Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

An iteration procedure for solving integral equations related to the American put options

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submitted: 4th April 2006

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No. 1105
Berlin 2006

2000 *Mathematics Subject Classification.* 65D15, 91B28, 60G40, 65D30, 60J60, 60J65.

Key words and phrases. American put option, Black-Scholes model, optimal stopping, Picard iterations, upper and lower bounds.

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Abstract

A new algorithm for pricing American put option in the Black-Scholes model is presented. It is based on a time discretization of the corresponding integral equation. The proposed iterative procedure for solving the discretized integral equation converges in a finite number of steps and delivers in each step a lower or an upper bound for the price of discretized option on the whole time interval. The method developed can be easily implemented and carried over to the case of more general optimal stopping problems.

1 Introduction

Pricing American options is one of the interesting and important problems in the mathematical theory of modern finance. This problem was first studied by McKean [13] who derived a free-boundary problem for the price and the optimal exercise boundary of an early exercise American option and obtained a countable system of nonlinear integral equations for the boundary. Kim [11], Jacka [9] and Carr, Jarrow and Myneni [3] (see also Myneni [14]) have independently arrived at a nonlinear integral equation for the exercise boundary of the American put option which follows from the more general early exercise premium (EEP) representation. The uniqueness of solution has been recently proven by Peskir [17].

Since the arbitrage-free price and the optimal boundary of an early exercise American put option cannot be found in an explicit form, some different numerical procedures for calculating the price and the boundary have been proposed. Carr [2] presented a method based on the randomization of the maturity time using the Erlang distribution which is equivalent to taking the Laplace transform of the initial price of an American put option. In that case the solution of the related free-boundary problem can be derived in a closed form. Hou, Little and Pant [8] have established a new representation for the American put option and proposed an efficient numerical algorithm for solving the corresponding nonlinear integral equation for the optimal exercise boundary. Pedersen and Peskir [16] (see also [5]-[6]) have used the backward induction method and simple time discretization of the nonlinear integral equation for obtaining the optimal stopping boundary. Kolodko and Schoenmakers [12] presented a policy iteration method for computing the optimal Bermudan stopping time. In recent years, Monte Carlo based methods have become rather popular (see e.g. Rogers [18] and Glasserman [7] for an overview). In [1] an iterative Monte-Carlo procedure has been proposed which makes use of the earlier exercise premium representation for American and Bermudan options. The method of [1] can be considered as an analogue to the classical Picard iteration method applied for the proof of existence of solutions of integral equations (cf. e.g. Tricomi [22]) having the advantage that it allows to obtain an upper bound for the price from a lower one and the lower bound from an upper one. In this paper we propose a modification of this method which employees along with the European option price the arbitrage-free price of the perpetual American put option derived by McKean [13] (see also Shiryaev et al [19], Shiryaev [21], Novikov and Shiryaev [15]). Moreover, the convergence of the new algorithm is established and the rates of convergence are obtained.

The paper is organized as follows. In Section 2 we recall some known results related to the American put option pricing problem and discuss different forms of EEP representation. In Section 3 we construct a simple time discretization of the corresponding integral equation and propose a numerical iteration procedure for solving it which produces in each step low or upper bounds for the solution and arrives at it in a finite number of steps. We stress that as opposite to the backward induction, in each step the procedure delivers an approximation on the whole time interval and not only for the several last time intervals. The main results of the paper are formulated in Lemma 3 and Theorem 4.

2 Formulation of the problem

In this section we recall the results from [13], [11], [9], [3] and [17] and formulate the problem of estimating the value function of the corresponding optimal stopping problem.

2.1. For a precise formulation of the American put option pricing problem, let us consider a probability space $(\Omega, \mathcal{F}, Q)$ with a standard Brownian motion $B = (B_t)_{0 \le t \le T}$ started at zero. Suppose that the stock price process $S = (S_t)_{0 \le t \le T}$ is defined by:

$$S_t = s \, \exp\Big(\big(r - \sigma^2/2\big)\, t + \sigma\, B_t\Big) \tag{2.1}$$

and hence solves the stochastic differential equation:

$$dS_t = r S_t \, dt + \sigma S_t \, dB_t \quad (S_0 = s) \tag{2.2}$$

where $s > 0$ is given and fixed. Here $r > 0$ is a continuously compounded interest rate and $\sigma > 0$ is a volatility coefficient.

It follows from the results of general arbitrage theory (see e.g. [21] or [10]) that the *arbitrage-free price* of an American put option with the strike $K > 0$ is given by

$$P(t, s) = \sup_{0 \le \tau \le T-t} E_{t,s}\big[e^{-r\tau}\,(K - S_{t+\tau})^+\big] \tag{2.3}$$

where the supremum is taken over all stopping times τ of the process S (i.e. with respect to the natural filtration $(\mathcal{F}_{t+u})_{0 \le u \le T-t}$ generated by the process $(S_{t+u})_{0 \le u \le T-t}$).

Here $E_{t,s}$ denotes the expectation with respect to the initial martingale measure $Q_{t,s}$ when the process $(S_{t+u})_{0\le u\le T-t}$ starts at $S_t = s$. It is known (see [13] and [3]) that the optimal stopping time in (2.3) is given by

$$\begin{aligned}\tau_b &= \inf\{0 \le u \le T-t \mid S_{t+u} \le b(t+u)\} \\ &= \inf\{0 \le u \le T-t \mid P(t+u, S_{t+u}) \le (K - S_{t+u})^+\}\end{aligned} \tag{2.4}$$

and that the value function (2.3) admits the following early exercise premium representation

$$\begin{aligned}P(t,s) \;&=\; e^{-r(T-t)}\, E_{t,s}\big[(K-S_T)^+\big] \\ &\quad + rK \int_0^{T-t} e^{-ru}\, Q_{t,s}\big[S_{t+u} \le b(t+u)\big]\, du \\ &=\; e^{-r(T-t)}\, E_{t,s}\big[(K-S_T)^+\big] \\ &\quad + rK \int_0^{T-t} e^{-ru}\, Q_{t,s}\big[P(t+u, S_{t+u}) \le (K - S_{t+u})^+\big]\, du.\end{aligned} \tag{2.5}$$

It is also known (see [11] and [9]) that the optimal exercise boundary $b(t)$ of the early exercise American put option solves the nonlinear integral equation

$$\begin{aligned}K - b(t) \;&=\; e^{-r(T-t)}\, E_{t,b(t)}\big[(K-S_T)^+\big] \\ &\quad + rK \int_0^{T-t} e^{-ru}\, Q_{t,b(t)}\big[S_{t+u} \le b(t+u)\big]\, du\end{aligned} \tag{2.6}$$

for all $0 \le t \le T$ and $s > 0$. By using the change-of-variable formula with local times on curves, it was proven in [17] that the equation (2.6) admits a unique solution. Note that the nonlinear integral equation (2.5) is preferable over the equation involving the boundary since it allows a generalization to the multidimensional case. Generally, the equations (2.5) and (2.6) cannot be solved in an explicit form and numerical methods have to be used.

2.2. By means of standard arguments based on the strong Markov property it can be shown that the arbitrage-free price (2.3) solves the following parabolic free-boundary problem (see [13])

$$(P_t + rsP_s + (\sigma^2/2)s^2 P_{ss})(t,s) = rP(t,s) \quad \text{for} \quad s > b(t) \tag{2.7}$$

$$P(t,s)\big|_{s=b(t)} = K - b(t) \quad \text{(instantaneous stopping)} \tag{2.8}$$

$$P_s(t,s)\big|_{s=b(t)} = -1 \quad \text{(smooth fit)} \tag{2.9}$$

$$P(t,s) > (K-s)^+ \quad \text{for} \quad s > b(t) \tag{2.10}$$

$$P(t,s) = (K-s)^+ \quad \text{for} \quad s < b(t) \tag{2.11}$$

where the condition (2.8) holds for all $0 \le t < T$.

Note that the superharmonic characterization of the value function (see [4] and [20]) implies that (2.3) is the smallest function satisfying (2.7)-(2.8) and (2.10)-(2.11).

2.3. Letting T tend to infinity in (2.5) and (2.6), we obtain

$$\begin{aligned}\overline{P}(t,s) &= rK \int_0^\infty e^{-ru}\, Q_{t,s}\big[S_{t+u} \le \overline{b}(t+u)\big]\, du \\ &= rK \int_0^\infty e^{-ru}\, Q_{t,s}\big[\overline{P}(t+u,S_{t+u}) \le (K-S_{t+u})^+\big]\, du \end{aligned} \tag{2.12}$$

and

$$K - \overline{b}(t) = rK \int_0^\infty e^{-ru}\, Q_{t,\overline{b}(t)}\big[S_{t+u} \le \overline{b}(t+u)\big]\, du \tag{2.13}$$

where functions $\overline{P}(t,s)$ and $\overline{b}(t)$ are uniquely determined by the equations (2.12) and (2.13), respectively. By means of straightforward calculations it can be verified that $\overline{P}(t,s) \equiv \overline{P}(s)$ and $\overline{b}(t) \equiv \overline{b}$ coincide with the arbitrage-free price and the optimal exercise boundary respectively of the perpetual American put option. From the formulas (2.5) and (2.12) it follows that

$$\begin{aligned} P(t,s) &= \widetilde{P}(t,s) + rK \int_0^{T-t} e^{-ru}\, Q_{t,s}\big[\overline{b} < S_{t+u} \le b(t+u)\big]\, du \\ &= \widetilde{P}(t,s) + rK \int_0^{T-t} e^{-ru} \\ &\quad \times Q_{t,s}\big[P(t+u,S_{t+u}) \le (K-S_{t+u})^+ < \overline{P}(S_{t+u})\big]\, du \end{aligned} \tag{2.14}$$

where we denote

$$\widetilde{P}(t,s) = \overline{P}(s) + e^{-r(T-t)}\, E_{t,s}\big[(K-S_T)^+\big] - rK \int_{T-t}^\infty e^{-ru}\, Q_{t,s}\big[S_{t+u} \le \overline{b}\big]\, du \tag{2.15}$$

for all $0 \le t \le T$ and $s > 0$. The expressions (2.5) and (2.14) are in fact basis for our algorithm. Note that (2.14) has an advantage over (2.5) for it involves probabilities of S_t belonging to a bounded intervals which are numerically (using Monte Carlo) easier to compute than those for unbounded intervals.

3 Main result and proofs

In this section we approximate the initial model by discretizing the integral equation (2.14) and propose an iteration procedure which solves the discretized integral equation in a finite number of steps. We prove uniform convergence of this solution to the initial value function as the discretization becomes finer and determine the rate of convergence.

3.1. In order to construct an approximation for the equation (2.14) let us fix some arbitrary $0 \le t \le T$ and $n \in \mathbb{N}$ and introduce a partition of the time interval $[0, T-t]$. Let $u_0 = 0$ and $u_i = i\Delta_n$ with $\Delta_n = (T-t)/n$ implying that $u_i - u_{i-1} = \Delta_n$ for every $i = 1, \ldots, n$. Taking into account the structure of expression (2.14), let us define the approximation $\widehat{P}_n(t+u,s)$ for the price $P(t+u,s)$ as a solution of the equation

$$\begin{aligned}
\widehat{P}_n(t+u,s) &= \widetilde{P}(t+u,s) + rK \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-ru_i} \\
&\quad \times Q_{t,s}\big[\overline{b} < S_{t+u_i} \le \widehat{b}_n(t+u_i)\big]\,\Delta_n \\
&= \widetilde{P}(t+u,s) + rK \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-ru_i} \qquad (3.1)\\
&\quad \times Q_{t,s}\big[\widehat{P}_n(t+u_i,S_{t+u_i}) \le (K-S_{t+u_i})^+ < \overline{P}(S_{t+u_i})\big]\,\Delta_n,
\end{aligned}$$

where the estimate $\widehat{b}_n(t+u)$ for the boundary $b(t+u)$ is defined as the maximum of the intersection curve of $\widehat{P}_n(t+u,s)$ with $(K-s)^+$ and the perpetual option boundary $\overline{b}$. Here $\lceil x\rceil = \lfloor x\rfloor + 1$ and $\lfloor x\rfloor$ denotes the integer part of a positive number $x \in \mathbb{R}$. It is clear that the equation (3.1) has a unique solution which can be obtained by means of backward induction in a finite number of steps. This implies that the (piecewise constant) function $\widehat{P}_n(t+u,s)$ is uniquely determined by (3.1) for all $0 \le u \le T-t$ and $s > 0$. Let us now define sequentially the functions $\widehat{P}_n^m(t+u,s)$ by

$$\begin{aligned}
\widehat{P}_n^m(t+u,s) &= \widetilde{P}(t+u,s) + rK \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-ru_i} \qquad (3.2)\\
&\quad \times Q_{t,s}\big[\widehat{P}_n^{m-1}(t+u_i,S_{t+u_i}) \le (K-S_{t+u_i})^+ < \overline{P}(S_{t+u_i})\big]\,\Delta_n.
\end{aligned}$$

Here we set $\widehat{P}_n^0(t+u,s) = (K-s)^+$ for all $0 \le u \le T-t$ and $s > 0$.

Remark 1 *It is easily seen from (3.1) that, by construction*

$$\widehat{P}_n^{2k-1}(t+u,s) \ge \widehat{P}_n(t+u,s), \quad 0 \le u \le T-t, \quad s > 0, \quad k \in \mathbb{N}$$

and

$$\widehat{P}_n^{2k}(t+u,s) \le \widehat{P}_n(t+u,s), \quad 0 \le u \le T-t, \quad s > 0, \quad k \in \mathbb{N}.$$

More generally, any low estimate $\widehat{P}_n^{m-1}$, $m \in \mathbb{N}$, for $\widehat{P}_n$ produces an upper one $\widehat{P}_n^m$ and vice versa.

Remark 2 *For each $m < n$ the function $\widehat{P}_n^m(t+u,s)$ is an estimate for $\widehat{P}_n(t+u,s)$ on the whole interval $[0,T-t]$. This fact shows the advantage of this method over the standard backward induction.*

3.2. Let us now show that the sequence of functions $(\widehat{P}_n^m(t+u,s))_{m\in\mathbb{N}}$ from (3.2) converges to the function $\widehat{P}_n(t+u,s)$ in n steps for all $0 \le u \le T-t$ and $s > 0$.

Lemma 3 *For each $0 \le t \le T$ fixed we have $\widehat{P}_n^m(t+u,s) = \widehat{P}_n(t+u,s)$ for every $m \ge n$ and all $0 \le u \le T-t$, $s > 0$.*

Proof. Let us fix $0 \le t \le T$ and $n \in \mathbb{N}$. Then by construction of $\widehat{P}_n^m(t+u,s)$ the equalities

$$\begin{aligned}\widehat{P}_n^{2k+1}(t+u,s) - \widehat{P}_n^{2k}(t+u,s) = rK \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-ru_i} \\ \times\ Q_{t,s}\big[\widehat{P}_n^{2k}(t+u_i,S_{t+u_i}) \le (K-S_{t+u_i})^+ < \widehat{P}_n^{2k-1}(t+u_i,S_{t+u_i})\big]\,\Delta_n \quad (3.3)\end{aligned}$$

and

$$\begin{aligned}\widehat{P}_n^{2k+2}(t+u,s) - \widehat{P}_n^{2k+1}(t+u,s) = rK \sum_{i=\lceil un/(T-t)\rceil}^{n} e^{-ru_i} \\ \times\ Q_{t,s}\big[\widehat{P}_n^{2k}(t+u_i,S_{t+u_i}) \le (K-S_{t+u_i})^+ < \widehat{P}_n^{2k+1}(t+u_i,S_{t+u_i})\big]\,\Delta_n \quad (3.4)\end{aligned}$$

hold for all $0 \le u \le T-t$ and $s > 0$ and every $k \in \mathbb{N}$.

In order to prove the desired assertion we should use the mathematical induction principle. First, note that $\widehat{P}_n^m(T,s) = (K-s)^+$ for all $s > 0$ and $m \in \mathbb{N}$. For checking the induction basis it is enough to observe that if $m = 2k$ with $k = 0$ then (3.4) implies equality

$$\begin{aligned}\widehat{P}_n^{2}(t+u,s) - \widehat{P}_n^{1}(t+u,s) = rK\, e^{-ru_n} \\ \times\ Q_{t,s}\big[\widehat{P}_n^{0}(t+u_n,S_{t+u_n}) \le (K-S_{t+u_n})^+ < \widehat{P}_n^{1}(t+u_n,S_{t+u_n})\big]\,\Delta_n = 0 \quad (3.5)\end{aligned}$$

which holds for all $(n-1)(T-t)/n \le u \le T-t$ where by definition of the partition $t + u_n = T$. ■

3.3. Now we prove that the solution of the discretized equation (3.1) converges to $P(t+u,s)$ uniformly on $[0,T-t]$ as n tends to infinity.

Theorem 4 *Let $\widehat{P}_n(t+u,s)$ be a solution of the discretized equation (3.1). Then there exists some $t \in [0,T]$ close enough to T such that the sequence $(\widehat{P}_n(t+u,s))_{n\in\mathbb{N}}$ converges to $P(t+u,s)$ uniformly for $0 \le u \le T-t$ and $s > 0$ with the rate $1/n$ when n tends to infinity.*

Proof. First, the representations (2.5) and (3.1) imply

$$\begin{aligned} \left|\widehat{P}_n(t,s) - P(t,s)\right| \;\le\; & \Bigg| rK \sum_{i=1}^{n} e^{-ru_i}\, Q_{t,s}\big[S_{t+u_i} \le b(t+u_i)\big]\, \Delta_n \\ & - rK \int_0^{T-t} e^{-ru}\, Q_{t,s}\big[S_{t+u} \le b(t+u)\big]\, du \Bigg| \\ & + rK \sum_{i=1}^{n} e^{-ru_i} \Big| Q_{t,s}\big[S_{t+u_i} \le \widehat{b}_n(t+u_i)\big] \\ & - Q_{t,s}\big[S_{t+u_i} \le b(t+u_i)\big] \Big|\, \Delta_n \end{aligned} \tag{3.6}$$

for all $0 \le t \le T$ and $s > 0$. In order to deal with the first term on the right-hand side of (3.6) we can use the estimate for Riemann sum approximation and obtain

$$\begin{aligned} \Bigg| \int_0^{T-t} & e^{-ru}\, Q_{t,s}\big[S_{t+u} \le b(t+u)\big]\, du \\ & - \sum_{i=1}^{n} e^{-ru_i}\, Q_{t,s}\big[S_{t+u_i} \le b(t+u_i)\big]\, \Delta_n \Bigg| \le \frac{C_1}{rKn} \end{aligned} \tag{3.7}$$

for $n \ge N_1$ and $C_1 > 0$ fixed. As to the second term in (3.6), we can make use of the mean value theorem and get

$$\begin{aligned} & \Big| Q_{t,s}\big[S_{t+u_i} \le \widehat{b}_n(t+u_i)\big] - Q_{t,s}\big[S_{t+u_i} \le b(t+u_i)\big] \Big| \\ & = \left| \Phi\left(\frac{\log[\widehat{b}_n(t+u_i)/s]}{\sigma\sqrt{u_i}} - \Big(r - \frac{\sigma^2}{2}\Big) u_i \right) - \Phi\left(\frac{\log[b(t+u_i)/s]}{\sigma\sqrt{u_i}} - \Big(r - \frac{\sigma^2}{2}\Big) u_i \right) \right| \\ & \qquad = \frac{\Phi'(\xi_i)}{\sigma\sqrt{u_i}} \left| \log \frac{\widehat{b}_n(t+u_i)}{b(t+u_i)} \right| = \frac{\Phi'(\xi_i)}{\sigma\sqrt{u_i}} \left| \log \frac{K - \widehat{P}_n(t+u_i, \widehat{b}_n(t+u_i))}{K - P(t+u_i, b(t+u_i))} \right| \end{aligned} \tag{3.8}$$

for some $\xi_i \in \mathbb{R}$, $i = 1, \dots, n$ and $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-y^2/2} dy$. The last equality in (3.8) follows directly from (2.8) and (2.11). Using the obvious fact that $b(t) \ge \overline{b} > 0$ for all $0 \le t \le T$ and $s > 0$, from (3.6) we obtain

$$\begin{aligned} & \left| \log \frac{K - \widehat{P}_n(t+u_i, \widehat{b}_n(t+u_i))}{K - P(t+u_i, b(t+u_i))} \right| \\ & = \log \left(1 + \left| \frac{\widehat{P}_n(t+u_i, \widehat{b}_n(t+u_i)) - P(t+u_i, b(t+u_i))}{K - P(t+u_i, b(t+u_i))} \right| \right) \\ & \le \frac{|\widehat{P}_n(t+u_i, \widehat{b}_n(t+u_i)) - P(t+u_i, b(t+u_i))|}{K - P(t+u_i, b(t+u_i))} \\ & \le \frac{|\widehat{P}_n(t+u_i, \widehat{b}_n(t+u_i)) - P(t+u_i, b(t+u_i))|}{\overline{b}} \\ & \le \frac{|\widehat{P}_n(t+u_i, s_i) - P(t+u_i, s_i)|}{\overline{b}} \end{aligned} \tag{3.9}$$

for some $s_i \in (\widehat{b}_n(t) \wedge b(t), \widehat{b}_n(t) \vee b(t))$, and hence

$$|\widehat{P}_n(t+u_i, s_i) - P(t+u_i, s_i)| \leq \sup_{u_i \in [0,T-t]} \sup_{s_i > 0} |\widehat{P}_n(t+u_i, s_i) - P(t+u_i, s_i)| \quad (3.10)$$

for all $0 \leq t \leq T$ and every $i = 1, \dots, n$. By virtue of the fact that the function $e^{-ru}/\sqrt{u}$ is decreasing, straightforward calculations show that the inequalities

$$\sum_{i=1}^{n} \frac{e^{-ru_i}}{\sigma \sqrt{u_i}} \, \Phi'(\xi_i) \, \Delta_n \leq \frac{1}{\sqrt{2\pi}} \int_0^{T-t} \frac{e^{-ru}}{\sigma \sqrt{u}} \, du \leq \frac{C_2 \overline{b}}{rK} \sqrt{T-t} \quad (3.11)$$

hold for all $0 \leq t \leq T$ and some $C_2 > 0$ fixed. Therefore, combining (3.7)-(3.11), from (3.6) we obtain

$$\begin{aligned} \big|\widehat{P}_n(t,s) - P(t,s)\big| &\leq \frac{C_1}{n} + C_2 \sqrt{T-t} \\ &\qquad \times \sup_{u_i \in [0,T-t]} \sup_{s_i > 0} \big|\widehat{P}_n(t+u_i, s_i) - P(t+u_i, s_i)\big| \quad (3.12) \end{aligned}$$

for all $0 \leq t \leq T$ and $s > 0$. Hence

$$\begin{aligned} &\sup_{u \in [0,T-t]} \sup_{s > 0} \big|\widehat{P}_n(t+u, s) - P(t+u, s)\big| \quad (3.13) \\ &\qquad \leq \frac{C_1}{n} + C_2 \sqrt{T-t} \sup_{u \in [0,T-t]} \sup_{s > 0} \big|\widehat{P}_n(t+u, s) - P(t+u, s)\big| \end{aligned}$$

for all $0 \leq t \leq T$ and $s > 0$.

Let us now choose some $t \in [0,T]$ such that $C_2 \sqrt{T-t} \leq 1/2$. Then it follows from (3.13) that:

$$\sup_{u \in [0,T-t]} \sup_{s > 0} \big|\widehat{P}_n(t+u, s) - P(t+u, s)\big| \leq \frac{2C_1}{n} \quad (3.14)$$

for all $n \in \mathbb{N}$ such that $n \geq N_1$. This completes the proof of the theorem. $\square$

3.4. In principle, one could construct directly the estimate for the price function (2.3) without use of discretization by the iterative scheme

$$\begin{aligned} P^m(t,s) &= \widetilde{P}(t,s) + rK \int_0^{T-t} e^{-ru} \quad (3.15) \\ &\qquad \times Q_{t,s}\big[P^{m-1}(t+u, S_{t+u}) \leq (K - S_{t+u})^+ \leq \overline{P}(S_{t+u})\big] \, du \end{aligned}$$

where we set $P^0(t,s) = (K-s)^+$ for all $0 \leq t \leq T$ and $s > 0$.

Remark 5 *Again, by the construction*

$$P^{2k-1}(t,s) \geq P(t,s), \quad 0 \leq t \leq T, \quad s > 0, \quad k \in \mathbb{N}$$

and

$$P^{2k}(t,s) \leq P(t,s), \quad 0 \leq t \leq T, \quad s > 0, \quad k \in \mathbb{N}.$$

This means that an upper estimate (3.15) for (2.5) can be obtained from a lower one and a lower estimate (3.15) can be obtained from an upper one.

Acknowledgments. This paper was written during the time when the second author was visiting Weierstraß Institute for Applied Analysis and Stochastics (WIAS) Berlin and he is thankful for the financial support and hospitality.

References

[1] Belomestny, D. and Milstein, G. N. (2005). Adaptive simulation algorithms for pricing American and Bermudan options by local analysis of the financial market. *WIAS Preprint* **1022**. To appear in *J. Comput. Finance.*

[2] Carr, P. (1998). Randomization and the American put. *Review of Financial Studies* **11** (597–626).

[3] Carr, P., Jarrow, R. and Myneni, R. (1992). Alternative characterization of American put options. *Math. Finance* **2** (78–106).

[4] Dynkin, E. B. (1963). The optimum choice of the instant for stopping a Markov process. *Soviet Math. Dokl.* **4** (627–629).

[5] Gapeev, P. V. and Peskir, G. (2003). The Wiener disorder problem with finite horizon. *Research Report* No. 435, Dept. Theoret. Statist. Aarhus (22 pp).

[6] Gapeev, P. V. and Peskir, G. (2004). The Wiener sequential testing problem with finite horizon. *Stochastics and Stochastic Reports* **76** (59–75).

[7] Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering.* Springer, New York.

[8] Hou, C., Little, T. and Pant, V. (2000). A new integral representation of the early exercise bounday for American put options. *J. Comput. Finance* **3** (73–96).

[9] Jacka, S. D. (1991). Optimal stopping and the American put. *Math. Finance* **1** (1–14).

[10] Karatzas, I. and Shreve, S. E. (1998). *Methods of Mathematical Finance.* Springer, New York.

[11] Kim, I. J. (1990). The analytic valuation of American options. *Rev. Financial Stud.* **3** (547–572).

[12] Kolodko, A. and Schoenmakers, J. (2005). Iterative construction of the optimal Bermudan stopping time. *Finance Stochast.* **10**(1) (27–49).

[13] McKean, H. P. Jr. (1965). Appendix: A free boundary problem for the heat equation arising form a problem of mathematical economics. *Ind. Management Rev.* **6** (32–39).

[14] MYNENI, R. (1992). The pricing of the American option. *Ann. Appl. Probab.* **2**(1) (1–23).

[15] NOVIKOV, A. A. and SHIRYAEV, A. N. (2004). On an effective solution of the optimal stopping problem for random walks. *Theor. Probab. Appl.* **49**(2) (373–382).

[16] PEDERSEN, J. L. and PESKIR, G. (2002). On nonlinear integral equations arising in problems of optimal stopping. *Proc. Functional Anal.* VII (Dubrovnik 2001), *Various Publ. Ser.* **46** (159–175).

[17] PESKIR, G. (2005). On the American option problem. *Math. Finance.* **15**(1) (169-181).

[18] ROGERS, L. C. G. (2002). Monte Carlo valuation of American options. *Math. Finance* **12** (271–286).

[19] SHIRYAEV, A. N., KABANOV, Y. M., KRAMKOV, D. O. and MELNIKOV, A. V. (1994). On the pricing of options of European and American types, II. Continuous time. *Theory Probab. Appl.* **39** (61–102).

[20] SHIRYAEV, A. N. (1978). *Optimal Stopping Rules.* Springer, Berlin.

[21] SHIRYAEV, A. N. (1999). *Essentials of Stochastic Finance.* World Scientific, Singapore.

[22] TRICOMI, F. G. (1957). *Integral Equations.* Interscience Publishers, London and New York.