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Random Walk on Fixed Spheres method for electro- and elastostatics problems

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Abstract

Stochastic algorithms for solving Dirichlet boundary value problems for the Laplace and Lamé equations governing 2D elasticity problems are developed. The approach presented is based on the Poisson integral formula written for each disc of a domain consisting of a family of overlapping discs. The original differential boundary value problem is reformulated in the form of equivalent system of integral equations defined on the intersection surfaces, i.e., arcs in 2D. A Random Walk algorithm can be applied then directly to the obtained system of integral equations where the random walks are living on the intersecting surfaces. We develop also a discrete random walk technique for solving the system of linear equations approximating the system of integral equations. We construct a randomized version of the successive over relaxation (SOR) method.

In [6] we have demonstrated that in the case of classical potential theory our method considerably improves the convergence rate of the standard Random Walk on Spheres method. In this paper we extend the algorithm to the system of Lamé equations which cannot be solved by the conventional Random Walk on Spheres method. Illustrating computations for 2D Laplace and Lamé equations, and comparative analysis of different stochastic algorithms are presented.

1. Introduction

Recently we have suggested a new version of the well known Random Walk on Spheres (RWS) method where the spheres of the random walk are prefixed. The method is obviously well adapted to domains which can be represented as a union of overlapping spheres. However the method is flexible, and often, quite general domains can be approximated by this kind of sphere families.

In this paper, we further develop the Random Walk on Fixed Spheres method (RWFS) for two dimensional Dirichlet problems for the Laplace and Lamé equations. The method differs from the conventional Random Walk on Spheres method (e.g., see [5]) in the following points: (1) The spheres are not randomly chosen, instead, they are deterministically fixed so that the original domain is well approximated by this set of spheres; (2) The randomized evaluation of the solution via the iterations of integrals follows not the Neumann Ulam scheme, but a different iterative method, e.g., the Chebyshev iterations, or the successive over relaxation (SOR) method; (3) Since the phase space is fixed, one may introduce its discretization, and construct discrete Random Walks. It should be noted that one can think of different combinations of (1)-(3), e.g., (1) means the Random Walk is constructed on the set of fixed spheres; (1) and (2) means that the relevant iteration procedure is constructed directly for integral operators, and finally, with (1) - (3) we turn to the discrete random walk approach. In [8], we have developed this approach by using the Poisson integral formula for overlapping spheres, and considered the relevant system of integral equations. The kernel of the Poisson integral formula was the generating transition probability density function of the Markov chain. The iterative procedure was actually a randomized method of simple iterations. Generally, this iterative procedure diverges in the case of Lamé equation (e.g., see [4],[5], [7]). Therefore, we suggest different iteration methods, in particular, a randomized version of SOR.

2. 2D Dirichlet problem and the Poisson kernel

For simplicity, we will explain here the main idea of the method for the two-dimensional Dirichlet problem for the Laplace equation.

2.1 Integral formulation for two discs

Let us consider the Dirichlet boundary value problem

$$\Delta u(x) = 0,\ x \in D, \quad u(y) = \varphi, \quad y \in \Gamma = \partial D, \tag{2.1}$$

where the domain D consists of two overlapping discs $K(x_0^{(1)}, R_1)$ and $K(x_0^{(2)}, R_2)$ centered at $O_1 = x_0^{(1)}$ and $O_2 = x_0^{(2)}$. We denote by γ_2 the part of the circle $S(x_0^{(1)}, R_1)$ which belongs to the second disc while Γ_1 is the part of the circle $S(x_0^{(1)}, R_1)$ not belonging to the second disc; analogously γ_2 and Γ_2 are defined. So the boundary of the domain D consists of Γ_1 and Γ_2, and $\gamma_1 \cup \gamma_2$ is the phase space of the integral equation to be constructed. We introduce also angles θ_1^* and θ_2^* as follows: $2\theta_1^*$ is the angle of view of the arc γ_2 from the center O_1, and $2\theta_2^*$ is the angle of view of the arc γ_1 from the center O_2.

The regular solution to the harmonic equation satisfies the spherical mean value relation in each of the two discs:

$$u(x) = \int\limits_{S(O_i,R)} p(y;x)\,u(y)dS_y\,, \quad p(y;x) = \frac{R^2 - r^2}{2\pi R} \cdot \frac{1}{|x-y|^2}\,. \tag{2.2}$$

Here $R = R_1$ in the first, and $R = R_2$ in the second disc, while $r = |x - x_0^{(i)}|$ is the distance from x to the circle's center, $i = 1,2$.

It is not difficult to find out that the function $p(y;x)$ is a probability density function of the variable $y \in S(x_0^{(1)}, R_1)$, for all $x \in K(x_0^{(1)}, R_1)$. This immediately follows from the representation of the solution $u = 1$ to the Dirichlet problem for the Laplace equation $\Delta u(x) = 0$, $u(y) = 1$ through the Poisson integral. From the probabilistic representation of the Dirichlet boundary value problem considered the density $p(y;x)$ coincides with the pdf of the first passage on $S(x_0^{(1)}, R)$ of a Wiener process starting at $x \in K(x_0^{(1)}, R)$.

Let us introduce the notation: $v_1(x) = u(x)$ for $x \in \gamma_1$, and $v_2(x) = u(x)$ for $x \in \gamma_2$. Then, (2.2) reads

$$v_1(x) = \int\limits_{\gamma_2} p(y;x)v_2(y)\,dS_y + f_1(x), \quad v_2(x) = \int\limits_{\gamma_1} p(x';x)v_1(x')\,dS_{x'} + f_2(x), \tag{2.3}$$

where $f_1(x) = \int\limits_{\Gamma_1} p(y,x)\varphi(y)\, dS_y, \quad f_2(x) = \int\limits_{\Gamma_2} p(x',x)\varphi(x')\, dS_{x'}$.

It is convenient to rewrite the system (2.3) in the matrix form $v = \mathbf{G}v + f$ where $v = (v_1, v_2)^T$, $f = (f_1, f_2)^T$, and $\mathbf{G}$ is the matrix-integral operator which acts on v as follows

$$\mathbf{G}v \quad = \quad \begin{pmatrix} 0 & G_{12} \\ G_{21} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \int\limits_{\gamma_2} p(y,x) v_2(y)\, dS_y \\ \int\limits_{\gamma_1} p(x',x) v_1(x')\, dS_{x'} \end{pmatrix} .$$

The system of integral equations with the integral operator $\mathbf{G}$ has nice properties. First of all, the L_1-norm of $\mathbf{G}$ is less than 1, for any configuration of the two overlapping discs, since $\int\limits_{S(x,R)} p(y;x) dS_y = 1$. Hence $(E - \mathbf{G})^{-1}$ exists and is represented as a convergent Neumann series. In constructions of iterative numerical procedures, we will need the information about the principal eigen-value of the integral operator. In the next theorem we give this eigen-value explicitly. This result can be found in our previous paper [6].

Theorem 1. *The integral operator* $\mathbf{G}$ *is a Fredholm operator with the kernels* $p(y;x)$, $p(x';y)$, *continuous on* $x \in \gamma_1$, $y \in \gamma_2$, *with integrable singularities at the points of intersection of* γ_1 *and* γ_2 *of the type* $p(y;x) \simeq \frac{sin(\theta_1^* + \theta_2^*)}{\pi |x-y|}$ *as* $x \to y$. *The eigen-values of* $\mathbf{G}$, λ_i, *are all real, and moreover,* $\lambda_i = \pm \sigma_i \rho(\mathbf{G})$ *where* $\rho(\mathbf{G})$ *is the spectral radius of* $\mathbf{G}$ ($\sigma_i \leq 1$ *are positive constants) given explicitly by* $\rho(\mathbf{G}) = \frac{\theta_{12}^*}{\pi}$ *where* θ_{12}^* *is the angle of view of the segment* (O_1, O_2) *from the points of intersection of the two discs. The integral equation with the operator* $\mathbf{G}$ *has a unique solution which solves the Dirichlet problem (2.1).*

2.2 Extension to domains consisting of a set of overlapping discs

Generalization to domains consisting of n arbitrarily overlapping discs is not difficult, the main problem is to choose a convenient numbering of the arcs inside the discs. Generally this is a tricky problem but not too difficult for computer implementation. We have written a code which automatically generates a numeration, and the relevant matrix of the generated system of linear equations.

There are domains for which the structure of the matrix kernel of the system of integral equations (or the block-matrix of the relevant approximating linear system) is very simple and convenient for both theoretical analysis and practice.

Definition. *A* DS_2*-domain is defined as follows: (1) the domain is a connected union of overlapping discs, (2) each disc may overlap with an arbitrary number of discs but each intersection is a result of overlapping of only two discs, (3) any subset of discs which is a closed family of discs (a chain of successively overlapping discs where the first disc overlaps with the last disc, e.g., see Figure 1) consists of an even number of discs.*

This class of domains has a remarkable property: the numbering of the arcs can be chosen so that the generating matrix kernel G of the operator $\mathbf{G}$ is a cyclic matrix of order 2 and hence has the form: $G = \begin{pmatrix} 0 & G_{12} \\ G_{21} & 0 \end{pmatrix}$. Indeed, the relevant numbering is chosen as follows. Let us index the discs successively following say a clockwise direction, as $1, 2, \ldots, n$. We divide all the discs in two different classes: the first class (say, "red" discs) includes the discs with odd indices

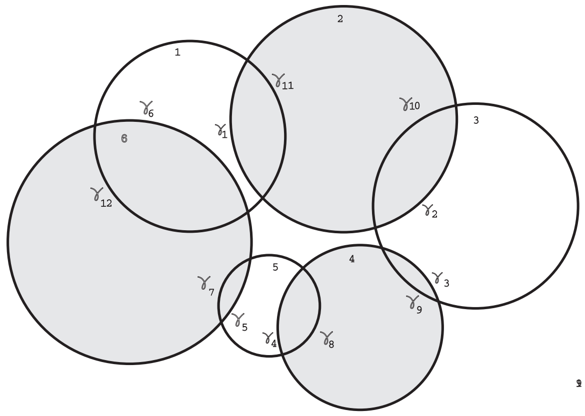

Figure 1: “Red-Black” indexation

$1,3,5,\ldots$, and the second class includes the discs (say “black” discs) with even indices. Now, the numbering of arcs is as follows: first we number successively along the chosen direction the arcs of the discs belonging to the first class, and then turn, in the last disc, to numbering successively the arcs of the discs in the second class, see Figure 1. Now, the matrix G has the desired block structure simply by the construction.

Let us take the decomposition $G = L + U$ where L is the left, and U the right triangular operators. Important properties of matrices for DS_2-domains are the following identities $(E - \omega L)^{-1} = E + \omega L$ and $L^2 = U^2 = 0$ where ω is an arbitrary parameter. This property will be used in section 4 where we construct a SOR (successive over relaxation) method for solving this kind of linear equations.

2.3 Discrete approximation

Let us consider our domain of two overlapping discs. The system of integral equations $v = \mathbf{G}v + f$ can be approximated by a system of linear algebraic equations $u = Au + g$. We choose a set of nodes $x_1, \ldots, x_{m_1+1}$ uniformly on the arc γ_1 and $y_1, \ldots, y_{m_2+1}$ on γ_2 generating by the uniform polar angles distributions (the end points are included). These meshes subdivide γ_1 and γ_2 in the set of arcs $\gamma_1^{(i)}$, $i = 1, \ldots, m_1$ and $\gamma_2^{(i)}, i = 1, \ldots, m_2$, respectively. Of course, the nodes can be chosen not uniformly, say, according to some distribution which generates the nodes more densely around the singular points where the arcs do intersect.

Since the Poisson kernel $p(y,x)$ has a singularity, it is convenient to take the approximation in the form:

$$\int\limits_{\gamma_1} p(y, y_k) v_2(y) dS_y = \sum_{i=1}^{m_1} p_i^{(1)}(x_i, y_k) v_2(x_i), \quad k = 1, \ldots m_2,$$

and analogously,

$$\int\limits_{\gamma_2} p(x', x_k) v_1(x') dS_{x'} = \sum_{i=1}^{m_2} p_i^{(2)}(y_i, x_k) v_1(y_i), \quad k = 1, \ldots m_1,$$

where

$$p_i^{(1)}(x_i, y_k) = \int\limits_{\gamma_1^{(i)}} p(y, y_k) dS_y\ , \quad p_i^{(2)}(y_i, x_k) = \int\limits_{\gamma_2^{(i)}} p(x', x_k) dS_{x'}. \tag{2.4}$$

These coefficients can be evaluated explicitly, using the relevant formula given in [6]. The same approximation is used to calculate the right hand sides f_1 and f_2 in all grid points. Thus we come to a discrete approximation in the form a system of linear algebraic equations: $u = Au + g$.

To ensure that the system of linear algebraic equations is a good approximation to the exact system of integral equations it is enough to prove that $(E - A)^{-1}$ exists. This is ensured by the fact that our matrices are all substochastic, and their spectral radii are all less than 1.

Let D be a DS_2 domain with the boundary $\Gamma = \partial D$. Let us consider an arbitrary disc of this domain, say, $K(x_k, R_k)$ with $S(x_k, R_k) = \partial K(x_k, R_k)$ which is overlapped with say m_k other discs. Then, $S(x_k, R_k)$ consists of two sets of arcs: one set (we denote it by $\tilde{\gamma}_k$) consists of arcs lying in the overlappings, and the second one (we denote it by $\tilde{\Gamma}_k$) is a part of the boundary Γ.

So assume that we have fixed some numbering of the arcs in our domain D, and the relevant numbering of the functions: the functions $v_j(x)$ are defined on the arcs γ_j. This numbering generates the block matrix A, so that the entries are constructed as follows. So let us fix an arc $\gamma_j \in K(x_k, R_k)$, then the j-th block row consists of $m_j = m_k$ blocks; we denote the integer set of numbering these blocks by J_j, so that $\sum\limits_{l \in J_j}$ means that the sum is taken over all blocks in j-th block row.

Now we estimate the difference between $v_j(x_i)$, the exact solution of the system of integral equations taken on j-th arc at a point $x_i \in \gamma_j$, and the approximation $u_i^{(j)}$ taken as the i-th component of the solution of our linear equation (ith row in the j-th block of the matrix A): $\varepsilon_i^{(j)} = u_i^{(j)} - v_j(x_i)$. Hence the error vector ε has in j-th block the components $\varepsilon_i^{(j)}$.

Let us also define the error vectors δ and δ^f with entries $\delta_i^{(\tilde{\gamma}_j)}$ and $\delta_i^{(\tilde{\Gamma}_j)}$, the errors of approximation of the Poisson integrals (of the function v and of the boundary function φ, respectively) for j-th arc taken over the set of arcs $\tilde{\gamma}_l$ and $\tilde{\Gamma}_l$. Thus we can write for the i-th row in the jth block:

$$\begin{aligned} u_i^{(j)} - v_j(x_i) &= \sum_{l \in J_j} \sum_{k=1}^{n_l} a_{ik}^{(l)} u_k^{(l)} - \sum_{l \in J_j} \int\limits_{\gamma_l} p(y; x_i) v_l(y) dS(y) + \delta_i^{(\tilde{\Gamma}_j)} \\ &= \sum_{l \in J_j} \sum_{k=1}^{n_l} a_{ik}^{(l)} (u_k^{(l)} - v_l(x_k)) + \delta_i^{(\tilde{\gamma}_j)} + \delta_i^{(\tilde{\Gamma}_j)}\ . \end{aligned}$$

Thus written in the matrix form these relations are $\varepsilon = A\varepsilon + \delta + \delta^f$. Let $\Delta\varphi = \max\limits_i(\varphi_{i+1} - \varphi_i)$ be the maximum difference taken over the all angular meshes. For simplicity we take simple estimations $\|\delta\| < C_1 \Delta\varphi$ and $\|\delta^f\| < C_2 \Delta\varphi$. Therefore, we have $\|\varepsilon\| \leq \|(E - A)^{-1}\| \, (C_1 + C_2) \Delta\varphi$.

3. The system of Lamé equations

Suppose a homogeneous isotropic medium $D \subset \mathbf{R}^n$ with a boundary Γ is given, whose state in the absence of body forces is governed by the classical static equation, the Lamé equation, see, e.g., [7]:

$$\Delta \mathbf{u}(x) + \alpha \,\mathrm{grad}\,\mathrm{div}\, \mathbf{u}(x) = 0, \quad x \in D, \tag{3.5}$$

where $\mathbf{u}(x) = (u_1(x_1,\dots,x_n),\dots,u_n(x_1,\dots,x_n))$ is a vector of displacements, whose components are real-valued regular functions. The elastic constant $\alpha = \frac{\lambda+\mu}{\mu}$ is expressed through the Lamé constants of elasticity λ and μ. It can be expressed through the Poisson ratio $\nu = \lambda/2(\lambda+\mu)$ as follows: $\alpha = 1/(1-2\nu)$. The Poisson ratio characterizes the relative amount of the change of the transverse to longitudinal displacements. It is known that due to thermodynamical reasons ν is bounded between $-1 \le \nu < 0.5$. This implies for α: $1/3 \le \alpha < \infty$. So there are materials with negative values of ν (α varies in $1/3 \le \alpha \le 1$), and materials with $\nu \approx 0.5$. The last case is very difficult for computational treating.

The first boundary value problem for the Lamé equation consists in finding a vector function $\mathbf{u} \in C^2(D) \cap C(\bar{D})$ satisfying the boundary condition $\mathbf{u}(y) = \varphi(y), \quad y \in \Gamma$ where $\varphi \in C(\Gamma)$ is a given vector-function.

In a full analogy with the Laplace equation, we will use the integral formulation of the given boundary value problem which is based on the spherical mean value relation. In what follows we deal with the two-dimensional case.

Consider an arbitrary point x with polar coordinates (r,φ') inside a disk $K(x_0,R)$. The point y situated on the circle $S(x_0,R)$ has the coordinates (R,θ), where $\theta = \varphi' + \alpha$, and z is defined by $z = y - x$; note that α is the angle between the vectors x and y; ψ is the angle between x and z. Define also the angle φ by $\varphi = \varphi' + \psi$.

The generalization of the Poisson formula we use [7] has the form

$$u_i(x) = \frac{R^2 - |x-x_0|^2}{2\pi R} \sum_{j=1}^{2} \int\limits_{S(x_0,R)} \frac{b_{ij}(x,y)\, u_j(y)}{|x-y|^2}\, dS_y\,, \quad i = 1,2, \tag{3.6}$$

where b_{ij} are functions of x, y, explicitly represented as the entries of the following matrix

$$B = \frac{\alpha}{\alpha+2} \begin{pmatrix} \frac{2}{\alpha} + 2\cos^2\varphi + \frac{|x-y|}{R}\cos(\theta+\varphi) & 2\cos\varphi\sin\varphi + \frac{|x-y|}{R}\sin(\theta+\varphi) \\ 2\cos\varphi\sin\varphi + \frac{|x-y|}{R}\sin(\theta+\varphi) & \frac{2}{\alpha} + 2\sin^2\varphi - \frac{|x-y|}{R}\cos(\theta+\varphi) \end{pmatrix}$$

where $\cos\theta = (y_1 - x_{01})/R$, $\sin\theta = (y_2 - x_{02})/R$, $\cos\varphi = (y_1 - x_1)/|x-y|$, $\sin\varphi = (y_2 - x_2)/|x-y|$.

In the notation of the Poisson kernel $p(y;x)$ the relation (3.6) reads in the matrix form:

$$\mathbf{u}(x) = \int\limits_{S(x_0,R)} p(y;x)\, B\mathbf{u}(y)\, dS(y)\,. \tag{3.7}$$

Taking this representation for two overlapping discs, we can derive a system of 4 integral equations defined on the arcs γ_1 and γ_2. Indeed, let us introduce the notations: $v_1^{(1)}(x) = u_1(x)$ and $v_1^{(2)}(x) = u_2(x)$ for $x \in \gamma_2$, and $v_2^{(1)}(x) = u_1(x)$ and $v_2^{(2)}(x) = u_2(x)$ for $x \in \gamma_1$. Then the analog of the linear system we derived above for the Laplace equation can be written as $v = \mathbf{G}v + f$, or in more details,

$$\begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \\ v_2^{(1)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & B_{22} \\ \hat{B}_{11} & \hat{B}_{12} & 0 & 0 \\ \hat{B}_{21} & \hat{B}_{22} & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \\ v_2^{(1)} \\ v_2^{(2)} \end{pmatrix} + \begin{pmatrix} f_1^{(1)} \\ f_1^{(2)} \\ f_2^{(1)} \\ f_2^{(2)} \end{pmatrix} \tag{3.8}$$

where the integral operators B_{ij}, $i, j = 1,2$ are defined, according to (3.6), for the points of the first disc $x \in K(x_0^{(1)})$:

$$B_{ij} v_2^{(j)}(x) = \int_{\gamma_1} p(y;x)\, b_{ij}(x,y)\, v_2^{(j)}(y)\, dS(y), \quad i, j = 1,2\ ,$$

while the integral operators $\hat{B}_{ij}$, $i, j = 1,2$ are defined for the points of the second disc $x \in K(x_0^{(2)})$:

$$\hat{B}_{ij} v_1^{(j)}(x) = \int_{\gamma_2} p(y;x)\, b_{ij}(x,y)\, v_1^{(j)}(y)\, dS(y), \quad i, j = 1,2\ .$$

The functions f_i^j are defined analogously:

$$f_i^{(j)}(x) = \sum_{k=1}^{2} \int_{\Gamma_i} p(y;x)\, b_{jk}(x,y)\, \varphi_k(y)\, dS(y), \quad i, j = 1,2\ .$$

It should be noted that the equivalence of the system (3.8) and the original Dirichlet problem is not evident, in contrast to the case of the Laplace equation. The L_1-norm of the integral operator is generally larger than 1, so we have to use finer properties. Indeed, let us estimate the L_1-norm. It can be shown that $\|\mathbf{G}\|_{L_1} \leq \frac{2}{2+\alpha}\left\{1 + \alpha 2\sqrt{2}\right\}\frac{\theta_{12}^*}{\pi}$. This estimation shows that $\|\mathbf{G}\|_{L_1}$ can be made less than 1 for any α by a proper choice of θ_1^*, θ_2^* which would imply a restriction of the overlapping configuration. To be free of such a restriction, we need an estimation of the spectral radius.

Theorem 2. *The integral operator* $\mathbf{G}$ *of the system (3.8) is a Fredholm operator with kernels continuous on* $x \in \gamma_1$ *and* $y \in \gamma_2$*, with the same type of singularities at the points of intersections of the arcs* γ_1 *and* γ_2 *as the singularities in the case of Laplace equation. The spectral radius of* $\mathbf{G}$ *is less than* 1 *for any nonempty overlapping, which ensures the equivalence of the system (3.8) and the original Dirichlet problem for the Lamé equation (3.5).*

The first part of the statement immediately follows from the fact that the kernel functions of the integral operator $\mathbf{G}$ are represented as products of bounded functions b_{ij} and the function $p(y;x)$. The property $\rho(\mathbf{G}) < 1$ can be derived as a consequence of the result obtained by S. L. Sobolev in [9]: the Schwarz alternation procedure for the overlapped domains, for the Dirichlet problem (3.5) constructed by the simple iteration of the Green functions for these domains, is convergent. For details of this derivation see [7]. The discrete approximation of the system of integral equations (3.7) in the form of linear system of algebraic equations is straightforward: using exactly the same nodes we obtain the system of equations in a fully analogy to the case of Laplace equation. The estimation for the error vector will have also the same form. This follows from the structure of the matrix kernel represented as a product of the Laplace kernel $p(y;x)$ and the matrix B whose entries are bounded functions $b_{ij}(x,y)$. The difference with the Laplace equation is that instead of a scalar element a_{ij} we have a 2x2 matrix B. So all the stochastic iteration procedures we present in the next sections are equally applicable to systems of linear algebraic equations generated by both the Laplace and Lamé integral kernels. The principal difference of the Laplace and Lame integral kernels is that the system of algebraic equations generated by the Lamé integral kernel is not substochastic, in contrast to the case of the Laplace equation.

4. SOR method

Let us consider a DS_2-domain, and the relevant system of integral equations. The matrix integral operator $\mathbf{G}$ can be represented as $\mathbf{G} = L+U$ where L and U are the lower and upper - triangular operators, respectively.

Introducing a scalar parameter ω we rewrite our integral equation $v = \mathbf{G}v + F$ in the form:

$$v = (E-\omega L)^{-1}[(1-\omega)E+\omega U]v+\omega(E-\omega L)^{-1}F \tag{4.9}$$

As mentioned above in section 2.2, for the DS_2-domains we have $(E-\omega L)^{-1} = E+\omega L$, therefore, our equation has the following simple form:

$$v = \mathbf{T}_\omega v + d \tag{4.10}$$

where

$$\mathbf{T}_\omega = (E+\omega L)[(1-\omega)E+\omega U], \quad d = \omega(E+\omega L)F \; .$$

If $\mathbf{T}_\omega$ is a contracting operator, we can apply a standard collision estimator. This can be applied directly to the integral form, or to the approximating system of linear algebraic equations. Here it is convenient to use a Markov chain of length n, to evaluate the n-th approximation.

Thus for iteration methods, we need to have information about the spectral properties of the integral operator $\mathbf{T}_\omega$. This can be extracted from a natural generalization of the well known result in linear algebra obtained for consistently ordered matrices (e.g., see [1], [2]). In our case of system of integral equations we give the relevant generalization.

Theorem 3. *Assume that the Dirichlet problem for the Laplace or the Lamé equations is solved for a DS_2-domain. Then the numbering of arcs can be chosen so that the matrix kernel G is cyclic, of index 2, and the property $(E-\omega L)^{-1} = E+\omega L$ holds true. The non-zero eigen-values λ_T of the integral operator $\mathbf{T}_\omega$ and the eigen-values μ of the original integral operator $\mathbf{G} = L+U$ are related by*

$$(\lambda_T+\omega-1)^2 = \lambda_T\,\omega^2\mu^2 \; . \tag{4.11}$$

Hence, for each i, the two numbers

$$\lambda^{\pm}_{T,i} = \left\{\frac{\omega\mu_i \pm \sqrt{\omega^2\mu_i^2-4(\omega-1)}}{2}\right\}^2 = 1-\omega+\frac{\mu_i^2\omega^2}{2} \pm \frac{\mu_i\omega}{2}\sqrt{\mu_i^2\omega^2+4-4\omega} \tag{4.12}$$

are eigen-values of the operator $\mathbf{T}_\omega$.

Proof. Indeed, our kernel matrix G is a cyclic matrix of index 2, by the construction of the DS_2-domain and for the numbering described above in section 2.2. It can be therefore written in the block form $G = \begin{pmatrix} 0 & G_{12} \\ G_{21} & 0 \end{pmatrix}$. For this structure of the system of integral equations the relation $(E - \omega L)^{-1} = E + \omega L$ is proved by simple evaluations. Simple transformation of the eigen-value problem

$$\mathbf{T}_\omega v = \lambda_T v$$

yields:

$$(1 - \omega - \lambda_T) E v + \omega U v + \lambda_T \omega L v = 0,$$

hence,

$$(U + \lambda_T L) v = \frac{\lambda_T + \omega - 1}{\omega} v\,. \tag{4.13}$$

Second iteration of (4.13) results in

$$(U + \lambda_T L)^2 v = \lambda_T (UL + LU) v = \Big(\frac{\lambda_T + \omega - 1}{\omega}\Big)^2 v\,. \tag{4.14}$$

Now we note that the second iteration of the original eigen-value problem $(L + U) w = \mu w$ reads:

$$(U + L)^2 w = (LU + UL) w = \mu^2 w. \tag{4.15}$$

Comparison of (4.14) and (4.15) yields

$$\frac{1}{\lambda_T} \Big(\frac{\lambda_T + \omega - 1}{\omega}\Big)^2 = \mu^2 \tag{4.16}$$

which proves (4.11). The relation (4.12) follows immediately from (4.11).

Simple arguments show (e.g., see [1]) that the minimum of λ_T is attained at

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(G)}},$$

and the relevant minimal value of $\rho(\mathbf{T}_\omega)$ is

$$\rho(\mathbf{T}_\omega) = \begin{cases} \Big[\omega\rho(G) + \sqrt{\omega^2\rho^2(G) - 4(\omega - 1)}\Big]^2 / 4 & if\ \ 0 < \omega \le \omega_{opt}, \\ \omega - 1 & if\ \ \omega_{opt} \le \omega < 2\,. \end{cases} \tag{4.17}$$

5. Discrete Random Walk method based on SOR

As mentioned above, the derived system of integral equations and the transformed system of integral equations with the operator $\mathbf{T}_\omega$ both can be solved by a Monte Carlo collision estimator. When we turn to the discrete approximations in the form of systems of linear algebraic equations, we come to discrete Random Walk methods which have some interesting advantages.

Suppose that D is a DS_2-domain, so that $(E - \omega L)^{-1} = E + \omega L$, and hence our system can be rewritten in the form of a system of algebraic equations

$$x = T x + f\,, \tag{5.1}$$

where T is the matrix $T = (E + \omega\hat{L})((1-\omega)E + \omega\hat{U})$, and $f = \omega(E + \omega\hat{L})g$.

Here $\hat{L}$ and $\hat{U}$ are the relevant left- and right triangular matrices related to the operators integral L and U, respectively. In what follows we denote by t_{ij} the entries of the matrix T.

The algorithm for calculation of n-th approximation is based on a direct randomized calculation of the finite number of iterations of the operator T, i.e., by evaluation of the Neumann series $f + Tf + T^2 f + \ldots + T^n f$. We do not introduce absorptions in our Markov chain. So to calculate a component x_l of the solution to (5.1), we suggest the following algorithm

1. Fix n, the number of iterations to be made, and choose the parameter ω, say, equal to 1 as in Seidel's method, or to ω_{opt}. The initial score is set to zero: $S = 0$. Calculate the matrix T, and the vector f.

2. Set the initial weight $Q = 1$, the number of iteration $j = 1$, and the current state of the Markov chain $i = l$. The initial value of the estimator is set as $v = f(l)$.

3. Simulate the transition from the state i to the new state k according to the density $p(i \to k)$ which is chosen, e.g., as follows

$$p(i \to k) = p_{ik} = \frac{|t_{ik}|}{\sum_{j=1}^{n} |t_{ij}|}$$

Recalculate the weight and the random estimator:

$$Q := Q t_{ik}/p_{ik}\ , \quad v := v + Qf(k)$$

then, renew the state as $i = k$, and go to the next iteration, i.e., $j := j+1$, and go to p.3, if j, the number of iterations is less than n; otherwise make a score: $S := S + v$, and start the new statistics from p.2.

4. Averaging the estimator over statistics of size N gives the result: $x_l \approx S/N$.

Sampling from the discrete distribution p_{ik} generated by the density $p(x;y)$ is carried out by Walker's alias method [3]. The alias arrays are prepared once, and then the method is extremely efficient: just one call of the random generator per a sample.

6. Numerical simulations

We present here first the results of numerical experiments for the Laplace equation which illustrate the convergence acceleration in comparison to the conventional Random Walk on Spheres method. The main calculation results however concern with the Lamé equation: here we give a detailed numerical analysis of the new methods suggested. In particular, we analyze the behaviour of the spectral radii of the SOR-based integral operators, in particular, how they depend on the parameter α, the rate of overlapping, the number of discs. For both Laplace and Lamé equations we analyze the error of the method as a function of the number of iterations.

6.1 Laplace equation

The domain consists of four discs of radii 1 equally overlapped with $\theta^* = \theta_1^* + \theta_2^* = 0.5$. The Laplace equation is solved by the standard random walk on spheres method (RWS), and by

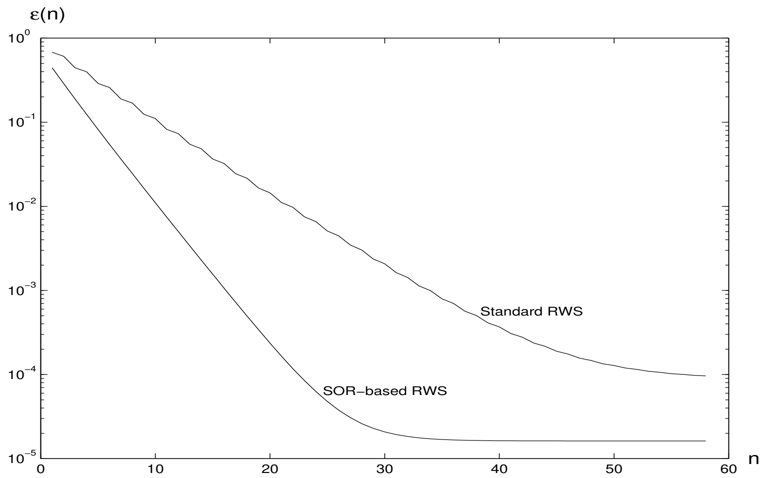

Figure 2: The relative error $\varepsilon(n)$ as a function of n, the number of iterations, for the standard Random Walk on Spheres method (RWS) - upper curve, and for the SOR-based RWS. Geometry: 4 equal discs of radii 1, equally overlapped with $\theta_1^* + \theta_2^* = 0.5$. Laplace equation, with Dirichlet boundary conditions.

the SOR-based RWS. In Figure 2 we show the relative error ε as a function of the number of iterations. It is clearly seen that the SOR-based RWS reaches its steady-state error ε_0 more than 2 times faster than that of standard RWS. In addition, the steady state error ε_0 of the SOR-based RWS is considerably smaller than that of the standard RWS.

6.2 Lamé equation

We solved the following model problem:

$$\Delta \mathbf{u}(x) + \alpha \,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}(x) = 0, \quad x \in D, \tag{6.2}$$

with the Dirichlet boundary conditions $\mathbf{u}(y) = \varphi(y)$, for $y \in \partial D$, the domain D consists of 5 overlapping discs. We have chosen the case with the exact solution $u_i(x_1,x_2) = 1 + \frac{1}{2}\frac{\alpha}{1+\alpha}x_i^2 - x_1x_2 + x_j$, $i = 1,2$, with $j \neq i$.

The first study concerns the rate of convergence of the SOR-based RWS in dependence of the parameter ω. In Figure 3 we show the spectral radii of the following operators: A - the original (untransformed) operator which generates the standard RWS, with the relevant operator A^2/p, - a matrix with the entries $\{a_{ij}^2/p_{ij}\}$, where p_{ij} is the relevant transition probability. Analogously is defined the matrix T^2/p where T is the matrix of the SOR-based RWS.

As seen from the results of Figure 3, the standard RWS diverges, since $\rho(A^2/p) \approx 1.11$. For the SOR-based RWS, the spectral radius $\rho(T)$ monotonically decreases with ω, however $\rho(T^2/p)$ reaches its minimum $\rho(T^2/p) \approx 0.6$ at $\omega = 1$. Thus the SOR method with $\omega = 1$, the Gauss-Seidel method gives the best results.

A detailed information about the spectral radii is presented in table 1: here α varies from $1/32$ to 15, the number of overlapping discs is $2,5$, and 10. It is seen that the spectral radii very slowly depend on the number of discs. The dependence on α is more pronounced.

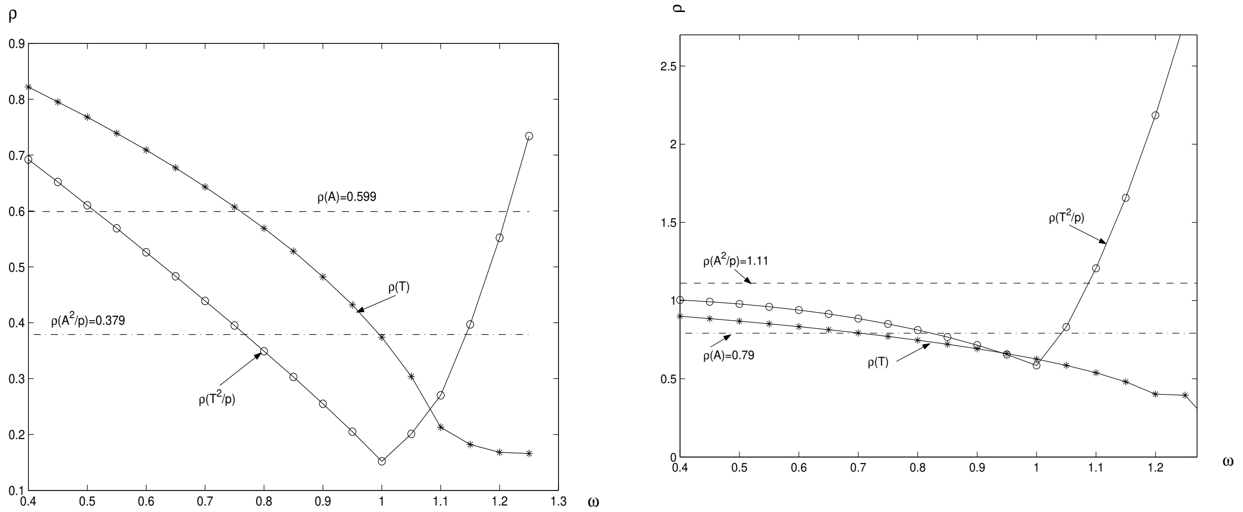

Figure 3: The dependence of the spectral radii $\rho(T)$ and $\rho(T^2/p)$ on the parameter ω. Geometry: 5 ovelapping discs. Left panel: Laplace equation, right panel: Lamé equation, $\alpha = 5$. For comparison, the spectral radii $\rho(A)$ and $\rho(A^2/p)$ are shown where A is the matrix of the untransformed system.

α	N_{discs}	$\rho(T)$	$\rho(T^2/p)$	$\rho(A)$	$\rho(A^2/p)$
	2	0.547	0.322	0.740	0.718
$\frac{1}{32}$	5	0.549	0.333	0.741	0.731
	10	0.536	0.319	0.732	0.713
	2	0.569	0.355	0.755	0.783
1.	5	0.576	0.375	0.759	0.801
	10	0.565	0.361	0.751	0.782
	2	0.737	0.705	0.858	1.28
5.	5	0.781	0.805	0.884	1.34
	10	0.779	0.797	0.883	1.32
	2	0.791	0.863	0.889	1.43
9.	5	0.847	0.997	0.920	1.51
	10	0.853	1.00	0.924	1.51
	2	0.826	0.983	0.909	1.54
15.	5	0.890	1.14	0.944	1.62
	10	0.898	1.15	0.947	1.63

Table 1: The spectral radii of the simple iteration operator and the SOR operator.

7. Conclusion

A new type of Random Walk on Spheres methods is developed for domains represented as a family of overlapping discs (2D). It is based on the following two steps: (1) transformation of the original differential boundary value problem into a system of integral equations, and (2) construction of a stochastic iterative procedure for solving the system of integral equations. We have constructed a randomized SOR-based iteration procedure for these integral equations. The calculations have shown that the new SOR-based RWS considerably accelerate the convergence of the standard Random Walk on Spheres method, and provide much higher accuracy. However the most interesting result of the paper is that the new SOR-based Random Walk on Fixed Spheres method is the first convergent method with finite variance for solving the system of Lamé equations governing the elastic deformation of a finite body.

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