Weierstraß–Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Resolvent estimates in $W^{-1,p}$ related to strongly coupled linear parabolic systems with coupled nonsmooth capacities

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submitted: 4th January 2006

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Preprint No. 1086

Berlin 2006

2000 *Mathematics Subject Classification.* 35K45, 35R05, 35B65.

Key words and phrases. Linear parabolic systems, coupled capacities, strong coupling, nonsmooth data, resolvent estimates, regularity, existence, uniqueness.

Edited by
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Abstract

We investigate linear parabolic systems with coupled nonsmooth capacities and mixed boundary conditions. We prove generalized resolvent estimates in $W^{-1,p}$ spaces. The method is an appropriate modification of a technique introduced by Agmon to obtain L^p estimates for resolvents of elliptic differential operators in the case of smooth boundary conditions. Moreover, we establish an existence and uniqueness result.

1 Introduction

We are interested in the investigation of strongly coupled linear parabolic systems with coupled nonsmooth capacities and mixed boundary conditions of the form

$$\begin{aligned} \frac{\partial}{\partial t}\sum_{k=1}^m e_{jk}u_k - \sum_{k=1}^m\sum_{\beta=0}^N \left(\sum_{\alpha=1}^N D^\alpha\Big(a_{\alpha\beta}^{jk}D^\beta u_k\Big) - a_{0\beta}^{jk}D^\beta u_k\right) &= f_j \quad \text{on } \Omega\times\mathbb{R}_+,\\ \sum_{k=1}^m\sum_{\alpha=1}^N\sum_{\beta=0}^N a_{\alpha\beta}^{jk}D^\beta u_k\,\nu^\alpha &= g_j \quad \text{on } \Gamma_N\times\mathbb{R}_+,\\ u_j &= 0 \quad \text{on } \Gamma_D\times\mathbb{R}_+,\\ \sum_{k=1}^m e_{jk}u_k(0) &= w_j \quad \text{on } \Omega,\\ j &= 1,\dots,m, \end{aligned} \tag{1}$$

where $(D^0v, D^1v, \dots, D^Nv) = (v, \frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_N})$, and ν^α denotes the α-th component of the outer unit normal vector. In our system the coefficient functions e_{jk} in the terms with the time derivative as well as the components of the diffusion coefficients $a_{\alpha\beta}^{jk}$ are discontinuous space functions. The aim of the paper are (modified) resolvent estimates related to the system (1) in the scale of $W_0^{1,p}$, $W^{-1,p}$ spaces where $p > 2$.

For the special case of (1) with only one parabolic equation ($m = 1$), $e_{11} = 1$ and mixed boundary conditions a corresponding result can be found in a paper of Gröger, Rehberg [5].

Writing (1) in form of an operator equation

$$(Eu)' + Au = F, \quad Eu(0) = w_0,$$

where the operator E corresponds to the multiplication by the $m \times m$ matrix e_{jk} of $L^\infty(\Omega)$ coefficients (see (8)) and A represents the linear second order elliptic differential operator (see (5)) we intend to prove in the paper (modified) resolvent estimates of the form

$$\begin{aligned}
&\|(A+\lambda E)^{-1}\|_{\mathcal{L}(W^{-1,p}(\Omega\cup\Gamma_N)^m, W_0^{1,p}(\Omega\cup\Gamma_N)^m)} \le c,\\
&\|E(A+\lambda E)^{-1}\|_{\mathcal{L}(W^{-1,p}(\Omega\cup\Gamma_N)^m, W^{-1,p}(\Omega\cup\Gamma_N)^m)} \le \frac{c}{|\lambda|},\\
&\|(A+\lambda E)^{-1}E\|_{\mathcal{L}(W_0^{1,p}(\Omega\cup\Gamma_N)^m, W_0^{1,p}(\Omega\cup\Gamma_N)^m)} \le \frac{c}{|\lambda|} \qquad \text{if } \operatorname{Re}\lambda \ge 0.
\end{aligned} \tag{2}$$

In Section 2 we introduce the notation and some auxiliary results. Section 3 contains some results for linear elliptic systems with complex coefficients. A Hilbert space formulation of the instationary problem and an existence and uniqueness result for this formulation are given in Section 4. Section 5 is devoted to the main result of the paper. There we establish the resolvent estimates. In this section we apply techniques used in [1, 5]. Moreover, we derive conclusions which allow us to to apply results of [2] for evolution problems of parabolic type in Banach spaces. This then is done in Section 6, where we provide a regularity result for the corresponding parabolic system.

2 Notation

Let $G = \Omega\cup\Gamma_N$ be a bounded regular subset of $\mathbb{R}^N$ (see [5, Section 2], [4, Definition 2]). We denote by G°, ∂G and $\overline{G}$ the interior, the boundary and the closure of G, respectively. We use different function spaces defined on G. All functions are considered to be complex-valued. For $p\in[1,\infty)$ we introduce the spaces $W_0^{1,p}(G)$ to be the closure of the set

$$\big\{u|_{G^\circ}\colon u\in C_0^\infty(\mathbb{R}^N), \operatorname{supp} u\cap(\overline{G}\setminus G)=\emptyset\big\}$$

in the space $W^{1,p}(G^\circ)$. By $W^{-1,p}(G)$ we denote the dual space of $W_0^{1,p'}(G)$, where p' is related to p by $\frac{1}{p}+\frac{1}{p'}=1$. We introduce the abbreviations

$$\begin{aligned}
&\mathcal{V}^{1,p} := W_0^{1,p}(G)^m, \quad \mathcal{V}^{-1,p} := W^{-1,p}(G)^m,\\
&Y^p := L^p(G,\mathbb{C})^m, \quad Z^p := L^p(G,\mathbb{C}^{N+1})^m.
\end{aligned}$$

On $\mathcal{V}^{1,p}$ we use the norm

$$\|u\|^p_{\mathcal{V}^{1,p}} = \int_G \left(\sum_{j=1}^m\sum_{\alpha=0}^N |D^\alpha u_j|^2\right)^{p/2} \mathrm{d}x, \quad u=(u_1,\dots,u_m)\in\mathcal{V}^{1,p}.$$

As in [6] we suppose the anti-linear or conjugate-linear forms to form the dual spaces. We define the map $J\in\mathcal{L}(\mathcal{V}^{1,2},\mathcal{V}^{-1,2})$ by

$$\langle Ju,v\rangle_{\mathcal{V}^{1,2}} = \int_G\sum_{j=1}^m\sum_{\alpha=0}^N D^\alpha u_j D^\alpha v_j\,\mathrm{d}x, \quad \forall u,v\in\mathcal{V}^{1,2},$$

where $Du_j = (D^0 u_j, D^1 u_j, \dots, D^N u_j) = (u_j, \frac{\partial u_j}{\partial x_1}, \dots, \frac{\partial u_j}{\partial x_N})$ for $u_j \in W_0^{1,p}(G)$. With the notation $\mathcal{D}u = (Du_1, \dots, Du_m)$ we obtain

$$\langle Ju, v\rangle_{\mathcal{V}^{1,2}} = \int_G \mathcal{D}u \cdot \mathcal{D}v \,\mathrm{d}x, \quad \forall u, v \in \mathcal{V}^{1,2}.$$

If it is necessary to indicate the subset $G \subset \mathbb{R}^N$ on which the functions spaces are considered we write J_G instead of J. Let us remark that J corresponds to an m component map from $W_0^{1,2}(G)$ to $W^{-1,2}(G)$ which is used in [5] for the treatment of one single equation. Therefore we can carry over the corresponding arguments and results from [5, Section 2] and cite them here.

First, for $p \geq 2$, J maps $\mathcal{V}^{1,p}$ continuously into $\mathcal{V}^{-1,p}$. Second, for $p \in [2, \infty)$, let $\mathcal{R}_p$ denote the class of all regular subsets $G \subset \mathbb{R}^N$ such that J_G maps $W_0^{1,p}(G)^m$ onto $W^{-1,p}(G)^m$. For $G \in \mathcal{R}_p$ we introduce the number

$$\gamma_{p,G} := \sup\Big\{\|u\|_{W_0^{1,p}(G)^m} : u \in W_0^{1,p}(G)^m, \quad \|J_G u\|_{W^{-1,p}(G)^m} = 1\Big\}. \tag{3}$$

We will write γ_p instead of $\gamma_{p,G}$ if the choice of G is clear. Third, we have $\gamma_2 = 1$ and, according to the Open Mapping Theorem, $\gamma_p < \infty$. Fourth, the following results can be found in [4] (for real-valued functions) and in [5].

Lemma 2.1 i) *For every regular subset $G \subset \mathbb{R}^N$ there exists a $p_0 > 2$ such that $G \in \mathcal{R}_{p_0}$.*
ii) *If $G \in \mathcal{R}_{p_0}$ for some $p_0 > 2$, then $G \in \mathcal{R}_p$ for all $p \in [2, p_0]$ and*

$$\gamma_p \leq \gamma_{p_0}^{\theta} \quad \textit{where} \quad \frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{2}.$$

3 Results for linear elliptic systems

To write down formulas more concise we will write quantities $y \in \mathbb{C}^{m(N+1)}$ in the form $y = \{y_j^\alpha\}_{j=1,\dots,m,\, \alpha=0,\dots,N}$.

$$\begin{aligned}
&\text{For } j,k = 1,\dots,m \text{ let } (a^{jk}_{\alpha\beta})_{\alpha,\beta=0,\dots,N} = (a^{kj}_{\alpha\beta})_{\alpha,\beta=0,\dots,N} \text{ be measurable}\\
&\text{complex valued } (N+1)\times(N+1) \text{ matrix functions with}\\
&a^{jk}_{\alpha\beta} = a^{jk}_{\beta\alpha} \in L^\infty(G), \quad \alpha,\beta = 0,1,\dots,N,\\
&\mathrm{Re}\left(\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{N} a^{jk}_{\alpha\beta}(x) y_k^\beta\, \overline{y}_j^\alpha\right) \geq a_0 |y|^2,\\
&\sum_{j=1}^{m}\sum_{\alpha=0}^{N}\left|\sum_{k=1}^{m}\sum_{\beta=0}^{N} a^{jk}_{\alpha\beta}(x) y_k^\beta\right|^2 \leq a_1^2 |y|^2 \quad \text{f.a.a. } x \in G, \quad \forall\, y \in \mathbb{C}^{m(N+1)},
\end{aligned}\tag{4}$$

where $a_1 \geq a_0 > 0$ are suitable constants. By means of these coefficient functions we define the linear continuous operator A from $\mathcal{V}^{1,2}$ into $\mathcal{V}^{-1,2}$ by

$$\langle Au, v\rangle_{\mathcal{V}^{1,2}} := \int_G \sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{N} a_{\alpha\beta}^{jk}\, D^\beta u_k D^\alpha v_j \,\mathrm{d}x, \quad u, v \in \mathcal{V}^{1,2}. \tag{5}$$

The restriction of the operator A to $\mathcal{V}^{1,p}$ is continuous from $\mathcal{V}^{1,p}$ into $\mathcal{V}^{-1,p}$, too.

Theorem 3.1 *We assume that $G \in \mathcal{R}_q$ for some $q > 2$ and that the coefficients $a_{\alpha\beta}^{jk}$ fulfil* (4)*. Let $K := \big(1 - \big(\frac{a_0}{a_1}\big)^2\big)^{1/2}$ and let $p \in [2, q]$ be such that $\gamma_p K < 1$, where γ_p is defined in* (3)*. Then the operator A maps $\mathcal{V}^{1,p}$ onto $\mathcal{V}^{-1,p}$. Moreover, there holds true the estimate*

$$\|A^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p},\mathcal{V}^{1,p})} \leq \frac{a_0}{a_1^2}\, \frac{\gamma_p}{1 - \gamma_p K}. \tag{6}$$

Proof. Let $\tau := a_0\, a_1^{-2}$. We define an operator $A_\tau \colon Z^2 \to Z^2$ by

$$(A_\tau z)_j^\alpha = z_j^\alpha - \tau \sum_{k=1}^{m} \sum_{\beta=0}^{N} a_{\alpha\beta}^{jk}\, z_k^\beta$$

for $z = \{z_j^\alpha\}_{j=1,\dots,m,\ \alpha=0,\dots,N} \in Z^2$. For arbitrarily fixed $f \in \mathcal{V}^{-1,p}$ we introduce the operator $Q_f \colon \mathcal{V}^{1,p} \to \mathcal{V}^{1,p}$,

$$Q_f u := J^{-1}(\mathcal{D}^* A_\tau \mathcal{D} u + \tau f), \quad u \in \mathcal{V}^{1,p}.$$

Here $\mathcal{D}^*$ means the adjoint of $\mathcal{D} \colon \mathcal{V}^{1,2} \to Z^2$. Remembering the definition of A and A_τ we find

$$Q_f u = u - \tau J^{-1}(Au - f), \quad u \in \mathcal{V}^{1,p}.$$

Our aim is to prove that the equation $Au = f$ can be solved. For this purpose we show that the operator $Q_f \colon \mathcal{V}^{1,p} \to \mathcal{V}^{1,p}$ is strictly contractive. For $z \in Z^p$ we obtain

$$\begin{aligned} |(A_\tau z)(x)|^2 &= |z(x)|^2 - 2\tau \,\mathrm{Re}\Big(\sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{N} a_{\alpha\beta}^{jk}(x)\, z_k^\beta(x)\, \overline{z}_j^\alpha(x)\Big) \\ &\qquad + \tau^2 \sum_{j=1}^{m} \sum_{\alpha=0}^{N} \Big| \sum_{k=1}^{m} \sum_{\beta=0}^{N} a_{\alpha\beta}^{jk}(x)\, z_k^\beta(x)\Big|^2 \\ &\leq (1 + a_1^2\tau^2 - 2a_0\tau)|z(x)|^2 = K^2\, |z(x)|^2. \end{aligned}$$

Thus we conclude that $\|A_\tau z\|_{Z^p} \leq K\|z\|_{Z^p}$ for all $z \in Z^p$. Note that for the adapted spaces of domain and range the estimates $\|\mathcal{D}^*\|_{\mathcal{L}(Z^p,\mathcal{V}^{-1,p})} \leq \|\mathcal{D}\|_{\mathcal{L}(\mathcal{V}^{1,p},Z^p)} = 1$ and $\|J^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p},\mathcal{V}^{1,p})} = \gamma_p$ hold true. We estimate

$$\begin{aligned} &\|Q_f u - Q_f v\|_{\mathcal{V}^{1,p}} \\ &\quad = \|J^{-1}\mathcal{D}^* A_\tau \mathcal{D}(u - v)\|_{\mathcal{V}^{1,p}} \\ &\quad \leq \|J^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p},\mathcal{V}^{1,p})} \|\mathcal{D}^*\|_{\mathcal{L}(Z^p,\mathcal{V}^{-1,p})} \|A_\tau\|_{\mathcal{L}(Z^p,Z^p)} \|\mathcal{D}\|_{\mathcal{L}(\mathcal{V}^{1,p},Z^p)} \|u - v\|_{\mathcal{V}^{1,p}} \\ &\quad \leq \gamma_p\, K\, \|u - v\|_{\mathcal{V}^{1,p}} \quad \forall u,\, v \in \mathcal{V}^{1,p}. \end{aligned}$$

Since by assumption $\gamma_p K < 1$ the operator Q_f is strictly contractive and the fixed point $u \in \mathcal{V}^{1,p}$ is a solution to $Au = f$. Thus A maps $\mathcal{V}^{1,p}$ onto $\mathcal{V}^{-1,p}$. Furthermore, if for arbitrarily given $f, g \in \mathcal{V}^{-1,p}$ the fixed points of Q_f and Q_g are u_f and u_g, respectively, then

$$\begin{aligned}\|u_f - u_g\|_{\mathcal{V}^{1,p}} = \|Q_f u_f - Q_g u_g\|_{\mathcal{V}^{1,p}} &\le \|Q_f u_f - Q_f u_g\|_{\mathcal{V}^{1,p}} + \|Q_f u_g - Q_g u_g\|_{\mathcal{V}^{1,p}} \\ &\le \gamma_p K \,\|u_f - u_g\|_{\mathcal{V}^{1,p}} + \tau\, \gamma_p \|f - g\|_{\mathcal{V}^{-1,p}}.\end{aligned}$$

And we obtain

$$\|u_f - u_g\|_{\mathcal{V}^{1,p}} \le \frac{a_0}{a_1^2}\, \frac{\gamma_p}{1 - \gamma_p K}\, \|f - g\|_{\mathcal{V}^{-1,p}},$$

which proves the norm estimate (6). $\square$

4 The instationary problem

Let G be a bounded regular subset of $\mathbb{R}^N$, $S = [0, T]$. We assume that $(e_{jk})_{j,k=1,\dots,m}$ is a real-valued $m \times m$ matrix function on G with the properties

$$\begin{aligned}& e_{jk} = e_{kj} \in L^\infty(G), \quad j,k = 1, \dots, m, \\ & \sum_{j=1}^m \Big| \sum_{k=1}^m e_{jk}(x) y_k \Big|^2 \le e_1^2 |y|^2, \\ & \mathrm{Re} \Big(\sum_{j,k=1}^m e_{jk}(x) y_k \overline{y}_j \Big) \ge e_0 |y|^2 \quad \text{f.a.a. } x \in G, \quad \forall\, y \in \mathbb{C}^m.\end{aligned} \tag{7}$$

By means of this matrix we define the operator E from $\mathcal{V}^{1,2}$ into $\mathcal{V}^{-1,2}$ by

$$\langle Eu, v\rangle_{\mathcal{V}^{1,2}} = \int_G \sum_{j,k=1}^m e_{jk}\, u_k v_j \,\mathrm{d}x, \quad u, v \in \mathcal{V}^{1,2}. \tag{8}$$

For right hand sides $F \in L^2(S, \mathcal{V}^{-1,2})$ and initial values $w_0 \in Y^2$ we consider the linear instationary problem

$$(Eu)' + Au = F, \quad Eu(0) = w_0, \quad u \in L^2(S, \mathcal{V}^{1,2}), \quad Eu \in H^1(S, \mathcal{V}^{-1,2}). \tag{9}$$

Theorem 4.1 *Let G be a bounded regular subset of $\mathbb{R}^N$ and let the coefficients $a^{jk}_{\alpha\beta}$ and e_{jk} fulfil the properties* (4) *and* (7)*, respectively. Then, for all $F \in L^2(S, \mathcal{V}^{-1,2})$ and all initial values $w_0 \in Y^2$ there is a unique solution u to the initial value problem* (9)*.*

Main ideas of the proof. Applying techniques as used in [3, Hilfssatz 2.84] (for one component) and the properties (7) we can show that the operator

$$\Lambda\colon \{u \in L^2(S, \mathcal{V}^{1,2}),\, Eu \in H^1(S, \mathcal{V}^{-1,2}),\, Eu(0) = w_0\} \subset L^2(S, Y^2) \to L^2(S, \mathcal{V}^{-1,2}),$$

$$\Lambda u = (Eu)'$$

is maximal monotone. According to (4) the operator $A\colon L^2(S,\mathcal{V}^{1,2}) \to L^2(S,\mathcal{V}^{-1,2})$ is Lipschitz continuous and strongly monotone. Therefore by a theorem of Browder (see [8, vol. II/B]), for all $F \in L^2(S,\mathcal{V}^{-1,2})$ and all initial values $w_0 \in Y^2$ there is a unique solution u to the initial value problem (9). $\square$

Now we are interested in assertions concerning higher regularity of the solution to (9). For this purpose we will deal with resolvent estimates and will apply results of Favini and Yagi [2].

5 Resolvents

We denote by $\mathcal{H}$ the complex half plane

$$\mathcal{H} := \{\lambda \in \mathbb{C} \colon \mathrm{Re}\ \lambda \geq 0\}.$$

Lemma 5.1 *Let G be a bounded regular subset of $\mathbb{R}^N$ and let the coefficients $a^{jk}_{\alpha\beta}$ and e_{jk} fulfil the properties* (4) *and* (7)*, respectively. Then there exists a $q > 2$ such that for every $p \in [2, q]$ and all $\lambda \in \mathcal{H}$*
i) *the mapping $(A + \lambda E)|_{\mathcal{V}^{1,p}}$ is a continuous bijection from $\mathcal{V}^{1,p}$ onto $\mathcal{V}^{-1,p}$ and*
ii) *the mapping $(Id_{\mathcal{V}^{-1,p}} + \lambda E A^{-1})$ is a continuous bijection from $\mathcal{V}^{-1,p}$ onto itself.*

Proof. Let $\lambda \in \mathcal{H}$ be fixed. We set $\kappa = 1 - \frac{a_0}{2a_1}\mathrm{sgn}\,(\mathrm{Im}\ \lambda)\, i$. Then $|\kappa|^2 < 2$ and $\mathrm{Re}\ (\kappa\lambda) = \mathrm{Re}\ \lambda + \frac{a_0}{2a_1}|\mathrm{Im}\ \lambda|$. Furthermore, note that $\mathrm{Im}\ \langle Eu, \overline{u}\rangle = 0$ for $u \in \mathcal{V}^{1,2}$. Then, for $u \in \mathcal{V}^{1,2}$ we can estimate

$$\begin{aligned}
2\|Au + \lambda Eu\|_{\mathcal{V}^{-1,2}}\|u\|_{\mathcal{V}^{1,2}} &\geq |\kappa\langle Au + \lambda Eu, \overline{u}\rangle| \geq \mathrm{Re}\ (\kappa\langle Au + \lambda Eu, \overline{u}\rangle) \\
&= \mathrm{Re}\ \langle Au, \overline{u}\rangle - \mathrm{Im}\ \kappa\, \mathrm{Im}\ \langle Au, \overline{u}\rangle + \mathrm{Re}\ (\kappa\lambda)\, \mathrm{Re}\ \langle Eu, \overline{u}\rangle - \mathrm{Im}\ (\kappa\lambda)\, \mathrm{Im}\ \langle Eu, \overline{u}\rangle \\
&\geq \mathrm{Re}\ \langle Au, \overline{u}\rangle - \frac{a_0}{2a_1}|\mathrm{Im}\ \langle Au, \overline{u}\rangle| + (\mathrm{Re}\ \lambda + \frac{a_0}{2a_1}|\mathrm{Im}\ \lambda|)\, \mathrm{Re}\ \langle Eu, \overline{u}\rangle \\
&\geq a_0\|u\|^2_{\mathcal{V}^{1,2}} - \frac{a_0}{2}\|u\|^2_{\mathcal{V}^{1,2}} + \frac{a_0}{2a_1}|\lambda| e_0 \|u\|^2_{Y^2} \\
&\geq \frac{a_0}{2}\left(\|u\|^2_{\mathcal{V}^{1,2}} + \frac{e_0}{a_1}|\lambda|\|u\|^2_{Y^2}\right).
\end{aligned}$$

Here we have used the properties (4), (7). In summary we obtain

$$\|u\|_{\mathcal{V}^{1,2}} \leq \frac{4}{a_0}\|Au + \lambda Eu\|_{\mathcal{V}^{-1,2}} \quad \forall u \in \mathcal{V}^{1,2}. \tag{10}$$

Since the mappings $A|_{\mathcal{V}^{1,p}}$ as well as $E|_{\mathcal{V}^{1,p}}$ are linear and continuous from $\mathcal{V}^{1,p}$ into $\mathcal{V}^{-1,p}$ for all $p \in [2,\infty)$, the continuity of $(A + \lambda E)|_{\mathcal{V}^{1,p}}$ is obvious and injectivity results from (10).

By Theorem 3.1 there exists a $q > 2$ such that for all $p \in [2, q]$ the operator A from $\mathcal{V}^{1,p}$ onto $\mathcal{V}^{-1,p}$ is linear and continuous, and $A^{-1}\colon \mathcal{V}^{-1,p} \to \mathcal{V}^{1,p}$ is linear and continuous, too. Therefore, $(Id_{\mathcal{V}^{-1,p}} + \lambda E A^{-1})$ is linear and continuous from $\mathcal{V}^{-1,p}$ into itself. Injectivity can be shown as follows: Let $v + \lambda E A^{-1} v = 0$ for some $v \in \mathcal{V}^{1,p}$. Then $u := A^{-1} v \in \mathcal{V}^{1,p}$ fulfills $Au + \lambda E u = 0$ which by the injectivity of $A + \lambda E$ leads to $u = 0$ and $v = 0$.

Next we show the surjectivity. Let $f \in \mathcal{V}^{-1,p}$ arbitrarily be given. We want to solve the equation $Au + \lambda E u = f$. We set $v = Au$, $u = A^{-1} v$ and obtain the problem

$$v + \lambda E A^{-1} v = f. \tag{11}$$

Since $A^{-1}\colon \mathcal{V}^{-1,p} \to \mathcal{V}^{1,p}$ is continuous and the embedding $W_0^{1,p}(G) \hookrightarrow L^p(G)$ is compact the operator $A^{-1}\colon \mathcal{V}^{-1,p} \to Y^p$ is completely continuous. On the other hand, E considered as mapping from Y^p to $\mathcal{V}^{-1,p}$ is continuous. Therefore $EA^{-1}\colon \mathcal{V}^{-1,p} \to \mathcal{V}^{-1,p}$ is completely continuous. Hence, by the Riesz-Schauder Theory $Id_{\mathcal{V}^{-1,p}} + \lambda E A^{-1}$ could fail to be an operator from $\mathcal{V}^{-1,p}$ onto $\mathcal{V}^{-1,p}$ only if $\frac{1}{\lambda}$ is an eigenvalue of $-EA^{-1}$. If $\frac{1}{\lambda}$ would be an eigenvalue and $v^* \neq 0$, $v^* \in \mathcal{V}^{-1,p}$ would be the corresponding eigenfunction we would find $u^* = A^{-1} v^* \neq 0$, $u^* \in \mathcal{V}^{1,p}$ (since A is linear and surjective). We apply (10) to u^* and obtain

$$\|u^*\|_{\mathcal{V}^{1,2}} \leq c \|Au^* + \lambda E u^*\|_{\mathcal{V}^{-1,2}} = c \|v^* + \lambda E A^{-1} v^*\|_{\mathcal{V}^{-1,2}}.$$

The last term is zero if $(\frac{1}{\lambda}, v^*)$ is an eigenpair which gives the contradiction to $u^* \neq 0$. Thus, $Id_{\mathcal{V}^{-1,p}} + \lambda E A^{-1}$ is a mapping from $\mathcal{V}^{-1,p}$ onto $\mathcal{V}^{-1,p}$ and for all $f \in \mathcal{V}^{-1,p}$ there is a solution $v \in \mathcal{V}^{1,p}$ to (11). Setting $u = A^{-1} v$ we get a solution to $Au + \lambda E u = f$. Thus, $(A + \lambda E)|_{\mathcal{V}^{1,p}}\colon \mathcal{V}^{1,p} \to \mathcal{V}^{-1,p}$ is surjective, too. $\square$

Theorem 5.1 *Let G be a bounded regular subset of $\mathbb{R}^N$ and let the coefficients $a_{\alpha\beta}^{jk}$ and e_{jk} fulfil the properties* (4) *and* (7), *respectively. Then there exists a $q > 2$ such that for every $p \in [2, q]$*

$$\sup_{\lambda \in \mathcal{H}} \|(A + \lambda E)^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p}, \mathcal{V}^{1,p})} < \infty,$$
$$\sup_{\lambda \in \mathcal{H}} \|\lambda E (A + \lambda E)^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p}, \mathcal{V}^{-1,p})} < \infty,$$
$$\sup_{\lambda \in \mathcal{H}} \|(A + \lambda E)^{-1} \lambda E\|_{\mathcal{L}(\mathcal{V}^{1,p}, \mathcal{V}^{1,p})} < \infty.$$

Proof. 1. We define the set $\widetilde{G} := G \times (-1, 1)$ which becomes a regular subset in $\mathbb{R}^{N+1}$. Thus we find a $\widetilde{q} > 2$ such that $\widetilde{G}$ belongs to $R_{\widetilde{q}}(\mathbb{R}^{N+1})$ (see Section 2).

For $\lambda \in \mathcal{H}$ we define the operator $\widetilde{A}_\lambda\colon W_0^{1,2}(\widetilde{G})^m \to W^{-1,2}(\widetilde{G})^m$,

$$\langle \widetilde{A}_\lambda \widetilde{u}, \widetilde{v} \rangle_{W_0^{1,2}(\widetilde{G})^m} := \int_{\widetilde{G}} \sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{N+1} \widetilde{a}_{\alpha\beta}^{jk} \, D^\beta \widetilde{u}_k D^\alpha \widetilde{v}_j \, \mathrm{d}x, \quad \widetilde{u}, \widetilde{v} \in W_0^{1,2}(\widetilde{G})^m, \tag{12}$$

where for $j,k=1,\dots,m$ the $(N+2)\times(N+2)$ matrix functions $\widetilde{a}^{jk}$ are given by

$$\begin{aligned}
&\widetilde{a}^{jk}_{\alpha\beta}(\widetilde{x}) := \kappa\, a^{jk}_{\alpha\beta}(x) \quad \text{for } \alpha,\beta=0,\dots,N,\\
&\widetilde{a}^{jk}_{\alpha(N+1)}(\widetilde{x}) = \widetilde{a}^{jk}_{(N+1)\,\alpha}(\widetilde{x}) := 0 \quad \text{for } \alpha=0,\dots,N,\\
&\widetilde{a}^{jk}_{(N+1)(N+1)}(\widetilde{x}) := \frac{\kappa\,\lambda\, a_1\, e_{jk}(x)}{|\lambda|\, e_0}, \quad \widetilde{x}=(x,x_{N+1})\in\widetilde{G},
\end{aligned}$$

κ is the same as in the proof of Lemma 5.1.

2. Then, by (4), (7)

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{\alpha=0}^{N+1}\Bigl|\sum_{k=1}^{m}\sum_{\beta=0}^{N+1}\widetilde{a}^{jk}_{\alpha\beta}\,y_k^\beta\Bigr|^2 &= \sum_{j=1}^{m}\sum_{\alpha=0}^{N}\Bigl|\kappa\sum_{k=1}^{m}\sum_{\beta=0}^{N}a^{jk}_{\alpha\beta}\,y_k^\beta\Bigr|^2+\sum_{j=1}^{m}\Bigl|\sum_{k=1}^{m}\frac{\kappa\lambda a_1 e_{jk}}{|\lambda|e_0}\,y_k^{N+1}\Bigr|^2\\
&\le 2a_1^2\left\{\sum_{k=1}^{m}\sum_{\beta=0}^{N}|y_k^\beta|^2+\left(\frac{e_1}{e_0}\right)^2\sum_{k=1}^{m}|y_k^{N+1}|^2\right\}\\
&\le 2a_1^2\left(1+\left(\frac{e_1}{e_0}\right)^2\right)|y|^2 \quad \forall y\in\mathbb{C}^{m(N+2)}.
\end{aligned}$$

3. Furthermore, we estimate

$$\begin{aligned}
&\mathrm{Re}\,\left(\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{N+1}\widetilde{a}^{jk}_{\alpha\beta}\,y_k^\beta\,\overline{y}_j^\alpha\right)\\
&=\mathrm{Re}\,\left(\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{N}\kappa a^{jk}_{\alpha\beta}\,y_k^\beta\,\overline{y}_j^\alpha+\sum_{j,k=1}^{m}\frac{\kappa\lambda a_1 e_{jk}}{|\lambda|e_0}\,y_k^{N+1}\,\overline{y}_j^{N+1}\right)\\
&\ge \mathrm{Re}\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{N}a^{jk}_{\alpha\beta}\,y_k^\beta\,\overline{y}_j^\alpha-|\mathrm{Im}\,\kappa|\,\Bigl|\mathrm{Im}\sum_{j,k=1}^{m}\sum_{\alpha,\beta=0}^{N}a^{jk}_{\alpha\beta}\,y_k^\beta\,\overline{y}_j^\alpha\Bigr|\\
&+\frac{\mathrm{Re}\,(\kappa\lambda)a_1}{|\lambda|e_0}\mathrm{Re}\sum_{j,k=1}^{m}e_{jk}\,y_k^{N+1}\,\overline{y}_j^{N+1}-\frac{\mathrm{Im}\,(\kappa\lambda)a_1}{|\lambda|e_0}\mathrm{Im}\sum_{j,k=1}^{m}e_{jk}\,y_k^{N+1}\,\overline{y}_j^{N+1}\\
&\ge a_0\sum_{j=1}^{m}\sum_{\alpha=0}^{N}|y_j^\alpha|^2-\frac{a_0}{2}\sum_{j=1}^{m}\sum_{\alpha=0}^{N}|y_j^\alpha|^2\\
&+\frac{a_1\mathrm{Re}\,\lambda+\frac{a_0}{2}|\mathrm{Im}\,\lambda|}{|\lambda|e_0}\mathrm{Re}\sum_{j,k=1}^{m}e_{jk}\,y_k^{N+1}\,\overline{y}_j^{N+1}\\
&\ge \frac{a_0}{2}\sum_{j=1}^{m}\sum_{\alpha=0}^{N}|y_j^\alpha|^2+\frac{a_0}{2}\sum_{j=1}^{m}|y_j^{N+1}|^2\\
&\ge \frac{a_0}{2}\sum_{j=1}^{m}\sum_{\alpha=0}^{N+1}|y_j^\alpha|^2 \qquad \forall y\in\mathbb{C}^{m(N+2)}.
\end{aligned}$$

4. According to the last two steps we can apply Theorem 3.1 to the operator $\widetilde{A}_\lambda$ with the constants $\frac{a_0}{2}$ and $2a_1(1+e_1/e_0)$ instead of a_0 and a_1. Therefore there exists an exponent $\widetilde{q} > 2$ such that for all $p \in [2, \widetilde{q}]$ the estimate

$$\|\widetilde{u}\|_{W_0^{1,p}(\widetilde{G})^m} \le c \inf_{\lambda \in \mathcal{H}} \|\widetilde{A}_\lambda \widetilde{u}\|_{W^{-1,p}(\widetilde{G})^m} \quad \forall \widetilde{u} \in W_0^{1,p}(\widetilde{G})^m \tag{13}$$

is fulfilled. We denote the minimal exponent of q and $\widetilde{q}$ again by q.

5. We fix some function $\phi \in C_0^\infty((-1,1))$ with the properties $0 \le \phi(s) \le 1$ and $\phi(s) = 1$ for $s \in [-\frac{1}{2}, \frac{1}{2}]$. We enlarge functions $u \in \mathcal{V}^{1,p}$ to functions $\widetilde{u}$ defined on $\widetilde{G}$ by the rule

$$\widetilde{u}(\widetilde{x}) = u(x)\phi(s)\mathrm{e}^{i\mu s}, \quad \widetilde{x} = (x,s) \in \widetilde{G}, \quad \mu = \left(\frac{|\lambda| e_0}{a_1}\right)^{1/2}.$$

Then we can validate the estimate

$$\|\widetilde{u}\|^p_{W_0^{1,p}(\widetilde{G})^m} \ge \int_{-1/2}^{1/2} \int_G \left(\sum_{j=1}^m \sum_{\alpha=0}^N |D^\alpha u_j|^2 \right)^{p/2} \mathrm{d}x\, \mathrm{d}s = \|u\|^p_{\mathcal{V}^{1,p}}. \tag{14}$$

Moreover, for $\widetilde{v} \in W_0^{1,p'}(\widetilde{G})^m$ we reconstruct functions $v \in \mathcal{V}^{1,p'}$ by

$$v(x) := \int_{-1}^1 \widetilde{v}(x,s)\phi(s)\mathrm{e}^{i\mu s}\, \mathrm{d}s, \quad x \in G,$$

and obtain

$$\begin{aligned} \|v\|^{p'}_{\mathcal{V}^{1,p'}} &= \int_G \left(\sum_{j=1}^m \sum_{\alpha=0}^N |D^\alpha v_j|^2 \right)^{p'/2} \mathrm{d}x \\ &\le c \int_{\widetilde{G}} \left(\sum_{j=1}^m \sum_{\alpha=0}^N |D^\alpha \widetilde{v}_j|^2 \right)^{p'/2} \mathrm{d}x \le c \|\widetilde{v}\|^{p'}_{W_0^{1,p'}(\widetilde{G})^m}. \end{aligned}$$

Since $\phi \in C_0^\infty((-1,1))$ we can calculate

$$\begin{aligned} &\int_{-1}^1 \frac{d}{ds}[\phi(s)\mathrm{e}^{i\mu s}] D^{N+1}\widetilde{v}_j \, \mathrm{d}s \\ &\quad = -\int_{-1}^1 \frac{d^2}{ds^2}[\phi(s)\mathrm{e}^{i\mu s}]\widetilde{v}_j \, \mathrm{d}s \\ &\quad = -\int_{-1}^1 \big(\phi''(s)\mathrm{e}^{i\mu s} + 2\frac{d}{ds}(\mathrm{e}^{i\mu s})\phi'(s) - \mu^2 \mathrm{e}^{i\mu s}\phi(s)\big)\widetilde{v}_j \, \mathrm{d}s \\ &\quad = -\int_{-1}^1 \big(\phi''(s)\mathrm{e}^{i\mu s}\widetilde{v}_j - 2\mathrm{e}^{i\mu s}\frac{d}{ds}[\phi'(s)\widetilde{v}_j] - \mu^2 \mathrm{e}^{i\mu s}\phi(s)\widetilde{v}_j\big) \, \mathrm{d}s \\ &\quad = \int_{-1}^1 \mathrm{e}^{i\mu s}\big(\phi''(s)\widetilde{v}_j + 2\phi'(s)D^{N+1}\widetilde{v}_j + \mu^2\phi(s)\widetilde{v}_j\big) \, \mathrm{d}s. \end{aligned}$$

Using this identity we estimate

$$
\begin{aligned}
|\langle \widetilde{A}_\lambda \widetilde{u}, \widetilde{v}\rangle| &= \left| \int_{\widetilde{G}} \sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{N+1} \widetilde{a}^{jk}_{\alpha\beta} D^\beta \widetilde{u}_k D^\alpha \widetilde{v}_j \,\mathrm{d}\widetilde{x} \right| \\
&= \left| \int_G \left\{ \kappa \sum_{j,k=1}^{m} \sum_{\alpha,\beta=0}^{N} a^{jk}_{\alpha\beta} D^\beta u_k \int_{-1}^{1} \phi(s) \mathrm{e}^{i\mu s} D^\alpha \widetilde{v}_j(\cdot, s)\,\mathrm{d}s \right.\right. \\
&\qquad \left.\left. + \frac{\kappa\lambda}{\mu^2} \sum_{j,k=1}^{m} e_{jk} u_k \int_{-1}^{1} \frac{d}{ds}[\phi(s)\mathrm{e}^{i\mu s}] D^{N+1}\widetilde{v}_j \,\mathrm{d}s \right\} \mathrm{d}x \right| \\
&= \left| \kappa \int_G \sum_{j,k=1}^{m} \left\{ \sum_{\alpha,\beta=0}^{N} a^{jk}_{\alpha\beta} D^\beta u_k D^\alpha v_j \right.\right. \\
&\qquad \left.\left. + \frac{\lambda}{\mu^2} e_{jk} u_k \int_{-1}^{1} \mathrm{e}^{i\mu s} \big(2\phi'(s) D^{N+1}\widetilde{v}_j + \big(\mu^2\phi(s) + \phi''(s)\big)\widetilde{v}_j(\cdot, s)\big)\mathrm{d}s \right\} \mathrm{d}x \right| \\
&= |\kappa \langle Au + \lambda E u, v\rangle| \\
&\qquad + \left| \frac{\kappa\lambda}{\mu^2} \sum_{j,k=1}^{m} \int_G e_{jk} u_k \int_{-1}^{1} \mathrm{e}^{i\mu s}\big(2\phi'(s) D^{N+1}\widetilde{v}_j + \phi''(s)\widetilde{v}_j(\cdot, s)\big)\mathrm{d}s \,\mathrm{d}x \right| \\
&\le |\kappa\langle Au + \lambda E u, v\rangle| + c\|u\|_{Y^p} \|\widetilde{v}\|_{W^{1,p'}(\widetilde{G})^m}.
\end{aligned}
$$

In summary we end up with

$$
\|\widetilde{A}_\lambda \widetilde{u}\|_{W^{-1,p}(\widetilde{G})^m} \le c(\|Au + \lambda E u\|_{\mathcal{V}^{-1,p}} + \|u\|_{Y^p}). \tag{15}
$$

6. Now we combine the estimates (13), (14) and (15) and get

$$
\|u\|_{\mathcal{V}^{1,p}} \le c\big(\|Au + \lambda E u\|_{\mathcal{V}^{-1,p}} + \|u\|_{Y^p}\big) \quad \forall \lambda \in \mathcal{H}. \tag{16}
$$

According to Nečas [7, Lemma 2.6.1], for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that

$$
\|u\|_{Y^p} \le \varepsilon \|u\|_{\mathcal{V}^{1,p}} + c_\varepsilon \|u\|_{Y^2}.
$$

Therefore it results from (16) and (10) and the continuous embeddings $W^{1,2}(G) \hookrightarrow L^2(G)$ and $W^{-1,p}(G) \hookrightarrow W^{-1,2}(G)$ that

$$
\|u\|_{\mathcal{V}^{1,p}} \le c\|Au + \lambda E u\|_{\mathcal{V}^{-1,p}}, \tag{17}
$$

which proves the first assertion of the theorem.

7. Since $A\colon \mathcal{V}^{1,p} \to \mathcal{V}^{-1,p}$ is linear and continuous, and (17) holds, we estimate

$$
\begin{aligned}
\|\lambda E u\|_{\mathcal{V}^{-1,p}} &\le \|Au + \lambda E u\|_{\mathcal{V}^{-1,p}} + \|Au\|_{\mathcal{V}^{-1,p}} \\
&\le \|Au + \lambda E u\|_{\mathcal{V}^{-1,p}} + c\|u\|_{\mathcal{V}^{1,p}} \le c\|Au + \lambda E u\|_{\mathcal{V}^{-1,p}}.
\end{aligned} \tag{18}
$$

For $g \in \mathcal{V}^{-1,p}$ we define $u_g = (A+\lambda E)^{-1} g \in \mathcal{V}^{1,p}$. Using (18) we find

$$\begin{aligned}
&\|\lambda E(A+\lambda E)^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p},\mathcal{V}^{-1,p})} \\
&\quad = \sup\big\{\|\lambda E(A+\lambda E)^{-1} g\|_{\mathcal{V}^{-1,p}} \,\big|\, g \in \mathcal{V}^{-1,p},\ \|g\|_{\mathcal{V}^{-1,p}} \le 1\big\} \\
&\quad = \sup\big\{\|\lambda E u_g\|_{\mathcal{V}^{-1,p}} \,\big|\, g \in \mathcal{V}^{-1,p},\ \|g\|_{\mathcal{V}^{-1,p}} \le 1\big\} \\
&\quad \le c \sup\big\{\|(A+\lambda E) u_g\|_{\mathcal{V}^{-1,p}} \,\big|\, g \in \mathcal{V}^{-1,p},\ \|g\|_{\mathcal{V}^{-1,p}} \le 1\big\} \\
&\quad = c \sup\big\{\|(A+\lambda E)(A+\lambda E)^{-1} g\|_{\mathcal{V}^{-1,p}} \,\big|\, g \in \mathcal{V}^{-1,p},\ \|g\|_{\mathcal{V}^{-1,p}} \le 1\big\} \\
&\quad = c \sup\big\{\|g\|_{\mathcal{V}^{-1,p}} \,\big|\, g \in \mathcal{V}^{-1,p},\ \|g\|_{\mathcal{V}^{-1,p}} \le 1\big\} \le c,
\end{aligned}$$

which gives the second assertion.

8. For $u \in \mathcal{V}^{1,p}$ we can estimate

$$\begin{aligned}
\|(A+\lambda E)^{-1}\lambda E u\|_{\mathcal{V}^{1,p}} &\le \|(A+\lambda E)^{-1}(A+\lambda E)u\|_{\mathcal{V}^{1,p}} + \|(A+\lambda E)^{-1} A u\|_{\mathcal{V}^{1,p}} \\
&\le \big(1 + \|(A+\lambda E)^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p},\mathcal{V}^{1,p})} \|A\|_{\mathcal{L}(\mathcal{V}^{1,p},\mathcal{V}^{-1,p})}\big) \|u\|_{\mathcal{V}^{1,p}} \\
&\le c\|u\|_{\mathcal{V}^{1,p}},
\end{aligned}$$

where we used the first assertion of the theorem and (4). Thus

$$\|(A+\lambda E)^{-1}\lambda E\|_{\mathcal{L}(\mathcal{V}^{1,p},\mathcal{V}^{1,p})} = \sup_{u\in\mathcal{V}^{1,p},\, \|u\|_{\mathcal{V}^{1,p}} \le 1} \|(A+\lambda E)^{-1}\lambda E u\|_{\mathcal{V}^{1,p}} \le c$$

proves the last assertion. □

Next we formulate a result which ensures all requirements of [2, Theorem 3.8, p.56]. Our Theorem 5.2 guarantees that (in the setting $M = EA^{-1}$, $L = -Id_{\mathcal{V}^{-1,p}}$) [2, Theorem 3.8] can be applied.

Theorem 5.2 *Let G be a bounded regular subset of $\mathbb{R}^N$ and let the coefficients $a^{jk}_{\alpha\beta}$ and e_{jk} fulfil the properties (4) and (7), respectively. Moreover, let q be given by Theorem 5.1. Then for every $p \in [2,q]$ the operator $EA^{-1} \colon \mathcal{V}^{-1,p} \to \mathcal{V}^{-1,p}$ is a closed linear operator. Moreover, the generalized resolvent set*

$$\rho_{EA^{-1}}(Id_{\mathcal{V}^{-1,p}}) = \Big\{\lambda \in \mathbb{C}\colon Id_{\mathcal{V}^{-1,p}} + \lambda EA^{-1} \text{ has a single valued bounded inverse on } \mathcal{V}^{-1,p}\Big\}$$

contains a sector

$$\Sigma = \Big\{\lambda \in \mathbb{C} :\ \lambda = r(\cos\varphi + i\sin\varphi),\ r \ge 0,\ |\varphi| < \frac{\pi}{2} + \delta\Big\}$$

for a suitable $\delta > 0$, and the generalized resolvent fulfils

$$\|EA^{-1}(Id_{\mathcal{V}^{-1,p}} + \lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \le \frac{c}{|\lambda|+1} \quad \forall \lambda \in \Sigma.$$

Proof. 1. Let $p \in [2, q]$ be arbitrarily fixed. We denote

$$I = Id_{\mathcal{V}^{-1,p}}.$$

The operators I and EA^{-1} are closed linear operators defined on the whole space $\mathcal{V}^{-1,p}$. According to the proof of Lemma 5.1, $EA^{-1}\colon \mathcal{V}^{-1,p} \to \mathcal{V}^{-1,p}$ is completely continuous, $I + \lambda EA^{-1}$ could fail to be an operator from $\mathcal{V}^{-1,p}$ onto $\mathcal{V}^{-1,p}$ only if $\frac{1}{\lambda}$ is an eigenvalue of $-EA^{-1}$, and $\mathcal{H}$ lies in the generalized resolvent set $\rho_{EA^{-1}}(I)$.

2. Next, we prove two generalized resolvent estimates for $\lambda \in \mathcal{H}$. Using the second inequality in Theorem 5.1 and the boundedness of the linear operator $A\colon \mathcal{V}^{1,p} \to \mathcal{V}^{-1,p}$ we can estimate

$$\begin{aligned}
\|(I + \lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} &= \|[(A + \lambda E)A^{-1}]^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \\
&= \|A(A + \lambda E)^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \\
&\le \|A\|_{\mathcal{L}(\mathcal{V}^{1,p},\mathcal{V}^{-1,p})}\|(A + \lambda E)^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p},\mathcal{V}^{1,p})} \\
&\le c_1 \quad \forall \lambda \in \mathcal{H}.
\end{aligned} \tag{19}$$

Moreover, we find from (19) that

$$\begin{aligned}
&\|\lambda EA^{-1}(I + \lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \\
&= \|(I + \lambda EA^{-1})(I + \lambda EA^{-1})^{-1} - (I + \lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \\
&\le \|I\|_{\mathcal{L}(\mathcal{V}^{-1,p})} + \|(I + \lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \\
&\le c_2 \quad \forall \lambda \in \mathcal{H}.
\end{aligned} \tag{20}$$

3. Next, we prove that a resolvent estimate of type (19) (with a changed constant) holds true for λ in a suitable sector $\Sigma \supset \mathcal{H}$, too. Let $\delta > 0$ be a constant such that $\sqrt{\cos^2\varphi + (1 - \sin\varphi)^2} \le \frac{1}{2c_2}$ for all φ with $\frac{\pi}{2} < |\varphi| \le \frac{\pi}{2} + \delta$. We define

$$\Sigma = \Big\{\lambda \in \mathbb{C} :\ \lambda = r(\cos\varphi + i\sin\varphi),\ r \ge 0,\ |\varphi| < \frac{\pi}{2} + \delta\Big\}.$$

Let $\lambda = r(\cos\varphi + i\sin\varphi) \in \Sigma \setminus \mathcal{H}$ be arbitrarily given. Then $\frac{\pi}{2} < |\varphi| \le \frac{\pi}{2} + \delta$ and $\lambda_0 = ir \in \mathcal{H}$. We write

$$\begin{aligned}
(I + \lambda EA^{-1})^{-1} &= [I + \lambda_0 EA^{-1} + (\lambda - \lambda_0)EA^{-1}]^{-1} \\
&= [\{I + (\lambda - \lambda_0)EA^{-1}(I + \lambda_0 EA^{-1})^{-1}\}(I + \lambda_0 EA^{-1})]^{-1} \\
&= (I + \lambda_0 EA^{-1})^{-1}\{I + (\lambda - \lambda_0)EA^{-1}(I + \lambda_0 EA^{-1})^{-1}\}^{-1}.
\end{aligned}$$

Since $\lambda \in \Sigma \setminus \mathcal{H}$, $\lambda_0 \in \mathcal{H}$ the inequality (20) guarantees that

$$\|(\lambda - \lambda_0)EA^{-1}(I + \lambda_0 EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})} \le r\sqrt{\cos^2\varphi + (1 - \sin\varphi)^2}\,\frac{c_2}{|\lambda_0|} \le \frac{1}{2}.$$

Therefore, the operator $I+(\lambda-\lambda_0)EA^{-1}(I+\lambda_0 EA^{-1})^{-1}$ possesses a bounded inverse with

$$\begin{aligned}&\|I+(\lambda-\lambda_0)EA^{-1}(I+\lambda_0 EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &\qquad\le \sum_{n=0}^{\infty}|\lambda-\lambda_0|^n\|EA^{-1}(I+\lambda_0 EA^{-1})^{-1}\|^n_{\mathcal{L}(\mathcal{V}^{-1,p})}\le \sum_{n=0}^{\infty}\Big(\frac{1}{2}\Big)^n=2.\end{aligned}$$

In summary, using (19), we obtain

$$\begin{aligned}&\|(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &\le \|I+(\lambda-\lambda_0)EA^{-1}(I+\lambda_0 EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\|(I+\lambda_0 EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &\le 2c_1\quad \forall\lambda\in\Sigma.\end{aligned}\tag{21}$$

Thus, the generalized resolvent set $\rho_{EA^{-1}}(I)$ contains the set Σ.

4. Now we carry over the estimate of type (20) to $\lambda\in\Sigma$. We find

$$\begin{aligned}|\lambda|\|EA^{-1}(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}&=\|\lambda EA^{-1}(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &=\|I-(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &\le(\|I\|_{\mathcal{L}(\mathcal{V}^{-1,p})}+2c_1)\le c_3\quad\forall\lambda\in\Sigma.\end{aligned}\tag{22}$$

5. Using the inequalities (21) and (22) we obtain for all $\lambda\in\Sigma$ the estimate

$$\begin{aligned}&(|\lambda|+1)\|EA^{-1}(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &\qquad\le|\lambda|\|EA^{-1}(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}+\|EA^{-1}(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\\ &\qquad\le c_3+\|EA^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\|(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\le c_3+2cc_1\le c_5.\end{aligned}$$

This ensures

$$\|EA^{-1}(I+\lambda EA^{-1})^{-1}\|_{\mathcal{L}(\mathcal{V}^{-1,p})}\le\frac{c_5}{|\lambda|+1}\quad\forall\lambda\in\Sigma,\tag{23}$$

which completes the proof. □

6 Regularity results for the solution of the instationary problem

Lemma 6.1 *Let $G\in\mathbb{R}^N$ be a regular bounded set and let $p\in[2,q]$ where q be given by Theorem 5.1. Moreover, we assume (7) for the coefficients e_{jk}. Then the closure of the set $E[\mathcal{V}^{1,p}]$ in $\mathcal{V}^{-1,p}$ is the whole space $\mathcal{V}^{-1,p}$.*

Proof. 1. It suffices to prove that for any real-valued function

$$b\in L^\infty(G)\text{ with }\quad 0<\frac{1}{\tau}\le b\le\tau\quad\text{a.e. on }G\text{ for some }\tau>0$$

the set $b\,[W_0^{1,p}(G)]$ is dense in $W^{-1,p}(G)$. Then the result can be carried over to the m component case where the operator E describes the multiplication by a symmetric, positive definite $m \times m$ matrix of real-valued $L^\infty(G)$ coefficients (see (7)).

2. Let $f \in W^{-1,p}(G)$ and $\varepsilon > 0$ be arbitrarily given. Since $W^{1,p'}(G)$ is dense in $L^{p'}(G)$ and $W^{1,p'}(G)$ is reflexive, $L^p(G)$ is dense in $W^{-1,p}(G)$. Let $I_p \colon L^p(G) \to W^{-1,p}(G)$ denote the corresponding embedding and let c_p be its norm. Thus there exists an $u \in L^p(G)$ such that $\|f - I_p u\|_{W^{-1,p}(G)} < \frac{\varepsilon}{2}$. Then $\frac{1}{b}u \in L^p(G)$, too. Since $C_0^\infty(\Omega) \subset W_0^{1,p}(G)$ is dense in $L^p(G)$, we find some $y \in C_0^\infty(\Omega)$ with

$$\|\frac{1}{b}u - y\|_{L^p(G)} < \frac{1}{c_p \tau}\,\frac{\varepsilon}{2}.$$

Finally, we can conclude that

$$\begin{aligned}\|f - I_p b y\|_{W^{-1,p}(G)} &\le \|f - I_p u\|_{W^{-1,p}(G)} + \|I_p u - I_p b y\|_{W^{-1,p}(G)} \\ &< \frac{\varepsilon}{2} + c_p \|u - by\|_{L^p(G)} \le \frac{\varepsilon}{2} + c_p \|b\|_{L^\infty(G)} \|\frac{1}{b}u - y\|_{L^p(G)} < \varepsilon\end{aligned}$$

which proves the lemma. $\square$

Theorem 6.1 *Let G be a bounded regular subset of $\mathbb{R}^N$ and let the coefficients $a^{jk}_{\alpha\beta}$ and e_{jk} fulfil the properties (4) and (7), respectively. Moreover, let q be given by Theorem 5.1. Then for every $p \in [2, q]$ and $\sigma \in (0, 1]$ the following assertions hold: For any $F \in C^\sigma([0,T]; \mathcal{V}^{-1,p})$ and any $w_0 \in \mathcal{V}^{-1,p}$ there is a unique solution to the problem*

$$\begin{aligned}(Eu)'(t) + Au(t) = F(t) \quad &\text{in } \mathcal{V}^{-1,p}, \quad t \in (0, T], \\ (Eu)(0) = w_0&.\end{aligned} \tag{24}$$

This solution owns the regularity properties $Eu \in C^1((0,T]; \mathcal{V}^{-1,p}) \cap C([0,T]; \mathcal{V}^{-1,p})$ and $u \in C((0,T]; \mathcal{V}^{1,p})$.

Proof. 1. First, we consider the instationary problem

$$(EA^{-1}v)'(t) + v(t) = F(t) \quad \text{in } \mathcal{V}^{-1,p}, \quad t \in (0, T] \tag{25}$$

with an initial condition which is to be understood in the seminorm sense that

$$\|EA^{-1}\left\{EA^{-1}v(t) - w_0\right\}\|_{\mathcal{V}^{-1,p}} \to 0 \quad \text{as } t \to 0.$$

[2, Theorem 3.8, p. 56] guarantees the following existence result for problem (25). For any $F \in C^\sigma([0,T]; \mathcal{V}^{-1,p})$ $(0 < \sigma \le 1)$ and any $w_0 \in \mathcal{V}^{-1,p}$ equation (25) possesses a unique strict solution v such that

$$EA^{-1}v \in C^1((0,T]; \mathcal{V}^{-1,p}), \quad v \in C((0,T]; \mathcal{V}^{-1,p}).$$

2. Moreover, (see [2, Theorem 3.9, p. 56]) if $w_0 \in \overline{EA^{-1}[\mathcal{V}^{-1,p}]} = \overline{E[\mathcal{V}^{1,p}]}$ then $EA^{-1}v(t)$ is continuous at $t = 0$ in the norm of $\mathcal{V}^{-1,p}$, i.e. $EA^{-1}v \in C([0,T];\mathcal{V}^{-1,p})$ and $EA^{-1}v(0) = w_0$.

3. According to Lemma 6.1 we have $\overline{E[\mathcal{V}^{1,p}]} = \mathcal{V}^{-1,p}$ such that for any $w_0 \in \mathcal{V}^{-1,p}$ the solution v of (25) fulfills $EA^{-1}v \in C([0,T];\mathcal{V}^{-1,p})$ and $EA^{-1}v(0) = w_0$.

4. Next, we take this solution v of (25), define $u = A^{-1}v$ and find that the function u is a solution of the problem

$$(Eu)'(t) + Au(t) = F(t) \quad \text{in } \mathcal{V}^{-1,p}, \quad t \in (0,T].$$

This solution u fulfills $Eu \in C^1((0,T];\mathcal{V}^{-1,p})$ and $Au \in C((0,T];\mathcal{V}^{-1,p})$. By the isomorphism property of A we get $u \in C((0,T];\mathcal{V}^{1,p})$. Moreover, since $w_0 \in \mathcal{V}^{-1,p} = \overline{E[\mathcal{V}^{1,p}]}$ we get $Eu \in C([0,T];\mathcal{V}^{-1,p})$ and $(Eu)(0) = w_0$. $\square$

For $\theta \in (0,1)$ we consider the interpolation spaces (cf. [2, (3.17)])

$$\left[\mathcal{V}^{-1,p}\right]^\theta = \left\{ z \in \mathcal{V}^{-1,p} \colon \sup_{\zeta > 0} \zeta^\theta \|(\zeta EA^{-1} + I)^{-1} z\|_{\mathcal{V}^{-1,p}} < \infty \right\}.$$

[2, Theorem 1.12] ensures that

$$\left[\mathcal{V}^{-1,p}\right]^\theta = \left(\mathcal{V}^{-1,p}, D(AE^{-1})\right)_{\theta,\infty}, \quad \theta \in (0,1),$$

where $(\mathcal{V}^{-1,p}, D(AE^{-1}))_{\theta,\infty}$ denotes the real interpolation spaces and $D(AE^{-1})$ is the domain of definition of the operator AE^{-1}.

Remark 6.1 *[2, Theorem 3.17, p. 62] ensures the following regularity properties of the solutions to* (25) *and* (24)*, respectively.*

If the right hand side fulfills $F \in C^\theta([0,T];\mathcal{V}^{-1,p})$ *and* $F(0) \in \left[\mathcal{V}^{-1,p}\right]^\theta$ *for some* $\theta \in (0,1)$ *and if additionally* $w_0 = 0$ *then the solution* v *to problem* (25) *enjoys the regularity*

$$(EA^{-1}v)' \in C^\theta([0,T];\mathcal{V}^{-1,p}) \cap B([0,T];\left[\mathcal{V}^{-1,p}\right]^\theta)$$

where $B([0,T];X)$ *denotes the set of bounded functions* $f : [0,t] \to X$*. Moreover we obtain*

$$v = F - (EA^{-1}v)' \in C^\theta([0,T];\mathcal{V}^{-1,p}).$$

Under the same assumptions the corresponding solution u *to problem* (24) *possesses the property that*

$$(Eu)' \in C^\theta([0,T];\mathcal{V}^{-1,p}) \cap B([0,T];\left[\mathcal{V}^{-1,p}\right]^\theta), \quad Au = F - (Eu)' \in C^\theta([0,T];\mathcal{V}^{-1,p}).$$

Using the isomorphism property of A *this yields for* u *itself the regularity* $u \in C^\theta([0,T];\mathcal{V}^{1,p})$.

References

1. S. Agmon, *On the eigenfunctions of the general elliptic boundary value problems*, Comm. Pure Appl. Math. **15** (1962), 119–147.

2. A. Favini and A. Yagi, *Degenerate differential equations in banach spaces*, Marcel Dekker, Inc., New York, Basel, Hong Kong, 1999.

3. J. A. Griepentrog, *Zur Regularität linearer elliptischer und parabolischer Randwertprobleme mit nichtglatten Daten*, Ph.D. thesis, Humboldt-Universität zu Berlin, 2000.

4. K. Gröger, *A $W^{1,p}$–estimate for solutions to mixed boundary value problems for second order elliptic differential equations*, Math. Ann. **283** (1989), 679–687.

5. K. Gröger and J. Rehberg, *Resolvent estimates in $W^{-1,p}$ for second order elliptic differential operators in case of mixed boundary conditions*, Math. Ann. **285** (1989), 105–113.

6. T. Kato, *Perturbation theorie for linear operators*, Springer, Berlin, Heidelberg, New York, Tokyo, 1984.

7. J. Nečas, *Les méthodes directes en théorie des équations elliptiques*, Prague, 1967.

8. E. Zeidler, *Nonlinear functional analysis and its applications, vol. i-iv*, Springer, Berlin, Heidelberg, 1985-1990.