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Oscillatory instability in systems with delay

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Abstract

Any biological or physical system, which incorporates delayed feedback or delayed coupling, can be modeled by a dynamical system with delayed argument. We describe a standard oscillatory destabilization mechanism, which occurs in such systems.

Differential equations with delayed argument

$$\dot{y} = \mathcal{F}(y, y_\tau), \tag{1}$$

where $y_\tau = y(t-\tau)$, turned out to be a very useful tool for studying many physical and biological systems. Such models frequently appear in laser physics [1], different biological feedback systems, such as dynamic diseases [2], neural networks [3], population ecology [4], etc. Therefore, understanding common properties of all above mentioned physical systems, is an important interdisciplinary problem. In this letter, we describe the basic oscillatory destabilization mechanism, which leads to the appearence of nonstationary behavior in delayed systems.

When the delay τ is sufficiently small, the effective dynamics is still low-dimensional and can be considered as a small correction to the dynamics with $\tau = 0$. Some rigorous results in this direction are given in [5]. On the other hand, for larger τ, the additional degrees of freedom introduced by the delay become relevant and the dynamics becomes multidimensional. In many cases it may even display features of spatially extended systems [6, 7, 8, 9].

In this letter, we reveal generic features of the oscillatory instability in systems with large delay. It appears to be twofold: after the bifurcation, the originally stable stationary state becomes weakly unstable and the Eckhaus instability develops, which leads to the appearance of multiple coexistent stable periodic attractors. With further changing of the bifurcation parameter, the stationary state becomes strongly unstable and the system exhibits single periodic attractor. The bifurcation parameter mediates the transition between these two states, one of which is essentially multidimensional and the other is low-dimensional.

The plan of our paper is as follows: first, we describe the conditions, which lead to the appearance of such instability by analyzing the linearized equation. Then we show, that close to the bifurcation, the Eckhaus phenomenon with multiple coexisting periodic attractors occurs. We show that this phenomenon can be well described by the amplitude equations approach [6]. In particular, we derive a complex Ginzburg-Landau (GL) equation, which, equipped with corresponding boundary conditions, describes the dynamics of a slowly varying amplitude. The existence of Eckhaus instability in the amplitude equations then implies a similar behavior in the delay system. With further increasing of the bifurcation parameter, the stationary state becomes strongly unstable with only one dominant mode. As a result, the above mentioned spatiotemporal representation fails, the multistability vanishes and a unique periodic attractor remains as in the case of the usual Hopf bifurcation.

We consider the following class of systems

$$z' = (\alpha + i\beta)\, z + z_\tau - z|z|^2, \tag{2}$$

where z is the complex variable and α, β are real parameters. System (2) can be obtained from the normal form of a supercritical Hopf bifurcation for finite-dimensional systems (ordinary differential equations) by adding the delayed feedback term z_τ. It has the stationary state $z = 0$. Our goal is to explain in details the mechanism of destabilization of this state as the parameter α, which governs the Hopf bifurcation in the non-delayed case, varies. As we will see, it shows some interesting generic features.

Linear stability analysis: Let us start with the linear stability analysis of the steady state. The growth rate λ of small perturbations of type $e^{\lambda t}$ is determined by solutions of the characteristic equation

$$\lambda - (\alpha + i\beta) - e^{-\lambda\tau} = 0. \tag{3}$$

Eq. (3) has infinitely many solutions, which can be expressed, for instance, via the Lambert function [10]. Since we consider the case when the delay τ is large, we are interested in the asymptotics of these roots as $\tau \to \infty$. Therefore, it is convenient to introduce a small parameter $\varepsilon = 1/\tau$. As has been shown in [11, 12] for more general cases, the characteristic equation (3) has two types of solutions, which have different asymptotical properties with respect to ε:

• Strongly unstable eigenvalues $\lambda_S = \alpha + i\beta + \mathcal{O}(\varepsilon)$ for $\alpha > 0$, which originate from the instantaneous terms;

• Pseudo-continuous spectrum of eigenvalues, which, up to the leading order in ε, can be approximated as $\lambda_P(\omega) = i\omega + \varepsilon\gamma(\omega)$. Here, the parameter ω admits a countable set of values $\omega = \omega_k = \omega_0 + 2\pi k\varepsilon$, $k = 0, \pm1, \pm2, \ldots$ By substituting λ_P into (3), we obtain

$$\gamma(\omega) = -\frac{1}{2}\ln\left(\alpha^2 + (\omega - \beta)^2\right). \tag{4}$$

Since $\operatorname{Re}\lambda_P = \varepsilon\gamma(\omega)$, the function $\gamma(\omega)$ determines the stability of the stationary state for $\alpha < 0$. Recall, that for $\alpha > 0$ the strongly unstable eigenvalue appears. The pseudo-continuous spectrum is illustrated in Fig. 1(a). The corresponding eigenvalues are located along the curves $(\operatorname{Re}\lambda = \varepsilon\gamma(\omega), \operatorname{Im}\lambda = \omega)$ at discrete positions, corresponding to the values $\omega = \omega_k$ with small distances $2\pi\varepsilon$ between each other. As delay is increased, the curves $\gamma(\omega)$ persist being filled more and more densely with eigenvalues.

One can see that the pseudo-continuous spectrum implies instability for $|\alpha| < 1$. At $|\alpha| = 1$, the curve touches the imaginary axis at the critical frequency β ($\beta = 1$ in the

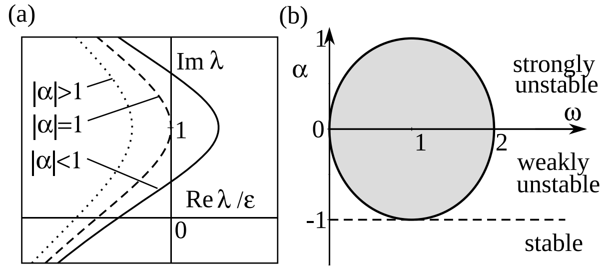

Figure 1: Linear stability analysis of system (2). (a) Curves of pseudo-continuous spectrum, along which the eigenvalues are accumulated as delay increases, shown for three different parameter values: solid line corresponds to $\alpha = 0.8$, dashed to $\alpha = 1$, and dotted to $\alpha = 1.2$. The destabilization occurs at $|\alpha| = 1$. (b) The interval of unstable frequencies ω is shown in gray for different values of parameter α. These frequencies correspond to the interval of Imλ, for which the pseudo-continuous spectrum from Fig. (a) is unstable. In addition, the stability properties of the stationary state for different values of α are indicated. $\beta = 1$ is fixed.

figure). Hence, in contrast to the Hopf bifurcation in the system without delay, stability loss happens already for $\alpha = -1$. With increasing control parameter α, the stationary state becomes unstable to perturbations of the form $e^{i\omega t}$, where ω belongs to some interval around β. These unstable frequencies can be obtained from (4) as those, which satisfy $\gamma(\omega) > 0$. We obtain $\alpha^2 + (\omega - \beta)^2 < 1$. This set of unstable frequencies of the stationary state is illustrated in Fig. 1(b). This figure also summarizes the stability properties of the stationary state:

- for $\alpha < -1$ it is stable;
- for $-1 < \alpha < 0$ it is weakly unstable, i.e. the unstable eigenvalues belong to the pseudo-continuous spectrum and their real parts are of order ε;
- for $\alpha > 0$ it is strongly unstable possessing the eigenvalue $\lambda_S \approx \alpha + i\beta$.

Numerical results: In our numerical simulations, we fix $\beta = 1$. First, we have chosen the bifurcation parameter $\alpha = -0.8$ such that the stationary state is already unstable. The results of the integration for system (2) are shown in Fig. 2. For convenience, we show the orbit of the delay system using "spatio-temporal" representation [6]. Roughly speaking the horizontal axis corresponds to the spacelike coordinate ranging from 0 to τ and the vertical axis to some rescaled slow time t/τ^3. The precise meaning of the axes will become clear further in the text, when the amplitude equations are introduced. Such a representation is useful, since it shows the solution over a time interval of order τ^3. We observe, that the system can approach different periodic states depending on initial conditions [1]. The solutions in Figs. 2(a) and (b) are obtained for the initial functions

[1] The transient should be of order τ^3. This rule will be confirmed by the amplitude equations, since the scale of the slow time for the amplitude variable is $\varepsilon^3 t$.

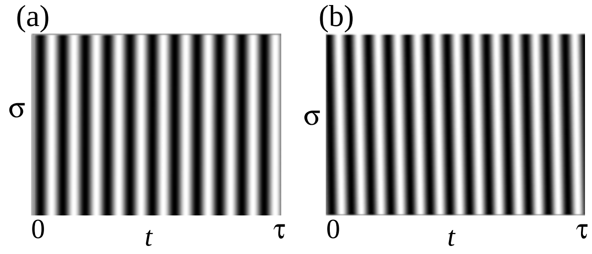

Figure 2: "Space-time" representation of the asymptotic states of system (2) for $\alpha = -0.8$ shows numerically the coexistence of stable solutions with different frequencies. The real part of z is plotted. The horizontal axis represents the spacelike direction ranging from 0 to τ. Solutions in (a) and (b) have different number of maxima per delay interval. They are obtained by choosing different initial conditions (see details in the text). $\tau = 80$, $\beta = 1$.

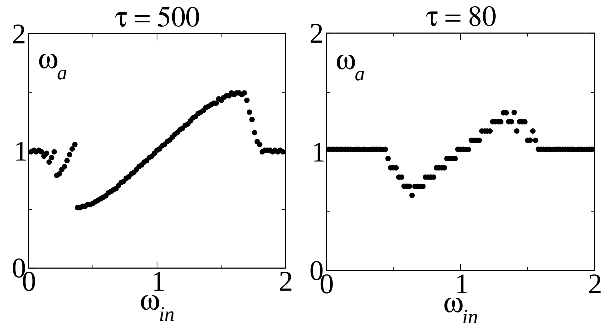

Figure 3: Frequency of the periodic attractor ω_a as a function of the frequency of the initial condition ω_{in}, from which the attractor is approached. Initial conditions are chosen as follows $z_0(s) = 0.01(1+i)\cos(\omega_{in}s)$. $\alpha = -0.2$, $\beta = 1$.

$z_0(s) = 0.01(1+i)\cos(\omega_{in}s/\tau)$ $(-\tau \leq s \leq 0)$ with $\omega_{in} = 0.6$ and $\omega_{in} = 1.6$, respectively. The figure shows, that the asymptotic solutions have different frequencies as well. This demonstrates the coexistence of periodic attractors with different frequencies.

In order to find out which coexistent periodic attractors are admitted in our system, we perform a numerical integration with different frequencies ω_{in} of the initial conditions. The results are shown in Fig. 3 for two different values of delay $\tau = 80$ and $\tau = 500$. The frequencies ω_a of the asymptotic states are plotted versus ω_{in}. We observe that about ten different frequencies appear for $\tau = 80$, which can be realized depending on initial conditions. For $\tau = 500$, this frequency discretization still persists, but no longer visible due to the small distance between the neighboring frequencies.

Note, that Fig. 3 is obtained for a fixed value of the bifurcation parameter $\alpha = -0.2$. Changing α, the range of available frequencies ω_a is varying as well. This dependence of ω_a on α is summarized in Fig. 4 where α varies from the bifurcation point $\alpha = -1$ up to

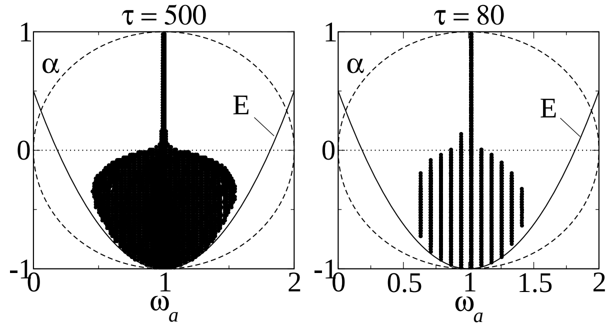

Figure 4: Dependence of the frequencies of the asymptotic states ω_a on the bifurcation parameter α. Close to the destabilization point at $\alpha = -1$, the system exhibits typical Eckhaus instability (E denotes the Eckhaus parabola, which delineates the moments of stabilization of periodic states). At $\alpha > 0$ the zero state is strongly unstable, which leads to dominating of low-dimensional dynamics and appearance of the typical Hopf scenario (no splitting of frequencies). More details are given in the text.

$\alpha = 1$. We observe that the range of available frequencies of periodic attractors ω_a first increases quadratically as α is increased from -1 and then shrinks to the single frequency $\omega_a \approx \beta$ for $\alpha > 0$. Figure 4 shows how the dynamics of the system changes as it goes through the bifurcation. We are going now to describe analytically important features of this destabilization process.

Eckhaus instability and amplitude equations: The phenomena, which are discovered numerically in Figs. 2, 3 and 4 can be partially understood using the correspondence between delayed systems and spatially extended systems, see, e.g. [6]. In fact, for α close to -1, we observe the Eckhaus instability scenario for a delay system. In order to show this, let us first derive the amplitude equation for (2), which describes the dynamics of the amplitude of destabilized oscillations close to the bifurcation. We assume $\alpha = -1 + \mu\varepsilon^2$, where μ is a new parameter, which controls deviation from this bifurcation. Hence, we have

$$z' = \left(-1 + \mu\varepsilon^2 + i\beta\right) z + z_\tau - z|z|^2 \tag{5}$$

Let us introduce different time scales $T_j = \varepsilon^j t$. Denote $\overline{T} = (T_1, T_2, T_3)$. Now we apply the standard multiscale slow-amplitude ansatz:

$$z(t) = \varepsilon e^{i\beta T_0} \left(u\left(\overline{T}\right) + \varepsilon v\left(\overline{T}\right) + \varepsilon^2 w\left(\overline{T}\right)\right). \tag{6}$$

After substituting (6) into (5) we obtain a set of equations for different orders of ε. It is known [7, 13] that the solvability conditions up to the ε^3 order give the necessary amplitude equation.

In the first order in ε, we obtain

$$u\left(T_1, T_2, T_3\right) = e^{i\varphi} u\left(T_1 - 1, T_2, T_3\right),$$

where $\varphi = -\beta\tau \bmod 2\pi$ is the phase change within the delay interval.

At the order ε^2,the solvability condition reads

$$v = e^{i\varphi} v(T_1 - 1, T_2, T_3) - \partial_{T_1} u - \partial_{T_2} u. \tag{7}$$

Eq. (7) can be considered as a mapping with respect to the argument T_1. For physically relevant situations, this mapping must have bounded solutions. Therefore, the resonant terms in Eq. (7) have to vanish $\partial_{T_1} u + \partial_{T_2} u = 0$. This implies that u is a function of $T_1 - T_2$, i.e. $u = \xi(T_1 - T_2, T_3) := \xi(x, T)$. Another consequence of Eq. (7) is that v must satisfy the boundary condition $v = e^{i\varphi} v(T_1 - 1, T_2, T_3)$.

The solvability condition for ε^3 order implies

$$\begin{aligned} w &= e^{i\varphi} w\,(T_1 - 1, T_2, T_3) - \partial_{T_1} v - \partial_{T_2} v \\ &\quad -\partial_{T_3} u + \frac{1}{2}\partial^2_{T_2 T_2} u - \partial_{T_2} u + \mu u - u|u|^2 \end{aligned} \tag{8}$$

Vanishing of resonant terms now leads to the final result. In particular, it implies that u satisfies the GL equation $\partial_{T_3} u = \frac{1}{2}\partial^2_{T_2 T_2} u - \partial_{T_2} u + \mu u - u|u|^2$. In terms of function $\xi(x, T)$ it reads

$$\partial_T \xi = \frac{1}{2}\partial^2_{xx}\xi + \partial_x \xi + \mu\xi - \xi|\xi|^2 \tag{9}$$

with the corresponding boundary condition

$$\xi(x, T) = e^{i\varphi}\xi(x - 1, T). \tag{10}$$

As follows from our derivation, the relation between a solution $\xi(x, T)$ of (9-10) and $z(t)$ of the delay system (5) is given by

$$z(t) = \varepsilon e^{i\beta t}\xi(\varepsilon t - \varepsilon^2 t, \varepsilon^3 t), \tag{11}$$

which is expected to be accurate for a time interval of order ε^{-3}. Note, that the convective term in (9) can be eliminated by a suitable change of variables. Thus, we obtain

$$\partial_T \xi = \frac{1}{2}\partial^2_{xx}\xi + \mu\xi - \xi|\xi|^2. \tag{12}$$

The obtained GL system (12) has been shown (cf. [14]) to explain the Eckhaus instability at the bifurcation point $\mu = 0$. This phenomenon was first reported in [15]. Like in the case of the delay system under consideration, in spatially extended systems it is characterized by the loss of stability of the trivial solution to a periodic pattern of the form $e^{i\beta_c x}$ with the wavenumber β_c. For systems on unbounded domain, the trivial state becomes unstable to all periodic patterns e^{iqx}, whose wavenumber satisfies $(q - \beta_c)^2 \leq 2\mu$. However, these periodic solutions are themselves unstable, unless q belongs to the smaller interval $(q -$

$\beta_c)^2 \leq \frac{2}{3}\mu^{\ 2}$. The Eckhaus region is the parabolic region in the (q, μ) plane containing stable plane waves. This region is bounded by the Eckhaus parabola

$$\mu_E(q) = \frac{3}{2}\left(q - \beta_c\right)^2. \tag{13}$$

Note that in the theory of amplitude equations for spatially extended systems, Eq. (12) describes the dynamics of the complex amplitude of the pattern via, e.g.

$$w(x, T) = \xi(x, T)e^{i\beta_c x} + \xi^*(x, T)e^{-i\beta_c x},$$

while for the delay system we have the relationship (11).

It has been shown in [14], that a similar scenario of the Eckhaus instability occurs for systems in a large but finite domain. The main qualitative differences are as follows:

• The set of allowed frequencies is discretized due to the restrictions imposed by boundary conditions;

• The Eckhaus parabola is shifted downwards

$$\mu_E = \frac{3}{2}(q - \beta_c)^2 - \frac{1}{4}. \tag{14}$$

We refer to the more detailed analysis of the Eckhaus phenomenon for Eq. (12) in a finite domain to [14].

Coming back to our delay system (5) we recall the correspondence between the bifurcation parameters $\varepsilon^2\mu = \alpha + 1$ and the frequencies $\omega_a - \beta = \varepsilon(q - \beta_c)$. Using these scalings, we obtain the Eckhaus parabola for the delay system as

$$\alpha_E + 1 = \frac{3}{2}(\omega_a - \beta)^2 - \frac{\varepsilon^2}{4} \approx \frac{3}{2}(\omega_a - \beta)^2. \tag{15}$$

We would like to emphasize the following interesting feature: due to the scaling restrains, our final formula for the Eckhaus region for the delay system (15) is, in fact, approximated by the corresponding formula for the unbounded domain (13) in spite of the fact, that the amplitude dynamics is governed by the system on a finite domain.

The Eckhaus curve $\alpha_E(\omega_a)$ is plotted in Fig. 4 (the line with label E). One can note, that there is perfect matching of the theoretically predicted results from the amplitude GL model (12) and the numerically obtained results for the delay equation (2) in the region, which is close to the bifurcation point $\alpha \approx -1$.

When spatio-temporal representation fails: With further increasing control parameter α, the multiple periodic attractors of the delay system no longer exist, except for that one with the frequency closest to the destabilization frequency β. This can be explained by the fact, that for $\alpha > 0$ the local dynamics in the vicinity of the zero state is dominated

[2] Given formulas for the Eckhaus region differ from the formulas in [14] by the factor 2, since we do not rescale the coefficient 1/2 at ξ_{xx}.

by the single strongly unstable mode with the eigenvalue $\lambda_S \approx \alpha + i\beta$. This appears to be true even in the presence of the unstable pseudo-continuous spectrum for $0 < \alpha < 1$.

Summary: We have shown that the characteristic development of the oscillatory instability for delay systems follows two stages:
– On the first stage the Eckhaus phenomenon occurs and multiple coexisting periodic attractors appear. This stage can be nicely approximated by the complex GL equation with boundary conditions of the form (10). We conclude also, that this phenomenon is generic, since amplitude equations of such type will appear generically for any delay system with large delay, where the instantaneous terms give rise to a Hopf bifurcation.
– On the second stage, the domain with multiple periodic attractors shrinks and one attractor with frequency close to β survives, see Fig. 4. This stage can no longer be explained by the amplitude equations. Instead, a low-dimensional approximation should be used.

Note, that the only conditions for the described phenomenon to occur are the presence of the delay and the control parameter, which mediates the transition from stationary to a nonstationary regime. Therefore, it is common for models describing the dynamics of different physical systems, see, for example [1, 2, 3, 4].

Finally, we would like to remark, that systems with large delay exhibit many interesting phenomena, which are usually accompanied by a high degree of multistability [16, 17]. As a rule, the number of coexisting attractors growths as delay is increased. We show in this letter, that such a multistability is an inherent feature of large-delay systems, since it generically occurs already at the basic oscillatory destabilization bifurcation.

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