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Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise

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Abstract. Several results concerning asymptotical mean square stability of an equilibrium point (here the null solution) of specific linear stochastic systems given at discrete time–points are presented and proven. It is shown that the mean square stability of the implicit Euler method, taken from the monograph of Kloeden and Platen (1992) and applied to linear stochastic differential equations, is necessary for the mean square stability of the corresponding implicit Mil'shtein method (using the same implicitness parameter). Furthermore, a sufficient condition for the mean square stability of the implicit Euler method can be verified for autonomous systems, while the principle of 'monotonic nesting' of the mean square stability domains holds for linear systems. The Euler method taking any integration step size with drift–implicitness 0.5 is able to indicate mean square stability of any equilibrium point of the continuous time system. As a practicable alternative for controlling the temporal mean square evolution, the class of Balanced methods with deterministic, positive scalar correction provides the most mean square stable numerical solution known under 'low smoothness conditions' so far. The paper summarizes and continues the stability examinations of Schurz (1993). The results can also be used to deduce recommendations for the practical implementation of numerical methods solving nonlinear systems in terms of their linearization. Finally, effects of the presented mean square calculus are shown by the Kubo oscillator perturbed by white noise and a simplified system of noisy Brusselator equations.

Contents

1. A brief introduction and some motivation

Stability is one of the most fascinating and desireable properties of objects in natural sciences. Stable long–term behaviour of both exact and numerical solutions is required in many applications, e.g. for numerical computation of Lyapunov exponents (Pardoux and Talay [65] or Talay [80, 82]), for the approximation of stochastic oscillators (Wedig [86]), in quantum optical systems (Smith and Gardiner [75]), in approximate Markov chain filtering (Kloeden et al. [46]) or for simulation of discretized parametric estimators (Kloeden et al. [47]) relying on the theory of exponential families (cf. Küchler and Sørensen [53] or Heyde [32]). The stability of any equilibrium point of them ensures us that small perturbations of the initial values or perturbations during the temporal evolution are not influencing decisively the further dynamical behaviour of the considered object. Particularly, in the field of numerical integration of stochastic differential equations we do need some guarantee to have this kind of stability too. Here, the dynamical behaviour of the exact and numerical solutions is fairly complicated, even more complicated than in deterministic analysis.

In stochastic analysis we have a large variety of different stability concepts. For excellent surveys see Khas'minskij [42], Kozin [49] or Kushner [50]. The works of Bunke, Car'kov, Kac and Krasovskij, Levit and Jakubovič, Mil'shtein and Repin, Sasagawa, Willems, Wonham, among many others, treat the mean square stability problem only for continuous time systems, especially for the object of stochastic differential equations. A pioneer work for more general stochastic stability theory of such continuous systems has been done by Khas'minskij [42].

Of course, several attempts concerning the examination of discrete stochastic systems has also been made. These investigations go back up to the early works of Furstenberg [20] on random matrix products and were intensively continued by many authors, e.g. Bougerol and Lacroix [8], Cohen and Newman [15], Morozan [61] or Willems [87]. In this paper we restrict ourselves to stability examinations for discrete time systems solving numerically stochastic differential equations. These very specific stochastic systems are naturally fixed at discrete time points. See the monographs of Mil'shtein [58], Kloeden and Platen [44], the survey paper of Talay [80] or in the succeeding section for numerical integration methods solving these equations.

For numerical solutions with additive noise, Mil'shtein [58] has examined, as one of the first, the stability in probability. Further attempts have been made by Hernandez and Spigler [30], Kloeden and Platen [45]. In these contributions the investigation draws back to an one-dimensional complex–valued test equation and the increment function of the corresponding numerical method. This increment function is decomposed in a deterministic and stochastic part. Then, only the deterministic part is investigated in concern with asymptotical stability. This approach may be considered as representative for stochastic dynamics with 'small noise', especially with very small additive noise. For multiplicative and more general noise, the neglect of the stochastic parts is not possible and would lead to inadequate conclusions. In many applications the dynamical behaviour is decisively influenced on the stochastic terms. Also some attempts to investigate multiplicative noise cases and stability have been done. For example, Mitsui and Saito [60], Hernandez and Spigler [31], Hofmann and Platen [33] and Schurz [71]. They only considered one-dimensional complex–valued test equations. Hofmann and Platen [33] have examined the situation with a very specific equation using a very strong stability criterion (via the 'essup criterion for increment functions' of special numerical methods). In this concept one must be able to control more or less all moments what surely is seldom possible. More general approaches concerning mean square stability could be found by Willems [87], Artemiev [6] and Schurz [72]. These authors could already state some sufficient conditions for mean square stability of numerical methods.

This paper summarizes, extends and proves further results of contributions [71, 72] on mean square stability of equilibria in order to gain more insight of the stability behaviour of these discrete systems. Throughout the paper the examination goes back to linear systems, keeping in mind that they are obtained as the linearization of corresponding nonlinear stochastic systems. Note that we will not discuss and enlighten the relation between the original nonlinear system and its corresponding linearized one. Khas'minskij [42] has done it for stochastic differential equation to a great extent, but it is still an open question for the discrete systems to be mentioned here. It is assumed that the original nonlinear system has components (drift and diffusion) vanishing simultaneously at any constant. This is the case, e.g. in population dynamics, if Lotka–Volterra systems are disturbed by multiplicative noise, see Gard [22]. For simplicity we suppose that the null solution is an equilibrium solution, i.e. a stationary solution of the considered system.

We remind the reader that there are two basic methods to examine the stability of general nonlinear systems (as well as linear ones), the method of stochastic Lyapunov functions and that of examination of the corresponding linearized system about a stationary solution. In this paper we try to avoid applying directly the technique of stochastic Lyapunov functions. Nevertheless, a corresponding discrete Lyapunov technique for stochastic numerical methods still has to be explored. Then one could even treat more fancy and accurately some nonlinear systems. The direct approach is preferable for mean square stability examinations of linear numerical methods. Mean square stability is just the subject where exact computations can be done. The requirement of almost sure stability represents a relatively strong stability assertion. Thus, it is natural to look at weaker stability concepts. Furthermore, in many physical phenomena, engineering and environmental problems only the moment evolution is of interest, e.g. see Karmeshu and Bansal [39] in Nuclear Science or Karmeshu and Schurz [40, 41] in Hydrology and Seismology. Under these circumstances a natural and weaker concept is that of moment stability. For a sophisticated approach to moment stability, e.g. see Baxendale [7]. Arnold [3] or Arnold, Oeljeklaus and Pardoux [4] among many other authors have also enlightened the relation between almost sure and moment stability. The latter works focus on continuous time systems of stochastic differential equations.

The paper consists of eight further sections. In the next section we introduce two discrete time systems (numerical methods) to be examined for the class of bilinear stochastic differential equations. There some basic definitions and results concerning the mean square stability of the continuous time systems are also assembled.

In section 3 we formulate and prove a necessary condition for the stability of one of the discrete systems (Mil'shtein methods). The main result in this section also gives an occasion to concentrate the examinations on implicit Euler methods, hence methods of lowest convergence order, and to think new about the relation (balance) between convergence and stability. After it, a simple example taken from Schurz [71] illustrates some observed effects, together with numerical experiments for the Kubo oscillator. Sections 5 and 6 are devoted to the study of linear autonomous systems. In section 5 we state sufficient conditions for the mean square stability of the simplest numerical methods (implicit Euler methods). Section 6 explains the principle of 'monotonic nesting' of stability domains discovered by Schurz [71, 72] for the two discrete linear systems. In the succeeding section a simple and practicable alternative to the implicit methods examined before is presented by Balanced methods with deterministic, positive scalar correction (weights) for control on the numerical mean square evolution. Finally, in section 8 the noisy, planar Brusselator equations are used to demonstrate effects of the presented mean square stability calculus. At the end we give a summary and conclusions in section 9.

2. Two iterated linear systems and mean square stability

Mainly, throughout this paper we consider the following two iterated linear systems

$$Y_{n+1}^{(M)} = Y_n^{(M)} + \left(\alpha A_{n+1} Y_{n+1}^{(M)} + (1-\alpha) A_n Y_n^{(M)}\right)\Delta + \sum_{j=1}^{m} B_n^j Y_n^{(M)} \xi_n^j \sqrt{\Delta} + \\ + \sum_{j,k=1}^{m} B_n^j B_n^k Y_n^{(M)} \int_0^{\Delta}\int_0^{s} d\xi_n^j(r)\, d\xi_n^k(s)\, \Delta \tag{2.1}$$

and

$$Y_{n+1}^{(E)} = Y_n^{(E)} + \left(\alpha A_{n+1} Y_{n+1}^{(E)} + (1-\alpha) A_n Y_n^{(E)}\right)\Delta + \sum_{j=1}^{m} B_n^j Y_n^{(E)} \xi_n^j \sqrt{\Delta}, \tag{2.2}$$

for nonrandom real–valued $d \times d$ - matrices A_n and B_n^j, $(d = 1, 2, \dots)$ and standard Gaussian distributed random variables $\xi_n^j (i.i.d., n = 0, 1, 2, \dots, j = 1, \dots, m)$ starting at $Y_0 \in \mathbb{R}^d$ for a given fixed $\Delta > 0$. System 2.1 is often called implicit Mil'shtein method and 2.2 implicit Euler method with implicitness parameter $\alpha \in [0, 1]$. For the resolution of the system of algebraic equations 2.1 as well as 2.2 we have to require the existence of the inverse of $C_{n+1}(\alpha) := I - \alpha \Delta A_{n+1}$ at any step n in the case of $\alpha > 0$. For the same parameter value α we also call the method 2.2 the Euler method corresponding to the Mil'shtein method 2.1.
System 2.1 as well as 2.2 can be interpreted as equidistant numerical solution of the stochastic differential equation

$$dX_t \;=\; A(t)\, X_t\, dt \;+\; \sum_{j=1}^{m} B^j(t)\, X_t\, dW_t^j \tag{2.3}$$

at time $t_n = n \cdot \Delta$. Here $(W_t^j)_{j=1,2,\dots,m}$ are independent identically distributed Wiener processes with $W_t^j = \xi_t^j \sqrt{t}$ and $\xi_{t_n}^j = \sum_{k=0}^{n-1}(\xi_{k+1}^j - \xi_k^j)$; $(W_0^j = \xi_0^j = 0)$. The solution of 2.3 always exists and is unique under appropriate boundedness conditions on the matrices $A(t)$ and $B^j(t)$ over the time interval $[0, T]$.
Often stochastic differential equations play an important role to discribe the dynamical evolution. For theory and applications see Arnold [2], Gard [22], Gardiner [23], Gikhman and Skorokhod [24], Honerkamp [34], Horsthemke and Lefever [35], Ikeda and Watanabe [36], Øksendal [63], Soong [77] or Stroock and Varadhan [78]. The solving of such objects requires numerical techniques and recipes. For convenient qualitative analysis, these stochastic systems, both continuous and discrete time ones, are linearized at a stationary point. Eventually one examines the qualitative behaviour of them in a neighbourhood of this equilibrium. Thus, it is reasonable to suppose that system 2.3 represents such a system linearized at the equilibrium point $0 \in \mathbb{R}^d$. The bilinear system 2.3 turns out to be already quite general and complicated one. In contrast to deterministic analysis, most of the linear stochastic differential equations are not explicitly solvable (Moreover, the only exception is the case of mutually commutative drift A and diffusion matrices B^j). Consequently, we do need a detailed stochastic analysis (such as stability analysis) in order to 'crystallize out' robust and reliable numerical methods for solving them.
The methods 2.1 and 2.2 enable us to construct the simplest numerical solution of 2.3. Other numerical solutions can be systematically constructed by appropriate truncation of the stochastic Taylor formula (methods of higher convergence order) which is due to Wagner and Platen [85] or by replacing the occured derivatives by corresponding difference quotients (Runge–Kutta methods). For general reviews and recipes on stochastic numerical analysis, we refer to Artemiev [6], Greiner et al. [26], Helfand [28], Mil'shtein [58], Kloeden and Platen [44], Newton [62], Rümelin [68], Smith and Gardiner [75], Talay [80] or Kloeden, Platen and Schurz [48]. Furthermore, Hernandez and Spigler [29, 30, 31] especially consider implicit Runge–Kutta methods. Effects of structural peculiarities on the numerical methods have also examined by Klauder and Petersen [43] as well as Drummond and Mortimer [17]. The latter two papers especially deal with the situation of multiplicative noise. Petersen [66] himself enlightens stability and accuracy of stochastic numerical methods together, a view point which is basic for numerical analysis and attracts our attention throughout this exposition, with more emphasis on stability. Particularly, Shkurko [74] and Török [83] have studied the numerical solution of linear stochastic differential equations.
In passing we remark that there are a few alternative approaches to treat numerically stochastic differential equations. One is the usage of Monte Carlo methods (Russian school, e.g. Sabelfeld) and the other via construction of Markov chain approximations. The latter method has been intensively investigated by Kushner [51] in the context of stochastic control theory. Moreover he constructs efficient Markov chain approximations for the heavy traffic problem. We leave the decision to the reader to find out which approach is more convenient and appropriate. Anyway, this choice depends on the modelling issue and practical purpose. However, the numerical methods mentioned throughout this paper have the advantage to be very close to those of the deterministic analysis of ordinary differential equation.

They can be considered more or less as natural extension of their deterministic counterparts.
Corresponding convergence results justify the application and meaningfulness of methods such as 2.1 and 2.2 as numerical approximation of systems such as 2.3 at all. In L^1 (the space of mean absolutely integrable functions) or in L^2 (the space of mean square integrable functions) the method 2.1 possesses the same (by definition) convergence order Δ and the method 2.2 order $\Delta^{1/2}$, but in the distributional or weak sense both methods are converging with the same order Δ. The following results shall be independent of these convergence notions. Convergence and asymptotical stability together yield reasonable and well–behaving numerical solutions, as in ordinary numerical analysis. Our contribution is based on the concept of mean square stability.

Definition 2.1. Let $X_t(x_0, t_0)$ denote the solution of equation 2.3 starting in x_0 at time t_0. Then the null solution $X \equiv 0$ is called **(asymptotically) mean square stable** iff

$$\exists \delta > 0 \ \ \forall t_0 \geq 0 \ \ \forall x_0 \in \mathbb{R}^d \ \ ||x_0|| < \delta \quad : \quad \lim_{t \to \infty} \mathbb{E} \, ||X_t(x_0, t_0)||^2 = 0, \qquad (2.4)$$

where $|| \cdot ||$ denotes the Euclidean vector norm in $\mathbb{R}^d$, and **exponentially mean square stable** iff

$$\exists c_1, c_2 > 0 \ \ \forall t_0 \geq 0 \quad : \quad \mathbb{E} \, ||X_t(x_0, t_0)||^2 \ \leq \ c_1 ||x_0||^2 \exp(-c_2(t - t_0)). \qquad (2.5)$$

Many authors examined mean square stability of stochastic systems, e.g. Willems [87]. Exponential stability of continuous time systems has particularly studied in papers of Car'kov [12], Sasagawa [69] or Sasagawa and Willems [70]. In Khas'minskij [42] one also finds a large collection of remarkable results on these subjects.
The following theorem taken from Arnold [2] (see [42] for a more general proof by Lyapunov functions) yields necessary and sufficient conditions on the matrices A and B^j in order to guarantee an exponentially mean square stable null solution of 2.3. It also ensures sufficient conditions for the existence of asymptotically mean square stable solutions of linear systems, and shows for which systems it makes sense to look at them concerning mean square stability.

Theorem 2.1. *Assume that the matrix-valued functions $A(t)$ and $B^j(t)$ in equation 2.3 are bounded on $[t_0, \infty)$. Then, for exponential stability of the null solution in the mean square sense it is necessary that for any, and sufficient that for a particular symmetrical, positive definite, continuous and bounded $d \times d$ - matrix $C(t)$ with $x^T C(t) x \geq k_1 |x|^2$ $(k_1 > 0)$ for all $t \geq 0$ the matrix–valued differential equation*

$$\frac{dD(t)}{dt} + A(t)D(t) + D(t)A^T(t) + \sum_{j=1}^{m} B^j(t)D(t){B^j}^T(t) = -C(t) \qquad (2.6)$$

possesses a solution matrix $D(t)$ with the same properties as the matrix $C(t)$.

Here $()^T$ denotes the transposed of the inscribed vector or matrix.

Remark. For autonomous systems, equation 2.6 obtains a simpler structure

$$\mathcal{H}D := AD + DA^T + \sum_{j=1}^{m} B^j D B^{j^T} = -Q, \tag{2.7}$$

i.e. the bounded linear operator $\mathcal{H}$ possesses always a positive definite inverse for any positive definite matrix $Q \in \mathbb{S}_{d\times d}$ (the space of symmetrical $d \times d$ - matrices).

With these facts in mind, we introduce the notion of the mean square stable null solution of discrete time systems, such as 2.1 or 2.2, in an analogous way. For the sake of simplicity we only consider equidistant numerical solutions throughout this paper, i.e. there is a unique integration step size $\Delta = t_{n+1} - t_n$ for the time–discretization $(t_n)_{n=0,1,2,...}$ of a given time interval. Of course, all the results can be carried over to nonequidistant discretizations with bounded supremum of local integration step sizes.

Definition 2.2. A numerical solution $(Y_n)_{n\in\mathbb{N}}$ using fixed step size Δ starting in y_0 at time t_0 has an **(asymptotically) mean square stable** null solution iff

$$\exists\delta > 0\ \forall t_0 \geq 0\ \forall y_0 \in \mathbb{R}^d \quad ||y_0|| < \delta\ :\ \lim_{n\to\infty} \mathbb{E}\ ||Y_n||^2 = 0. \tag{2.8}$$

The limit in 2.8 is understood only at discrete times t_n (by definition). In the following we examine the time evolution of the symmetrical 2nd moment matrices

$$P(t) = \mathbb{E}\ X_t(x_0,t_0)X_t^T(x_0,t_0) = (\mathbb{E}\ X_t^i X_t^j),\ t \geq t_0 \tag{2.9}$$

for continuous time system 2.3 and

$$P_n = \mathbb{E}\ Y_n Y_n^T = (\mathbb{E}\ Y_n^i Y_n^j). \tag{2.10}$$

for discrete time systems 2.1 and 2.2 in order to make assertions about mean square stability of the null solution. Since

$$||P(t)||_{d\times d} \leq K \cdot \mathbb{E}\ ||X_t||^2 \quad \text{and} \quad \mathbb{E}\ ||X_t||^2 = \text{ tr } (P(t)) = \sum_{i=1}^{d} p_{ii}(t) \tag{2.11}$$

mean square stability is obviously equivalent to the stability of the corresponding matrix system 2.9 or 2.10 (Here $||\cdot||$ is the Euclidean vector norm on $\mathbb{R}^d$, tr is the trace and $||\cdot||_{d\times d}$ any compatible matrix norm on $\mathbb{R}^{d\times d}$, $K > 0$).

3. A NECESSARY CONDITION FOR MEAN SQUARE STABILITY OF SYSTEM (2.1)

In this section we formulate and prove a relation between the mean square evolution of discrete systems 2.1 and 2.2. In stating our main result below we note that the inequality character between matrices is understood in the sense of positive semi-definite matrices S, i.e.

$$x^T S x \geq 0 \quad \forall x \in \mathbb{R}^d$$

or in other words, if between two positive semi-definite matrices S_1 and S_2 the symbolic relation $\leq_{(+)}$ is used then the following relation is meant

$$\forall x \in \mathbb{R}^d \times \mathbb{R}^d : x^T(S_2 - S_1)x = x^T S_2 x - x^T S_1 x \geq 0 \iff S_1 \leq_{(+)} S_2 .$$

In the following $\mathbb{S} = \mathbb{S}_{d\times d}$ denotes the space of symmetrical real $d \times d$ matrices S, and $\mathbb{S}^+ = \mathbb{S}^+_{d\times d}$ the space of positive semi-definite $d \times d$ matrices (also symmetrical by definition). Thus, $\mathbb{S}^+$ attached with the relation $\leq_{(+)}$ is an ordered space.

Theorem 3.1. *Assume that* $\mathbb{E}\, Y_0^{(E)} Y_0^{(E)^T} = \mathbb{E}\, Y_0^{(M)} Y_0^{(M)^T} = \mathbb{E}\, X_0 X_0^T \in \mathbb{R}^{d\times d}$ *is a positive semi-definite matrix with* Y_0 *being independent of the random variables* ξ_n*, and that the inverses of matrices*

$$C_{n+1} := I - \alpha \Delta A_{n+1} \qquad (n = 0, 1, 2, \ldots)$$

always exist.
Then, for the linear stochastic systems 2.1 *and* 2.2 *the following inequality holds*

$$\mathbb{E}\, Y_n^{(E)} Y_n^{(E)^T} \quad \leq_{(+)} \quad \mathbb{E}\, Y_n^{(M)} Y_n^{(M)^T} \quad : \quad \forall n \in \mathbb{N} . \tag{3.1}$$

Remarks. From this one immediately concludes that if the implicit Mil'shtein method possesses a mean square stable null solution then the corresponding Euler method possesses it too. The invertibility of matrices $I - \alpha \Delta A_{n+1}$ is ensured if $||\alpha \Delta A(t)|| < 1$ uniformly in $t \in [0, \infty)$, or if the matrices $A(t)$ have only eigenvalues with nonpositive real parts, as it is the case in mean square stable, autonomous systems 2.3.

Proof. We prove this theorem by induction. For $n = 0$ the assertion is obviously true by assumption. Suppose the relation

$$P_n^{(E)} := \mathbb{E}\, Y_n^{(E)} Y_n^{(E)^T} \quad \leq_{(+)} \quad \mathbb{E}\, Y_n^{(M)} Y_n^{(M)^T} =: P_n^{(M)} \tag{3.2}$$

is satisfied for a fixed $n \geq 1$ where $P_n^{(E)}$ and $P_n^{(M)}$ are positive semi-definite.
Now we show the validity for $n + 1$. Systems 2.1 and 2.2 can be rewritten to the equivalent form

$$Y_{n+1}^{(M)} = C_{n+1}^{-1} \left(I + (1-\alpha)\Delta A_n + \sum_{j=1}^{m} B_n^j \xi_n^j \sqrt{\Delta} + \sum_{j,k=1}^{m} B_n^j B_n^k V_n^{j,k} \Delta \right) Y_n^{(M)} \tag{3.3}$$

and

$$Y_{n+1}^{(E)} = C_{n+1}^{-1}\left(I + (1-\alpha)\Delta A_n + \sum_{j=1}^{m} B_n^j \xi_n^j \sqrt{\Delta}\right) Y_n^{(E)} \tag{3.4}$$

where

$$C_{n+1} = I - \alpha\Delta A_{n+1} \quad \text{and} \quad V_n^{j,k} = \int_0^{\Delta}\int_0^{s} d\xi_n^j(r)\, d\xi_n^k(s)\ . \tag{3.5}$$

Consequently, one obtains the matrix equations

$$\begin{aligned}
P_{n+1}^{(M)} &= \mathbb{E}\, Y_{n+1}^{(M)} Y_{n+1}^{(M)T} \\
&= \mathbb{E}\, C_{n+1}^{-1}\left(I + (1-\alpha)A_n + \sum_{j=1}^{m} B_n^j \xi_n^j \sqrt{\Delta} + \sum_{j,k=1}^{m} B_n^j B_n^k V_n^{j,k}\Delta\right) Y_n^{(M)} Y_n^{(M)T} \cdot \\
&\quad \left(I + (1-\alpha)\Delta A_n^T + \sum_{j=1}^{m} B_n^{j\,T} \xi_n^j \sqrt{\Delta} + \sum_{j,k=1}^{m} B_n^{k\,T} B_n^{j\,T} V_n^{j,k}\Delta\right) C_{n+1}^{-1\,T} \\
&= C_{n+1}^{-1}(I + (1-\alpha)\Delta A_n) P_n^{(M)} (I + (1-\alpha)\Delta A_n)^T C_{n+1}^{-1\,T} \\
&\quad + \sum_{j=1}^{m} C_{n+1}^{-1} B_n^j P_n^{(M)} B_n^{j\,T} C_{n+1}^{-1\,T}\Delta + \sum_{j,k=1}^{m} C_{n+1}^{-1} B_n^j B_n^k P_n^{(M)} B_n^{k\,T} B_n^{j\,T} C_{n+1}^{-1\,T}\Delta^2/2.
\end{aligned} \tag{3.6}$$

Here have used the following relations. Firstly, it is well–known that

$$\mathbb{E}\,\xi_n^j = 0 \quad \text{and} \quad \mathbb{E}\, V_n^{j,k} = 0 \quad \forall j,k = 1,\ldots,m,$$

where the latter represents the martingale property of Itô–integrals. In [44] (Lemma 5.7.2, p. 191) one encounters with

$$\begin{aligned}
\mathbb{E}\, I_{j_1,k_1}(\Delta) I_{j_2,k_2}(\Delta) &= \mathbb{E}\left(\int_0^{\Delta}\int_0^{s} dW^{j_1}(r)\, dW^{k_1}(s) \cdot \int_0^{\Delta}\int_0^{s} dW^{j_2}(r)\, dW^{k}(s)\right) \\
&= \Delta^2 \mathbb{E}\left(\int_0^{\Delta}\int_0^{s} d\xi^{j_1}(r)\, d\xi^{k_1}(s) \cdot \int_0^{\Delta}\int_0^{s} d\xi^{j_2}(r)\, d\xi^{k_2}(s)\right) = \delta_{j_1,j_2}\cdot\delta_{k_1,k_2}\cdot\Delta^2/2
\end{aligned}$$

(cf. also p. 223 in [44]). Analogously one argues with the remainder terms. Lemma 5.12.3 from [44] (p. 221) provides us the relation

$$I_j(\Delta) I_{j_1,k_1}(\Delta) = \int_0^{\Delta} I_{j_1,k_1}(s)\, dW_s^j + \int_0^{\Delta} I_{j_1}(s) I_j(s)\, dW_s^{k_1} + \int_0^{\Delta} I_{j_1}(s)\cdot \mathbb{1}_{\{j=k_1\neq 0\}}\, ds$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function of the inscribed set. Thus $\mathbb{E}\,(I_j \cdot I_{j_1,k_1}) = 0$ follows for all $j, j_1, k_1 = 1,\ldots,m$. The relation 3.6 is finally confirmed after rearranging the matrix products and applying the moment properties mentioned above. Returning to 3.6, we introduce the abbreviation

$$P_{n+1}^{(M)} = \mathcal{L} P_n^{(M)} \tag{3.7}$$

as the operator equation defined via right hand side of 3.6. This operator $\mathcal{L}$ mapping $\mathbb{S}_{d\times d}$ onto $\mathbb{S}_{d\times d}$ is linear and bounded. Furthermore $\mathcal{L}$ is nonnegative, i.e. $\mathcal{L}\mathbb{S}^+ \subseteq \mathbb{S}^+$. This can be easily verified as follows. Since $P_n^{(M)}$ is positive semi-definite, it can be decomposed by the Cholesky factorization such that

$$P_n^{(M)} = L_n^{(M)} L_n^{(M)^T}$$

where $L_n^{(M)}$ is triangular. Thus, there exist matrices Q_n^l satisfying

$$\begin{aligned} P_{n+1}^{(M)} &= \mathcal{L}P_n^{(M)} = \sum_{l=0}^{m(m+1)} Q_n^l \cdot Q_n^{l^T} \\ \text{with} \quad Q_n^0 &= C_{n+1}^{-1}(I + (1-\alpha)\Delta A_n)L_n^{(M)}, \quad Q_n^j = C_{n+1}^{-1} B_n^j L_n^{(M)} \cdot \sqrt{\Delta} \\ \text{and} \quad Q_n^l &\in \{C_{n+1}^{-1} B_n^i B_n^k L_n^{(M)} \cdot \Delta \frac{\sqrt{2}}{2} \; : \; i,k = 1,2,\ldots,m\} \end{aligned}$$

($j = 1,2,\ldots,m,\quad l = m+1,\ldots m(m+1)$). The sum of positive semi-definite matrices is again positive semi-definite. So we can conclude that $\mathcal{L}\mathbb{S}^+ \subseteq \mathbb{S}^+$. The difference $P_n^{(M)} - P_n^{(E)}$ must be positive semi-definite according to the induction assumption, hence the relation

$$\mathcal{L}(P_n^{(M)} - P_n^{(E)}) \geq_{(+)} \mathcal{O} \qquad \left(\Longleftrightarrow \mathcal{L}P_n^{(M)} \geq_{(+)} \mathcal{L}P_n^{(E)}\right)$$

follows ($\mathcal{O}$ is the null matrix in $\mathbb{S}$). Finally, we obtain

$$\begin{aligned} P_{n+1}^{(M)} = \mathcal{L}P_n^{(M)} &= \mathcal{L}P_n^{(M)} - \mathcal{L}P_n^{(E)} + \mathcal{L}P_n^{(E)} \\ &= \mathcal{L}P_n^{(E)} + \mathcal{L}(P_n^{(M)} - P_n^{(E)}) \geq_{(+)} \mathcal{L}P_n^{(E)} \geq_{(+)} P_{n+1}^{(E)}\,, \end{aligned} \tag{3.8}$$

on $\mathbb{S}^+$ where we have used the identity

$$\mathcal{L}P_n^{(E)} = P_{n+1}^{(E)} + \sum_{j,k=1}^{m} C_{n+1}^{-1} B_n^j B_n^k P_n^{(E)} B_n^{k^T} B_n^{j^T} C_{n+1}^{-1^T} \Delta^2/2$$

which follows from 3.6 via the definition of the operator $\mathcal{L}$ in 3.7. This completes the proof of the theorem. □

Remarks. As we know about the meaning of the relation 3.1, we have obtained that even the difference $P_n^{(M)} - P_n^{(E)}$ is positive semi-definite for all $n \in \mathbb{N}$, provided that the assumption of Theorem 3.1 is satisfied. The property of positive semi-definiteness yields nonnegative diagonal elements of the considered matrix. Consequently, Theorem 3.1 also implies that the relation

$$\mathbb{E}\,(Y_{n,i}^{(M)})^2 \;\geq\; \mathbb{E}\,(Y_{n,i}^{(E)})^2$$

holds for each component of the subscribed vectors. The assumption that the matrix $\mathbb{E}\,X_0X_0^T$ is positive semi-definite can be considered as reasonable and naturally fulfilled, for example for independent initial random variables X_0^i with $\mathbb{E}\,X_0^i = 0$. Moreover, Theorem 2.1 justifies that the requirement of positive semi-definiteness of the initial moment matrix $P(0)$ is not restrictive, cf. also Theorem 8.5.5 in [2].

4. A SIMPLE COMPLEX–VALUED EXAMPLE (KUBO OSCILLATOR)

For the real Wiener process W_t, the one–dimensional complex-valued stochastic differential equation

$$dX_t = \lambda X_t dt + \gamma X_t dW_t \ , \quad X_0 = x_0 \tag{4.1}$$

has the exact solution

$$X_t = X_0 \cdot \exp((\lambda - \gamma^2/2)t + \gamma W_t)$$

with its second moment

$$\begin{aligned} \mathbb{E}\, X_t X_t^* &= \mathbb{E}\, \exp(2(\lambda - \gamma^2/2)_r t + 2\gamma_r W_t) \cdot ||x_0||^2 \\ &= ||x_0||^2 \cdot \exp(2(\lambda_r - \gamma_r^2/2 + \gamma_i^2/2)t + 2\gamma_r^2 t) \\ &= ||x_0||^2 \cdot \exp((2\lambda_r + ||\gamma||^2)t) \end{aligned}$$

where $x_0 \in \mathbb{C}$ is nonrandom (z_r is the real part, z_i the imaginary part of $z \in \mathbb{C}$) and $*$ denotes the complex conjugate value. The trivial solution $X \equiv 0$ of 4.1 is mean square stable for the process $\{X_t : t \geq 0\}$ iff $2\lambda_r + ||\gamma||^2 < 0$.
Applied to equation 4.1 the implicit Mil'shtein 2.1 and Euler methods 2.2 are given by

$$Y_{n+1}^{(M)} = \frac{1 + (1-\alpha)\lambda\Delta + \gamma\xi_n\sqrt{\Delta} + \gamma^2(\xi_n^2 - 1)\Delta/2}{1 - \alpha\lambda\Delta} \cdot Y_n^{(M)} \tag{4.2}$$

and

$$Y_{n+1}^{(E)} = \frac{1 + (1-\alpha)\lambda\Delta + \gamma\xi_n\sqrt{\Delta}}{1 - \alpha\lambda\Delta} \cdot Y_n^{(E)} \ , \tag{4.3}$$

respectively. Their second moments $P_n^{(E/M)} = \mathbb{E}\, Y_n^{(E/M)} Y_n^{(E/M)^*}$ satisfy the relations

$$\begin{aligned} P_{n+1}^{(M)} &= \left(\mathbb{E}\, ||\frac{1 + (1-\alpha)\lambda\Delta + \gamma\xi_n\sqrt{\Delta}}{1 - \alpha\lambda\Delta}||^2 + \mathbb{E}\, ||\frac{\gamma^2(\xi_n^2 - 1)}{1 - \alpha\lambda\Delta}||^2 \cdot \Delta^2/4 \right) \cdot P_n^{(M)} \\ &= P_0^{(M)} \cdot \left(\frac{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta + ||\gamma||^4\Delta^2/2}{||1 - \alpha\lambda\Delta||^2} \right)^{n+1} \\ &> P_0^{(E)} \cdot \left(\frac{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta}{||1 - \alpha\lambda\Delta||^2} \right)^{n+1} = P_{n+1}^{(E)} \qquad (n = 0, 1, 2, \ldots) \end{aligned}$$

provided that $P_0^{(M)} = \mathbb{E}\, Y_0^{(M)} Y_0^{(M)^*} \geq \mathbb{E}\, Y_0^{(E)} Y_0^{(E)^*} = P_0^{(E)}$, and

$$\begin{aligned} P_{n+1}^{(M)} &= P_{n+1}^{(E)} \cdot \left(\frac{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta + ||\gamma||^4\Delta^2/2}{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta} \right)^{n+1} \\ &= P_{n+1}^{(E)} \cdot \left(1 + \frac{||\gamma||^4\Delta^2/2}{||1 + (1-\alpha)\lambda\Delta||^2 + ||\gamma||^2\Delta} \right)^{n+1} . \end{aligned}$$

while assuming identical initial values $P_0^{(M)} = P_0^{(E)}$.
Hence, if the implicit Mil'shtein method 4.2 possesses a mean square stable null

solution then the corresponding Euler method 4.3 possesses it too. Moreover, as it was already shown in [71], the mean square stability domain of 4.2 is smaller than the corresponding mean square stability domain of 4.3 for any implicitness $\alpha \in [0,1]$. Also, it can be concluded that the implicit Euler method 4.3 has a mean square stable null solution if $\alpha \geq \frac{1}{2}$ and $2\lambda_r + ||\gamma||^2 < 0$. The latter condition coincides with the necessary and sufficient condition for the mean square stability of the null solution of complex-valued equation 4.1. Thus, the Euler method 4.2 with implicitness 0.5 is useful to indicate mean square stability of the equilibrium solution of 4.1. For visualization purposes we add figure 1 to this exposition. It shows the boundary hyperplane of the mean square stability domain for the implicit Euler method with implicitness 0.5. The corresponding region is located below this hyperplane. Furthermore, it reflects exactly the domain of mean square stability of the null solution of complex-valued test equation 4.1, as already mentioned.

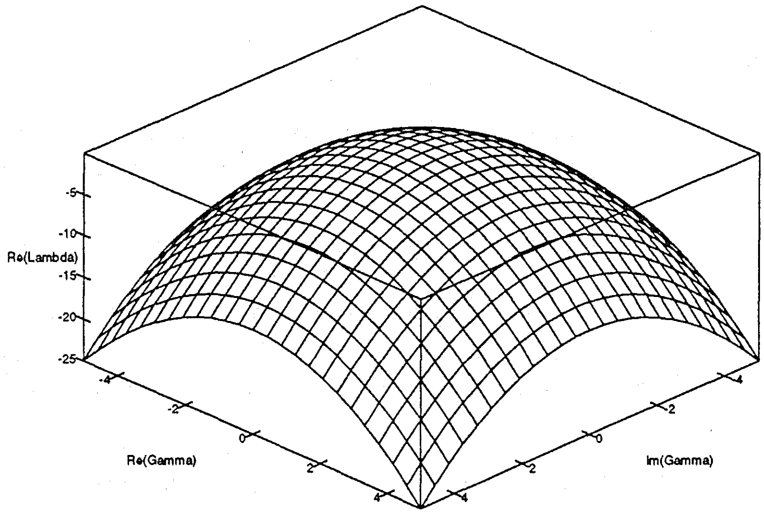

Figure 1. Boundary hyperplane of the mean square stability domain for the implicit Euler methods using implicitness 0.5 applied to bilinear equation 4.1

Experiments for the Kubo oscillator. Model 4.1 includes the model of the Kubo oscillator with white noise perturbations. For a corresponding reference, see [34]. In contrast to the model with white noise, the Kubo oscillator with more realistic coloured noise perturbations is common to use for modelling of random oscillations in nuclear reactions. In this paper we make use of the white noise perturbed model in order to demonstrate what can happen with numerical solutions if they are not choosen carefully enough. Just as well we suggest this model as one favourable test model for numerical methods concerning their mean square stability behaviour (in general for p-th mean stability). Note this test procedure is only appropriate for one-dimensional stochastic differential equations.
The following system of the Kubo oscillator perturbed by white noise has been taken from Honerkamp [34]. Driven by one-dimensional real-valued Wiener process $W(t)$, the system is given by a complex-valued Stratonovich stochastic differential

equation of the form ($\imath$ denotes the imaginary unit, i.e. $\imath^2 = -1$)

$$dX(t) \;=\; \imath X(t)\,dt \;+\; \imath\rho X(t) \circ dW(t) \tag{4.4}$$

for the variable $X(t)$ and parameter $\rho \in \mathbb{R}^1$ on the time interval $[0, T]$. Equation 4.4 is explicitly solvable and has the solution

$$X(t) \;=\; X(0)\exp\{\imath\rho W(t) + \imath t\} \;=\; X(0)\Big(\cos(\rho W(t) + t) + \imath\sin(\rho W(t) + t)\Big)\,.$$

Obviously, this system describes a stochastic movement on the circle with radius $\|X(0)\|$, i.e. it holds $\|X(t)\| = \|X(0)\|$ for all $t \geq 0$. Another interesting fact occurs, with the Kubo oscillator we study a system where all p-th mean Lyapunov exponents $l(x_0; \mathrm{p}) = 0$ ($x_0 \neq 0$). These exponents establish the exponential rates of the p-th moments and determine with their sign exploding or declining behaviour of the p-th absolute moments of dynamical systems, for a corresponding definition and theory see Arnold and Wihstutz (1986). To check the condition for p-th mean stability (= stability of p-th absolute moments) one receives

$$\frac{l(\mathrm{p})}{\mathrm{p}} \;=\; \lambda_r + \frac{1}{2}\gamma_i^2 + \frac{1}{2}\gamma_r^2(p-1) = -\frac{1}{2}\rho^2 + \frac{1}{2}\rho^2 = 0\,, \tag{4.5}$$

for system 4.4. Therefore its null solution is not asymptotically p-th mean stable for any $\mathrm{p} > 0$, in particular not asymptotically mean square stable. However, because of

$$\mathbb{E}(X(t))^{\mathrm{p}} \;=\; (x_0)^{\mathrm{p}}\exp\{-\frac{\mathrm{p}^2}{2}\rho^2 t + \mathrm{p}\imath t\}$$

with nonrandom $X(0) = x_0$, one knows that the p-th moments are converging to zero as $t \to \infty$, i.e. in the mean sense the null solution is stable for this system. For the sake of numerical investigation we state the Itô version corresponding to 4.4. It has the form

$$dX(t) \;=\; (\imath - \frac{1}{2}\rho^2)X(t)\,dt \;+\; \imath\rho X(t)\,dW(t)$$

or in componentwise description

$$\begin{aligned} dX^1(t) \;&=\; \Big(-\frac{1}{2}\rho^2 X^1(t) - X^2(t)\Big)\,dt \;-\; \rho X^2(t)\,dW(t) \\ dX^2(t) \;&=\; \Big(X^1(t) - \frac{1}{2}\rho^2 X^2(t)\Big)\,dt \;+\; \rho X^1(t)\,dW(t)\quad. \end{aligned} \tag{4.6}$$

That is, in our system notation of drift and diffusion matrices we find

$$A \;=\; \begin{pmatrix} -\frac{1}{2}\rho^2 & -1 \\ +1 & -\frac{1}{2}\rho^2 \end{pmatrix} \quad\text{and}\quad B \;=\; \begin{pmatrix} 0 & -\rho \\ \rho & 0 \end{pmatrix}\,. \tag{4.7}$$

For experiments we choose the following three methods applied to the model system 4.6 with $\rho \neq 0$:

(i) The Euler method with implicitness $\alpha = \frac{1}{2}$

$$\begin{aligned} Y_{n+1}^{(1)} &= Y_n^{(1)} - \frac{1}{2}\left(\frac{1}{2}\rho^2(Y_{n+1}^{(1)} + Y_n^{(1)}) + Y_{n+1}^{(2)} + Y_n^{(2)}\right)\Delta - \rho Y_n^{(2)}\Delta W_n \\ Y_{n+1}^{(2)} &= Y_n^{(2)} + \frac{1}{2}\left(Y_{n+1}^{(1)} + Y_n^{(1)} - \frac{1}{2}\rho^2(Y_{n+1}^{(2)} + Y_n^{(2)})\right)\Delta + \rho Y_n^{(1)}\Delta W_n \end{aligned}$$

(ii) The Mil'shtein method with implicitness $\alpha = \frac{1}{2}$

$$\begin{aligned} Y_{n+1}^{(1)} &= Y_n^{(1)} - \frac{1}{2}\left(\frac{1}{2}\rho^2(Y_{n+1}^{(1)} + Y_n^{(1)}) + Y_{n+1}^{(2)} + Y_n^{(2)}\right)\Delta - \rho Y_n^{(2)}\Delta W_n \\ &\quad -\frac{1}{2}\rho^2 Y_n^{(1)}((\Delta W_n)^2 - \Delta) \\ Y_{n+1}^{(2)} &= Y_n^{(2)} + \frac{1}{2}\left(Y_{n+1}^{(1)} + Y_n^{(1)} - \frac{1}{2}\rho^2(Y_{n+1}^{(2)} + Y_n^{(2)})\right)\Delta + \rho Y_n^{(1)}\Delta W_n \\ &\quad -\frac{1}{2}\rho^2 Y_n^{(2)}((\Delta W_n)^2 - \Delta) \end{aligned}$$

(iii) The Balanced implicit method (cf. [59])

$$Y_{n+1} = Y_n + (I + c^0\Delta + c^1|\Delta W_n|)^{-1}(A\,Y_n\,\Delta + B\,Y_n\,\Delta W_n)$$

in vector resp. matrice notation with weight–matrices

$$c^0 = \begin{pmatrix} \frac{1}{4}\rho^2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4}\rho^2 \end{pmatrix} \quad \text{or} \quad c^0 = \begin{pmatrix} \frac{1}{4}\rho^2 + \frac{1}{\rho^2} & 0 \\ 0 & \frac{1}{4}\rho^2 + \frac{1}{\rho^2} \end{pmatrix} \quad \text{and} \quad c^1 \equiv 0\,.$$

For a little more detailed explanation and references for method (iii) we refer to section 8. Method (iii) using the first choice of c^0 is identical with method (i) for this model. Thus, in the following simulations we will draw more attention to the second form of the balanced method stated in (iii), just the balanced method with the 'pure–diagonal correction' c^0.

Methods (i) and (iii) remain close to the circle with radius $\|x_0\|$ and do not provide asymptotically mean square stable numerical solutions. It seems that they possess no 'explosions in mean square sense'. Thus, they replicate accurately enough the behaviour of the norm of the exact solution of model 4.4. In contrast to them the method (ii) produces mean square instable numerical solutions for all step sizes (except for one step size!). Even for the 'completely drift–implicit' Mil'shtein method applied to system 4.4 it is not getting better. This can be theoretically predicted by so–called stability indicators, cf. [71].

To confirm the statements above by experiments with methods (i) - (iii) we plotted estimates for the second moments $\mathbb{E}\|Y_n\|^2$ at the time points τ_n on the time interval $[0,1]$ and interpolated linearly the data to be visualized. The corresponding results are visible in figure 2. There the dotted line corresponds to the exact level to be expected trivially at the height 1.0. Distinctly, the methods (i) and (iii) provide better approximations concerning the mean square evolution. They are able to control the second moment much longer than the method (ii). Moreover, the second moment of the implicit Mil'shtein approximation even seems to 'explode'. Of course, the difference depends on the amount of the parameter ρ, but is still observable for a quite large range of these parameters. Note that these experiments bear more experimental character, demonstrating some numerical effects (Explicit solution is known here) which may occur in quite more general models.

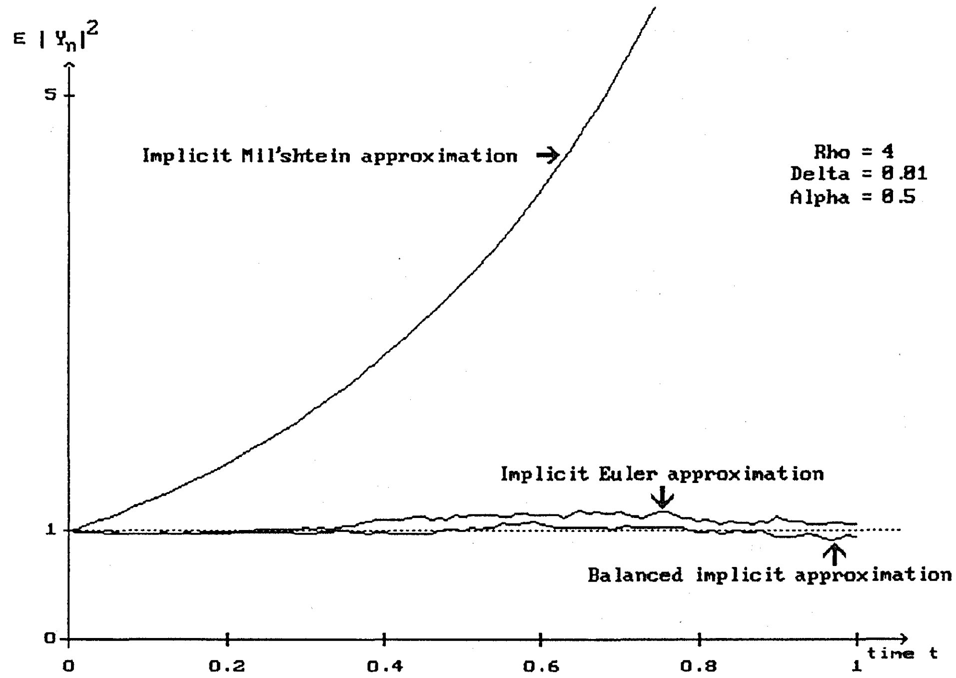

Figure 2. Estimates for the mean square evolution of Kubo oscillator approximated by methods (i) – (iii) using step size $\Delta = 10^{-2}$ with $\rho = 4$ started in $(1,0)$.

Additionally, for the special model 4.4 one could be tricky. Concerning the information $\|X(t)\| = \|x_0\|$ for all $t \in [0,T]$, by the use of $\hat{Y}_n := \|x_0\| Y_n / \|Y_n\|$ as the approximation value for the Kubo oscillator at the time τ_n the condition $\|\hat{Y}(\tau_n)\| = \|x_0\|$ would be trivially fulfilled. However, it assumes that $\|X(t)\|$ is always a constant (above all the information about it). Consequently, only in movements on the circle one can apply this trick. In practice of more general modells, such a 'normalization' does not help. Moreover, it would ruin the 'goodness' of the approximation.

5. Mean square stability of (2.2) solving autonomous systems (2.3)

In fact, an interesting result concerning mean square stability for the implicit Euler methods 2.2 can be formulated. To simplify the considerations we restrict our attention to autonomous systems, i.e. systems 2.2 and 2.3 with time–independent matrices A and B^j. Furthermore, the validity of relation 2.7 is required. Thereby we examine systems 2.2 where the corresponding differential equation 2.3 has a mean square stable null solution. Assume that all real parts of eigenvalues of matrix A are negative. This requirement is necessary for mean square stability of the null solution of system 2.3. It additionally implies the existence of inverse of $I - \alpha\Delta A$ for all $\alpha \geq 0$ and $\Delta > 0$. For such autonomous systems, the notation

$$\begin{aligned} Y_{n+1} &= (I-\alpha\Delta A)^{-1}(I+(1-\alpha)\Delta A+\sum_{j=1}^{m} B^j\sqrt{\Delta}\xi_n^j)\,Y_n \\ &= C^{-1}(\alpha)(C(\alpha)+\Delta A+\sqrt{\Delta}\sum_{j=1}^{m} B^j\xi_n^j)\,Y_n \end{aligned} \tag{5.1}$$

is used for the Euler method with implicitness $\alpha \in [0,1]$.
Let $\overset{\circ}{\mathbb{S}}{}^+$ be the interior of the set $\mathbb{S}^+$, hence $\overset{\circ}{\mathbb{S}}{}^+$ is identical with the open set of positive definite matrices in $\mathbb{S}$. Besides, $\mathcal{H}^{-1}(-\overset{\circ}{\mathbb{S}}{}^+)$ denotes the inverse image of the negative definite matrices with respect to the operator $\mathcal{H}$ defined by 2.7 on $\mathbb{S}^+$. The system of second moments $P_n = (p^n_{i,j}) = (\mathbb{E}\, Y_n^i Y_n^j)$ of method 5.1 satisfies the inequality

$$\begin{aligned} \mathcal{O} \leq_{(+)} P_{n+1} &= \mathbb{E}\, Y_{n+1}Y_{n+1}^T = (\mathbb{E}\, Y_{n+1}^i Y_{n+1}^j) = \mathcal{L}\, P_n \\ &= C^{-1}(\alpha)\Big(P_n+(1-\alpha)^2\Delta^2 AP_nA^T+(1-\alpha)\Delta(AP_n+P_nA^T) \\ &\qquad +\Delta\sum_{j=1}^{m} B^jP_n\, {B^j}^T\Big)\, {C^{-1}}^T(\alpha) \\ &= C^{-1}(\alpha)(P_n+\alpha^2\Delta^2AP_nA^T-\alpha\Delta(AP_n+P_nA^T)){C^{-1}}^T(\alpha) \\ &\qquad +\Delta C^{-1}(\alpha)(AP_n+P_nA^T+\sum_{j=1}^{m} B^jP_n{B^j}^T){C^{-1}}^T(\alpha) \\ &\qquad +(1-2\alpha)\Delta^2C^{-1}(\alpha)AP_nA^T{C^{-1}}^T(\alpha) \\ &= C^{-1}(\alpha)\Big(C(\alpha)P_nC^T(\alpha)-\Delta Q+(1-2\alpha)\Delta^2AP_nA^T\Big)\,{C^{-1}}^T(\alpha) \\ &<_{(+)} P_n+(1-2\alpha)\Delta^2C^{-1}(\alpha)AP_nA^T{C^{-1}}^T(\alpha) \end{aligned} \tag{5.2}$$

provided that $P_n \in \mathcal{H}^{-1}(-\overset{\circ}{\mathbb{S}}{}^+)$. where the positive definiteness of matrix Q follows from the condition of mean square stability for 2.3 stated in relation 2.7. The inequality in 5.2 is understood once again in terms of positive definiteness of the $d\times d$–matrices S, i.e. $0 < x^TS_1x < x^TS_2x \iff S_1 <_{(+)} S_2$ holds for all vectors $x \in \mathbb{R}^d$. Suppose that system 5.1 starts with a positive definite matrix of second moments $(\mathbb{E}\, Y_0^iY_0^j)$ what is naturally fulfilled for mean square stable systems 2.3.

Now, if one chooses $\alpha \geq \frac{1}{2}$ the relation

$$\mathcal{O} \leq_{(+)} \mathcal{L} P_n = P_{n+1} <_{(+)} P_n <_{(+)} P_0 \qquad (P_0 \in \mathcal{H}^{-1}(-\overset{\circ}{\mathbb{S}}{}^{+})) \tag{5.3}$$

follows for all $n = 1, 2, \ldots$. Consequently, $\lim_{k\to\infty} \mathcal{L}P_k <_{(+)} \mathcal{L}P_0 <_{(+)} P_0$ is valid. The limit moment matrix $P := \lim_{k\to\infty} \mathcal{L}^k P_0$ must be positive semi–definite because the space $\mathbb{S}^+$ of positive semi–definite $d \times d$–matrices is closed. Therefore we obtain

$$x^T P x < x^T P_n x < x^T P_0 x \qquad (P_0 \in \mathcal{H}^{-1}(-\overset{\circ}{\mathbb{S}}{}^{+}))$$

for all vectors $x \in \mathbb{R}^d$.
Consider now the operator $\mathcal{L}$ on the set $\mathbb{K}(S)$ in $\mathbb{S}^+$ defined by

$$\mathbb{K}(S) = \{S, \mathcal{L}\,S, \mathcal{L}^2\,S, \mathcal{L}^3\,S, \ldots, \mathcal{L}^n\,S, \ldots, \lim_{k\to\infty} \mathcal{L}^k\,S\} \subset \mathcal{H}^{-1}(-\mathbb{S}^+) \subset \mathbb{S}^+ \tag{5.4}$$

for a fixed positive definite matrix $S \in \mathcal{H}^{-1}(-\overset{\circ}{\mathbb{S}}{}^{+})$. This set is naturally closed by its limits. In the following we argue with the principle of Banach–Caccioppoli (cf. Theorem 2 (1.XVI) proved in Kantorowitsch and Akilow [38], p. 512). To make use of it, we recall that the operator $\mathcal{L}$ is linear and bounded, hence continuous on the finite–dimensional space $\mathbb{S}^+$. Subsequences of $(\mathcal{L}^n\,S)_{n\in\mathbb{N}}$ must converge to a limit $\lim_{k\to\infty} \mathcal{L}^k\,S$ in $\mathbb{K}$. It is relative easy to see that such limits are fixpoints of the operator $\mathcal{L}$. Furthermore we know the zero matrix $\mathcal{O}$ as trivial fixpoint of $\mathcal{L}$. Now we show the uniqueness of the fixpoint on the set $\mathbb{K}$. For this purpose the metric $\rho(.,.)$ with

$$\rho(S_1, S_2) \;=\; \sum_{j=1}^{d} |f_j^T(S_1 - S_2)f_j| \tag{5.5}$$

is introduced on the space $\mathbb{S}^+$ for a basis $(f_j)_{j=1,\ldots,d}$ of $\mathbb{R}^d$. Then it holds

$$\rho(\mathcal{L}\,S_1, \mathcal{L}\,S_2) \quad < \quad \rho(S_1, S_2) \tag{5.6}$$

for all $S_1, S_2 \in \mathbb{K}$ with $S_1 \neq S_2$. $\mathcal{L}\,\mathbb{K}$, the image of the compact set $\mathbb{K}$, is again compact and $\mathcal{L}\,\mathbb{K} \subset \mathbb{K}$, i.e. the operator $\mathcal{L}$ itself is compact (hence fully–continuous). Thus, with 5.6 and compactness of $\mathcal{L}\,\mathbb{K}$ the assumptions of Theorem $2(1.XVI)$ from [38] are satisfied. From this the uniqueness of the fixpoint on $\mathbb{K}$ follows as well as that any iteration $\mathcal{L}^n\,S$ must converge to this unique fixpoint. Additionally, this fixpoint is given by the zero matrix $\mathcal{O} \in \mathbb{S}^+$. This can be seen from the inequality

$$P \;=\; \mathcal{L}\,P \;=\; P + \Delta C^{-1}(\alpha)\Big(\mathcal{H}\,P + (1-2\alpha)\Delta A P A^T\Big){C^{-1}}^T(\alpha) \;\leq_{(+)} P$$

which is true for all $\Delta > 0$, and the conclusion $\mathcal{H}\,P = \mathcal{O}$ for $P = \lim_{k\to\infty} \mathcal{L}^k\,P_0$. Note, by theorem 2.1 operator $\mathcal{H}$ has a continuous inverse on $\mathcal{H}^{-1}(-\mathbb{S}^+)$. Hence, it holds

$$ker(\mathcal{H}) := \{S \in \mathcal{H}^{-1}(-\mathbb{S}^+) \,:\, \mathcal{H}\,S = \mathcal{O}\} = \{\mathcal{O}\}\,.$$

Thus, from $\mathcal{H}\,P \leq_{(+)} \mathcal{O}$ must follow $P = \mathcal{O}$. Consequently, the asymptotical mean square stability of the null solution of the implicit Euler method using implicitness $\alpha \geq 0.5$ is obvious, hence the following assertion has been proved.

Theorem 5.1. *Assume that the null solution is mean square stable for the autonomous stochastic differential equation* 2.3, *i.e. with constant matrices A and B^j (independent of time).*
Then the implicit Euler method 5.1 *with $\alpha \geq \frac{1}{2}$ possesses a mean square stable null solution provided that it starts with a positive definite initial moment matrix $P_0 = \mathbb{E}\, Y_0 Y_0^T \in \mathcal{H}^{-1}(-\mathbb{S}^+)$ (the image of the inverse of operator $\mathcal{H}$).*

Therefore we know numerical methods ($\alpha \geq \frac{1}{2}$) which provide mean square stable solutions under appropriate conditions. Moreover, the result we verified above says that independently of the size of coefficients of the initial matrix P_0 any sequence $(\mathcal{L}^n P_0)$ must converge against the fixpoint $\mathcal{O}$ while 2.6. Such a property is called stability in large. Furthermore, it has been proven due to this theorem that for linear systems there is no need to correct the Euler method by stochastic weights in the Balanced methods. These numerical methods were introduced in Mil'shtein et al. [59] and give alternative means of achieving numerical stability. A similar result to Theorem 5.1 could be simultaneously formulated by Artemiev [6]. In [71] one also finds this result for one-dimensional linear complex equations which are numerically solved by Balanced methods.

6. Monotonic nesting principle of mean square stability domains

In this section we present a further interesting result for linear autonomous systems. For the numerical methods defined by the iterated linear systems 2.1 and 2.2, the property of **'monotonic nesting'** of the sequel of mean square stability domains $(\Gamma_\alpha)_{\alpha \geq 0}$ is discovered, i.e. if $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ then the mean square stability domain Γ_{α_2} includes the domain Γ_{α_1}. These domains can be expressed via the operator $\mathcal{L}$ having eigenvalues smaller than one. The following result is established.

Theorem 6.1. *Consider autonomous system* 2.1 *or* 2.2 *with its mean square operator $\mathcal{L}_\alpha$. Then, for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$ the implication*

$$\mathcal{L}_{\alpha_1} P \;\leq_{(+)}\; P \quad\Longrightarrow\quad \mathcal{L}_{\alpha_2} P \;\leq_{(+)}\; P \tag{6.1}$$

holds, provided that $P \in \mathbb{S}^+$, and the matrix A has only eigenvalues with nonpositive real parts.

Proof. We prove the assertion only for the implicit Euler methods. The proof for the implicit Mil'shtein methods follows similarily. From section 5 we recall that

$$\begin{aligned}\mathcal{L}_\alpha P_n^{(\alpha)} &= P_{n+1}^{(\alpha)} \\ &= C^{-1}(\alpha)(I+(1-\alpha)\Delta A)P_n^{(\alpha)}(I+(1-\alpha)\Delta A)^T {C^{-1}}^T(\alpha) \\ &\quad +\Delta\sum_{j=1}^{m} C^{-1}(\alpha)B^j P_n^{(\alpha)} {B^j}^T {C^{-1}}^T(\alpha)\end{aligned} \tag{6.2}$$

with $\mathcal{L}_\alpha \mathbb{S}^+ \subseteq \mathbb{S}^+$ is valid for the implicit Euler method 2.2 with $C(\alpha) = I - \alpha\Delta A$. Now, suppose that $\mathcal{L}_\alpha P \leq_{(+)} P$ holds for any $P \in \mathbb{S}^+$ in the sense of positive semi-definite matrices. Then

$$\begin{aligned}(I-\alpha\Delta A)(\mathcal{L}_\alpha P - P)(I-\alpha\Delta A)^T &= C(\alpha)\,(\mathcal{L}_\alpha P - P)\,C^T(\alpha)\\ &= (I+(1-\alpha)\Delta A)P(I+(1-\alpha)\Delta A)^T + \Delta\sum_{j=1}^{m} B^j P B^{j^T} - C(\alpha)PC^T(\alpha)\\ &= \Delta(AP + PA^T + \sum_{j=1}^{m} B^j P B^{j^T}) + (1-2\alpha)\Delta^2 APA^T \leq_{(+)} \mathcal{O} \qquad (6.3)\end{aligned}$$

follows. Assume $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. From this we conclude that

$$\begin{aligned}&(I-\alpha_2\Delta A)(\mathcal{L}_{\alpha_2} P - P)(I-\alpha_2\Delta A)^T\\ &\quad= \Delta(AP + PA^T + \sum_{j=1}^{m} B^j P B^{j^T}) + (1-2\alpha_2)\Delta^2 APA^T\\ &\quad\leq_{(+)} \Delta(AP + PA^T + \sum_{j=1}^{m} B^j P B^{j^T}) + (1-2\alpha_1)\Delta^2 APA^T \qquad (6.4)\\ &\quad= (I-\alpha_1\Delta A)(\mathcal{L}_{\alpha_1} P - P)(I-\alpha_1\Delta A)^T \leq_{(+)} \mathcal{O} \qquad (6.5)\end{aligned}$$

because of positive semi-definiteness of APA^T and the relation $(1-2\alpha_2) \leq (1-2\alpha_1)$. Thereby, we obtain $(I-\alpha_2\Delta A)(\mathcal{L}_{\alpha_2}P - P)(I-\alpha_2\Delta A)^T \leq_{(+)} \mathcal{O}$, what the relation $\mathcal{L}_{\alpha_2}P \leq_{(+)} P$ implies. □

Remarks. The proof of Theorem 6.1 is essentially based on the fact that transformations CPC^T with any invertible matrix C do not change the positive or negative semi-definiteness of matrices. See, e.g. Usmani [84]. Consequently, the 'most stable null solution in mean square sense' is provided by the completely drift–implicit Euler method (Mil'shtein method) with $\alpha = 1$ within the class of implicit Euler methods (Mil'shtein methods, resp.) with implicitness $\alpha \in [0,1]$, at least for linear autonomous systems. The assertion of the Theorem 6.1 can be carried over to nonautonomous systems if one additionally requires monotonically decreasing real parts of eigenvalues of negative semi-definite matrices A_n or $A(t)$, respectively, as well as this is possible for Theorem 5.1.

7. A PRACTICABLE ALTERNATIVE - BIMS WITH SCALAR CORRECTION

To simplify the implementation of mean square stable numerical methods we proceed on a little with the examination of a more general class of implicit methods, the Balanced implicit methods (BIMs) introduced firstly in Mil'shtein et al. [59]. BIMs allow to introduce even implicitness making use of stochasticity. To achieve control on the moment evolution, to bound the increments or to effect some positivity of numerical solutions they are appropriate numerical methods (cf. also a forthcoming paper of the author about properties of BIMs). We have already examined such methods with the implicit Euler methods and seen that they possess mean square stable null solutions if one appropriately chooses the implicitness degree. BIMs provide a numerical solution which is weakly and strongly converging to the exact solution under appropriate conditions on their correction weights. For proof, see [59]. However, they represent nothing else than some 'linearly corrected Euler methods'. For the sake of simplification and reduction of computational efforts in scheme evaluation while keeping mean square stability control, the further main attention is drawn to a simple subclass of BIMs. It only uses purely deterministic weights with nonnegative scalars. Thus one avoids the costly inversion of matrices. For equidistant time step size Δ this class has the scheme

$$\begin{aligned} Y_{n+1}^{(B)} &= Y_n^{(B)} + AY_n^{(B)}\Delta + \sum_{j=1}^{m} B^j Y_n^{(B)}\Delta W_n^j + (Y_n^{(B)} - Y_{n+1}^{(B)})\alpha_n\Delta \\ &= (1+\alpha_n\Delta)^{-1}(I(1+\alpha_n\Delta) + A\Delta + \sum_{j=1}^{m} B^j\Delta W_n^j)\, Y_n^{(B)} \end{aligned} \tag{7.1}$$

for the bilinear SDE 2.3. The numbers $\alpha_n \geq 0$ represent the sequence of implicitness degrees to be choosen appropriately. Similarly to previous sections we describe the mean square evolution by operator $\mathcal{L}$ for BIM 7.1. Considering autonomous systems 2.3 one encounters with the following assertion.

Theorem 7.1. *Assume that the autonomous system* 2.3 *possesses a mean square stable null solution, and the condition*

$$\exists \hat{\alpha} \in \mathbb{R}^+ \, : \, \alpha_n \geq \hat{\alpha}\ \forall n \ \wedge\ \hat{\alpha}(AS + SA^T) + ASA^T \leq_{(+)} \mathcal{O}, \quad S \in \mathbb{S}^+ \tag{7.2}$$

is satisfied for the sequence $(\alpha_n)_{n=0,1,2,...}$ *of positive reals* α_n.
Then it holds

(*i*) $\mathcal{L}_\alpha^{(B)} S <_{(+)} S$ *if* $S \in \mathcal{H}^{-1}(-\mathring{\mathbb{S}}^+), \alpha \geq \hat{\alpha} > 0$,

(*ii*) *The operator* $\mathcal{L}_\alpha^{(B)} : \mathbb{S}^+ \longrightarrow \mathbb{S}^+$ *only has the fixpoint* $\mathcal{O}$ *on* $\mathbb{K}$ *defined as in* 5.4 *if* $\alpha \geq \hat{\alpha}$, *hence under* 2.7 *the BIMs* 7.1 *with positive scalar correction possess a mean square stable null solution,*

(*iii*) $\mathcal{L}_{\alpha_1}^{(B)} S \leq_{(+)} S \Longrightarrow \mathcal{L}_{\alpha_2}^{(B)} S \leq_{(+)} S, \quad S \in \mathbb{S}^+, \quad 0 \leq \alpha_1 \leq \alpha_2,$

(*iv*) $\mathcal{L}_{\alpha_B}^{(E)} S \leq_{(+)} S \Longrightarrow \mathcal{L}_{\alpha_B + 2\alpha_B\hat{\alpha}}^{(B)} S \leq_{(+)} S, \quad S \in \mathbb{S}^+, \quad \alpha_E, \alpha_B \in [0, \infty)$

(*v*) $\mathcal{L}_{\|A\|/2+\alpha}^{(B)} S <_{(+)} S, \quad S \in \mathcal{H}^{-1}(-\mathring{\mathbb{S}}^+), \quad \alpha \geq 0,$

(*vi*) $\Gamma_{\|A\|/2+\alpha}^{(B),\Delta} := \{S \in \mathbb{S}^+ \, : \, \mathcal{L}_{\|A\|/2+\alpha}^{(B)} S <_{(+)} S\} \supset \mathcal{H}^{-1}(-\mathring{\mathbb{S}}^+), \alpha \geq 0$

where $\mathcal{L}_\alpha^{(B)}$ *denotes the mean square stability operator corresponding to BIM* 7.1 *and* $\mathcal{L}_\alpha^{(E)}$ *the operator corresponding to the implicit Euler method* 2.2 *as stated in* 6.2 *in the previous section.*

Proof. It is not hard to verify that

$$\mathcal{L}_{\alpha_n}^{(B)} S = S + \Delta\, c_n^{-2} \left(AS + SA^T + \sum_{j=1}^{m} B^j S {B^j}^T + \Delta\Big(\alpha_n(AS+SA^T) + ASA^T\Big)\right) \tag{7.3}$$

for the methods 7.1 with $c_n = 1 + \alpha_n \Delta$, $S \in \mathbb{S}^+$ and $\alpha_n \geq 0$, while

$$P_{n+1}^{(B)} = \mathbb{E}\, Y_{n+1}^{(B)} {Y_{n+1}^{(B)}}^T = \mathcal{L}_{\alpha_n}^{(B)} \left(\mathcal{L}_{\alpha_{n-1}}^{(B)} \left(\ldots \mathcal{L}_{\alpha_0}^{(B)} \left(\mathbb{E}\, Y_0^{(B)} {Y_0^{(B)}}^T \right) \ldots \right)\right).$$

The expression $AS + SA^T + \sum_{j=1}^m B^j S {B^j}^T$ in 7.3 must be negative definite because of Theorem 2.1 due to Khas'minskij. Furthermore, $\alpha_n(AS + SA^T) + ASA^T$ is negative semi–definite for $S \in \mathbb{S}^+$, as assumed in 7.2. Thus, the expression $\mathcal{L}_{\alpha}^{(B)} S - S$ can only be a negative definite matrix in $\mathcal{H}^{-1}(-\overset{\circ}{\mathbb{S}}{}^+)$, and hence item (i) is true. Using the same fixpoint arguments on the set $\mathbb{K}$ like in section 5, the mean square stability of the null solution for BIMs 7.1 satisfying 7.2, and hence (ii) turns out to be obvious. For the verification of (iii) one encounters with the inequality

$$\begin{aligned}
\mathcal{O} &\geq_{(+)} (1+\alpha_1\Delta)\left(\mathcal{L}_{\alpha_1}^{(B)} S - S\right)(1+\alpha_1\Delta) \\
&= \Delta(AS + SA^T + \sum_{j=1}^{m} B^j S {B^j}^T) + \Delta^2(\alpha_1(AS+SA^T) + ASA^T) \\
&\geq_{(+)} \Delta(AS + SA^T + \sum_{j=1}^{m} B^j S {B^j}^T) + \Delta^2(\alpha_2(AS+SA^T) + ASA^T) \\
&= (1+\alpha_2\Delta)\left(\mathcal{L}_{\alpha_2}^{(B)} S - S\right)(1+\alpha_2\Delta)
\end{aligned}$$

provided that $\mathcal{L}_{\alpha_1}^{(B)} S \leq_{(+)} S$. Thus, it follows the validity of (iii). Assertion (iv) is confirmed by the relation

$$\begin{aligned}
&(I - \alpha_E \Delta A)\left(\mathcal{L}_{\alpha_E}^{(E)} S - S\right)(I - \alpha_E\Delta A)^T \\
&= \Delta\Big(AS + SA^T + \sum_{j=1}^{m} B^j S {B^j}^T + (1-2\alpha_E)\Delta ASA^T\Big) \\
&\geq_{(+)} \Delta\Big(AS + SA^T + \sum_{j=1}^{m} B^j S {B^j}^T + \Delta\big(ASA^T + (\alpha_B + 2\alpha_E\hat{\alpha})(AS+SA^T) \\
&\qquad\qquad -2\alpha_E(\hat{\alpha}(AS+SA^T) + ASA^T)\big)\Big) \\
&\geq_{(+)} \Delta\left(AS + SA^T + \sum_{j=1}^{m} B^j S {B^j}^T + \Delta\big(ASA^T + (\alpha_B + 2\alpha_E\hat{\alpha})(AS+SA^T)\big)\right) \\
&= (1 + (\alpha_B + 2\alpha_E\hat{\alpha})\Delta)^2\left(\mathcal{L}_{(\alpha_B+2\alpha_E\hat{\alpha})}^{(B)} S - S\right)
\end{aligned}$$

under the requirement 7.2. Finally, (v) and (vi) are obvious and follow from (i) with the special choice $\hat{\alpha} = \|A\|/2$ directly. Consequently, the verification of Theorem 7.1 has completed. □

Remarks. BIMs 7.1 represent an useful alternative to implicit Euler methods examined in the previous sections for achieving control on the mean square evolution of the numerical solution of SDEs. In fact Theorem 7.1 shows that the costly matrix inversion in the implicit Euler methods can be circumvented by BIMs with simple scalar correction without loosing mean square stability control. For this purpose one takes BIMs 7.1 with the choice $\alpha_n \geq \hat{\alpha} \geq \|A\|/2$, and the requirement 7.2 is naturally fulfilled under the assumption 2.7. Thereby their application only assumes the knowledge about the norm of matrix A or its estimate. Additionally item (iii) of the theorem above reflects the 'monotonic nesting property' of mean square stability domains for BIMs 7.1, whereas (iv) the dominating role of these numerical methods in comparison with methods of lower convergence order with respect to their mean square stability behaviour. Consequently we have found simply implementable and implicit methods to be efficient for mean square control. In passing, we note that BIMs and the Euler method strongly converge with the same order $\gamma = 0.5$ (i.e. in L^1), see [59]. Thus, roughly speaking, BIMs can achieve a balance between mean square stability and convergence requirements on numerical methods.

8. An application and experiments for the noisy Brusselator

For modelling unforced periodic oscillations in chemical reactions it is common to use the model of the Brusselator. Therein, after neglecting spatial diffusion and centering at an equilibrium point, the following system of deterministic nonlinear equations (planar Brusselator) occurs

$$\begin{aligned} \frac{dx_1}{dt} &= (\hat{a}-1)x_1 + \hat{a}x_1^2 + (x_1+1)^2 x_2 \\ \frac{dx_2}{dt} &= -\hat{a}x_1 - \hat{a}x_1^2 - (x_1+1)^2 x_2 \end{aligned} \tag{8.1}$$

where $\hat{a}$ is a positive real parameter. When $\hat{a} < 2$ the zero solution $(x_1, x_2) = (0,0)$ is globally asymptotically stable, but looses stability in a Hopf bifurcation point $\hat{a} = 2$. The system also possesses a limit cycle for $\hat{a} > 2$.
Now we are especially interested in stability results under stochastic perturbations. When the parameter $\hat{a}$ is stochastically perturbed one encounters with

$$dX_t^{(1)} = \Big((a-1)X_t^{(1)} + a(X_t^{(1)})^2 + (X_t^{(1)}+1)^2 X_t^{(2)}\Big)\, dt + \sigma X_t^{(1)}(1+X_t^{(1)}) \circ dW_t$$

$$dX_t^{(2)} = \Big(-aX_t^{(1)} - a(X_t^{(1)})^2 - (X_t^{(1)}+1)^2 X_t^{(2)}\Big) dt - \sigma X_t^{(1)}(1+X_t^{(1)}) \circ dW_t \tag{8.2}$$

where $\hat{a} = a + \sigma\xi_t$. ξ_t represents Gaussian white noise, i.e. $W_t = \int_0^t \xi_s ds$. We deliberately interprete the system 8.2 in Stratonovich sense, as it is preferable in practical modelling, cf. also Wong and Zakai [88]. Ehrhardt [18] has already considered the noisy Brusselator equations. We continue his considerations with some numerical stability analysis for the Brusselator model. Obviously $(X_t^{(1)}, X_t^{(2)}) = (0,0)$ is an equilibrium of the stochastic system too. After linearizing system 8.2 at this steady state one obtains

$$\begin{pmatrix} dZ_t^{(1)} \\ dZ_t^{(2)} \end{pmatrix} = \begin{bmatrix} a-1 & 1 \\ -a & -1 \end{bmatrix} \begin{pmatrix} Z_t^{(1)} \\ Z_t^{(2)} \end{pmatrix} dt + \sigma \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} Z_t^{(1)} \\ Z_t^{(2)} \end{pmatrix} \circ dW_t \tag{8.3}$$

which can be equivalently rewritten to its Itô prescription

$$\begin{pmatrix} dZ_t^{(1)} \\ dZ_t^{(2)} \end{pmatrix} = \begin{bmatrix} a-1+\frac{\sigma^2}{2} & 1 \\ -a-\frac{\sigma^2}{2} & -1 \end{bmatrix} \begin{pmatrix} Z_t^{(1)} \\ Z_t^{(2)} \end{pmatrix} dt + \sigma \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} Z_t^{(1)} \\ Z_t^{(2)} \end{pmatrix} dW_t\,. \tag{8.4}$$

For simplification we denote with A the drift matrix and with B the diffusion matrix of system 8.4.

At first we deal with the stability analysis of first and second moments of the exact solution of the linearization 8.4 of system 8.2. Then it follows the analysis of the numerical systems.

Stability of first moments. The stability analysis for the first moment of the linearized Brusselator equations is done through the examination of the characteristic polynomial of the drift matrix A. One encounters with its polynomial

$$c_A(\lambda) \;=\; det(A-\lambda I) \;=\; \lambda^2 - (a-2+\frac{\sigma^2}{2})\lambda + 1 \;=\; \lambda^2 + b_1\lambda + b_2\,. \tag{8.5}$$

Its roots only possess negative real parts iff $b_1 = 2 - a - \frac{\sigma^2}{2} > 0$. Thereby the linearized Brusselator has asymptocially stable first moments iff $a + \frac{\sigma^2}{2} < 2$. Thus it is easy to see that Stratonovich noise introduced above destabilizes the dynamical behaviour in comparison with the deterministic linearization.

Stability of second moments. Following the school of Arnold and Khas'minskij we investigate the behaviour of the matrix differential equation

$$\dot{P} \;=\; AP + PA^T + BPB^T\,,\; P(0) = P_0 \in \mathbb{S}^+$$

for the system of second moments $P = (p_{i,j}(t)) = (\mathbb{E}Z^{(i)}Z^{(j)}(t))$. This system is equivalently rewritten to the three–dimensional vector differential equation

$$\dot{q} \;=\; \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} \;=\; \begin{bmatrix} 2(a-1+\sigma^2) & 0 & 2 \\ \sigma^2 & -2 & -2a-\sigma^2 \\ -a-\frac{3}{2}\sigma^2 & +1 & a-2+\frac{\sigma^2}{2} \end{bmatrix} q \;=:\; Q\,q \tag{8.6}$$

using the notation $q_1 = p_{1,1}, q_2 = p_{1,2}$ and $q_3 = p_{2,2}$. Thereby it remains to analyze the eigenvalues of matrix Q defined in 8.6. One obtains the characteristic polynomial

$$c_Q(\lambda) \;=\; det(Q-\lambda I) \;=\; \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 \tag{8.7}$$

where $b_1 = 6-(3a+\frac{5}{2}\sigma^2)$, $b_2 = 12-8a+2a^2+(3a-6)\sigma^2+\sigma^4$, $b_3 = 8-4(a+\sigma^2)$.

Applying the Routh–Hurwitz criterion to this polynomial, all real parts of its roots are negative iff

$$b_1 > 0, b_3 > 0 \quad \text{and} \quad \delta_2 = \begin{vmatrix} b_1 & b_3 \\ 1 & b_2 \end{vmatrix} = b_1b_2 - b_3 > 0\,. \tag{8.8}$$

It is clear that the behaviour of the Routh–Hurwitz determinant δ_2 plays the most important role for indicating mean square stability. Figure 3 shows several plots of the value of this determinant depending on the interaction of system parameters a and σ. There it is visible how increasing σ destabilizes the mean square behaviour of the linearized Brusselator system.

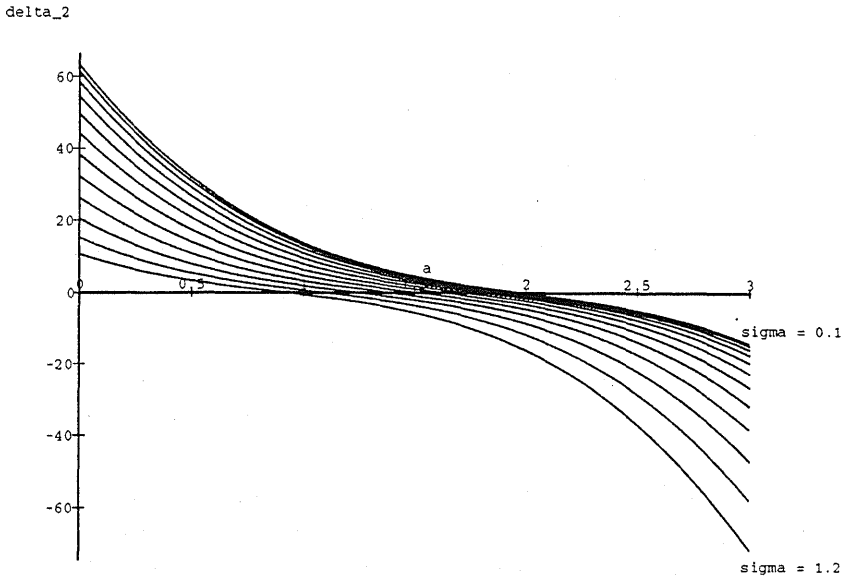

Figure 3. Dependence of Routh–Hurwitz determinant δ_2 on system parameters.

Numerical simulation and estimation of the Euclidean norm. The mean square stability of the described system can be indicated by the temporal behaviour of the Euclidean norm of the Brusselator process. Therefore we estimate this term in the following in order to demonstrate the effect of the results of the previous sections. For numerical simulation of this expression of second moments we make use of implicit Euler methods with implicitnesses $\alpha = 0, 0.5, 1$, just as well a Balanced method with scalar correction ($\alpha_B = \|A\|/2$). These methods are applied to the linearized Brusselator equations 8.3 with the same initial second moments and equidistant step size $\Delta = 0.01$. In figure 4 the estimates corresponding to the implicit Euler methods are plotted for increasing time t. Thus the growth of the estimates for decreasing implicitness α confirms the monotonicity of the mean square evolution. Furthermore one easily recognizes the destabilizing effect of Stratonovich noise in comparison with the deterministic evolution. In this figure the 'Balanced estimate' has not drawn to avoid confusions. By computer simulation it can be seen that it also declines for increasing time t, and it is close to the 'Euler estimates'. In passing we note that for $a = 1.7$ and $\sigma = 0.5$ the exact solution of the linearization possesses a mean square stable null solution, hence the Euclidean norm must decline. This can be easily checked with MAPLE or other algebraic packages. A similar effect of monotonicity of the estimates for the mean Euclidean norm could be observed with the implicit Mil'shtein methods. However these estimates are larger than those corresponding to the implicit Euler methods, cf. also [71].

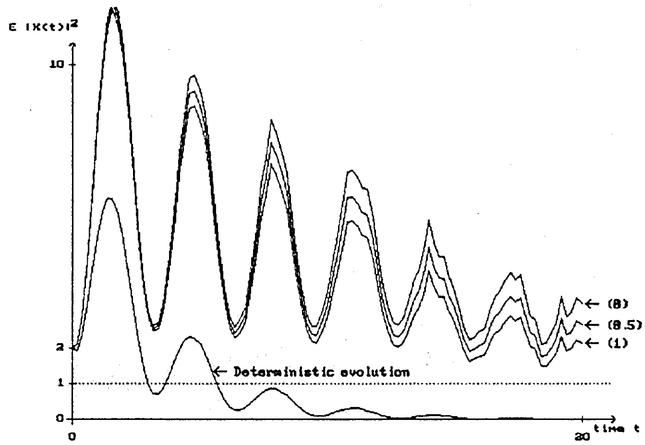

Figure 4. Estimates of the Euclidean norm of several 'Euler solutions'.

Now it could follow an application of theses methods to the nonlinear Brusselator system. However this would involve the algebraic resolution of implicit systems, e.g. by the Newton–Raphson iteration scheme at each time step. Instead of this, we prefer a Balanced method with simple scalar correction. In figure 5 the numerical results for the estimation of the Euclidean norm of the nonlinear Brusselator system is given for the same parameter choice as above. Thus we suspect that also under this kind of nonlinearity the system still behaves mean square stable.

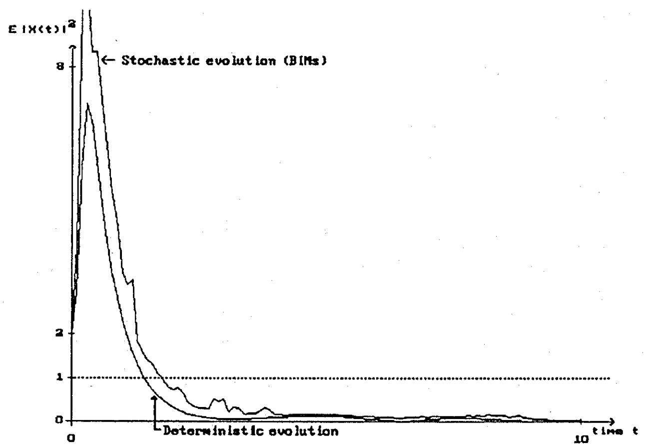

Figure 5. Evolution of the Euclidean norm of the nonlinear Brusselator in comparison with deterministic evolution.

9. Conclusions and remarks

The analysis of deterministic numerical methods is at a very high stage nowadays, cf. Hairer and Wanner [27]. In comparison with this state of the art the analysis of stochastic numerical methods possesses still 'infancy character'. Convergence results for them have been known fairly long since the early works of Maruyama and Mil'shtein, cf. also [44], but the knowledge about stability of these stochastic methods is particularly underdeveloped up to now. In so far this paper can be considered as a small contribution to explain the appearance of stability and instability of stochastic numerical methods. To some extent mean square stability analysis for methods with lower convergence order is now in a more satisfactory state, except for the problem of test equations to be discussed in the nearest future. The solution of the problem of stochastic test equations could considerably reduce the computational effort in stability examinations, cf. Dahlquist [16] for the solution of this problem in deterministic analysis. Although the hope to get completely solved this problem in stochastic analysis is little, even a partly answere on it could lead to decisive simplifications. However, to gain a satisfactory answere seems to be very complicated within the framework of stochastic analysis. In the one–dimensional situation we may refer to the Kubo oscillator perturbed by white noise for testing mean square stability behaviour of numerical methods.

While examing linear stochastic systems one encounters with multiplicative and additive noise. Purely multiplicative noise systems can have deterministic equilibria, whereas purely additive noise systems can possess stochastic equilibria. Thus, the stability investigation under purely additive noise really requires a new approach (more stochastically oriented). This fact has been already noted by Artemiev [5] (concept of asymptotical unbiasedness) and in [73] (concept of asymptotical preservation of probabilistic characteristics by numerical solutions).

We interpreted the stochastic systems in Itô sense. This assumption is not essential for the application of corresponding stability results, but note the alternative of Stratonovich interpretation can destabilize the dynamical behaviour. Of course, our results are applicable after transforming the Stratonovich-interpreted system into the corresponding equivalent Itô prescription.

This paper provides us with several conclusions arising from four main contributions to asymptotical mean square analysis of numerical methods for bilinear systems with purely multiplicative noise. By Theorem 3.1 one sees that a higher order method (higher convergence order) does not improve the mean square stability behaviour in comparison with that of a corresponding lower order method. Thus, it is not recommendable to look for a higher order mean square stable numerical solution before the class of lower order methods, such as implicit Euler or more general Balanced methods (see [59]) has not been carefully examined. The proof of Theorem 3.1 can be directly generalized to the case of weak approximations or weak numerical solutions, because it only uses the independence of the random variables ξ_n^i and $V_n^{j,k}$ for $i \neq j, k$, respectively, as well as some moment properties of these random variables. Furthermore, it should be possible to extend this result to other higher order both weak and strong numerical solutions arising from Taylor methods proposed in [44, 48, 58] in a similar way. Note, thereby it has not been

proven that mean square stability behaviour is always worsening with increasing convergence order in general.
A further basic result (Theorem 5.1) concerning mean square stability of the null solution of the implicit Euler method solving autonomous linear stochastic differential equations could be obtained. For $\alpha \in [\frac{1}{2}, 1]$ the Euler method possesses a mean square stable null solution under the assumption that the corresponding continuous linear system possesses one. That means, it is not necessary to add a stochastic term in Balanced methods proposed by Mil'shtein et al. [59] in order to achieve control in mean square sense. Note, the validity of this fact for simple linear complex–valued systems has been already shown in [71].
Furthermore, a 'monotonic nesting principle' of mean square stability domains (Theorem 6.1) could be verified. For increasing implicitness $\alpha \in [0, 1]$ the mean square stability domains Γ_α of the implicit Euler method as well as of the implicit Mil'shtein method increase monotonically. Thus, while assuming mean square stability of the corresponding continuous system the use of completely drift–implicit method ($\alpha = 1$) is recommended for those practical implementations where one wants to achieve more stable numerical behaviour than that of the exact solution. Note, for $\alpha = 0.5$ and autonomous systems we have just the situation that exponential mean square stability of the linear stochastic differential equation 2.3 is ensured iff the null solution is mean square stable for the corresponding Euler method 2.2. Thereby this, and hence a numerical method, can be used to test mean square stability of continuous time systems (a possibility for construction of mean square stability indicators!). Consequently, our first recommendation is to choose $\alpha = 0.5$ while using an implicit Euler method for linear systems.
In passing, Theorems 5.1 and 6.1 are also valid for $\alpha > 1$. This case covers a special class of Balanced methods, the class which does not use stochastic weights (a sum of matrices multiplied by the current absolute increment of the Wiener process), i.e. only with deterministic weight matrix $(-\alpha)A$ ($\alpha \geq 0$, provided that the matrix A is negative semi-definite). See in [59] for their structure. However, for very large $\alpha > 1$ one looses convergence speed, hence also the value of these approximations, and finally the numerical solution drifts away from the exact one while gaining numerical moment stability with increasing α. Thus, it exists the task to get the right balance between stability and convergence requirements on numerical methods in practical modelling.
Finally, for easier and practicable implementation of implicit methods, we presented a result (Theorem 7.1) on Balanced implicit methods (BIMs) with deterministic, positive scalar correction factor (weights). It turned out that they are appropriate for mean square stability control, i.e. an equilibrium point is mean square stable for these methods. For example, BIMs with scalar weights $c_n = 1 + \alpha_n \Delta_n$ where $\alpha_n \geq ||A||/2$. They avoid the problem of costly inversion of correction matrices or, in general, the resolution of implicit algebraic equations in numerical methods. Thus, such specific BIMs considerably reduce the computational effort while using implicit methods.
As it has been seen before, the paper enlightened the mean square stability analysis of some numerical solutions of stochastic differential equations by linearization around their equilibria. Although progress concerning mean square stability of dis-

crete linear systems could be made it is still necessary to extend the examination to nonlinear systems. However, the reader has already received recommendations through this paper in nonlinear situations. One linearizes the nonlinear equation, checks mean square stability of the obtained system, works out appropriate numerical solutions, and finally, one applies a corresponding numerical method (being preferable for the linearized equation) to the nonlinear system. This procedure has been demonstrated with the system of noisy Brusselator equations, as one example for nonlinear applications.

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A note of the author. This paper is a result of completely own work started in 1992, resulting in the papers [71, 72] in 1993 and summarized in the current version in order to document current author's view on mean square stability analysis of numerical methods with lower convergence order by linearizing them about stationary points. All ideas presented here are originally included in [71], where a simple complex–valued one–dimensional test equation has been considered. Eventually author's examinations end up in the today's paper to be placed in the Ph. D. thesis. During that time the author received no decisive assistance from any of the specialists, except for languagual improvements by **Dr. R. Funke** and **Prof. P. E. Kloeden**. Special thanks also to my colleague **Dipl.-math. N. Hofmann** who gave some useful comments and an opportunity for discussions. These latter explanations are not stated here for showing off or drawing attention to the author himself. It should be understood simply as an official statement and trial for protection of his authority against any unpolite and untrue claims of third persons.

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