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Analysis of profile functions for general linear regularization methods

Bernd Hofmann and Peter Mathé

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. The stable approximate solution of ill-posed linear operator equations in Hilbert spaces requires regularization. Tight bounds for the noise-free part of the regularization error are constitutive for bounding the overall error. Norm bounds of the noise-free part which decrease to zero along with the regularization parameter are called profile functions and are subject of our analysis. The interplay between properties of the regularization and certain smoothness properties of solution sets, which we shall describe in terms of source-wise representations is crucial for the decay of associated profile functions. On the one hand, we show that a given decay rate is possible only if the underlying true solution has appropriate smoothness. On the other hand, if smoothness fits the regularization, then decay rates are easily obtained. If smoothness does not fit, then we will measure this in terms of some distance function. Tight bounds for these allow us to obtain profile functions. Finally we study the most realistic case when smoothness is measured with respect to some operator which is related to the one governing the original equation only through a link condition. In many parts the analysis is done on geometric basis, extending classical concepts of linear regularization theory in Hilbert spaces. We emphasize intrinsic features of linear ill-posed problems which are frequently hidden in the classical analysis of such problems.

1. Introduction

We study noisy linear operator equations

$$y^\delta = Ax^\dagger + \delta\xi \qquad (\|\xi\| \le 1)\,, \tag{1.1}$$

where $A : X \to Y$ is some bounded linear operator mapping between infinite-dimensional separable Hilbert spaces X and Y and $\delta > 0$ denotes the noise level. The spaces X and Y are equipped with norms $\|\cdot\|$. The same norm symbol is also used for associated operator norms.

We assume that A is *injective* and that the range $\mathcal{R}(A)$ is *not closed* in Y. Then the *linear operator equation* $Ax = y$ has a unique solution $x = x^\dagger \in X$, for every $y \in \mathcal{R}(A)$, but the equation is *ill-posed* since A^{-1} is an unbounded operator. Thus regularization is required in order to find stable approximate solutions of the operator equation based on noisy data $y^\delta \in Y$. We consider general linear regularization schemes based on a family of piecewise continuous functions $g_\alpha(t) \quad (0 < t \le a := \|A^*A\|)$ for regularization parameters $0 < \alpha \le \overline{\alpha}$. The family g_α determines the *regularization method*. Once a regularization g_α is chosen, the approximate solution to (1.1) is given by

$$x_\alpha^\delta = g_\alpha(A^*A)A^*y^\delta.$$

For such approximate solution x_α^δ we obtain an obvious error bound, using the intermediate quantity $x_\alpha = g_\alpha(A^*A)A^*y = g_\alpha(A^*A)A^*Ax^\dagger$, as

$$e(x^\dagger,\alpha,\delta) := \|x_\alpha^\delta - x^\dagger\| \le \|x^\dagger - x_\alpha\| + \delta\|g_\alpha(A^*A)A^*\| \qquad \text{for all} \quad 0 < \alpha \le \overline{\alpha}. \tag{1.2}$$

The second summand on the right is independent of the underlying true solution. Therefore the accuracy of the regularized solution is basically determined by tight bounds on the norm of the residual $\|x^\dagger - x_\alpha\| = \|r_\alpha(A^*A)x^\dagger\|$, where we denote by $r_\alpha(t) := 1 - t\,g_\alpha(t)$ $(0 < t \le a)$ the residual or bias functions related to the regularization method g_α. Bounds which are increasing functions in $\alpha > 0$ will give rise to what we call profile functions.

The outline is as follows. In Section 2 we recall the basic underlying quantities, namely general linear regularization methods for operator equations in Hilbert space and the concept of solution smoothness in terms of general source conditions. Then, in Section 3 we associate profile functions to any given regularization and to any set of smooth solutions and discuss their existence. The rate at which profile functions decay to zero turns out to be crucial and is the objective of our analysis. It will become clear that this rate depends on the underlying regularization as well as on the solution smoothness. In Section 4 we indicate situations when maximal rates of decay occur, regardless of the underlying solution smoothness, namely due to the limited qualification of the regularization method. We close this part by showing that decay rates imply solution smoothness.

The constructive part of obtaining explicit descriptions of profile functions, as dependent on the qualification of the regularization and smoothness properties of the solution with respect to the operator A is carried out in Sections 5 and 6 for several degrees of generality. We start in Section 5 with the easiest case, when solution smoothness is measured in terms of general source conditions given through functions of A^*A. This is then extended to the situation where a source condition is satisfied only approximately, measured in terms of a specific concept of distance functions. Tight upper bounds for such distance functions imply profile functions.

We close the analysis with Section 6 discussing the situation when solution smoothness is measured with respect to a self-adjoint operator $G : X \to X$ with non-closed range which is different from A^*A. In this case an assumption, linking A^*A and G, will allow us to draw conclusions on the decay rate of the associated profile functions.

In many parts the analysis is done on geometric basis, extending classical concepts as used in the theory of linear ill-posed equations in Hilbert space. By doing so we not only extend previous results to a more general situation, but we aim at emphasizing intrinsic features of the problems under consideration. Such features are often hidden in the classical analysis of linear ill-posed problems.

2. General linear regularization methods and general smoothness

As mentioned in the introduction, profile functions will be assigned to regularization methods and solution sets of equation (1.1). We start with the notion of a general linear regularization scheme. Then we turn to the description of solution smoothness in terms of general source conditions.

The basic underlying objects are index functions, and we recall the following definition, as known in the literature (e.g. [8, 16, 3]).

Definition 2.1. A real function $\varphi(t)$ $(0 < t \le \bar{t})$ is called *index function* if it is continuous, strictly increasing and satisfies the limit condition $\lim_{t\to 0+} \varphi(t) = 0$.

2.1. General regularization methods.

Definition 2.2. A family of functions $g_\alpha(t)$ $(0 < t \le a)$, defined for parameters $0 < \alpha \le \overline{\alpha}$, is called *regularization* if they are piece-wise continuous in α and the following three properties are satisfied:

(i) For each $0 < t \leq a$ there is convergence $|r_\alpha(t)| \to 0$ as $\alpha \to 0$.
(ii) There is a constant γ_1 such that $|r_\alpha(t)| \leq \gamma_1$ for all $0 < \alpha \leq \overline{\alpha}$.
(iii) There is a constant γ_* such that $\sqrt{t}\,|g_\alpha(t)| \leq \gamma_*/\sqrt{\alpha}$ for all $0 < \alpha \leq \overline{\alpha}$.

Example 2.3. The most famous method of regularization is the *Tikhonov method* with $g_\alpha(t) = 1/(t+\alpha)$, which satisfies the properties of Definition (2.2) for the constants $\gamma_1 = 1$ and $\gamma_* = 1/2$ and arbitrarily large $\overline{\alpha} > 0$.

Example 2.4. Another common regularization method is *spectral cut-off*, which is given as
$g_\alpha(t) = \begin{cases} 0 & (0 < t < \alpha) \\ 1/t & (\alpha \leq t \leq a) \end{cases}$ with respective residual $r_\alpha(t) = \begin{cases} 1 & (0 < t < \alpha) \\ 0 & (\alpha \leq t \leq a) \end{cases}$.
Obviously this obeys the properties from Definition 2.2 with $\gamma_1 = \gamma_* = 1$. Also for that method, the upper bound $\overline{\alpha}$ for the regularization parameter can be selected arbitrarily.

Example 2.5. Iterative regularization methods, as for instance *Landweber iteration*, where for some $0 < \mu < 1/\|A^*A\|$ we let

$$x_n^\delta := \sum_{j=0}^{n-1} (I - \mu A^*A)^j A^* y^\delta, \quad n = 1, 2, \dots,$$

are covered by this approach when assigning $n := \lfloor 1/\alpha \rfloor$ $(0 < \alpha < 1)$. Thus with this identification we obtain $g_\alpha(t) := 1/t\,(1 - (1-\mu t)^n)$ and the corresponding residual $r_\alpha(t) := (1-\mu t)^{\lfloor 1/\alpha \rfloor}$ $(0 < \alpha < 1)$, hence obviously $\gamma_1 = 1$. It remains to bound γ_*. Bernoulli's inequality yields $1 - n\mu t \leq (1-\mu t)^n$, which can be used to bound

$$\sqrt{t} g_\alpha(t) = 1/\sqrt{t}\,(1 - (1-\mu t)^n) \leq (1/t\,(1 - (1-\mu t)^n))^{1/2} \leq \sqrt{\mu n}\,.$$

By the definition of n this yields $\gamma_* = \sqrt{2\mu}$.

We mention the following technical result from [23, Lemma 2.1], see also [6, Proof of Proposition 4.13].

Lemma 2.6. *Let $g_\alpha(t)$ $(0 < t \leq a,\ 0 < \alpha \leq \bar{\alpha})$ be a regularization with constant γ_*. If $0 < t \leq \min\{\alpha, a\}$, then $\left|r_{(4\gamma_*^2\alpha)}(t)\right| \geq 1/2$.*

The above requirements (i) – (iii) are made to ensure convergence of regularization methods for any given element $x^\dagger \in X$. However, these are not enough to describe rates of convergence.

As introduced in the papers [15] and [16] – [18], we measure the *qualification* of any regularization method in terms of index functions ψ.

Definition 2.7. Let $\psi(t)$ $(0 < t \leq a)$ be an index function. A regularization g_α for the operator equation (1.1) is said to have qualification ψ with constant $\gamma \in (0, \infty)$ if

$$\sup_{0<t\leq a} |r_\alpha(t)|\,\psi(t) \;\leq\; \gamma\,\psi(\alpha) \qquad \text{for all} \quad 0 < \alpha \leq a\,. \tag{2.1}$$

This definition generalizes the concept of qualification of a regularization method as a finite number or infinity, as for example used in [6]. We remark that a first systematic discussion of the interrelations between solution smoothness and that traditional concept of qualification was given in [28, 29].

For Tikhonov regularization (see Example 2.3) we can give sufficient conditions for ψ being a qualification in different ways, as this is formulated in the following proposition. For more details and proofs we refer to [17, 18] and [3].

Proposition 2.8. *The index function $\psi(t)$ $(0 < t \le a)$ is a qualification of Tikhonov regularization with constant $\gamma = 1$ if either (a) $\psi(t)/t$ is non-increasing on $(0, a]$ or (b) $\psi(t)$ is concave on $(0, a]$.*

If there exists an argument $\hat{t} \in (0, a)$ such that (c) $\psi(t)/t$ is non-increasing on $(0, \hat{t}\,]$ or (d) $\psi(t)$ is concave on $(0, \hat{t}\,]$, then ψ is a qualification with constant $\gamma = \psi(a)/\psi(\hat{t}\,)$.

2.2. **Measuring solution smoothness.** In a wide sense the smoothness of expected solutions $x^\dagger$ to (1.1) can be written as a property of the form $x^\dagger \in M$ with $M \subseteq \mathcal{R}(G)$ for some 'smoothing' linear operator $G : X \to X$, where G is assumed to be positive self-adjoint with non-closed range $\mathcal{R}(G)$ (see also [3], [20]). Specifically, here we shall assume that the solution $x^\dagger$ belongs to a set

$$G_\tau(R) := \{x \in X : \ x = \tau(G)w, \quad \|w\| \le R\} \tag{2.2}$$

with some index function $\tau(t)$ $(0 < t \le \|G\|)$.

As the following lemma asserts such set is closed in X and even compact whenever G is compact.

Lemma 2.9. *For a positive self-adjoint bounded linear operator $G : X \to X$ and an index function $\tau(t)$ $(0 < t \le \|G\|)$ the set $G_\tau(R)$ from (2.2) is closed in X. Moreover, $G_\tau(R)$ is a compact subset of X whenever G is a compact operator.*

Proof. First we show that $G_\tau(R)$ is a closed subset in X. We show that the image $\{x \in X : x = Gw, \ w \in X, \ \|w\| \le R\}$ of the centered ball with radius R in X with respect to any bounded positive self-adjoint linear operator $G : X \to X$ is a closed subset of X. Since $\tau(G)$ has the same properties as a consequence of the boundedness of any index function τ, this shows the closedness of $G_\tau(R)$. Consider a convergent sequence of images $Gx_n \to y_0 \in X$ with $\|x_n\| \le R$. Since any closed ball in X is weakly precompact and weakly closed, there is a weakly convergent subsequence $x_{n_k} \rightharpoonup x_0$ with $\|x_0\| \le R$. Since every continuous operator G is also weakly continuous and hence weakly closed, this implies the weak convergence $Gx_{n_k} \rightharpoonup Gx_0$ thus $y_0 = Gx_0$ which shows the required closedness. Moreover, for compact G it is evident that $\tau(G) : X \to X$ is a compact operator and then $G_\tau(R)$ is a precompact subset of X. Since $G_\tau(R)$ is closed in X, this implies the compactness and proves the lemma. $\square$

In our analysis below for index functions τ we shall assign pairs (G, τ) Hilbert spaces X_τ^G having $G_\tau(1)$ as their unit balls. In particular, we use the shortcut $H := A^*A$

and consider Hilbert spaces X_φ^H for index functions φ with the set $H_\varphi(1)$ as unit ball, where we define

$$H_\varphi(R) := \{x \in X : \; x = \varphi(A^*A)w, \quad \|w\| \le R\}\,. \tag{2.3}$$

Corresponding norms will be denoted by $\|\cdot\|_{X_\tau^G}$ and $\|\cdot\|_{X_\varphi^H}$, respectively. This construction is basically due to [7].

3. Profile functions

In this section we shall introduce the notion of a profile function, discuss the problem of existence and show that their decay is related to smoothness of the underlying solution $x^\dagger$ of equation (1.1).

3.1. **Definition and existence.** Having chosen a linear regularization method g_α, and having fixed a set $M \subset X$ of possible solutions to (1.1) we assign profile functions as follows.

Definition 3.1. An index function $f\colon (0,\overline{\alpha}] \to (0,\infty)$ is called *profile function for* (M, g_α) whenever

$$\sup_{x\in M} \|r_\alpha(A^*A)x\| \le f(\alpha) \qquad \text{for all} \quad 0 < \alpha \le \overline{\alpha}\,. \tag{3.1}$$

In the definition we suppress the dependence of profile functions f on the operator A, governing the equation (1.1). If $M := \{x\} \in X$ is a singleton, then we shall write (x, g_α), instead of $(\{x\}, g_\alpha)$. Note that the bound (3.1) is required only for $\alpha \le \overline{\alpha}$, which is useful for asymptotic considerations as $\delta \to 0$ in (1.1).

The character of possible profile functions f for (M, g_α) is closely connected with three ingredients and their interplay. In this context, properties of the regularization g_α as first component and of the set $M \subset X$ expressing the solution smoothness as second components meet as third component the smoothing behavior of the operator A in equation (1.1) which leads to the non-closedness of the range $\mathcal{R}(A)$.

Remark 3.2. Once a profile function $f(\alpha)$ as above is found, together with property (iii) of Definition 2.2 this allows us to continue the estimate (1.2) to derive

$$e(x^\dagger, \alpha, \delta) \le f(\alpha) + \frac{\gamma_*\,\delta}{\sqrt{\alpha}} \qquad \text{for all} \quad 0 < \alpha \le \overline{\alpha}\,, \tag{3.2}$$

uniformly for $x^\dagger \in M$. The bound on the right in (3.2) can be minimized with respect to the choice of α depending on δ. To this end we consider the index function

$$\Theta(\alpha) := \sqrt{\alpha}\, f(\alpha) \qquad (0 < \alpha \le \overline{\alpha}).$$

Let $\alpha_* = \alpha_*(\delta) = \Theta^{-1}(\delta)$ $(0 < \delta \le \Theta(\bar{\alpha}))$. Then we obtain uniformly for $x^\dagger \in M$ that

$$e(x^\dagger, \alpha_*, \delta) \le (1 + \gamma_*) f(\alpha_*), \tag{3.3}$$

Thus the function $f(\Theta^{-1}(\delta))$ yields a *convergence rate* of the regularization g_α for $x^\dagger$ as $\delta \to 0$. This rate is achieved by an a priori parameter choice $\alpha_* = \alpha_*(\delta)$.

First we shall establish that profile functions exist for any regularization g_α and compact subsets $M \subset X$.

Proposition 3.3. *Let g_α be any regularization and $M \subset X$ be compact. Then there is a profile function for (M, g_α).*

Proof. From the properties (i) and (ii) of Definition 2.2 we deduce for $\alpha \to 0$ pointwise convergence $r_\alpha(A^*A)x \to 0$ for all $x \in X$ (see, e.g., [6, Theorem 4.1]). This convergence is uniform on compact sets $M \subset X$. Hence we have

$$h(\alpha) := \sup_{x \in M} \|r_\alpha(A^*A)x\| \to 0 \quad \text{as } \alpha \to 0.$$

Its increasing majorant $\bar{h}(\alpha) := \sup_{0<s\le\alpha} h(s)$, which is well-defined for sufficiently small positive α, satisfies $\lim_{\alpha\to 0} \bar{h}(\alpha) = 0$. If $\bar{h}(\alpha)$ is continuous and non-vanishing, then it is a profile function. Otherwise, suppose $\bar{h}(s) = 0$ for some $s > 0$. We fix some $t > 0$ with $\bar{h}(t) > 0$ and let

$$\tilde{h}(x) := \begin{cases} \bar{h}(x), & x > t, \\ \bar{h}(t), & s < x \le t, \\ x/s\ \bar{h}(t), & 0 < x \le s, \end{cases}$$

which, when continuous, defines an index function. □

Thus if G is compact and τ is an index function, then for any regularization g_α there are profile functions for $(G_\tau(R), g_\alpha)$, where the sets $G_\tau(R)$ were defined in (2.2).

On the other hand, there cannot exist profile functions for (M, g_α), where $M := \{x \in X : \|x\| \le 1\}$ is the unit ball in X. Their existence would imply that $\|r_\alpha(A^*A)\| \to 0$ as $\alpha \to 0$ and hence that the range $\mathcal{R}(A)$ were closed, which would be contrary to the ill-posedness of the problem under consideration (see, e.g., [25] and [6, Chapter 3.1]). More generally, extending this argument, profile functions cannot exist for (M, g_α), whenever M possesses an interior point.

However, there are profile functions for non-compact sets. In Proposition 5.1 below profile functions for $(H_\varphi(R), g_\alpha)$ will be obtained, where the operator A may be compact (ill-posedness of type II in the sense of Nashed, [22]) or non-compact (ill-posedness of type I). In the latter case this yields non-compact sets $M = H_\varphi(R)$. Another specific example of profile functions for the non-compact set $M = \{x \in L^\infty(0,1) : \|x\|_{L^\infty(0,1)} \le R\} \subset X = L^2(0,1)$ for the Tikhonov regularization and multiplication operators A mapping in $L^2(0,1)$ can be taken from [11]. This is not by chance and some explanation will be given in Remark 5.2, below. Roughly speaking, if smoothness properties of M are appropriate for the underlying operator A from equation (1.1), then profile functions exist for (M, g_α), regardless of their compactness. In this respect, compactness of M may be viewed as universal (problem independent) smoothness.

3.2. **Decay rates yield solution smoothness.** To exhibit the fact that a decay rate of a profile function implies solution smoothness in the sense of Section 2.2 we start with the following result, which extends analysis in [23]. We recall that the operator $H = A^*A$ admits a spectral resolution with a family $(E_\lambda)_{0<\lambda\le a}$ of projections, which is assumed to be such that $\lambda \mapsto d\|E_\lambda x^\dagger\|^2$ is left continuous, thus represents a (spectral) measure. From Lemma 2.6 we derive the following estimate.

Lemma 3.4. *Let g_α be a regularization with constant γ_* as in property (iii) of Definition 2.2. The following estimate holds true.*

$$\|r_{(4\gamma_*^2\alpha)}x^\dagger\| \ge \frac{1}{2}\left(\int_0^\alpha d\|E_\lambda x^\dagger\|^2\right)^{1/2} \qquad \text{for all} \quad 0 < \alpha \le \min\left\{a, \bar{\alpha}/4\gamma_*^2\right\}. \tag{3.4}$$

Before turning to the main result of this section we state the following lemma.

Lemma 3.5. *Suppose $\varphi(t)$ $(0 < t \le \bar{t})$ is an index function. There is a sequence $f_n(t)$ $(0 < t \le \bar{t})$ of step functions of the form $\sum_{j=1}^m c_j\chi_{(0,\alpha_j)}(t)$ converging to $1/\varphi(t)$ point-wise and $f_n(t) \le 1/\varphi(t)$.*

Proof. Given any such φ and $n \in \mathbb{N}$ large enough $n \ge n_0$, we let $f(t) = 1/\varphi(t)$ and truncate at $t_n = f^{-1}(n) < \bar{t}$ to obtain $g^n(t)$ $(0 \le t \le \bar{t})$, which is a non-increasing bounded continuous function on the closed interval $[0, \bar{t}]$. Thus there is a step function $f_n(t)$ of the required form, satisfying $|f_n(t) - g^n(t)| \le 1/n$. The sequence $f_n(t)$ $(0 < t \le \bar{t})$, $n = n_0, n_0 + 1, \ldots$ converges point-wise to f. ☐

Given a regularization g_α with constant γ_* and any index function $h(t)$ $(0 < t \le a)$, we can assign a non-negative measure Φ_h on $(0, a]$ by letting

$$\Phi_h[0, \alpha) := h(4\gamma_*^2\alpha) \quad (0 < 4\gamma_*^2\alpha \le a).$$

With this notation we can formulate the following result.

Theorem 3.6. *Let $g_\alpha(t)$ $(0 < t \le a)$ for the parameters $0 < \alpha \le \bar{\alpha}$ be a regularization with constant γ_*. We assume that the index function $f(\alpha)$ $(0 < \alpha \le \bar{\alpha})$ is a profile function for $(x^\dagger, g_\alpha)$ with associated measure $\Phi = \Phi_{f^2}$, restricted to the interval $J_* := (0, \min\{a, \bar{\alpha}/4\gamma_*^2, a\}]$. Then the following assertions are true:*

(a) *If ψ is any index function such that $1/\psi \in L^2(J_*, d\Phi)$, then necessarily $x^\dagger \in X_\psi^H$.*
(b) *We have $x^\dagger \in X_\psi^H$ for every index function ψ for which $t \mapsto 1/(\psi^2((f^2)^{-1}(t))) \in L^1_{\mathrm{loc}}(J_*, dt)$, i.e., it is locally integrable.*

Proof. Using Lemma 3.4 and the fact that $f(\alpha)$ $(0 < \alpha \le \bar{\alpha})$ is assumed to be a profile function for $(x^\dagger, g_\alpha)$ we conclude that the estimate

$$\frac{1}{4}\int_0^\alpha d\|E_\lambda x^\dagger\|^2 \le \|r_{(4\gamma_*\alpha)}x^\dagger\|^2 \le f^2(4\gamma_*^2\alpha) = \int_0^\alpha d\Phi(\lambda) \quad (\alpha \in J_*) \tag{3.5}$$

is valid.

Now let ψ be any index function such that $1/\psi(t) \in L^2(J_*, d\Phi)$. By Lemma 3.5 we can find a sequence $f_n(t)$ of step functions on J_*, converging to $1/\psi^2(t)$ point-wise. Using (3.5) and the particular form of f_n we deduce that

$$\frac{1}{4}\int_{J_*} f_n(\lambda)d\|E_\lambda x^\dagger\|^2 \le \int_{J_*} f_n(\lambda)d\Phi(\lambda) \le \int_{J_*} \frac{1}{\psi^2(\lambda)}d\Phi(\lambda).$$

By Fatou's Lemma we conclude that also $1/\psi(t) \in L^2(J_*, d\|E_\lambda x^\dagger\|^2)$ and

$$\|1/\psi\|_{L^2(J_*, d\|E_\lambda x^\dagger\|^2)} \leq 2\|1/\psi\|_{L^2(J_*, d\Phi)}.$$

Consequently,

$$\begin{aligned}
\|x^\dagger\|^2_{X^H_\psi} &= \int_0^a \frac{1}{\psi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 \\
&= \int_{J_*} \frac{1}{\psi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 + \int_{(0,a]\setminus J_*} \frac{1}{\psi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 \\
&\leq 4\|1/\psi\|^2_{L^2(J_*, d\Phi)} + \frac{1}{\min_{\lambda\in(0,a]\setminus J_*} \psi^2(\lambda)} \|x^\dagger\|^2 < \infty, \qquad (3.6)
\end{aligned}$$

because the second summand on the right is finite, which proves assertion (a).

We may use a change of measure to establish assertion (b). The proof is complete. □

Remark 3.7. If the interval J_* coincides with $(0, a]$, then the second summand on the right in (3.6) does not appear and we get a bound $\|x^\dagger\|_{X^H_\psi} \leq 2\|1/\psi\|_{L^2((0,a], d\Phi)}$.

The following elementary observation is useful.

Lemma 3.8. *Suppose ψ, ψ_1 and f, f_1 are pairs of index functions which are related by some common strictly increasing function g as $f(t) = f_1(g(t))$ and $\psi(t) = \psi_1(g(t))$ on the respective domains of definition. Then it holds true that $f(\psi^{-1}(t)) = f_1(\psi_1^{-1}(t))$.*

Theorem 3.6 covers the cases which were known before.

Example 3.9 ([23]). If the profile function f for $(x^\dagger, g_\alpha)$ is a monomial $f(\alpha) = \alpha^\nu$ for some $\nu > 0$, then we we can draw the following conclusion. For every monomial $\psi(t) = t^\mu$ we obtain $1/\psi^2((f^2)^{-1}(t)) = t^{-\mu/\nu}$, which is integrable on every finite interval for $\mu < \nu$. Hence we deduce that necessarily $x^\dagger \in X^H_\psi$ for all $0 < \mu < \nu$.

Example 3.10 ([14, Theorem 8]). If the profile function f for $(x^\dagger, g_\alpha)$ is of logarithmic type, say $f(\alpha) = \log^{-\nu}(1/\alpha)$ $(0 < \alpha < 1)$ for some $\nu > 0$, then by using Lemma 3.8 we also deduce that necessarily $x^\dagger \in X^H_\psi$ for all functions $\psi(t) = \log^{-\mu}(1/t)$ $(0 < t < 1)$ with $\mu < \nu$, because both are related to the respective functions from Example 3.9 through $g(t) := \log^{-1}(1/t)$ $(0 < t < 1)$.

For the discussion of results of converse nature as presented in this subsection we also refer to the recent book [1].

4. Lower bounds for profile functions

In general profile functions $f(\alpha)$ can decrease to zero arbitrarily fast as α tends to zero. This is for instance the case when g_α is chosen as spectral cut-off in Example 2.4 and $x^\dagger$ is an eigenelement of A^*A, in which case $\|r_\alpha(A^*A)x^\dagger\| \equiv 0$ for α small enough.

However, for many regularization methods there is a maximal speed of convergence $\|r_\alpha(A^*A)x^\dagger\| \to 0$ as $\alpha \to 0$, for any $x^\dagger \neq 0$, regardless of its smoothness. This phenomenon is related to *saturation*, as this was studied e.g. in [23, 24], and in more generality in [15], from which the present approach is taken. The impact of limited qualification on profile functions can be seen under an additional convexity assumption.

Theorem 4.1. *Let g_α be any regularization with residual r_α. Suppose that for all $0 < t \leq a$ the functions*

$$\alpha \mapsto |r_\alpha(t)| \qquad (0 < \alpha \leq \overline{\alpha}) \tag{4.1}$$

are increasing, and for all $0 < \alpha \leq \overline{\alpha}$ the functions

$$t \mapsto |r_\alpha(t)|^2 \quad (0 < t \leq a) \tag{4.2}$$

are convex. Let $\bar{\psi}$ be given as

$$\bar{\psi}(\alpha) := \inf_{0<t\leq a} |r_\alpha(t)| \quad (0 < \alpha \leq \overline{\alpha}). \tag{4.3}$$

Then for each $0 \neq x \in X$ we have

$$\bar{\psi}(\alpha) \leq \frac{1}{\|x\|} \|r_\alpha(A^*A)x\| \qquad \text{for all} \quad 0 < \alpha \leq \overline{\alpha}. \tag{4.4}$$

Hence $\bar{\psi}$ is a non-decreasing lower bound to any profile function for (x_0, g_α) uniformly for all elements $x_0 \in X$ of the unit sphere, i.e., with $\|x_0\| = 1$.

Sketch of a proof. To prove that $\bar{\psi}$ is a lower bound to any profile function for (x_0, g_α) we use a Jensen-type inequality (see e.g. [15]), which yields that under (4.2) we have

$$\bar{\psi}(\alpha) \leq \left| r_\alpha(\|Ax\|^2/\|x\|^2) \right| \leq \frac{\|r_\alpha(A^*A)x\|}{\|x\|} \qquad \text{for all} \quad 0 < \alpha \leq \overline{\alpha}.$$

Moreover, under (4.1) the function $\bar{\psi}$ is non-decreasing. This completes the proof. $\square$

Remark 4.2. In many cases, the above function $\bar{\psi}(\alpha)$ turns out to be a qualification of the regularization g_α. In such a case it is *maximal qualification*.

We shall exhibit the above result at some examples.

Example 4.3. For Tikhonov regularization as in Example 2.3 we easily verify that the assumptions are satisfied. We conclude that $\bar{\psi}(\alpha) = \alpha/(\alpha + a)$ with $\bar{\psi}(\alpha) \geq \alpha/(2a)$ $(0 < \alpha < a)$. In this case this corresponds to the maximal qualification which is $\psi(\alpha) = \alpha$.

Example 4.4. The n-fold iterated Tikhonov regularization, which has $r_\alpha(t) = (\alpha/(\alpha+t))^n$ as its residual function also satisfies the assumptions from Theorem 4.1, and $\bar{\psi}(\alpha) = (\alpha/(\alpha + a))^n \geq (\alpha/(2a))^n$. This method corresponds to the maximal known qualification $\psi(\alpha) = \alpha^n$.

As in [15] we close with the following example, which is interesting as it shows that regularization, which has arbitrary classical qualification in the form $\psi(t) = t^q$ for any $0 < q < \infty$, still has a limited rate of decay for the profile functions, although these can decay exponentially fast.

Example 4.5. Landweber iteration from Example 2.5 also satisfies all the assumptions. The function $\bar{\psi}$, letting $0 < b := 1/(1-\mu a) < \infty$, turns out to be $\bar{\psi}(\alpha) = (1-\mu a)^{\lfloor 1/\alpha \rfloor} \geq \exp(-b/\alpha)$ $(0 < \alpha < 1)$.

Finally we stress that spectral cut-off as in Example 2.4 does not fulfill the above assumptions. Moreover, formally we would obtain the lower bound $\bar{\psi}(\alpha) \equiv 0$, which is trivial.

Remark 4.6. Lower bounds for profile functions are related to the saturation phenomenon as we shall briefly sketch. The following estimate is shown in the cause of the proof of the theorem in [15].

$$\sup_{\|\xi\|\leq 1} e(x^\dagger, g_\alpha, \delta) \geq \max\left\{\|r_\alpha(A^*A)x^\dagger\|, \delta/\sqrt{\alpha}\right\} \quad (0 < \alpha \leq \bar{\alpha}). \tag{4.5}$$

Thus, if $\bar{\psi}(\alpha)$ is a lower bound as in (4.4), then for any $x^\dagger$ with $\|x^\dagger\| = 1$ we derive that

$$\sup_{\|\xi\|\leq 1} e(x^\dagger, g_\alpha, \delta) \geq \max\left\{\bar{\psi}(\alpha), \delta/\sqrt{\alpha}\right\} \geq \bar{\psi}(\Theta^{-1}(\delta)) \quad (0 < \alpha \leq \bar{\alpha})$$

with $\Theta(t) := \sqrt{t}\,\bar{\psi}(t)$ $(0 < t \leq \overline{\alpha})$. Hence, the function $\bar{\psi}(\Theta^{-1}(\delta))$ is a lower bound for the error at $x^\dagger$, no matter how smooth the true solution $x^\dagger \in X$ was.

The functions $\bar{\psi}$ derived in the Examples 4.3 – 4.5 can be seen to be the saturation rates caused by the limited qualifications of the underlying regularization methods.

5. Impact of solution smoothness

As stressed earlier, the behavior of profile functions is determined by both, the chosen regularization g_α and the underlying solution smoothness. As introduced in Section 2.2 we measure this in terms of smoothness conditions of the form $x^\dagger \in G_\tau(R)$, see (2.2), determined by an operator G and an index function τ. The impact of such smoothness assumption on the decay rate of profile functions is easiest seen if G is a function of A^*A.

5.1. G as a function of $\mathbf{A^*A}$.

To obtain profile functions f for the regularization method g_α the concept of *general source conditions*, as expressed in

$$x^\dagger = \psi(A^*A)\, w \qquad (w \in X,\ \|w\| \leq R), \tag{5.1}$$

for some index functions $\psi(t)$ $(0 < t \leq a)$ was used recently (see, e.g., [14, 16, 17, 27]). We note that (5.1) is a specific smoothing condition (2.2) with $\tau(G) = \psi(A^*A)$ (cf. [3] for further discussion of such conditions).

We are going to find profile functions f uniformly for sets $H_\psi(R)$, as defined by formula (2.3), provided the corresponding function ψ is a qualification of the chosen regularization g_α.

Proposition 5.1. *Let the index function ψ be a qualification of the regularization method g_α with constant $0 < \gamma < \infty$. Then uniformly for each $x^\dagger \in H_\psi(R)$ the inequality*

$$\|x_\alpha - x^\dagger\| \leq \gamma\, R\, \psi(\alpha) \qquad \textit{for all} \quad 0 < \alpha \leq a \tag{5.2}$$

is valid. Hence $f(\alpha) := \gamma\, R\, \psi(\alpha)$ is a profile function for $(H_\psi(R), g_\alpha)$.

Proof. From spectral theory (see, e.g., [6, Formula (2.47)]) we have with (5.1) that

$$\|x_\alpha - x^\dagger\| = \|r_\alpha(A^*A)\, x^\dagger\| = \|r_\alpha(A^*A)\, \psi(A^*A)\, w\| \le R \sup_{0<t\le a} |r_\alpha(t)|\, \psi(t)\,.$$

Taking into account inequality (2.1) this yields (5.2), and proves the proposition. □

Remark 5.2. This proposition can be reformulated as follows. Suppose that we are given a pair (M, g_α) of a solution set M and a regularization g_α. If we can find an index function ψ on $(0, a]$ that is *both* a qualification for g_α and a smoothness for M, i.e., $M \subseteq H_\psi(R)$ for some R, then there is a profile function for (M, g_α). In addition the index function ψ provides a decay rate. Although this is a simple observation it explains the existence of profile functions for non-compact sets M, as discussed at the end of Section 3.1.

5.2. **Approximate source conditions.** An important extension of the above concept is obtained by relaxing requirement (5.1). In this context, we restrict ourselves to a fixed index function $\varphi(t)$ $(0 < t \le a)$ as *benchmark function*. We suppose that the solution $x^\dagger \in X$ of (1.1) is not smooth enough to satisfy a condition (5.1) with φ instead of ψ even if $R \ge 0$ is arbitrary large. The injectivity of A implies the injectivity of $\varphi(A^*A)$ for any index function φ. Hence the range $\mathcal{R}(\varphi(A^*A))$ is dense in X. Consequently, for all $0 \le R < \infty$ the element $x^\dagger$ satisfies such a general source condition in an approximate manner as $x^\dagger = \varphi(A^*A)\, w + \xi \quad (\|w\| \le R,\ \xi \in X)$, where the norm of the perturbation $\|\xi\|$ tends to zero as R tends to infinity.

In the following we shall confine to this situation, when

$$x^\dagger \notin \mathcal{R}(\varphi(A^*A)). \tag{5.3}$$

The quality of the approximation of $x^\dagger$ by elements from $H_\varphi(1)$ can be be expressed by favor of the *distance function*

$$\rho_{x^\dagger}(t) = \rho_{x^\dagger}^{(H,\varphi)} := \operatorname{dist}(tx^\dagger, H_\varphi(1)) \;=\; \inf\left\{\|tx^\dagger - \varphi(H)v\| : v \in X,\ \|v\| \le 1\right\} \quad (t > 0). \tag{5.4}$$

If the reference to the benchmark (H, φ) is clear, as in the following lemma, then we shall omit the super-script.

Lemma 5.3. *Under the assumption (5.3) the functions $\rho_{x^\dagger}(t)$ $(t > 0)$ and $\rho_{x^\dagger}(t)/t$ $(t > 0)$ are both index functions. Moreover, we have we have $\lim_{t\to\infty} \rho_{x^\dagger}(t) = \infty$.*

Proof. The idea of the proof is standard in regularization theory. For each $t > 0$ the value $\rho_{x^\dagger}(t)/t = \operatorname{dist}(x^\dagger, H_\varphi(1/t))$ is obtained from constrained minimization, and Lagrange multipliers can be used to determine this value. Hence, given $x^\dagger \in X$ let

$$F_{x^\dagger}(\lambda) := \|x^\dagger - \varphi(A^*A)v\|^2 + \lambda\|v\|^2.$$

At given λ its minimizer with respect to $v \in X$ is

$$v_\lambda := \left[\varphi^2(A^*A) + \lambda I\right]^{-1} \varphi(A^*A)x^\dagger,$$

which has to obey the side constraint $\chi(\lambda) = 1/t$, where setting

$$\chi(\lambda) := \| \left[\varphi^2(A^*A) + \lambda I\right]^{-1} \varphi(A^*A)x^\dagger \|. \tag{5.5}$$

Based on the injectivity of $\varphi(A^*A)$ spectral calculus yields that the function $\chi(\lambda)$ ($\lambda > 0$) is positive, continuous and strictly decreasing to zero as $\lambda \to \infty$. Moreover, under (5.3) we have $\lim_{\lambda \to 0+} \chi(\lambda) = \infty$. Therefore for all $t > 0$ the function $\lambda(t) := \chi^{-1}(1/t)$ exists and is an index function. Hence we obtain

$$\rho_{x^\dagger}(t)/t = \|x^\dagger - \varphi(A^*A)v_{\lambda(t)}\| = \lambda(t) \| \left[\varphi^2(A^*A) + \lambda(t)I\right]^{-1} x^\dagger\| \quad (t > 0), \tag{5.6}$$

which is the composition of two index functions in t. As a consequence, both functions $\rho_{x^\dagger}(t)/t$ and $\rho_{x^\dagger}(t)$ have that property. On the other hand, we have

$$\lim_{t \to \infty} \rho_{x^\dagger}(t) = \lim_{t \to \infty} t \left(\frac{\rho_{x^\dagger}(t)}{t}\right) = \infty\,,$$

since $\rho_{x^\dagger}(t)/t$ as an index function cannot tend to zero as $t \to \infty$. This completes the proof. □

Remark 5.4. By using distance functions of the form

$$d(R) := \mathrm{dist}(x^\dagger, H_\varphi(R)) \;=\; R\rho_{x^\dagger}(1/R) \quad (0 < R < \infty) \tag{5.7}$$

error estimates for for the Tikhonov regularization were already studied in [10] and [5], see also [2, 9] and [13] for variants thereof.

The fundamental estimate for profile functions under approximate source conditions is as follows:

Theorem 5.5. *Let g_α be a regularization method with qualification φ and constant γ. If the solution $x^\dagger$ to equation (1.1) obeys (5.3), then*

(5.8)

$$\|x_\alpha - x^\dagger\| \le \max\{\gamma, \gamma_1\}\, \frac{1}{t}\left(\rho_{x^\dagger}(t) + \varphi(\alpha)\right) \quad \textit{for all} \quad t > 0 \quad \textit{and} \quad 0 < \alpha \le a.$$

Thus the function

$$f(\alpha) := 2\max\{\gamma, \gamma_1\}\, \frac{\varphi(\alpha)}{\rho_{x^\dagger}^{-1}(\varphi(\alpha))} \quad (0 < \alpha \le a) \tag{5.9}$$

is a profile function for $(x^\dagger, g_\alpha)$.

Proof. First we establish (5.8). For any $v \in X$ with $\|v\| \le 1$ we can estimate

$$\begin{aligned}
\|x_\alpha - x^\dagger\| &= \frac{1}{t}\|r_\alpha(A^*A)tx^\dagger\| \\
&= \frac{1}{t}\|r_\alpha(A^*A)\, tx^\dagger - r_\alpha(A^*A)\,\varphi(A^*A)v + r_\alpha(A^*A)\,\varphi(A^*A)v\| \\
&\le \frac{1}{t}\left(\|r_\alpha(A^*A)(tx^\dagger - \varphi(A^*A)v)\| \;+\; \|r_\alpha(A^*A)\,\varphi(A^*A)v\|\right) \\
&\le \frac{1}{t}\left(\gamma_1\|tx^\dagger - \varphi(A^*A)v\| \;+\; \|r_\alpha(A^*A)\,\varphi(A^*A)\|\right) \\
&\le \frac{1}{t}\left(\gamma_1\,\|tx^\dagger - \varphi(A^*A)v\| + \gamma\,\varphi(\alpha)\right).
\end{aligned}$$

Since this estimate remains true if we substitute $\|tx^\dagger - \varphi(A^*A)v\|$ by its infimum over all v from the unit ball of X and since φ is a qualification of the used regularization method, we obtain

$$\|x^\dagger - x_\alpha\| \le \max\{\gamma, \gamma_1\} \frac{1}{t} \left(\rho_{x^\dagger}(t) + \varphi(\alpha)\right) \quad \text{for all} \quad t > 0 \quad \text{and} \quad 0 < \alpha \le a,$$

which proves estimate (5.8). Since this estimate is valid for all $t > 0$ and we have by Lemma 5.3 for the index function $\rho_{x^\dagger}$ the limit condition $\lim_{t\to\infty} \rho_{x^\dagger}(t) = \infty$, we can equate the two terms in brackets of the right-hand side of (5.8). Taking into account the strict monotonicity of function $\rho_{x^\dagger}(t)$ $(t > 0)$ this yields (5.9). □

Remark 5.6. We notice that the upper bound in (5.8) cannot be improved by other values of t, because it is the balance of a strictly increasing function $\rho_{x^\dagger}(t)/t$ and for any α under consideration a decreasing function $\varphi(\alpha)/t$ with respect to t.

We also mention that the same arguments yield a slightly different bound

$$\|x^\dagger - x_\alpha\| \le \frac{(\gamma + \gamma_1)}{t} \max\{\rho_{x^\dagger}(t), \varphi(\alpha)\} \quad \text{for all} \quad t > 0 \quad \text{and} \quad 0 < \alpha \le a,$$

which is better if the constants γ and γ_1 differ. This implies that in all estimates below the expression $2\max\{\gamma, \gamma_1\}$ can be replaced by $(\gamma + \gamma_1)$.

Remark 5.7. Since the denominator $\rho_{x^\dagger}^{-1}(\varphi(\alpha))$ in (5.9) expresses an index function tending to zero as α tends to zero, the decay rate of $f(\alpha) \to 0$ as $\alpha \to 0$ is always *lower* than the corresponding rate of the benchmark function φ, i.e., $\varphi(\alpha) = o(f(\alpha))$ as $\alpha \to 0$. In particular, one has to choose a sufficiently good benchmark function and a regularization with high enough qualification to achieve by that way the best possible rate for given $x^\dagger$.

5.3. **Approximate source conditions for solutions in source-wise representation.** It is worthwhile to discuss the situation when $x^\dagger$ has a source-wise representation (5.1) but the benchmark function φ is chosen in such a way that $x^\dagger \notin \mathcal{R}(\varphi(A^*A))$. This can happen in the following case only.

Lemma 5.8. *Suppose $x^\dagger$ obeys (5.1). If $x^\dagger \notin \mathcal{R}(\varphi(A^*A))$ then necessarily $(\varphi/\psi)(t) \to 0$ as $t \to 0$.*

Proof. Suppose $\varphi(t) \ne o(\psi(t))$. Then there is $C < \infty$ such that $\psi(t) \le C\varphi(t)$ for small $0 < t \le \bar{t}$. Given $0 < \varepsilon \le \bar{t}$ we can bound

$$\begin{aligned}
\int_\varepsilon^a \frac{1}{\varphi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 &= \int_\varepsilon^{\bar{t}} \frac{1}{\varphi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 + \int_{\bar{t}}^a \frac{1}{\varphi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 \\
&\le C \int_\varepsilon^{\bar{t}} \frac{1}{\psi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 + \sup_{\lambda \ge \bar{t}} \frac{\psi^2(\lambda)}{\varphi^2(\lambda)} \int_{\bar{t}}^a \frac{1}{\psi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 \\
&\le \max\left\{C, \sup_{\lambda \ge \bar{t}} \frac{\psi^2(\lambda)}{\varphi^2(\lambda)}\right\} \int_\varepsilon^a \frac{1}{\psi^2(\lambda)} d\|E_\lambda x^\dagger\|^2 \\
&\le \max\left\{C, \sup_{\lambda \ge \bar{t}} \frac{\psi^2(\lambda)}{\varphi^2(\lambda)}\right\} \|x^\dagger\|^2_{X_\psi^H}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $\|x^\dagger\|_{X^H_\varphi} < \infty$, thus $x^\dagger \in \mathcal{R}(\varphi(A^*A))$, which completes the proof. $\square$

If, slightly stronger but geometrically intuitive, we assume that the quotient $(\varphi/\psi)(t)$ is strictly increasing, then we can give a clear picture for the resulting function $\rho_{x^\dagger}(t)$ for $t > 0$ sufficiently small.

Theorem 5.9. *We suppose that $x^\dagger$ obeys (5.1) and that the quotient $(\varphi/\psi)(t)$ is an index functions for $0 < t \le a$. Then we can estimate the distance function as*

$$\rho_{x^\dagger}(t) \le \varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(Rt)\right) \qquad \text{for all} \quad 0 < t \le \frac{1}{R}\frac{\varphi(a)}{\psi(a)}. \tag{5.10}$$

Proof. The proof is carried out using the analysis from the proof of Lemma 5.3 and we shall make use of the notation introduced there. As there let $\lambda(t) := \chi^{-1}(1/t)$ $(t > 0)$ with function χ from (5.5). Then for $x^\dagger = \psi(A^*A)v$ with $\|v\| \le R$ representation (5.6) allows for the following bound

$$\begin{aligned}
\rho_{x^\dagger}(t) = t\lambda(t)\| \left[\varphi^2(A^*A) + \lambda(t)I\right]^{-1} x^\dagger\| &\le (Rt)\lambda(t)\| \left[\varphi^2(A^*A) + \lambda(t)I\right]^{-1} \psi(A^*A)\| \\
= (Rt) \sup_{0<s\le a} \frac{\lambda(t)\psi(s)}{\varphi^2(s) + \lambda(t)} &= (Rt) \sup_{0<u\le\varphi^2(a)} \frac{\lambda(t)}{u + \lambda(t)}\psi((\varphi^2)^{-1}(u)),
\end{aligned}$$

where we make the crucial observation that $u \mapsto \lambda(t)/(u + \lambda(t))$ is the residual of Tikhonov regularization. To continue we introduce the auxiliary function

$$\kappa(s) := \frac{\psi((\varphi^2)^{-1}(s))}{\sqrt{s}} = \left(\frac{\psi}{\varphi}\right)\left((\varphi^2)^{-1}(s)\right) \qquad (0 < s \le \varphi^2(a)). \tag{5.11}$$

It is clear that $1/\kappa(s)$ is an index function, hence $\lim_{s\to 0+} \kappa(s) = \infty$. Also, the function $\kappa(u)/\sqrt{u}$ is decreasing whenever κ is. Hence Proposition 2.8 (a) applies and allows us to conclude the estimate

$$\rho_{x^\dagger}(t) \le (Rt)\psi((\varphi^2)^{-1}(\lambda(t))) \qquad (t > 0), \tag{5.12}$$

noting that $\psi((\varphi^2)^{-1}(s))$ for sufficiently small $s > 0$ is an index function.

Next we shall establish for sufficiently small $t > 0$ an upper bound $\tilde\lambda(t)$ for $\lambda(t)$ which then will yield estimate (5.10). Indeed, let $\tilde\lambda(t)$ be obtained as inverse

$$\tilde\lambda(t) = \kappa^{-1}(1/(Rt)). \tag{5.13}$$

It is enough to show that $\lambda(t) \le \tilde\lambda(t)$. To this end notice that κ was decreasing, hence $u \mapsto (\psi((\varphi^2)^{-1}(u))\sqrt{u})/u$ is so, and we derive, again using arguments as above that for $0 < t \le \frac{1}{R}\frac{\varphi(a)}{\psi(a)}$ the estimate

$$\begin{aligned}
\kappa(\tilde\lambda(t)) \le \frac{1}{Rt} = \frac{\chi(\lambda(t))}{R} &\le \| \left[\varphi^2(A^*A) + \lambda(t)I\right]^{-1} \varphi(A^*A)\psi(A^*A)\| \\
&\le \frac{1}{\lambda(t)} \sup_{0<u\le\varphi^2(a)} \frac{\lambda(t)}{u + \lambda(t)}\psi((\varphi^2)^{-1}(u))\sqrt{u} \le \frac{1}{\lambda(t)}\psi((\varphi^2)^{-1}(\lambda(t)))\sqrt{\lambda(t)} \\
&= \kappa(\lambda(t)).
\end{aligned}$$

Consequently, $\lambda(t) \le \tilde{\lambda}(t)$, and we arrive at

$$\rho_{x^\dagger}(t) \le (Rt)\psi((\varphi^2)^{-1}(\lambda(t))) \le (Rt)\psi((\varphi^2)^{-1}(\tilde{\lambda}(t))) = \sqrt{\kappa^{-1}\left(\frac{1}{Rt}\right)}. \tag{5.14}$$

It is a routine matter to check that both versions in the right hand side of (5.14) are equal. Indeed, starting from the identity $\psi(u)/\varphi(u) = \psi(u)/\varphi(u)$, a variable substitution $u := (\varphi/\psi)^{-1}(Rt)$ yields

$$\frac{1}{Rt} = \frac{\psi\left((\varphi/\psi)^{-1}(Rt)\right)}{\varphi\left((\varphi/\psi)^{-1}(Rt)\right)} = \kappa\left(\varphi^2\left(\left(\frac{\varphi}{\psi}\right)^{-1}(Rt)\right)\right),$$

completing the proof. □

Corollary 5.10. *Suppose that φ is a qualification for g_α with constant γ. Under the assumptions of Theorem 5.9 there is some $\overline{\alpha} > 0$ such that*

$$f(\alpha) := 2\max\{\gamma, \gamma_1\}\, R\,\psi(\alpha) \qquad (0 < \alpha \le \overline{\alpha}) \tag{5.15}$$

is a profile function for $(H_\psi(R), g_\alpha)$.

Proof. For proving that (5.15) is a profile function for $(H_\psi(R), g_\alpha)$ we use the estimate (5.8) of Theorem 5.5 and the bound (5.10) which together yield for some sufficiently small $\bar{t} > 0$ the error bound

$$\|x_\alpha - x^\dagger\| \le \max\{\gamma, \gamma_1\}\,\frac{1}{t}\left(\varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(Rt)\right) + \varphi(\alpha)\right) \quad (0 < t \le \bar{t},\ 0 < \alpha \le a).$$

Then for sufficiently small $\alpha > 0$ there is some $t_* = t_*(\alpha) \in (0, \bar{t}\,]$ satisfying the equation
$\varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(Rt_*)\right) = \varphi(\alpha)$, namely $t_* = \varphi(\alpha)/(R\,\psi(\alpha))$ implying

$$\|x_\alpha - x^\dagger\| \le 2\,\max\{\gamma, \gamma_1\}\,\frac{\varphi(\alpha)}{t_*} = 2\,\max\{\gamma, \gamma_1\}\; R\,\psi(\alpha).$$

This, however, completes the proof. □

Example 5.11. For monomials $\varphi(t) = t^\nu$ and $\psi(t) = t^\eta$ with $\nu, \eta > 0$, everything can be made explicit. Lemma 5.8 states that (5.3) is valid if and only if $0 < \eta < \nu$, which in the case of monomials is equivalent to saying that $(\varphi/\psi)(t)$ is an index function. We obtain the bound $\rho_{x^\dagger}(t) \le (Rt)^{\nu/(\nu-\eta)}$.

The global properties required for the quotient function φ/ψ on $(0, a]$ are rather strong assumptions in Theorem 5.9 used for obtaining the estimate (5.15) in Corollary 5.10. On the other hand, in [13] and [5] by a completely different technique there have been developed error estimates of type (5.15) with some other constant which only need local properties of φ/ψ on an arbitrarily small interval $(0, \varepsilon]$. In order to show that our approach is powerful enough to work with such weaker assumptions, we conclude this section with a local variant of Theorem 5.9 yielding the results of Corollary 5.10 with different constant under the local assumption on the quotient function.

Theorem 5.12. *We suppose that $x^\dagger$ obeys (5.1) and that φ, ψ are index functions on $(0,a]$. Moreover, it is assumed that there exists some $0<\varepsilon\le a$ such that the quotient function φ/ψ is an index function on the interval $(0,\varepsilon]$. Then with the constants $C_\varepsilon = \frac{\psi(a)}{\psi(\varepsilon} \ge 1$ and $K_\varepsilon = \frac{\psi(a)}{\psi(\varepsilon}\frac{\varphi(a)}{\varphi(\varepsilon})\ge 1$ we can estimate the distance function as*

$$\rho_{x^\dagger}(t) \le \frac{C_\varepsilon}{K_\varepsilon}\,\varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(R\,K_\varepsilon\,t)\right) \qquad \text{for all} \quad 0<t\le\bar t \tag{5.16}$$

and sufficiently small $\bar t>0$. If, moreover, φ is a qualification for g_α with constant γ, then there is $\bar\alpha>0$ such that the function

$$f(\alpha) := 2\max\{\gamma,\gamma_1\}\,K_\varepsilon\,R\,\psi(\alpha) \quad (0<\alpha\le\bar\alpha) \tag{5.17}$$

is a profile function for $(H_\psi(R), g_\alpha)$.

Sketch of a proof. We follow the proof of Theorem 5.9, but the local version of the estimate (5.12) is obtained using Proposition 2.8 (c) with $\hat t = \varphi^2(\varepsilon)$ as

$$\rho_{x^\dagger}(t) \le (Rt)\,C_\varepsilon\,\psi((\varphi^2)^{-1}(\lambda(t)))$$

for sufficiently small $t>0$. Moreover, instead of (5.13) in the local variant we have to set

$$\tilde\lambda(t) = \kappa^{-1}(1/(R\,K_\varepsilon\,t))\,,$$

which is well-defined for $t\in(0,\bar t\,]$ with $\bar t>0$ sufficiently small. Then as in the original proof it can be shown that $\lambda(t)\le\tilde\lambda(t)$ for $0<t\le\bar t$ again based on Proposition 2.8 (c). Precisely, we have the estimate

$$\begin{aligned}\kappa(\tilde\lambda(t)) &= \frac{1}{RK_\varepsilon t} = \frac{\chi(\lambda(t))}{RK_\varepsilon} \le \frac{1}{K_\varepsilon}\,\|\left[\varphi^2(A^*A)+\lambda(t)I\right]^{-1}\varphi(A^*A)\psi(A^*A)\| \\ &= \frac{1}{K_\varepsilon\lambda(t)}\sup_{0<u\le\varphi^2(a)}\frac{\lambda(t)}{u+\lambda(t)}\psi((\varphi^2)^{-1}(u))\sqrt{u} \le \frac{1}{\lambda(t)}\psi((\varphi^2)^{-1}(\lambda(t)))\sqrt{\lambda(t)} \\ &= \kappa(\lambda(t))\,.\end{aligned}$$

Finally, we arrive at

$$\rho_{x^\dagger}(t) \le (Rt)\,C_\varepsilon\,\psi((\varphi^2)^{-1}(\lambda(t))) \le (Rt)\,C_\varepsilon\,\psi((\varphi^2)^{-1}(\tilde\lambda(t))) = \frac{C_\varepsilon}{K_\epsilon}\sqrt{\kappa^{-1}\left(\frac{1}{RK_\varepsilon t}\right)}$$

which proves (5.16). For proving (5.17) we use the estimate (5.8) of Theorem 5.5 yielding here for sufficiently small $t>0$ and $\alpha>0$, and since $\frac{C_\varepsilon}{K_\varepsilon}\le 1$,

$$\|x_\alpha - x^\dagger\| \le \max\{\gamma,\gamma_1\}\,\frac{1}{t}\left(\varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(R\,K_\varepsilon\,t)\right)+\varphi(\alpha)\right).$$

Now we choose $t_* = t_*(\alpha)$ such that the equation

$$\varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(R\,K_\varepsilon\,t_*)\right) = \varphi(\alpha)$$

holds. This is possible for sufficiently small $\alpha>0$ and yields $t_* = \frac{\varphi(\alpha)}{\psi(\alpha)}\frac{1}{R\,K_\varepsilon}$. Hence we obtain the profile function (5.17) as required. $\square$

6. Linking scales by range inclusions

Since the initial study of linear inverse problems in Hilbert scales (see [21]) it is well known that the operator G measuring smoothness of the solution $x^\dagger$ must be linked to the operator A governing equation (1.1) in order to obtain error bounds. There are various ways to establish such a link and we will investigate its impact on profile functions, next.

Again we start with the benchmark function φ and assume in addition that

$$x^\dagger \in G_\tau(R) \tag{6.1}$$

with $G_\tau(R)$ defined by (2.2). Moreover, we impose the following *link condition*, precisely that there are an index function $\sigma(t)$ $(0 < t \le \|G\|)$ and a constant $C < \infty$ such that

$$\|\sigma(G)v\| \le C\|\varphi(A^*A)v\| \qquad \text{for all} \quad v \in X. \tag{6.2}$$

Remark 6.1. There is an extensive analysis in [3] of linking conditions in various ways. In particular it is shown as Proposition 2.1 in [3] that the validity of condition (6.2) with some positive C is equivalent to the *range inclusion*

$$\mathcal{R}(\sigma(G)) \subseteq \mathcal{R}(\varphi(A^*A))\,. \tag{6.3}$$

We mention the following consequence of (6.2) (see e.g. [19]). Given Hilbert spaces X and Z with $Z \subset X$ let $J\colon Z \to X$ be the canonical embedding, leaving elements from Z invariant.

Lemma 6.2. *Under (6.2) the canonical embedding $J_{G,\sigma}^{H,\varphi}\colon X_\sigma^G \to X_\varphi^H$ is norm bounded by C and we have*

$$G_\sigma(R) \subseteq H_\varphi(C\,R). \tag{6.4}$$

Proof. It is well known that for any pair S, T of operators a relation $\|Sv\| \le \|Tv\|$ implies $\|T^{-1}v\| \le \|S^{-1}v\|$, whenever the right hand sides are finite. Thus (6.2) implies for any $x \in X_\sigma^G$ with $\|x\|_{X_\sigma^G} \le 1$ that $\|x\|_{X_\varphi^H} \le C$ and hence (6.4). $\square$

We will distinguish two scenarios and we start with the easier one and state

Proposition 6.3. *Let g_α be a regularization which has qualification φ with constant γ and assume that $x^\dagger$ obeys (6.1). If condition (6.2) is valid for an index function σ, and if there is $K < \infty$ such that $\tau(t)/\sigma(t) \le K$ $(0 < t \le \|G\|)$, then the function*

$$f(\alpha) := \gamma\, C\, K\, R\, \varphi(\alpha) \qquad (0 < \alpha \le a) \tag{6.5}$$

is a profile function for $(G_\tau(R), g_\alpha)$.

Proof. From $\tau(t)/\sigma(t) \le K$ $(0 < t \le a)$ we deduce from Lemma 6.2 that $G_\tau(R) \subseteq G_\sigma(KR)$, which is equivalent to $\|\tau(G)x\| \le K\,\|\sigma(G)x\|$ for all $x \in X$. Furthermore, in the light of Lemma 6.2, the link condition (6.2) implies $G_\sigma(KR) \subseteq H_\varphi(CKR)$, and any profile function for $H_\varphi(CKR)$ provides us with a profile function for $G_\tau(R)$, such that the proof can be completed using Proposition 5.1. $\square$

Thus we are left with the case when

$$(\sigma/\tau)(t) \to 0 \quad \text{as} \quad t \to 0. \tag{6.6}$$

Then we have $X_\sigma^G \subset X_\tau^G$ and the canonical embedding $J_{G,\sigma}^{G,\tau}\colon X_\sigma^G \to X_\tau^G$ is norm bounded. The question is whether one can use condition (6.2) to draw conclusions for the behavior of profile functions in this case.

Suppose we assume a linking condition (6.3), but smoothness is measured as $x^\dagger \in G_\tau(R)$ with respect to a different index function τ. Can we establish an index function ψ, assigned to a triplet (σ, τ, φ), such that the following range implication holds true:

$$\mathcal{R}(\sigma(G)) \subseteq \mathcal{R}(\varphi(H)) \quad \Longrightarrow \quad \mathcal{R}(\tau(G)) \subseteq \mathcal{R}(\psi(H))? \tag{6.7}$$

In specific situations this problem was already posed (cf. [13, Formula (5.10) on p. 815]) and partially answered previously (cf. [3, Corollary 2.3]). Most prominently, the Heinz-Kato inequality (see [6, Proposition 8.21]) yields

$$\mathcal{R}(G) \subseteq \mathcal{R}(H) \quad \Longrightarrow \quad \mathcal{R}(G^\mu) \subseteq \mathcal{R}(H^\mu)$$

for $0 < \mu \leq 1$, as a consequence of operator monotonicity. In fact this can be extended to more general situations in which operator monotone functions occur. It is convenient to draw the following diagram.

$$\begin{array}{ccccc} G{:}X_\sigma^G & \xrightarrow{J_{G,\sigma}^{G,\tau}} & X_\tau^G & \xrightarrow{J_{G,\tau}^{I}} & X \\ \big\downarrow J_{G,\sigma}^{H,\varphi} & & \big\downarrow J_{G,\tau}^{H,\psi} & & \big\downarrow I \\ H{:}X_\varphi^H & \xrightarrow{J_{H,\varphi}^{H,\psi}} & X_{\psi?}^H & \xrightarrow{J_{H,\psi}^{I}} & X \end{array} \tag{6.8}$$

Under (6.6) the upper row shows embeddings which are bounded. Using Lemma 6.2 the embedding $J_{G,\sigma}^{H,\varphi}$ is norm bounded by C, provided the link condition (6.2) holds true. Plainly the identity $I\colon X \to X$ has norm equal to one. The question addressed in this diagram is whether we can describe an index function ψ such that the corresponding embedding $J_{G,\tau}^{H,\psi}$ is norm bounded, say by some constant $L < \infty$. Diagram (6.8) also suggests that the resulting function ψ will describe smoothness, not covered by φ, and approximate source conditions must be used to obtain results.

Remark 6.4. If the embedding $J_{G,\tau}^{H,\psi}$ were norm bounded, say by some constant $L < \infty$, then $G_\tau(R) \subseteq H_\psi(LR)$, and any profile function for $(H_\psi(LR), g_\alpha)$ would also be a profile function for $(G_\tau(R), g_\alpha)$.

As the diagram (6.8) clearly indicates, interpolation properties may help to find suitable index function ψ. The implication (6.7) of range inclusions is indeed true if operator monotonicity occurs and we shall mention the following result from [19].

Theorem 6.5. *Let $x^\dagger$ obey (6.1). We assume that G and A^*A are linked by (6.2), where we suppose that σ is such that there is an extension $\sigma(t)$ $(0 < t \leq b)$ with $\sigma(b) \geq \varphi(a)$ and this extension is an index function. Moreover, given an index function $\tau(t)$ $(0 < t \leq \|G\|)$ we assign the index function*

$$\psi(t) := \tau(\sigma^{-1}(\varphi(t))) \quad (0 < t \leq a). \tag{6.9}$$

Then the implication (6.7) is satisfied whenever the function $\tau^2((\sigma^2)^{-1}(t))$ $(0 < t \le \varphi^2(a))$ *is operator monotone.*

Precisely, the norm bound

$$\|J^{H,\psi}_{G,\tau} \colon X^G_\tau \to X^H_\psi\| \le \max\{1, C\} \tag{6.10}$$

with C from (6.2) is valid.

Now we return to the analysis of profile functions. To establish these the full strength of the implication (6.7) is not necessary. But we shall also indicate its strength in Corollary 6.11, below. However, the function ψ from (6.9) will occur, nonetheless.

There are in principle two ways to use the link conditions (6.2) or (6.3), respectively, to obtain profile functions. One can either transfer all information to the scale generated by the operator G or to the scale generated by $H := A^*A$. Both ways finally provide the same asymptotic results but under assumptions of different strength. We start with the first approach which requires weaker assumptions.

Lemma 6.6. *The link condition (6.2) implies*

$$\rho^{(H,\varphi)}_{x^\dagger}(t) \le \frac{1}{C}\rho^{(G,\sigma)}_{x^\dagger}(Ct) \qquad \textit{for all} \quad 0 < t < \infty. \tag{6.11}$$

Proof. Plainly, condition (6.2) yields $G_\sigma(1/C) \subseteq H_\varphi(1)$ and we obtain

$$\begin{aligned}\rho^{(H,\varphi)}_{x^\dagger}(t) &= \operatorname{dist}(tx^\dagger, H_\varphi(1)) \le \operatorname{dist}(tx^\dagger, G_\sigma(1/C)) \\ &= \frac{1}{C}\operatorname{dist}(Ctx^\dagger, G_\sigma(1)) = \frac{1}{C}\rho^{(G,\sigma)}_{x^\dagger}(Ct).\end{aligned}$$

□

With this preparation we can state the main result of this section.

Theorem 6.7. *Let g_α be a regularization method with qualification φ and constant γ for the operator equation (1.1) with solution $x^\dagger$ the smoothness of which is characterized by the conditions (5.3) and (6.1) with some index functions φ and τ. We suppose the link condition (6.2) with some index function σ for connecting A^*A and G. If the function*

$$(\sigma/\tau)(t) \;\; (0 < t \le \|G\|) \quad \textit{is an index function}, \tag{6.12}$$

then there is some $\bar\alpha > 0$ for which the function $\psi(t)$ $(0 < t \le \bar\alpha)$ from (6.9) is an index function, and

$$f(\alpha) := 2\,\max\{\gamma, \gamma_1\}\,\max\{1, C\}R\;\psi(\alpha) \quad (0 < \alpha \le \bar\alpha) \tag{6.13}$$

is a profile function for $(G_\tau(R), g_\alpha)$.

Remark 6.8. If we assume (6.3) instead of (6.2), then $C := \|(\varphi(A^*A))^{-1}\tau(G)\| < \infty$ and the function f from (6.13) is a profile function with the constant C.

Proof of Theorem 6.7. For an arbitrary $x^\dagger \in G_\tau(R)$ using the bound (5.8) and Lemma 6.6 we obtain for all $0 < \alpha \le a$

$$\|x_\alpha - x^\dagger\| \le \max\{\gamma,\gamma_1\}\frac{1}{t}\left(\rho_{x^\dagger}^{(H,\varphi)}(t)+\varphi(\alpha)\right) \le \max\{\gamma,\gamma_1\}\frac{1}{t}\left(\frac{1}{C}\rho_{x^\dagger}^{(G,\sigma)}(Ct)+\varphi(\alpha)\right).$$

By exploiting Theorem 5.9 in the scale generated by G we can continue and bound

$$\|x_\alpha - x^\dagger\| \le \max\{\gamma,\gamma_1\}\frac{1}{t}\left(\frac{1}{C}\sigma\left(\left(\frac{\sigma}{\tau}\right)^{-1}(RCt)\right)+\varphi(\alpha)\right) \tag{6.14}$$

for $0 < \alpha \le \frac{1}{RC}\left(\frac{\sigma}{\tau}\right)(\|G\|)$. There is some $0 < \bar{\alpha} \le \|G\|/C$ for which we can equate both summands on the right of formula (6.14) whenever $0 < \alpha \le \bar{\alpha}$. This leads to

$$t_* = t_*(\alpha) := \frac{1}{R}\frac{\varphi(\alpha)}{\tau(\sigma^{-1}(C\varphi(\alpha)))} \quad (0 < \alpha \le \bar{\alpha}).$$

Moreover by (6.12) we have that $\tau(\sigma^{-1}(Ct)) \le \max\{1,C\}\,\tau(\sigma^{-1}(t))$ for $0 < t \le \overline{\alpha}$. Thus we can estimate for $0 < \alpha \le \bar{\alpha}$

$$\|x_\alpha - x^\dagger\| \le 2\,\max\{\gamma,\gamma_1\}\,\frac{\varphi(\alpha)}{t_*} \le 2\,\max\{\gamma,\gamma_1\}\,R\,\tau(\sigma^{-1}(C\varphi(\alpha)))\,.$$

Consequently we obtain

$$\|x_\alpha - x^\dagger\| \le 2\,\max\{\gamma,\gamma_1\}\,\max\{1,C\}\;R\,\psi(\alpha) \qquad (0 < \alpha \le \bar{\alpha}),$$

completing the proof. □

Remark 6.9. The results of Theorem 6.7 with an appropriately modified constant in (6.13) can also be obtained under the weaker assumption that

$$(\sigma/\tau)(t) \;\; (0 < t \le \varepsilon) \quad \text{is an index function}\,,$$

for arbitrarily small $\varepsilon > 0$ instead of the global assumption (6.12). This is an immediate result of the opportunity of localization as outlined in Theorem 5.12 and its proof.

As mentioned above, we can also try to transfer the information from the link conditions (6.2) and (6.3) to the scale generated by $H = A^*A$.

We recall the definition of the function ψ in formula (6.9) in the context of Theorem 6.5. The following observation is useful.

Lemma 6.10. *Let the functions τ,σ and φ,ψ be as in Theorem 6.5. If the quotient σ/τ is an index function on $(0,\|G\|]$ then φ/ψ is an index function on $(0,a]$.*

Proof. We assign $s = s(t) := \sigma^{-1}(\varphi(t))$ $(0 < t \le a)$, thus $s \in (0,b]$. With this identification we obtain

$$\frac{\varphi(t)}{\psi(t)} = \frac{\sigma(s)}{\psi(\varphi^{-1}(\sigma(s)))} = \frac{\sigma(s)}{\tau(\sigma^{-1}(\sigma(s)))} = \frac{\sigma(s)}{\tau(s)}.$$

□

Keeping this lemma in mind we can prove the following counterpart of Theorem 6.7.

Theorem 6.11. *Assume that the regularization g_α has qualification φ with constant γ and that σ/τ is an index function on $(0, \|G\|]$. Under the assumptions of Theorem 6.5, in particular the operator monotonicity of the function $\tau^2((\sigma^2)^{-1}(t))$ $(0 < t \le \varphi^2(a))$, the function*

$$f(\alpha) = 2\,\max\{\gamma, \gamma_1\}\,\max\{1, C\}\; R\psi(\alpha) \qquad (0 < \alpha \le a) \tag{6.15}$$

is a profile function for $(G_\tau(R), g_\alpha)$.

Proof. Let $L := \max\{1, C\}$. The estimate (6.10) of Theorem 6.5 yields the inclusion $G_\tau(R) \subset H_\psi(LR)$. Thus profile functions for $(H_\psi(LR), g_\alpha)$ are also profile functions for $(G_\tau(R), g_\alpha)$. By Lemma 6.10 the function $\varphi(t)/\psi(t)$ $(0 < t \le a)$ is an index function and we can apply Theorem 5.9 to bound the distance function $\rho_{x^\dagger}^{(H,\psi)}$ as

$$\rho_{x^\dagger}^{(H,\psi)}(t) \le \varphi\left(\left(\frac{\varphi}{\psi}\right)^{-1}(LRt)\right) \qquad (0 < t \le \frac{\varphi(a)}{LR\psi(a)}).$$

Corollary 5.10 provides us with the profile function as given in (6.15). □

Example 6.12. Again, let us discuss the situation when the index functions are in the form of monomials, precisely we assume that $\sigma(t) = t^\mu$, $\tau(t) = t$. Then the operator monotonicity as required in Corollary 6.11 is fulfilled whenever $\mu \ge 1$, which can be deduced from the Heinz-Kato inequality. If the link condition (6.2) is assumed to hold for $\varphi(t) = t^\nu$, and if the regularization has qualification φ, then we arrive at a profile function $f(\alpha) = C\alpha^{\nu/\mu}$, uniformly for $x^\dagger$ satisfying (5.3) and (6.1).

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Department of Mathematics, Chemnitz University of Technology, 09107 Chemnitz, Germany

E-mail address: `hofmannb@mathematik.tu-chemnitz.de`

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany

E-mail address: `mathe@wias-berlin.de`