpha}!} \Big\| \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbf{Z}^n} \Big| \Big(\frac{\cdot - \mathbf{m}}{\sqrt{\mathcal{D}}} \Big)^{\boldsymbol{\alpha}} \eta \Big(\frac{\cdot - \mathbf{m}}{\sqrt{\mathcal{D}}} \Big) \Big| \Big\|_{L_\infty(\mathbf{R}^n)} \, .$$

Proof. Denoting

$$U_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}) := N \int_0^1 s^{N-1} \, \partial^{\boldsymbol{\alpha}} u(s\mathbf{x} + (1-s)\mathbf{y}) ds$$

from the Taylor expansion of $u \in C^N(\mathbf{R}^n)$

$$u(\mathbf{y}) = \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{(\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \partial^{\boldsymbol{\alpha}} u(\mathbf{x}) + \sum_{|\boldsymbol{\alpha}|=N} \frac{(\mathbf{y} - \mathbf{x})^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \, U_{\boldsymbol{\alpha}}(\mathbf{x}, \mathbf{y}) \tag{2.6}$$

we obtain the representation

$$u_h(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x})}{\boldsymbol{\alpha}!} \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbf{Z}^n} (h\mathbf{m} - \mathbf{x})^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}} h} \Big)$$

$$+ \sum_{|\boldsymbol{\alpha}|=N} \frac{1}{\boldsymbol{\alpha}!} \, \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbf{Z}^n} (h\mathbf{m} - \mathbf{x})^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}} h} \Big) \, U_{\boldsymbol{\alpha}}(\mathbf{x}, h\mathbf{m}) \, .$$

Since the series

$$\sum_{\mathbf{m}\in\mathbf{Z}^n} (h\mathbf{m}-\mathbf{x})^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)$$

converge absolutely in view of (2.4) the Poisson summation formula

$$\mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)^{\boldsymbol{\alpha}} \eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big) = \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} \mathcal{F}(\mathbf{t}^{\boldsymbol{\alpha}}\eta(\mathbf{t}))(\sqrt{\mathcal{D}}\boldsymbol{\nu})\, e_{\boldsymbol{\nu}}(\frac{\mathbf{x}}{h}) \,,$$

holds (cf. Stein and Weiss (1971)). Using the moment condition (2.3) and the estimate

$$|U_{\boldsymbol{\alpha}}(\mathbf{x}, h\mathbf{m})| \le \|\partial^{\boldsymbol{\alpha}} u\|_{L_\infty(\mathbf{R}^n)}$$

the assertion follows immediately. □

Let us formulate an estimate showing the local character of the quasi–interpolation if $\mathcal{D}$ is fixed. By $B(\mathbf{x},\kappa)$ we denote the closed ball with radius κ centered at the point $\mathbf{x}$. In view of (2.2) for any $\mathcal{D}>0$ we can find $\kappa=\kappa(\mathcal{D})$ such that

$$\max_{\mathbf{x}\in\mathbf{R}^n} \sum_{h\mathbf{m}\notin B(\mathbf{x},\kappa h)} \left|\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right)\right| = \max_{\mathbf{x}\in\mathbf{R}^n} \sum_{|\mathbf{x}-\mathbf{m}|>\kappa} \left|\eta\left(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{\mathcal{D}}}\right)\right| \le \mathcal{D}^{n/2}\, \varepsilon_{\mathbf{0}}(\eta,\mathcal{D}) \,,$$

where we set

$$\varepsilon_{\boldsymbol{\alpha}}(\eta,\mathcal{D}) := \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} |\mathcal{F}(\mathbf{t}^{\boldsymbol{\alpha}}\eta(\mathbf{t}))(\sqrt{\mathcal{D}}\boldsymbol{\nu})| \,.$$

Lemma 2.2 (Maz'ya and Schmidt (1994)) *Let $u\in C^N(\Omega)$ in some domain $\Omega\subset\mathbf{R}^n$ and η is as in Lemma* 2.1. *Then for any $\mathbf{x}$ inside Ω such that $B(\mathbf{x},\kappa h)\subset\Omega$ the estimate*

$$\Big|\mathcal{D}^{-n/2} \sum_{h\mathbf{m}\in B(\mathbf{x},\kappa h)} u(h\mathbf{m})\eta\left(\frac{\mathbf{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\right) - u(\mathbf{x})\Big| \le \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{|\partial^{\boldsymbol{\alpha}}u(\mathbf{x})|}{\boldsymbol{\alpha}!}(\sqrt{\mathcal{D}}h)^{|\boldsymbol{\alpha}|}\varepsilon_{\boldsymbol{\alpha}}(\eta,\mathcal{D})$$
$$+\ c_N \left((\sqrt{\mathcal{D}}h)^N \max_{|\boldsymbol{\alpha}|=N} \frac{\rho_{\boldsymbol{\alpha}}(\eta,\mathcal{D})}{\boldsymbol{\alpha}!} + \varepsilon_{\mathbf{0}}(\eta,\mathcal{D})\right) \|u\|_{C^N(B(\mathbf{x},\kappa h))}$$

is valid, where the constant c_N depends only on N and n.

In Section 3 we employ that the matrix $\big(\boldsymbol{\beta}^{2\boldsymbol{\alpha}}\big)_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|=0}^{M}$, $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{Z}^n_{\ge 0}$, is nonsingular. This follows from a general result of Hakopian which we now formulate.
A set of multiindeces $J\subset\mathbf{Z}^n_{\ge 0}$ is said to be normal if $\boldsymbol{\alpha}\in J$ and $\boldsymbol{\beta}\le\boldsymbol{\alpha}$ imply $\boldsymbol{\beta}\in J$. The linear space of n-variate polynomials associated with J we denote by

$$\mathcal{P}_J = \mathrm{span}\,\{\mathbf{x}^{\boldsymbol{\alpha}} : \boldsymbol{\alpha}\in J\}.$$

Further, for $1\le j\le n$ let be given sequences $T_j=\{t_{j,l}\}_{l=0}^{\infty}$ of distinct real numbers. We introduce the lattice $\mathcal{T}_J$ determined by the sequences T_j and $J\subset\mathbf{Z}^n_{\ge 0}$:

$$\mathcal{T}_J = \{\mathbf{t}_{\boldsymbol{\alpha}} = (t_{1,\alpha_1},\ldots,t_{n,\alpha_n}) : \boldsymbol{\alpha}\in J\}.$$

Theorem 2.1 (Hakopian (1983)) *Let $J\subset\mathbf{Z}^n_{\ge 0}$ be a normal set. Then each polynomial $p\in\mathcal{P}_J$ is uniquely determined by its values on the lattice $\mathcal{T}_J$.*

3. Formulas for high order approximation

Let us first consider the case that in (2.1)

$$\eta(\mathbf{x}) = \pi^{-n/2}\exp(-|\mathbf{x}|^2) \quad \text{with} \quad \mathcal{F}\eta(\boldsymbol{\lambda}) = \exp(-\pi^2|\boldsymbol{\lambda}|^2)\,.$$

Then this generating function satisfies the moment condition for $N = 2$ and we get from Lemma 2.1

$$\begin{aligned}
&\Big|u(\mathbf{x}) - (\pi\mathcal{D})^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n} u(h\mathbf{m})\,\exp\Big(-\frac{|\mathbf{x}-h\mathbf{m}|^2}{\mathcal{D}h^2}\Big)\Big| \\
\le&(\sqrt{\mathcal{D}}h)^2\sum_{|\boldsymbol{\alpha}|=2}\rho_{\boldsymbol{\alpha}}\|\partial^{\boldsymbol{\alpha}}u\|_{C(\mathbf{R}^n)} + |u(\mathbf{x})|\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}\exp(-\mathcal{D}\pi^2|\boldsymbol{\nu}|^2) \\
&+\mathcal{D}h\,\pi\sum_{|\boldsymbol{\alpha}|=1}|\partial^{\boldsymbol{\alpha}}u(\mathbf{x})|\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}|\boldsymbol{\nu}^{\boldsymbol{\alpha}}|\,\exp(-\mathcal{D}\pi^2|\boldsymbol{\nu}|^2)\,.
\end{aligned}$$

Therefore, if $h\to 0$ then u_h does not converge to u, i.e. u_h does not approximate u in the usual sense. On the other hand, since $\exp(-\pi^2) = 0.51723\ldots\cdot 10^{-4}$ the factors

$$\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}\exp(-\mathcal{D}\pi^2|\boldsymbol{\nu}|^2) \quad \text{and} \quad \mathcal{D}\pi\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n}|\boldsymbol{\nu}^{\boldsymbol{\alpha}}|\,\exp(-\mathcal{D}\pi^2|\boldsymbol{\nu}|^2)\,,\quad |\boldsymbol{\alpha}| = 1\,,$$

are for $\mathcal{D} = 2$ or $\mathcal{D} = 4$ comparable with the machine accuracy in single and double precision floating point arithmetic, respectively. Hence, for our concrete example the quasi–interpolation (2.1) behaves in numerical calculations like a usual second order approximation, if $\mathcal{D}$ is appropriate chosen.

The aim of this section is to construct approximate approximations generated by Gaussian kernels providing a prescribed approximation order with controlled saturation errors. To this end we form the generating function η as a linear combination of translates of the Gaussian kernels so that the moment condition holds for large N. In the class of functions symmetric in each variable, i.e.

$$\eta(x_1,\ldots,x_j,\ldots,x_n) = \eta(x_1,\ldots,-x_j,\ldots,x_n)\,,\quad j=1,\ldots,n\,, \tag{3.1}$$

we will give the function with "minimal support". From (3.1) it is clear that for $N = 2M+2$ the new generating function must be subjected to the conditions

$$\int_{\mathbf{R}^n}\eta(\mathbf{t})\,d\mathbf{t} = 1\,,\ \int_{\mathbf{R}^n}\mathbf{t}^{2\boldsymbol{\alpha}}\eta(\mathbf{t})\,d\mathbf{t} = 0, \quad \forall\boldsymbol{\alpha}\,,\ 1\le|\boldsymbol{\alpha}|\le M\,.$$

Lemma 3.3 *For any $M\ge 0$ there exist uniquely determined coefficients $c_{\mathbf{k}}$, $\mathbf{k}\in\mathbf{Z}^n$ with $|\xi(\mathbf{k})| = \sum_{j=1}^{n}|k_j|\le M$, such that the generating function*

$$\eta(\mathbf{x}) = \sum_{|\xi(\mathbf{k})|\le M} c_{\mathbf{k}}\,\exp\Big(-\Big|\mathbf{x}-\frac{\mathbf{k}}{\sqrt{\mathcal{D}}}\Big|^2\Big) \tag{3.2}$$

satisfies (3.1) *and the moment condition* (2.3) *with* $N = 2M+2$.

Proof. From (3.2) we see that

$$\mathcal{F}\eta(\boldsymbol{\lambda}) = \pi^{n/2}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\sum_{|\xi(\mathbf{k})|\le M}c_{\mathbf{k}}\,e_{\mathbf{k}}\Big(-\frac{\boldsymbol{\lambda}}{\sqrt{\mathcal{D}}}\Big)\,.$$

Hence, if a trigonometric polynomial

$$P_M(\boldsymbol{\lambda}) = \sum_{|\xi(\mathbf{k})|\le M} c_{\mathbf{k}}\, e_{\mathbf{k}}\Big(-\frac{\boldsymbol{\lambda}}{\sqrt{\mathcal{D}}}\Big) \tag{3.3}$$

satisfies

$$\partial^{\boldsymbol{\alpha}} P_M(\mathbf{0}) = \pi^{-n/2}\partial^{\boldsymbol{\alpha}} \exp(\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\mathbf{0}} \ ,\ 0 \le |\boldsymbol{\alpha}| < 2M+2\ , \tag{3.4}$$

then obviously

$$\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\mathbf{0}) = \partial^{\boldsymbol{\alpha}}\big(\exp(-\pi^2|\boldsymbol{\lambda}|^2)\, P_M(\boldsymbol{\lambda})\big)\Big|_{\boldsymbol{\lambda}=\mathbf{0}} = \delta_{|\boldsymbol{\alpha}|0}\ ,\ 0 \le |\boldsymbol{\alpha}| < 2M+2\,,$$

i.e. η is the required function.

To find this polynomial we use that

$$\partial^{\boldsymbol{\alpha}} \exp(\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\mathbf{0}} = \begin{cases} (2\pi)^{2|\boldsymbol{\beta}|}\displaystyle\prod_{j=1}^{n}\Gamma(\beta_j+1/2) & ,\ \boldsymbol{\alpha}=2\boldsymbol{\beta}\ ,\\ 0 & ,\ \text{otherwise}\ ,\end{cases} \tag{3.5}$$

with $\boldsymbol{\beta} = (\beta_1,\dots,\beta_n) \in \mathbf{Z}^n_{\ge 0}$. Therefore it is possible to seek $P_M(\boldsymbol{\lambda})$ in the form

$$P_M(\boldsymbol{\lambda}) = \sum_{|\boldsymbol{\beta}|\le M} a_{\boldsymbol{\beta}} \prod_{j=1}^{n} \cos\frac{2\pi}{\sqrt{\mathcal{D}}}\beta_j\lambda_j\ ,\quad \boldsymbol{\beta}\in\mathbf{Z}^n_{\ge 0}\,, \tag{3.6}$$

From (3.4) and (3.5) we obtain the system of linear equations

$$\sum_{|\boldsymbol{\beta}|\le M} a_{\boldsymbol{\beta}}\boldsymbol{\beta}^{2\boldsymbol{\alpha}} = \pi^{-n}(-\mathcal{D})^{|\boldsymbol{\alpha}|}\prod_{j=1}^{n}\Gamma(\alpha_j+1/2) \quad ,\ 0\le|\boldsymbol{\alpha}|\le M\ . \tag{3.7}$$

From Theorem 2.1 we know that the matrix $\big(\boldsymbol{\beta}^{2\boldsymbol{\alpha}}\big)^M_{|\boldsymbol{\alpha}|,|\boldsymbol{\beta}|=0}$, $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbf{Z}^n_{\ge 0}$, is nonsingular. Thus there exists a unique solution $a_{\boldsymbol{\beta}}$, $|\boldsymbol{\beta}|\le M$, of this system. Note that

$$\prod_{j=1}^{n}\cos\frac{2\pi}{\sqrt{\mathcal{D}}}\beta_j\lambda_j = 2^{-\kappa(\boldsymbol{\beta})}\sum_{\xi(\mathbf{k})=\boldsymbol{\beta}} e_{\mathbf{k}}\Big(-\frac{\boldsymbol{\lambda}}{\sqrt{\mathcal{D}}}\Big)\ ,$$

where $\kappa(\boldsymbol{\beta})$ is the number of nonzero components of $\boldsymbol{\beta}$. Therefore the solution of (3.7) provides the required generating function

$$\eta(\mathbf{x}) = \sum_{|\xi(\mathbf{k})|\le M} 2^{-\kappa(\mathbf{k})}\, a_{\xi(\mathbf{k})}\, \exp\Big(-\,\big|\mathbf{x}-\frac{\mathbf{k}}{\sqrt{\mathcal{D}}}\big|^2\Big)\ . \tag{3.8}$$

It remains to show that $P_M(\boldsymbol{\lambda})$ is the unique trigonometric polynomial of the form (3.3) satisfying (3.4) and (3.1). Suppose that there exists

$$\overline{P}_M(\boldsymbol{\lambda}) = \sum_{|\xi(\mathbf{k})|\le M} \overline{c}_{\mathbf{k}}\, e_{\mathbf{k}}\Big(\frac{\boldsymbol{\lambda}}{\sqrt{\mathcal{D}}}\Big)\ ,$$

which is symmetric in each variable λ_j and such that

$$\partial^{2\boldsymbol{\alpha}}\overline{P}_M(\mathbf{0}) = \Big(-\frac{4\pi^2}{\mathcal{D}}\Big)^{|\boldsymbol{\alpha}|} \sum_{|\xi(\mathbf{k})|\le M} \overline{c}_{\mathbf{k}}\,\mathbf{k}^{2\boldsymbol{\alpha}} = 0\,, \quad 0 \le |\boldsymbol{\alpha}| \le M\,.$$

As before we get from Theorem 2.1 that

$$\sum_{\xi(\mathbf{k})=\boldsymbol{\beta}} \overline{c}_{\mathbf{k}} = 0\,, \quad \forall \boldsymbol{\beta}\,,\ 0 \le |\boldsymbol{\beta}| \le M\,.$$

But the symmetry of $\overline{P}_M(\boldsymbol{\lambda})$ implies that

$$\overline{c}_{\mathbf{k}_1} = \overline{c}_{\mathbf{k}_2} \quad \text{if} \quad \xi(\mathbf{k}_1) = \xi(\mathbf{k}_2)\,,$$

which proves that (3.8) is uniquely determined. □

Remark: *It can be easily seen, that the solution $a_{\boldsymbol{\beta}}$ of* (3.7) *is independent of any permutation $\sigma(\boldsymbol{\beta})$ of the components of the multiindex $\boldsymbol{\beta}$. Therefore the generating function* (3.8) *can be written in the form*

$$\eta(\mathbf{x}) = \sum_{\substack{|\boldsymbol{\beta}|\le M\\ \beta_1\ge\cdots\ge\beta_n}} 2^{-\kappa(\boldsymbol{\beta})} a_{\boldsymbol{\beta}} \sum_{\sigma(\xi(\mathbf{k}))=\boldsymbol{\beta}} \exp\Big(-\,|\mathbf{x} - \frac{\mathbf{k}}{\sqrt{\mathcal{D}}}|^2\Big)\,.$$

Now we consider the saturation error provided by the formula (2.1) if the generating function (3.8) is used.

Lemma 3.4 *For any multiindex $\boldsymbol{\alpha}$, $|\boldsymbol{\alpha}| < N = 2M+2$, and $\boldsymbol{\nu} \in \mathbf{Z}^n$ it holds*

$$\mathcal{F}(\mathbf{t}^{\boldsymbol{\alpha}}\eta(\mathbf{t}))(\sqrt{\mathcal{D}}\boldsymbol{\nu}) = (-\pi i\sqrt{\mathcal{D}})^{|\boldsymbol{\alpha}|}\,\boldsymbol{\nu}^{\boldsymbol{\alpha}}\,\exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)\,. \tag{3.9}$$

Proof. Since

$$\mathcal{F}\eta(\boldsymbol{\lambda}) = \pi^{n/2}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\,P_M(\boldsymbol{\lambda})\,,$$

the periodicity of P_M and the equalities (3.4) imply

$$\begin{aligned}
\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu}) &= \pi^{n/2}\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\,(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\beta}}P_M(\sqrt{\mathcal{D}}\boldsymbol{\nu})\,\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\sqrt{\mathcal{D}}\boldsymbol{\nu}}\\
&= \pi^{n/2}\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\,(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\beta}}P_\Lambda(\mathbf{0})\,\partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\sqrt{\mathcal{D}}\boldsymbol{\nu}}\\
&= \sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\,(\boldsymbol{\alpha}-\boldsymbol{\beta})!}\partial^{\boldsymbol{\beta}}\exp(\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\mathbf{0}}\ \partial^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\sqrt{\mathcal{D}}\boldsymbol{\nu}}\,.
\end{aligned}$$

In view of (3.5) we obtain

$$\frac{\partial^{2\boldsymbol{\beta}}\exp(\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\mathbf{0}}}{(2\boldsymbol{\beta})!} = \frac{\pi^{2|\boldsymbol{\beta}|}}{\boldsymbol{\beta}!}\,,$$

which leads to the equality

$$\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu}) = \sum_{2\boldsymbol{\beta}\le\boldsymbol{\alpha}} \frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\,(\boldsymbol{\alpha}-2\boldsymbol{\beta})!}\,\frac{1}{\pi^{|\boldsymbol{\alpha}|-2|\boldsymbol{\beta}|}}\,\partial^{\boldsymbol{\alpha}-2\boldsymbol{\beta}}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\Big|_{\boldsymbol{\lambda}=\sqrt{\mathcal{D}}\boldsymbol{\nu}}\,.$$

Now we use the identity

$$\frac{1}{2^m}\sum_{j=0}^{[m/2]}\frac{1}{j!\,(m-2j)!}\Big(\frac{d}{dy}\Big)^{m-2j}\exp(-y^2) = -\frac{1}{m!}\,y^m\,\exp(-y^2)$$

to deduce

$$\begin{aligned}
&\sum_{2\boldsymbol{\beta}\le\boldsymbol{\alpha}}\frac{\boldsymbol{\alpha}!}{\boldsymbol{\beta}!\,(\boldsymbol{\alpha}-2\boldsymbol{\beta})!}\,\frac{1}{\pi^{|\boldsymbol{\alpha}|-2|\boldsymbol{\beta}|}}\,\partial^{\boldsymbol{\alpha}-2\boldsymbol{\beta}}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\\
&=(2\pi)^{|\boldsymbol{\alpha}|}\prod_{j=1}^{n}\frac{\alpha_j!}{2^{\alpha_j}}\sum_{2\beta_j\le\alpha_j}\frac{1}{\beta_j!\,(\alpha_j-2\beta_j)!}\,\frac{1}{\pi^{\alpha_j-2\beta_j}}\,\Big(\frac{d}{d\lambda_j}\Big)^{\alpha_j-2\beta_j}\exp(-\pi^2\lambda_j^2)\\
&=(2\pi)^{|\boldsymbol{\alpha}|}\prod_{j=1}^{n}(-1)^{\alpha_j}(\pi\lambda_j)^{\alpha_j}\,\exp(-\pi^2\lambda_j^2) = (2\pi)^{|\boldsymbol{\alpha}|}\,(-\pi\boldsymbol{\lambda})^{\boldsymbol{\alpha}}\exp(-\pi^2|\boldsymbol{\lambda}|^2)\,.
\end{aligned}$$

Since

$$\mathcal{F}(\mathrm{t}^{\boldsymbol{\alpha}}\eta(\mathrm{t}))(\sqrt{\mathcal{D}}\boldsymbol{\nu}) = \Big(\frac{i}{2\pi}\Big)^{|\boldsymbol{\alpha}|}\,\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{\mathcal{D}}\boldsymbol{\nu})$$

the assertion is proved. □

Using the notations introduced in the proof of Lemma 3.3 we can formulate

Theorem 3.2 *Let $M\ge 0$ be a natural number, $N=2M+2$ and set for fixed $\mathcal{D}$*

$$u^h_{\mathbf{m}} := \sum_{|\xi(\mathbf{k})|\le M} 2^{-\kappa(\mathbf{k})}\,a_{\xi(\mathbf{k})}\,u(h(\mathbf{m}-\mathbf{k}))\,, \tag{3.10}$$

where $a_{\xi(\mathbf{k})}$ is the solution of the system (3.7). There exist positive constants $\rho_{\boldsymbol{\alpha}}$, $|\boldsymbol{\alpha}|=N$, such that for any $u\in C^N(\mathbf{R}^n)\cap W^N_\infty(\mathbf{R}^n)$ and all $h>0$ the estimate

$$\begin{aligned}
\Big|u(x)-\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}u^h_m\,\exp\Big(-\frac{|\mathrm{x}-h\mathbf{m}|^2}{\mathcal{D}h^2}\Big)\Big| &\le (\sqrt{\mathcal{D}}h)^N\sum_{|\boldsymbol{\alpha}|=N}\rho_{\boldsymbol{\alpha}}\|\partial^{\boldsymbol{\alpha}}u\|_{C(\mathbf{R}^n)}\\
&+\sum_{|\boldsymbol{\alpha}|=0}^{N-1}|\partial^{\boldsymbol{\alpha}}u(\mathrm{x})|\,\frac{(\mathcal{D}h\,\pi)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!}\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}|\boldsymbol{\nu}^{\boldsymbol{\alpha}}|\,\exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)
\end{aligned}$$

is valid.

Proof. From Lemmas 2.1, 3.3 and 3.4 we derive that the quasi–interpolant

$$\begin{aligned}
u_h(\mathrm{x}) :=&\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}u(h\mathbf{m})\,\eta\Big(\frac{\mathrm{x}-h\mathbf{m}}{\sqrt{\mathcal{D}}h}\Big)\\
=&\mathcal{D}^{-n/2}\sum_{\mathbf{m}\in\mathbf{Z}^n}\exp\Big(-\frac{|\mathrm{x}-h\mathbf{m}|^2}{\mathcal{D}h^2}\Big)\sum_{|\xi(\mathbf{k})|\le M}2^{-\kappa(\mathbf{k})}\,a_{\xi(\mathbf{k})}\,u(h(\mathbf{m}-\mathbf{k}))
\end{aligned}$$

satisfies

$$u_h(\mathrm{x})-u(\mathrm{x}) = R_h(\mathrm{x})+\sum_{|\boldsymbol{\alpha}|=0}^{N-1}\frac{(\mathcal{D}h\pi i)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!}\,\partial^{\boldsymbol{\alpha}}u(\mathrm{x})\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}\boldsymbol{\nu}^{\boldsymbol{\alpha}}\,\exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)\,e_{\boldsymbol{\nu}}(\frac{\mathrm{x}}{h})\,,$$

where R_h is bounded by (2.5) and

$$\rho_{\boldsymbol{\alpha}} = \Big\| \frac{\mathcal{D}^{-n/2}}{\boldsymbol{\alpha}!} \sum_{\mathbf{m}\in\mathbf{Z}^n} \Big| \Big(\frac{\cdot - \mathbf{m}}{\sqrt{\mathcal{D}}} \Big)^{\boldsymbol{\alpha}} \sum_{|\xi(\mathbf{k})|\le M} 2^{-\kappa(\mathbf{k})} a_{\xi(\mathbf{k})} \exp\Big(- \frac{|\cdot - \mathbf{m} - \mathbf{k}|^2}{\mathcal{D}} \Big) \Big| \Big\|_{L_\infty(\mathbf{R}^n)}.$$

$\square$

Remark: *It is obvious that any trigonometric polynomial $P(\boldsymbol{\lambda})$ of period $\sqrt{\mathcal{D}}$ satisfying the equations* (3.4) *can be used to construct a generating function by*

$$\eta(\mathrm{x}) = \pi^{n/2} \mathcal{F}^{-1}\big(\exp(-\pi^2 |\cdot|^2)\, P\big)(\mathrm{x})$$

such that the assertion of the previous theorem holds. But from Lemma 3.3 *follows that formula* (3.10) *gives the quasi–interpolant depending on the minimal number*

$$\sum_{|\boldsymbol{\beta}|\le M} 2^{\kappa(\boldsymbol{\beta})} = \sum_{j=\max(n-M,0)}^{n} \binom{M}{n-j} \binom{n}{j} 2^{n-j}$$

of function values $u(h(\mathbf{m}-\mathbf{k}))$, $|\xi(\mathbf{k})| \le M$. The quasi–interpolant constructed in Beatson and Light (1992) as tensor product of one-dimensional formulas depends on the values of u at $(2M+1)^n$ grid points.

Another advantage of the quasi–interpolant given by (3.10) *comes from the fixed parameter $\mathcal{D}$. Since the coefficients $a_{\xi(\mathbf{k})}$ solve the system* (3.7) *they depend on $\mathcal{D}$ but not on h. Hence, using the previously determined numbers $a_{\xi(\mathbf{k})}$ we can obtain high order approximations for different h, which is impossible in the case of $\mathcal{D}$ varying together with h.*

In the table below we give upper bounds for the coefficients in the saturation error

$$\frac{(\mathcal{D}\,\pi)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{0\}} |\boldsymbol{\nu}^{\boldsymbol{\alpha}}| \exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)$$

for different $|\boldsymbol{\alpha}|$ and $\mathcal{D}$ in the case of quasi–interpolation in $\mathbf{R}^3$.

$\lvert\boldsymbol{\alpha}\rvert$	$\mathcal{D}=3$	$\mathcal{D}=4$
0	$8.303 \cdot 10^{-13}$	$4.295 \cdot 10^{-17}$
1	$2.609 \cdot 10^{-12}$	$1.799 \cdot 10^{-16}$
2	$1.230 \cdot 10^{-11}$	$1.131 \cdot 10^{-15}$
3	$3.862 \cdot 10^{-11}$	$4.735 \cdot 10^{-15}$
4	$9.099 \cdot 10^{-11}$	$1.488 \cdot 10^{-14}$
5	$1.715 \cdot 10^{-10}$	$3.738 \cdot 10^{-14}$
6	$2.694 \cdot 10^{-10}$	$7.829 \cdot 10^{-14}$
7	$3.628 \cdot 10^{-10}$	$1.406 \cdot 10^{-13}$

Note that due to Theorem 3.2 these coefficients are multiplied by $h^{|\boldsymbol{\alpha}|}$.

4. Interpolation with Gaussian kernels

Here we consider briefly some error estimates for the interpolation with formula (1.1) at the lattice $\{h\mathrm{m}\,,\, m \in \mathbf{Z}^n\}$. We restrict ourselves to the case that $u \in L_1(\mathbf{R}^n)$ and its Fourier transform satisfies certain integrability conditions. Let us introduce the positive smooth function

$$g_h(\boldsymbol{\lambda}) := \Big(\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} \exp(-\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2)\Big)^{-1} .$$

Theorem 4.3 *Let u be a continuous function such that $\mathcal{F}u \in L_1(\mathbf{R}^n)$. Then*

$$Q_h u(\mathrm{x}) = \int\limits_{\mathbf{R}^n} e_{\mathrm{x}}(\boldsymbol{\lambda}) \exp(-\pi^2\mathcal{D}h^2|\boldsymbol{\lambda}|^2)\, g_h(\boldsymbol{\lambda}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} \mathcal{F}u(\boldsymbol{\lambda}+\frac{\boldsymbol{\nu}}{h})\, d\boldsymbol{\lambda}$$

is the sum (1.1) *interpolating $u(\mathrm{x})$ at the points $h\mathbf{m}$, $m \in \mathbf{Z}^n$.*

Proof. The series $\sum\limits_{\boldsymbol{\nu}\in\mathbf{Z}^n} \mathcal{F}u(\boldsymbol{\lambda}+\frac{\boldsymbol{\nu}}{h})$ converges absolutely for almost all $\boldsymbol{\lambda}$ to an h^{-1}-periodic L_1-function, thus

$$\exp(-\pi^2\mathcal{D}h^2|\cdot|^2)\, g_h \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} \mathcal{F}u(\cdot+\frac{\boldsymbol{\nu}}{h}) \in L_1(\mathbf{R}^n)$$

and $Q_h u$ is well defined. Since $g_h(\boldsymbol{\lambda})$ is h^{-1}-periodic we obtain

$$Q_h u(\mathrm{x}) = \int\limits_{\mathbf{R}^n} e_{\mathrm{x}}(\boldsymbol{\lambda})\, \mathcal{F}u(\boldsymbol{\lambda}) g_h(\boldsymbol{\lambda}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} e_{\mathrm{x}}(\frac{\boldsymbol{\nu}}{h}) \exp(-\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2)\, d\boldsymbol{\lambda} . \tag{4.1}$$

Applying the Poisson summation formula we derive the equality

$$e_{\mathrm{x}}(\boldsymbol{\lambda}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} e_{\mathrm{x}}(\frac{\boldsymbol{\nu}}{h}) \exp(-\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2) = (\pi\mathcal{D})^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \exp\Big(-\frac{|\mathrm{x}-h\mathbf{m}|^2}{\mathcal{D}h^2}\Big) e_{h\mathbf{m}}(\boldsymbol{\lambda}) ,$$

which holds since for fixed x both series converge absolutely. Therefore

$$Q_h u(\mathrm{x}) = (\pi\mathcal{D})^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \exp\Big(-\frac{|\mathrm{x}-h\mathbf{m}|^2}{\mathcal{D}h^2}\Big) \int\limits_{\mathbf{R}^n} \mathcal{F}u(\boldsymbol{\lambda}) g_h(\boldsymbol{\lambda})\, e_{h\mathbf{m}}(\boldsymbol{\lambda})\, d\boldsymbol{\lambda} .$$

Further, from (4.1) we get

$$u(\mathrm{x}) - Q_h u(\mathrm{x}) = \int\limits_{\mathbf{R}^n} e_{\mathrm{x}}(\boldsymbol{\lambda})\, \mathcal{F}u(\boldsymbol{\lambda}) g_h(\boldsymbol{\lambda}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} (1 - e_{\mathrm{x}}(\frac{\boldsymbol{\nu}}{h}))\, \exp(-\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2)\, d\boldsymbol{\lambda} ,$$

such that

$$e_{h\mathbf{m}}(\frac{\boldsymbol{\nu}}{h}) = e^{2\pi i\langle \mathbf{m},\boldsymbol{\nu}\rangle} = 1 , \quad \mathbf{m},\boldsymbol{\nu} \in \mathbf{Z}^n ,$$

proves the assertion. □

To estimate the approximation error we write

$$g_h(\boldsymbol{\lambda}) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} (1 - e_{\mathrm{x}}(\frac{\boldsymbol{\nu}}{h}))\, \exp(-\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2) = 1 - \frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathrm{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})} ,$$

where

$$\vartheta(\mathbf{z}) := \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n} \exp(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)\, e^{2i\langle \mathbf{z},\boldsymbol{\nu}\rangle} = \prod_{j=1}^n \vartheta_3(z_j|i\pi\mathcal{D}) \tag{4.2}$$

with the Theta-function ϑ_3 (see e.g. Whittaker and Watson (1962)). This is an integral function and quasi doubly-periodic

$$\vartheta_3(z+\pi+i\pi^2\mathcal{D}|i\pi\mathcal{D}) = \exp(\pi^2\mathcal{D})\, e^{-2iz}\,\vartheta_3(z|i\pi\mathcal{D})\ ,$$

which implies

$$\vartheta(\mathbf{z}+i\pi^2\mathcal{D}\boldsymbol{\nu}) = \exp(\pi^2\mathcal{D}|\boldsymbol{\nu}|^2)\, e^{-2i\langle\mathbf{z},\boldsymbol{\nu}\rangle}\,\vartheta(\mathbf{z})\ . \tag{4.3}$$

Further, the function $\vartheta_3(z|i\pi\mathcal{D})$ has simple zeros at the points

$$(k+\frac{1}{2})\,\pi+(m+\frac{1}{2})\,i\pi^2\mathcal{D}\ ,\quad k,m\in\mathbf{Z}\ . \tag{4.4}$$

Using (4.3) we derive

$$\begin{aligned}
u(\mathrm{x})-Q_h u(\mathrm{x}) &= \int\limits_{\mathbf{R}^n} e_{\mathrm{x}}(\boldsymbol{\lambda})\,\mathcal{F}u(\boldsymbol{\lambda})\Big(1-\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big)\,d\boldsymbol{\lambda}\\
&=\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n}\int\limits_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda}+\frac{\boldsymbol{\nu}}{h})\,\mathcal{F}u(\boldsymbol{\lambda}+\frac{\boldsymbol{\nu}}{h})\Big(1-e_{\mathrm{x}}(-\frac{\boldsymbol{\nu}}{h})\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big)\,d\boldsymbol{\lambda}\\
&=\int\limits_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda})\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n}\mathcal{F}u(\boldsymbol{\lambda}+\frac{\boldsymbol{\nu}}{h})\Big(e_{\mathrm{x}}(\frac{\boldsymbol{\nu}}{h})-\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big)\,d\boldsymbol{\lambda}\ ,
\end{aligned}$$

where C_h denotes the cube $[-\frac{1}{2h},\frac{1}{2h})^n$. Since obviously

$$\Big|\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big|\le 1$$

we obtain the estimate

$$\begin{aligned}
|u(\mathrm{x})-Q_h u(\mathrm{x})| &\le 2\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}}\int\limits_{C_h}|\mathcal{F}u(\boldsymbol{\lambda}+\frac{\boldsymbol{\nu}}{h})|\,d\boldsymbol{\lambda}\\
&\qquad+\Big|\int\limits_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda})\mathcal{F}u(\boldsymbol{\lambda})\Big(1-\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big)\,d\boldsymbol{\lambda}\Big|\\
&\le 2\int\limits_{\mathbf{R}^n\setminus C_h}|\mathcal{F}u(\boldsymbol{\lambda})|\,d\boldsymbol{\lambda}+\Big|\int\limits_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda})\mathcal{F}u(\boldsymbol{\lambda})\Big(1-\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big)\,d\boldsymbol{\lambda}\Big|
\end{aligned} \tag{4.5}$$

After this preparations one can formulate several estimates for the interpolation error. We give two examples.

Theorem 4.4 *Suppose that the function u is such that*

$$\|u\|'_N := \int\limits_{\mathbf{R}^n}|\mathcal{F}u(\boldsymbol{\lambda})|\,(1+|\boldsymbol{\lambda}|)^N\,d\boldsymbol{\lambda}<\infty$$

for some natural N. Then the estimate

$$|u(\mathrm{x})-Q_h u(\mathrm{x})|\le c_N\,(2h)^N\,\|u\|'_N+\sum_{|\boldsymbol{\alpha}|=0}^{N-1}|a_{\boldsymbol{\alpha}}(\mathrm{x})|\,\Big(\frac{\pi\mathcal{D}h}{2}\Big)^{|\boldsymbol{\alpha}|}\frac{|\partial^{\boldsymbol{\alpha}}u(\mathrm{x})|}{\boldsymbol{\alpha}!}$$

is valid, where

$$a_{\boldsymbol{\alpha}}(\mathrm{x}) = \partial_{\mathbf{z}}^{\boldsymbol{\alpha}}\Big(1 - \frac{\vartheta(\mathbf{z} + \frac{\pi \mathrm{x}}{h})}{\vartheta(\mathbf{z})}\Big)\bigg|_{\mathbf{z}=\mathbf{0}} \tag{4.6}$$

and the constant c_N does not depend on u.

Proof. We take the Taylor series of

$$1 - \frac{\vartheta(\mathbf{z} + \frac{\pi \mathrm{x}}{h})}{\vartheta(\mathbf{z})} = \sum_{|\boldsymbol{\alpha}|=0}^{\infty} a_{\boldsymbol{\alpha}}(\mathrm{x}) \, \frac{\mathbf{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \,,$$

which due to (4.2) and (4.4) converges absolutely and uniformly for all $\mathrm{x} \in \mathbf{R}^n$ and all $\mathbf{z} = (z_1, \ldots, z_n)$ with $|z_j| \leq \frac{\pi}{2}\sqrt{1 + \pi^2 \mathcal{D}^2} - \delta,\ \delta > 0$. Note that the functions $a_{\boldsymbol{\alpha}}(\mathrm{x})$ given by formula (4.6) are smooth and h-periodic.

Since for $\boldsymbol{\lambda} \in C_h$ it holds $|\pi^2 \mathcal{D} h \lambda_j| \leq \pi^2 \mathcal{D}/2$ we derive

$$\begin{aligned}
&\int_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda}) \mathcal{F}u(\boldsymbol{\lambda}) \Big(1 - \frac{\vartheta(i\pi^2 \mathcal{D} h \boldsymbol{\lambda} + \frac{\pi \mathrm{x}}{h})}{\vartheta(i\pi^2 \mathcal{D} h \boldsymbol{\lambda})}\Big) \, d\boldsymbol{\lambda} \\
&= \sum_{|\boldsymbol{\alpha}|=0}^{\infty} \frac{a_{\boldsymbol{\alpha}}(\mathrm{x})}{\boldsymbol{\alpha}!} (i\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|} \int_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda}) \, \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \, \mathcal{F}u(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda} \\
&= \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{a_{\boldsymbol{\alpha}}(\mathrm{x})}{\boldsymbol{\alpha}!} \Big(\Big(\frac{\pi \mathcal{D} h}{2}\Big)^{|\boldsymbol{\alpha}|} \partial^{\boldsymbol{\alpha}} u(\mathrm{x}) - (i\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|} \int_{\mathbf{R}^n \setminus C_h} e_{\mathrm{x}}(\boldsymbol{\lambda}) \, \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \, \mathcal{F}u(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda}\Big) \\
&\quad + \sum_{|\boldsymbol{\alpha}|=N}^{\infty} \frac{a_{\boldsymbol{\alpha}}(\mathrm{x})}{\boldsymbol{\alpha}!} (i\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|} \int_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda}) \, \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \, \mathcal{F}u(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda} \,.
\end{aligned}$$

But

$$\begin{aligned}
(\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|} \int_{\mathbf{R}^n \setminus C_h} |\boldsymbol{\lambda}^{\boldsymbol{\alpha}}| \, |\mathcal{F}u(\boldsymbol{\lambda})| \, d\boldsymbol{\lambda} &\leq (\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|} \, \|u\|'_N \max_{\boldsymbol{\lambda} \in \mathbf{R}^n \setminus C_h} \frac{|\boldsymbol{\lambda}^{\boldsymbol{\alpha}}|}{(1 + |\boldsymbol{\lambda}|)^N} \\
&\leq (\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|} \, \|u\|'_N \, (2h)^{N - |\boldsymbol{\alpha}|} = (2h)^N \Big(\frac{\pi \mathcal{D}}{2}\Big)^{|\boldsymbol{\alpha}|} \|u\|'_N \,,
\end{aligned}$$

such that in view of (4.5) it remains to estimate

$$\begin{aligned}
&\Big| \sum_{|\boldsymbol{\alpha}|=N}^{\infty} a_{\boldsymbol{\alpha}}(\mathrm{x}) \, \frac{(i\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \int_{C_h} e_{\mathrm{x}}(\boldsymbol{\lambda}) \, \boldsymbol{\lambda}^{\boldsymbol{\alpha}} \, \mathcal{F}u(\boldsymbol{\lambda}) \, d\boldsymbol{\lambda} \Big| \\
&\leq \sum_{|\boldsymbol{\alpha}|=N}^{\infty} |a_{\boldsymbol{\alpha}}(\mathrm{x})| \, \frac{(\pi^2 \mathcal{D} h)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \, \|u\|'_N \max_{\boldsymbol{\lambda} \in C_h} |\boldsymbol{\lambda}|^{|\boldsymbol{\alpha}| - N} \\
&= \Big(\frac{2h}{\sqrt{n}}\Big)^N \|u\|'_N \sum_{|\boldsymbol{\alpha}|=N}^{\infty} \Big(\frac{\sqrt{n}\pi^2 \mathcal{D}}{2}\Big)^{|\boldsymbol{\alpha}|} \frac{|a_{\boldsymbol{\alpha}}(\mathrm{x})|}{\boldsymbol{\alpha}!} \,.
\end{aligned}$$

The last series is uniformly bounded and therefore the constant c_N can be estimated by

$$c_N < 2 + \max_{\mathrm{x}} \sum_{|\boldsymbol{\alpha}|=0}^{\infty} \Big(\frac{\sqrt{n}\pi^2 \mathcal{D}}{2}\Big)^{|\boldsymbol{\alpha}|} \frac{|a_{\boldsymbol{\alpha}}(\mathrm{x})|}{\boldsymbol{\alpha}!} \,. \quad \square$$

Remark: *Using the function ϑ the saturation error for the quasi–interpolant of Theorem 3.2 can be written in the form*

$$\sum_{|\boldsymbol{\alpha}|=0}^{N-1} \Big(\frac{\pi\mathcal{D}h}{2}\Big)^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x})}{\boldsymbol{\alpha}!} \ \partial_{\mathbf{z}}^{\boldsymbol{\alpha}} (1-\vartheta(\mathbf{z}+\frac{\pi\mathbf{x}}{h}))\Big|_{\mathbf{z}=\mathbf{0}} .$$

Finally we show that the interpolation with (1.1) converges exponentially up to the saturation error.

Theorem 4.5 *If for some $a>0$*

$$\int\limits_{\mathbf{R}^n} |\mathcal{F}u(\boldsymbol{\lambda})| \exp(a|\boldsymbol{\lambda}|)\, d\boldsymbol{\lambda} =: \|u\|_a' < \infty$$

then for all $\mathbf{x}\in\mathbf{R}^n$ the error of the interpolation with (1.1) can be estimated by

$$|u(\mathbf{x}) - Q_h u(\mathbf{x})| \le \Big(c_1(\mathcal{D}) \exp\Big(-\frac{a}{2h}\Big) + c_2(\mathcal{D})\Big) \|u\|_a' ,$$

where

$$c_1(\mathcal{D}) \le 2\Big(1+\sum_{k=1}^{n} \binom{n}{k} 2^k \exp(-\pi^2\mathcal{D}(k^2-k)) \exp\Big(-\frac{a}{2h}(\sqrt{k}-1)\Big)\Big) ,$$

$$c_2(\mathcal{D}) \le 2\Big(\sum_{k=1}^{n} \binom{n}{k} 2^k \exp(-\pi^2\mathcal{D}k^2) + \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\mathbf{V}} \exp(-\pi^2\mathcal{D}\sum_{j=1}^{n}(|\nu_j|^2-|\nu_j|))\Big) .$$

and $\mathbf{V} := \{\boldsymbol{\nu}\in\mathbf{Z}^n : \ |\nu_j|\le 1\}$.

Proof. Again we use the estimate (4.5). Obviously

$$\int\limits_{\mathbf{R}^n\setminus C_h} |\mathcal{F}u(\boldsymbol{\lambda})|\, d\boldsymbol{\lambda} \le \exp\big(-\frac{a}{2h}\big) \int\limits_{\mathbf{R}^n} |\mathcal{F}u(\boldsymbol{\lambda})| \exp(a|\boldsymbol{\lambda}|)\, d\boldsymbol{\lambda} .$$

Using the definition of the function ϑ we obtain

$$\left| 1 - \frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\right| = \left| \frac{\sum\limits_{\boldsymbol{\nu}\in\mathbf{Z}^n} \exp\left(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\right) (1-e_{\mathbf{x}}(\frac{\boldsymbol{\nu}}{h}))}{\sum\limits_{\boldsymbol{\nu}\in\mathbf{Z}^n} \exp\left(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\right)} \right|$$

$$\le 2 \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{0\}} \exp\left(-\pi^2\mathcal{D}|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\right) .$$

Hence

$$\Big|\int\limits_{C_h} e_{\mathbf{x}}(\boldsymbol{\lambda})\mathcal{F}u(\boldsymbol{\lambda})\Big(1-\frac{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda}+\frac{\pi\mathbf{x}}{h})}{\vartheta(i\pi^2\mathcal{D}h\boldsymbol{\lambda})}\Big)\, d\boldsymbol{\lambda}\Big|$$

$$\le 2\int\limits_{C_h} |\mathcal{F}u(\boldsymbol{\lambda})| \exp(a|\boldsymbol{\lambda}|)\, d\boldsymbol{\lambda} \ \max_{\boldsymbol{\lambda}\in C_h}\Big(\exp(-a|\boldsymbol{\lambda}|) \sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{0\}} \exp(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle))\Big).$$

To estimate

$$\max_{\boldsymbol{\lambda}\in C_h}\Big(\exp(-a|\boldsymbol{\lambda}|)\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{0\}}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)\Big)$$

we introduce the subsets

$$\mathbf{V}:=\{\,\boldsymbol{\nu}\in\mathbf{Z}^n:\ |\nu_j|\le 1\}\quad\text{and}\quad \mathbf{V}_k:=\{\,\boldsymbol{\nu}\in\mathbf{V}:\ |\boldsymbol{\nu}|=k\}\ ,\quad k=1,\dots,n\ .$$

Then

$$\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\{0\}}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)=$$

$$\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\mathbf{V}}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)+\sum_{k=1}^{n}\sum_{\boldsymbol{\nu}\in\mathbf{V}_k}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)\ .$$

For $\boldsymbol{\lambda}\in C_h$ we have $|h\lambda_j|\le\frac12$ such that

$$|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle\ge\sum_{j=1}^{n}\big(|\nu_j|^2-|\nu_j|\big)\ ,$$

yielding

$$\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\mathbf{V}}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)<\sum_{\boldsymbol{\nu}\in\mathbf{Z}^n\setminus\mathbf{V}}\exp\big(-\pi^2\mathcal{D}\sum_{j=1}^{n}(|\nu_j|^2-|\nu_j|)\big)\ .$$

Now we estimate

$$\begin{aligned}
&\exp(-a|\boldsymbol{\lambda}|)\sum_{k=1}^{n}\sum_{\boldsymbol{\nu}\in\mathbf{V}_k}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)\\
&=\sum_{k=1}^{n}\exp(-\pi^2\mathcal{D}k^2)\,2^k\exp(-a|\boldsymbol{\lambda}|)\sum_{\boldsymbol{\beta}\in\mathbf{V}_k\cap\mathbf{Z}^n_{\ge0}}\prod_{j=1}^{n}\cosh\big(2\pi^2\mathcal{D}h\beta_j\lambda_j\big)\\
&\le\sum_{k=1}^{n}\exp(-\pi^2\mathcal{D}k^2)\,2^k\sum_{\boldsymbol{\beta}\in\mathbf{V}_k\cap\mathbf{Z}^n_{\ge0}}\prod_{j=1}^{n}\exp\Big(-\frac{a}{\sqrt{k}}\beta_j|\lambda_j|\Big)\cosh\big(2\pi^2\mathcal{D}h\beta_j\lambda_j\big)\,.
\end{aligned}$$

Note that for $0\le z\le\dfrac12$

$$\exp\Big(-\frac{az}{h\sqrt{k}}\Big)\cosh\big(2\pi^2\mathcal{D}z\big)\le\begin{cases}\exp\Big(\pi^2\mathcal{D}-\dfrac{a}{2h\sqrt{k}}\Big) & ,\quad 2\pi^2\mathcal{D}\ge\dfrac{a}{h\sqrt{k}}\ ,\\ 1 & ,\quad 2\pi^2\mathcal{D}\le\dfrac{a}{h\sqrt{k}}\ .\end{cases}$$

This implies that for $h\le\dfrac{a}{2\pi^2\mathcal{D}\sqrt{n}}$ and $\boldsymbol{\lambda}\in C_h$

$$\exp(-a|\boldsymbol{\lambda}|)\sum_{k=1}^{n}\sum_{\boldsymbol{\nu}\in\mathbf{V}_k}\exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2+2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big)\le\sum_{k=1}^{n}\binom{n}{k}\,2^k\,\exp(-\pi^2\mathcal{D}k^2)\ .$$

If $h \geq \dfrac{a}{2\pi^2\mathcal{D}}$ then

$$\begin{aligned}
&\exp(-a|\boldsymbol{\lambda}|) \sum_{k=1}^{n} \sum_{\boldsymbol{\nu}\in\mathbf{V}_k} \exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2 + 2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big) \\
&\leq \sum_{k=1}^{n} \binom{n}{k} 2^k \exp(-\pi^2\mathcal{D}k^2) \exp\Big(\pi^2\mathcal{D}k - \frac{a\sqrt{k}}{2h}\Big) \\
&= \exp\Big(-\frac{a}{2h}\Big) \sum_{k=1}^{n} \binom{n}{k} 2^k \exp\big(-\pi^2\mathcal{D}(k^2-k)\big) \exp\Big(-\frac{a}{2h}(\sqrt{k}-1)\Big) .
\end{aligned}$$

Finally, in the case $\dfrac{a}{2\pi^2\mathcal{D}\sqrt{j}} \leq h \leq \dfrac{a}{2\pi^2\mathcal{D}\sqrt{j-1}}$, $j = 2, \ldots, n$, we obtain

$$\begin{aligned}
&\exp(-a|\boldsymbol{\lambda}|) \sum_{k=1}^{n} \sum_{\boldsymbol{\nu}\in\mathbf{V}_k} \exp\big(-\pi^2\mathcal{D}(|\boldsymbol{\nu}|^2 + 2h\langle\boldsymbol{\nu},\boldsymbol{\lambda}\rangle)\big) \\
&\leq \sum_{k=1}^{j-1} \binom{n}{k} 2^k \exp(-\pi^2\mathcal{D}k^2) + \sum_{k=j}^{n} \binom{n}{k} 2^k \exp(-\pi^2\mathcal{D}k^2) \exp\Big(\pi^2\mathcal{D}k - \frac{a\sqrt{k}}{2h}\Big) .
\end{aligned}$$

Collecting all estimates we get the assertion. □

Observe that for $\mathcal{D} \geq 2$ the constants $c_1(\mathcal{D})$ and $c_2(\mathcal{D})$ can be given in machine precision by

$$c_1(\mathcal{D}) = 2 + 4n \quad , \quad c_2(\mathcal{D}) = 4n\exp(-\pi^2\mathcal{D}) ,$$

which yields in $\mathbf{R}^3$ and the case $\mathcal{D} = 3$ the estimate of the interpolation error

$$|u(\mathrm{x}) - Q_h u(\mathrm{x})| \leq \Big(14 \cdot \exp\Big(-\frac{a}{2h}\Big) + 1.6605 \cdot 10^{-12}\Big) \|u\|'_a .$$

We remark that the best L_2-approximant $P_h u$ of the form (1.1) to $u \in L_2(\mathbf{R}^n)$ has the Fourier transform

$$\mathcal{F}(P_h u)(\boldsymbol{\lambda}) = \exp(-\pi^2\mathcal{D}h^2|\boldsymbol{\lambda}|^2) \frac{\sum\limits_{\boldsymbol{\nu}\in\mathbf{Z}^n} \mathcal{F}u(\boldsymbol{\lambda} + \dfrac{\boldsymbol{\nu}}{h}) \exp(-\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2)}{\sum\limits_{\boldsymbol{\nu}\in\mathbf{Z}^n} \exp(-2\pi^2\mathcal{D}|h\boldsymbol{\lambda}+\boldsymbol{\nu}|^2)} ,$$

(see de Boor *et al* (1994)). Hence the same technique provides similar estimates for the error of the orthogonal projection.

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