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$W^{1,q}$ regularity results for elliptic transmission problems on heterogeneous polyhedra

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Abstract

Let Υ be a three-dimensional Lipschitz polyhedron, and assume that the matrix function μ is piecewise constant on a polyhedral partition of Υ. Based on regularity results for solutions to two-dimensional anisotropic transmission problems near corner points we obtain conditions on μ and the intersection angles between interfaces and $\partial\Upsilon$ ensuring that the operator $-\nabla \cdot \mu\nabla$ maps the Sobolev space $W_0^{1,q}(\Upsilon)$ isomorphically onto $W^{-1,q}(\Upsilon)$ for some $q > 3$.

1 Introduction

The aim of this paper is to provide conditions under which the operator

$$-\nabla \cdot \mu\nabla : W_0^{1,q}(\Upsilon) \longmapsto W^{-1,q}(\Upsilon) \tag{1.1}$$

is an isomorphy for some $q > 3$. The domain Υ is a three-dimensional Lipschitz polyhedron and the positive definite 3×3 matrix μ is piecewise constant on a polyhedral partition of Υ. As usual, $W_0^{1,q}(\Upsilon)$ denotes the Sobolev space with trace zero on $\partial\Upsilon$ and $W^{-1,q}(\Upsilon)$ is the dual of $W_0^{1,q'}(\Upsilon)$, $q' = q/(q-1)$. The results can be easily extended to the case of curved piecewise smooth boundaries and interfaces; in general it is only known that q can be chosen slightly larger than 2, see [25, 11, 2].

Operators of type (1.1) – which may be seen as the principal part of the homogenized version of an elliptic operator with inhomogeneous Dirichlet data – are of fundamental significance in many application areas from physics, chemistry and engineering. For an excellent overview concerning applications in mechanics including many numerical examples see [18]. Such operators also frequently appear in thermodynamics [33, 30], in electrodynamics [32], and in quantum mechanics – as the principal part of Schrödinger operators in effective mass approximation – see for example [35]. Finally, such operators also abound in reaction-diffusion systems [1], where the coefficient function μ often depends on the solution itself. In particular, in semiconductor device simulation by means of van Roosbroeck's equations (see for instance [31]), operators of type (1.1) are of relevance. Here heterostructures are the determining features of many fundamental effects (see for instance [14]). With ongoing miniaturization of electronic devices the resolution of material interfaces becomes ever more important, so that one definitely has to deal with discontinuous coefficient functions.

Starting with the pioneering work of Kondratiev [17], the regularity of solutions to elliptic boundary value problems near corners and edges has been treated mathematically by many authors. Transmission problems, where the coefficients are discontinuous at nonsmooth interfaces, have been studied for example in [5, 15, 28, 6, 7, 26, 27], mainly in the Hilbert scale $W^{s,2}$ and the isotropic (i.e. Laplacian and related operators) context. This scale has, however, the disadvantage that $W^{3/2,2}$ is principally a threshold in case of jumping coefficients and, hence, one cannot get an imbedding into L^∞ by this way, see [29] for further results. Here our result is of interest because for $q > 3$ the domain of the operator imbeds into $L^\infty(\Upsilon)$ (even in $C^\delta(\Upsilon)$) and gives in this spirit an enforced substitute for the usual $W_0^{1,2}(\Upsilon) \longleftrightarrow W^{-1,2}(\Upsilon)$ isomorphism. In addition, the space $W^{-1,q}$ is large enough to contain suitable, say bounded, surface densities and even not too singular measures, see [36, Ch. 4]. As carried out in [21], this allows to use the isomorphism (1.1) for the treatment of quasi-linear parabolic equations. Another important application of the information that the gradient of the solution belongs to a summability class larger than the space dimension is the possibility to obtain uniqueness results for associated nonlinear equations and systems, see for example [9, 10]. Note that our result is a certain complement to [8], where 3D-problems with mixed boundary conditions and without heterogeneities are treated.

Besides the $W^{1,q}$-scale our focus is on the anisotropy of the occurring materials. $W^{1,q}$-regularity results for equations in divergence form, where the coefficients jump at smooth interfaces (at least C^1), have been obtained in [20, 19, 3]. In [16] regularity results are derived for a class of quasi-linear elliptic transmission problems on polyhedral domains, using a difference quotient technique similar to that of [29]. In particular, the application of these results to linear anisotropic transmission problems leads to new results on the $W^{s,2}$-regularity of weak solutions.

Our result rests on the finding of [21] that if the gradient of the weak solution $u \in W_0^{1,2}(\Upsilon)$ of the Dirichlet problem for the elliptic equation

$$-\nabla \cdot \mu \nabla u = \nabla \cdot \vec{f}, \quad \vec{f} \in (L^p(\Upsilon))^3 ,$$

belongs to L^p for some $p > 3$ near each interior point of boundary and interior edges of the polyhedral partition of Υ, then the operator $-\nabla \cdot \mu \nabla$ maps $W_0^{1,q}(\Upsilon)$ isomorphically onto $W^{-1,q}(\Upsilon)$ for some $q \in (3, p]$. This allows to reduce the question of the isomorphy to the study of principal edge singularities of solutions and avoids the rather complicated discussion of the regularity near vertices. More precisely, the isomorphy (1.1) is valid for $2 < q < 3+\varepsilon$ with some $\varepsilon > 0$, provided that the minimal value $\widehat{\lambda}_\Upsilon$ of the real part of the singular exponents λ for all auxiliary plane problems satisfies $\widehat{\lambda}_\Upsilon > 1/3$.

Erratum: Unfortunately, there are some errors in our paper [21] – not in the proofs, but in the formulation of the linear regularity result and certain formulas. The assertion of [21, Theorem 2.3] that the exponent q can be taken from the interval $(2, 2/(1 - \widehat{\lambda}_\Upsilon))$ is erroneous, since we have overlooked the assumptions of [21, Theorem 2.4]. The correct formulation of the linear regularity result proved in [21] is given in Theorem 2.6 below. We also found that the signs in formulas for the coefficients of certain generalized Sturm-Liouville equations are not correct, in Remark 2.7 we give a detailed explanation.

The study of the exponents of corner singularities for auxiliary transmission problems in the plane with piecewise constant anisotropic coefficients is the main focus of the present paper. Although it suffices to find conditions on μ and the intersection angles along the edges of $\{\Upsilon_j\}$, which ensure that these exponents satisfy $\operatorname{Re}\lambda > 1/3$, most of our results are related with conditions ensuring the stronger inequalities $\operatorname{Re}\lambda \geq 1/2$ or $\operatorname{Re}\lambda > 1/2$. In particular, this is of interest for the regularity of the two-dimensional problems themselves, see [18].

Sufficient conditions that $\operatorname{Re}\lambda \geq 1/2$ for multimaterial corners are derived from [16], where it was shown that a quasi-monotonicity condition guarantees local $W^{3/2-\epsilon,2}$-regularity of weak solutions for arbitrary $\epsilon > 0$. In the case of bimaterial corners where two different materials meet, we study the exponents of corner singularities similarly to the isotropic case via Mellin transform. By inspecting the eigenvalues for the resulting operator pencils of generalized Sturm-Liouville operators, which are the roots of so-called characteristic equations, we derive lower bounds for $\operatorname{Re}\lambda$ depending only on the boundary angles. For interior edges of $\{\Upsilon_j\}$ we obtain the bound $\operatorname{Re}\lambda > 1/2$ under the quasi-monotonicity condition, and show that otherwise $\operatorname{Re}\lambda$ can be arbitrarily small. At any rate, the results apply to the case of layered materials if the angles between interfaces and outer boundary planes are not larger than π.

The problem of finding the roots of the characteristic equations or their distribution has been extensively studied for the standard boundary and transmission problems in the isotropic and related cases, see the literature cited above. The papers [28, 24], for example, give a rather complete characterization of the principal singular exponents for the Laplacian on multi-material angles. The situation is much worse for the anisotropic case. Since the knowledge of the singularity of solutions is crucial for the efficiency of numerical methods, there exist of course several numerical approaches to determine singular exponents of concrete anisotropic problems, see [18, 5, 34] and the references therein. But to our knowledge, characteristic equations of general anisotropic equations in the plane have been considered only by Il'in [12, 13] already more than 30 years ago. He derived these equations for model problems at boundary and interior corner points of bimaterial angles and studied the number of roots in the strip $0 < \operatorname{Re}\lambda < 1$, which determines the number of linearly independent weak but not strong solutions. In particular, this number can be arbitrarily large for interior corners points.

The outline of the paper is as follows. In Section 2 we give the formulation of the main result and a brief description of the approach from [21]. The proof of the isomorphy (1.1) follows from lower bounds for the real part of singular exponents of the solutions to plane anisotropic transmission problems, which are obtained in Sections 3 - 5 for different configurations.

2 The regularity result

2.1 Assumptions and formulation

We suppose that the Lipschitz polyhedron Υ is partitioned into a finite set of polyhedra $\Upsilon_j \subset \Upsilon$ such that the real, symmetric, and positive definite 3×3 matrix valued function μ is constant on each of the subsets Υ_j. Therefore μ has jumps at plane interfaces which intersect at certain interior or boundary edges. To each edge we associate a 2×2 matrix–function $\hat{\mu}_E$ in the following way:
Let E be one of the edges of the subdomains Υ_j. Choose a new orthogonal coordinate system (x_1, x_2, x_3) with origin at a point P in the interior of E such that the direction of E coincides with the x_3-axis. Denote by $\mu_{E,P}$ the piecewise constant matrix function which coincides in a neighborhood of P with $\mathcal{O}_E^{-1}\mu(\mathcal{O}_E^{-1}(x+P))\mathcal{O}_E$, where $\mathcal{O}_E$ denotes the corresponding orthogonal transformation matrix, and which satisfies $\mu_{E,P}(t\mathrm{x}, x_3) = \mu_{E,P}(\mathrm{x}, 0)$, $\mathrm{x} = (x_1, x_2)$, for all $x_3 \in \mathbb{R}$, $t > 0$.

Definition 2.1. The 2×2 matrix $\hat{\mu}_E(\mathrm{x})$ is the upper left 2×2 block of $\mu_{E,P}(\mathrm{x}, 0)$.

Remark 2.2. Even if the original coefficient matrix μ is diagonal (orthotropic), one is confronted with non-diagonal (anisotropic) matrices $\hat{\mu}_E$.

The matrix $\hat{\mu}_E$ belongs to a class $\mathcal{M}$ of real, piecewise constant 2×2 matrices with the property that there exist angles $\theta_0 < \theta_1 < \ldots < \theta_n \leq \theta_0 + 2\pi$, such that $\hat{\mu}_E$ is symmetric, positive definite, and constant on the sectors $\mathcal{K}_j = \{\mathrm{x} \in \mathbb{R}^2 : r > 0\,,\ \theta_{j-1} < \theta < \theta_j\}$, $j = 1, \ldots, n$. Here $r = |\mathrm{x}|, \theta$ are the polar coordinates in the x-plane, $(x_1, x_2) = r(\cos\theta, \sin\theta)$. Note that

$\theta_n = \theta_0 + 2\pi$ if $\hat{\mu}_E$ corresponds to an interior edge E, otherwise $\hat{\mu}_E$ is given on an infinite angle $\mathcal{K}_E = \{\mathrm{x} \in \mathbb{R}^2 : r > 0\,,\ \theta_0 < \varphi < \theta_n\}$, which coincides near P with the intersection of Υ with the x-plane. We say that $\hat{\mu}_E$ corresponds to a *multimaterial angle*. If for a given edge E the matrix $\hat{\mu}_E$ takes two different values, then the corresponding angle is called *bimaterial*.

In [16] Knees studied the regularity of weak solutions of nonlinear transmission problems on polyhedral domains. The basic assumption to establish $W^{3/2-\epsilon,2}$-regularity is a geometric quasi-monotonicity condition, which for two-dimensional linear anisotropic problems on multimaterial angles, i.e. $\hat{\mu} \in \mathcal{M}$, can be formulated as follows:

Definition 2.3. The matrix $\hat{\mu}$ is distributed *quasi-monotonely* on $\mathbb{R}^2$ if $\theta_n = \theta_0 + 2\pi$ and there exist indices $j_{\min}, j_{\max} \in \{1, \ldots, n\}$ such that

$$\hat{\mu}|_{\mathcal{K}_{j_{\max}}} \geq \hat{\mu}|_{\mathcal{K}_{j_{\max}+1}} \geq \cdots \geq \hat{\mu}|_{\mathcal{K}_{j_{\min}-1}} \geq \hat{\mu}|_{\mathcal{K}_{j_{\min}}} \leq \hat{\mu}|_{\mathcal{K}_{j_{\min}+1}} \leq \cdots \leq \hat{\mu}|_{\mathcal{K}_{j_{\max}-1}} \leq \hat{\mu}|_{\mathcal{K}_{j_{\max}}}$$

and there exist $\mathrm{x} \in \mathbb{R}^2$ such that $\mathrm{x} \in K_{j_{\max}}$ and $-\mathrm{x} \in K_{j_{\min}}$.

Definition 2.4. The matrix $\hat{\mu}$ is distributed *quasi-monotonely* on the angle $\mathcal{K}_E$ if $\theta_n < \theta_0 + 2\pi$ and there exists $j_{\min} \in \{1, \ldots, n\}$ such that

$$\hat{\mu}|_{\mathcal{K}_1} \geq \hat{\mu}|_{\mathcal{K}_2} \geq \cdots \geq \hat{\mu}|_{\mathcal{K}_{j_{\min}-1}} \geq \hat{\mu}|_{\mathcal{K}_{j_{\min}}} \leq \hat{\mu}|_{\mathcal{K}_{j_{\min}+1}} \leq \cdots \leq \hat{\mu}|_{\mathcal{K}_{n-1}} \leq \hat{\mu}|_{\mathcal{K}_n}$$

and there exist $\mathrm{x} \in \mathbb{R}^2$ such that $\mathrm{x} \in \mathbb{R}^2 \setminus \overline{\mathcal{K}_E}$ and $-\mathrm{x} \in K_{j_{\min}}$.

Now we can formulate assumptions on μ and intersection angles of interfaces, under which the isomorphy $W_0^{1,2}(\Upsilon) \mapsto W^{-1,2}(\Upsilon)$ of the operator $-\nabla \cdot \mu \nabla$ can be extended to the spaces $W_0^{1,q}(\Upsilon) \mapsto W^{-1,q}(\Upsilon)$, $q \in (2, 3 + \varepsilon)$ for some $\varepsilon > 0$.

A1: The matrix $\hat{\mu}_E$ is distributed quasi-monotonely on $\mathbb{R}^2$ at an interior edge E of the partitioning of Υ.

A2: The matrix $\hat{\mu}_E$ is distributed quasi-monotonely on $\mathcal{K}_E$ at a boundary edge E of the partitioning $\{\Upsilon_j\}$.

A3: E is a boundary edge of $\{\Upsilon_j\}$, the matrix $\hat{\mu}_E$ corresponds to a bimaterial angle and one of the following conditions is satisfied:
(i) the opening angles of the corresponding sectors $\mathcal{K}_1$ and $\mathcal{K}_2$ do not exceed π
(ii) the opening angle of one sector $\mathcal{K}_j$ exceeds π and and the interior normals ν_1, ν_2 to the sides of $\mathcal{K}_j$ satisfy $(\hat{\mu}_E|_{\mathcal{K}_j}\nu_1, \nu_2) \geq 0$.

Theorem 2.5. *Suppose that the partitioning of* Υ *and the symmetric positive definite coefficient matrix* μ *satisfy assumption* **A1** *for any interior edge and assumption* **A2** *or* **A3** *for any boundary edge belonging to more than one sub-polyhedron* Υ_j*. Then the operator* $-\nabla \cdot \mu \nabla$ *provides a topological isomorphism between* $W_0^{1,q}(\Upsilon)$ *and* $W^{-1,q}(\Upsilon)$ *for some* $q > 3$.

Note that we assume nothing about the intersection points of edges and boundary corners and about edges of Υ where μ is constant. So quite complicated geometrical configurations of Υ and its polyhedral partition are possible.

2.2 Preliminaries

The proof of Theorem 2.5 is based on an approach developed in [21], which we briefly describe in the following.

Assign to each edge E of the partition $\{\Upsilon_j\}$ the two-dimensional problem with the elliptic operator

$$-\nabla_{\mathrm{x}} \cdot \hat{\mu}_E \nabla_{\mathrm{x}} u = f\,, \quad \mathrm{x} \in \operatorname{supp} \hat{\mu}_E \tag{2.1}$$

with $\hat{\mu}_E(\mathrm{x})$ from Definition 2.1, compactly supported right-hand side $f \in L_2$ and homogeneous Dirichlet conditions on $\partial\mathcal{K}_E$ in case of a boundary edge E.

By applying the Mellin transform with respect to the radial variable r, the regularity problem of solutions to (2.1) leads to the following nonlinear eigenvalue problem:
Denote by $\Sigma_E = G_E \cap S^1$ the intersection of the unit circle S^1 in the x-plane with the support G_E of $\hat{\mu}_E$, i.e. $G_E = \mathbb{R}^2$ for interior and $G_E = \mathcal{K}_E$ for boundary edges. If E is an interior edge of Υ, then $\Sigma = S^1$ and we denote by $\mathcal{H} = H^1(S^1)$ the periodic Sobolev space on the unit circle. Otherwise we set $\mathcal{H} = H^1_0(\Sigma_E)$. Consider the family of sesquilinear forms

$$a_E(u,v;\lambda) \overset{\text{def}}{=} \frac{1}{\log 2} \int\limits_{\{1<r<2\}\cap G_E} \hat{\mu}_E(\mathrm{x})\, \nabla_{\mathrm{x}}\, r^{\lambda} u(\theta) \cdot \nabla_{\mathrm{x}}\, r^{-\lambda}\overline{v(\theta)}\, \mathrm{dx}\,, \quad u,v \in \mathcal{H}\,, \tag{2.2}$$

with $\mathrm{x} = r(\cos\theta, \sin\theta)$. For any $\lambda \in \mathbb{C}$, the form (2.2) generates a continuous linear operator $\Pi_E(\lambda) : \mathcal{H} \to \mathcal{H}'$ by

$$(\Pi_E(\lambda)u, v) \overset{\text{def}}{=} a_E(u,v;\lambda)\,, \quad u,v \in \mathcal{H}, \tag{2.3}$$

where $(\cdot,\cdot)$ denotes the (extended) $L^2(\Sigma_E)$ duality. It was shown in [21] that the pencil $\Pi_E(\lambda)$, $\lambda \in \mathbb{C}$, is an analytic Fredholm operator function which has only isolated eigenvalues with finite multiplicity. Denote by λ°_E an eigenvalue with smallest positive real part of Π_E, and set $\widehat{\lambda}_\Upsilon = \min(1, \operatorname{Re} \lambda^\circ_E)$, where the minimum is taken over all edges E of the partition $\{\Upsilon_j\}$. The correct assertion of [21, Theorem 2.3] should be formulated as follows:

Theorem 2.6. *If $\widehat{\lambda}_\Upsilon > 1/3$, then there exists $\varepsilon \in (0, \frac{2}{1-\widehat{\lambda}_\Upsilon} - 3]$ such that $-\nabla \cdot \mu \nabla$ maps $W^{1,q}_0(\Upsilon)$ isomorphically onto $W^{-1,q}(\Upsilon)$ for all $q \in (2, 3+\varepsilon)$.*

The eigenvalues of $\Pi_E(\lambda)$ can be determined from one-dimensional problems on Σ_E with parameter. Elementary calculations show that

$$\hat{\mu}_E \nabla r^{\lambda} u(\theta) \cdot \nabla r^{-\lambda}\overline{v(\theta)} = r^{-2}\big(b_2\, u'\, \overline{v'} + \lambda b_1 u\, \overline{v'} - \lambda b_1 u'\, \overline{v} - \lambda^2 b_0\, u\overline{v}\big) \tag{2.4}$$

with the functions

$$\begin{aligned} b_0(\theta) &= \hat{\mu}_{11} \cos^2\theta + 2\hat{\mu}_{12} \sin\theta\cos\theta + \hat{\mu}_{22}\sin^2\theta\,, \\ b_1(\theta) &= (\hat{\mu}_{22} - \hat{\mu}_{11}) \sin\theta\cos\theta + \hat{\mu}_{12}(\cos^2\theta - \sin^2\theta)\,, \\ b_2(\theta) &= \hat{\mu}_{11} \sin^2\theta - 2\hat{\mu}_{12}\sin\theta\cos\theta + \hat{\mu}_{22}\cos^2\theta\,. \end{aligned} \tag{2.5}$$

Here $\hat{\mu}_{jk}$ are the elements of the 2×2 matrix $\hat{\mu}_E$. By (2.2) and (2.4) we therefore obtain

$$a_E(u,v;\lambda) = \int\limits_{\Sigma_E} \big(b_2\, u'\, \overline{v'} + \lambda b_1 u\, \overline{v'} - \lambda b_1 u'\, \overline{v} - \lambda^2 b_0\, u\overline{v}\big)\, d\theta\,. \tag{2.6}$$

Partial integration shows that $\lambda \in \mathbb{C}$ is an eigenvalue of the operator pencil Π_E if there exists a nontrivial solution $u \in \mathcal{H}$ of the differential equation

$$-(b_2 u')' - \lambda(b_1 u)' - \lambda b_1 u' - \lambda^2 b_0 u = 0\,, \tag{2.7}$$

satisfying at the discontinuity points $\theta_k \in \Sigma_E$ of the coefficients b_j the transmission conditions

$$[u]_{\theta_k} = 0\ ,\ [b_2 u' + \lambda b_1 u]_{\theta_k} = 0\,. \tag{2.8}$$

As usual the symbol $[w]_{\theta_k}$ stands for $\lim_{\theta\searrow\theta_k} w(\theta) - \lim_{\theta\nearrow\theta_k} w(\theta)$. Note that the condition $u \in \mathcal{H}$ implies periodic or homogeneous Dirichlet boundary conditions at θ_0 and θ_n if E is an interior or boundary edge, respectively.

Remark 2.7. In [21, p. 240] we have used the wrong sign in the formula for the Mellin transform $\widetilde{r\partial_r u} = -\lambda\tilde{u}$, which has to be replaced by $\widetilde{r\partial_r u} = \lambda\tilde{u}$. Therefore the formulas [21, (3.33)] for the sesquilinear form $a(u, v; \lambda)$ and [21, (3.32)] for the corresponding differential problem differ in sign from the correct formulas (2.6) and (2.7), (2.8). But this does not affect the correctness of the other considerations in [21].

2.3 Corner singularities of plane anisotropic transmission problems

For the proof of Theorem 2.5 it suffices to estimate the eigenvalues of operator pencils corresponding to anisotropic transmission problems in $\mathbb{R}^2$ for interior edges and on infinite angles $\mathcal{K}_E$ for boundary edges E and Dirichlet boundary conditions. The rest of this paper is devoted to this problem in a slightly more general setting, since besides the equation (2.1) in an infinite angle $\mathcal{K}_E$ with homogeneous Dirichlet conditions (the D problem) we will also consider homogeneous Neumann conditions (the N problem) of the form

$$\hat{\mu}_E \nu \cdot \nabla u|_{\partial\mathcal{K}_E} = 0\,. \tag{2.9}$$

To simplify notation, in the following we omit the index E, which indicates the dependence on the edge E in (2.1) - (2.6).

Similarly to the D problem treated in [21] the operator pencil corresponding to the N problem are generated by the sesquilinear form (2.2), but with the underlying space $\mathcal{H} = H^1(\Sigma)$. The eigenvalues can be determined from the differential equation (2.7) on $\Sigma = S^1 \cap \mathcal{K}$ with the transmission conditions (2.8) and the boundary conditions

$$(b_2 u' + \lambda b_1 u)|_{\theta=\theta_0} = (b_2 u' + \lambda b_1 u)|_{\theta=\theta_n} = 0\,. \tag{2.10}$$

In the next sections we derive the following lower bounds for the positive real part of eigenvalues of $\Pi(\lambda)$ for different situations.

Lemma 2.8. *Let $\hat{\mu} \in \mathcal{M}$. The eigenvalue of $\Pi_{\hat{\mu}}(\lambda)$ with minimal positive real part satisfies* $\operatorname{Re}\lambda^\circ \geq 1/2$ *if*
(i) *$\hat{\mu}$ is quasi-monotonely distributed on $\mathbb{R}^2$,*
(ii) *$\hat{\mu}$ is quasi-monotonely distributed on an infinite angle $\mathcal{K}$ for the D problem,*
(iii) *$-\hat{\mu}$ is quasi-monotonely distributed on an infinite angle $\mathcal{K}$ for the N problem.*

Lemma 2.9. *If $\hat{\mu} \in \mathcal{M}$ is constant on the infinite angle $\mathcal{K}$, then* $\operatorname{Re}\lambda^\circ > 1/2$ *for both the D and N problems.*

Lemma 2.10. *Let $\hat{\mu} \in \mathcal{M}$ correspond to a boundary bimaterial angle. Then the eigenvalues of the operator pencil $\Pi_{\hat{\mu}}$ for both the D and N problems are real and satisfy $\lambda^\circ > 1/4$ for any opening angles of the two sectors $\mathcal{K}_j$. Moreover, assumption* **A3**(i) *implies $\lambda^\circ > 1/2$, and assumption* **A3**(ii) *implies $\lambda^\circ > 1/3$.*

Lemma 2.11. *If $\hat{\mu} \in \mathcal{M}$ corresponds to an interior bimaterial angle and satisfies assumption* **A1**, *then* $\operatorname{Re} \lambda^\circ > 1/2$.

In view of Theorem 2.6, the assertions of Theorem 2.5 follow immediately from Lemmas 2.8(i-ii), 2.9 and 2.10.

3 Proof of Lemmas 2.8 and 2.9

3.1 Multimaterial angles

The proof of Lemma 2.8 is based on recent results of Knees [16] and the observation that the optimal regularity of solutions to transmission problems for the operator (2.1) is determined by the number $\min(\operatorname{Re} \lambda^\circ, 1/2)$.

Let λ with $\operatorname{Re} \lambda \in (0, 1/2]$ be an eigenvalue and $v_\lambda \in \mathcal{H}$ a corresponding eigenfunction of the operator defined by (2.3). Then the function $w_\lambda(\mathrm{x}) = r^\lambda v_\lambda(\theta) \in H^1_{loc}(G)$ is, by construction, a solution of the homogeneous equation

$$-\nabla_{\mathrm{x}} \cdot \hat{\mu} \nabla_{\mathrm{x}} w_\lambda = 0$$

in the distributional sense. Recall that $G = \mathbb{R}^2$ for interior transmission problems and $G = \mathcal{K}$ for the boundary D or N problems.

Let η be a smooth cut-off function such that $\eta(r) = 1$ for $r \in [0, 1/2]$ and $\eta(r) = 0$ for $r \geq 2/3$ and introduce the function

$$u_\lambda(\mathrm{x}) \stackrel{\mathrm{def}}{=} \eta(r) w_\lambda(\mathrm{x})$$

with support in $G \cap B_1$, where $B_1 \subset \mathbb{R}^2$ denotes the unit disk. Since v_λ does not vanish identically on Σ, we have $u_\lambda \in W^{1+\operatorname{Re}\lambda-\epsilon,2}(G \cap B_1)$ for any $\epsilon > 0$, but $\notin W^{1+\operatorname{Re}\lambda,2}(G \cap B_1)$. Moreover, it is easily seen that u_λ is a variational solution of the homogeneous Dirichlet problem (if $G = \mathbb{R}^2$ or for the D problem) or of the homogeneous Neumann problem (for the N problem) on $G \cap B_1$ for the equation

$$-\nabla_{\mathrm{x}} \cdot \hat{\mu} \nabla_{\mathrm{x}} u = f_\lambda \quad \text{with some} \quad f_\lambda \in L^2(G \cap B_1)\,. \tag{3.1}$$

Thus we conclude that if $\operatorname{Re} \lambda^\circ \leq 1/2$, then solutions to the Dirichlet or Neumann problem on $G \cap B_1$ for the operator (2.1) belong in general to $W^{1+\operatorname{Re}\lambda^\circ-\epsilon,2}(G \cap B_1)$ for arbitrary $\epsilon > 0$, but not for $\epsilon = 0$.

On the other hand, the general Theorem 4.1 of [16], applied to two-dimensional anisotropic transmission problems, states the following:
If the matrix $\hat{\mu}$ is quasi-monotonely distributed on G, then the weak solution of the Dirichlet problem for (3.1) with arbitrary right-hand side $f \in L^2(G \cap B_1)$ satisfies $u \in W^{3/2-\epsilon,2}(G \cap B_1)$ for any $\epsilon > 0$.
If the matrix $-\hat{\mu}$ is quasi-monotonely distributed on G, then the weak solution of the Neumann problem for (3.1) with arbitrary right-hand side $f \in L^2(G \cap B_1)$ satisfies $u \in W^{3/2-\epsilon,2}(G \cap B_1)$ for any $\epsilon > 0$.

Hence the relation $\operatorname{Re}\lambda^\circ < 1/2$ is impossible under the assumptions of Lemma 2.8 on the coefficient matrix $\hat{\mu}$. □

3.2 Linear transformations

For the proofs of Lemmas 2.9 - 2.10 we have to study the characteristic equations of the corresponding differential problems (2.7, 2.8). These problems can be simplified due to the observation that the eigenvalues of the operator pencils $\Pi_{\hat{\mu}}(\lambda)$ are invariant under linear transformations $\mathrm{y} = L\mathrm{x}$, where $\mathrm{x}, \mathrm{y} \in \mathbb{R}^2$, and L is a nonsingular 2×2 matrix. The substitution $\mathrm{y} = L\mathrm{x}$ transforms the differential operator $-\nabla_{\mathrm{x}} \cdot \hat{\mu}\nabla_{\mathrm{x}}$ to $-\nabla_{\mathrm{y}} \cdot L\hat{\mu}L^T\nabla_{\mathrm{y}}$, where L^T is the transpose of L. Then the differential equation corresponding to $L\hat{\mu}L^T$ is of the form

$$-(\tilde{b}_2u')' - \lambda(\tilde{b}_1u)' - \lambda\tilde{b}_1u' - \lambda^2\tilde{b}_0u = 0 \tag{3.2}$$

with coefficients $\tilde{b}_j$ obtained by formulas (2.5) from the elements of the matrix $L\hat{\mu}L^T$. Furthermore, in case of a boundary edge, the support $L(G) = \{L\mathrm{x} : \mathrm{x} \in G = \operatorname{supp}\hat{\mu}\}$ of $L\hat{\mu}L^T$ gives rise to another interval of angles $\widetilde{\Sigma} \subset S^1$, whereas for the transmission conditions other discontinuity angles $\tilde{\theta}_j$ of $L\hat{\mu}L^T$ occur. In the following we denote by $\Pi_{L\hat{\mu}L^T}(\lambda)$ the operator corresponding to equation (3.2), together with the transmission condition

$$[u]_{\tilde{\theta}_j} = 0\,, \quad [\tilde{b}_2u' + \lambda\tilde{b}_1u]_{\tilde{\theta}_j} = 0\,, \quad j = 1, 2\,, \tag{3.3}$$

for interior edges and with homogeneous Dirichlet or Neumann boundary conditions on $\partial\widetilde{\Sigma}$ for the D or N problem, respectively.

Lemma 3.1. *For any nonsingular real matrix L, the eigenvalues of $\Pi_{\hat{\mu}}(\lambda)$ and $\Pi_{L\hat{\mu}L^T}(\lambda)$ coincide.*

Proof. We have to show that

$$a(u, v; \lambda) = \frac{1}{\log 2} \int\limits_{\{1<r<2\}\cap G} \hat{\mu}\,\nabla r^\lambda u(\theta) \cdot \nabla r^{-\lambda}\overline{v(\theta)}\,d\mathrm{x} = 0\,, \quad \forall\, v \in \mathcal{H}\,, \tag{3.4}$$

has a nontrivial solution $u \in \mathcal{H}$ if and only if there exists $\tilde{u} \in \widetilde{\mathcal{H}}$, $\tilde{u} \neq 0$, satisfying

$$\tilde{a}(\tilde{u}, v; \lambda) = \frac{1}{\log 2} \int\limits_{\{1<\rho<2\}\cap L(G)} L\hat{\mu}L^T\,\nabla \rho^\lambda \tilde{u}(\vartheta) \cdot \nabla \rho^{-\lambda}\overline{v(\vartheta)}\,d\mathrm{y} = 0\,, \quad \forall\, v \in \widetilde{\mathcal{H}}\,. \tag{3.5}$$

Here (ρ, ϑ) are the polar coordinates $\rho(\cos\vartheta, \sin\vartheta) = (y_1, y_2)$, and the space $\widetilde{\mathcal{H}}$ in (3.5) is either $H^1(S^1) = \mathcal{H}$ for interior edges, or otherwise $\widetilde{\mathcal{H}} = H^1_0(\widetilde{\Sigma})$ for the D problem and $\widetilde{\mathcal{H}} = H^1(\widetilde{\Sigma})$ for the N problem.

The linear transformation L maps the unit circle S^1 in the x-plane onto an ellipse centered at the origin in the y-plane and generates a smooth one-to-one function $\vartheta : S^1 \to S^1$ by

$$\theta = \frac{\mathrm{x}}{|\mathrm{x}|} \to \vartheta(\theta) = \frac{L\mathrm{x}}{|L\mathrm{x}|}$$

Let us denote its inverse function by $\theta(\vartheta) : S^1 \to S^1$ and introduce the positive smooth function $f(\theta) = |L\theta|$. Then the image of the sector $\{1 < r < 2\} \cap G$ under the transformation L is given by

$$L\big(\{1 < r < 2\} \cap G\big) = \big\{(\rho, \vartheta) : \vartheta \in S^1 \cap L(G)\,,\ f(\theta(\vartheta)) < \rho < 2f(\theta(\vartheta))\big\}\,, \tag{3.6}$$

and the substitution y $= L$x in the integral (3.4) gives

$$a(u,v;\lambda) = \frac{1}{|\det L|}\frac{1}{\log 2}\int\limits_{L(\{1<r<2\}\cap G)} L\hat{\mu}L^T\,\nabla_y\,\rho^\lambda\tilde{u}(\vartheta)\cdot\nabla_y\,\rho^{-\lambda}\,\overline{\tilde{v}(\vartheta)}\,d\mathrm{y}$$

with $\tilde{u} = (f\circ\theta)^{-\lambda}(u\circ\theta)$, $\tilde{v} = (f\circ\theta)^{\lambda}(v\circ\theta)$. Since $\tilde{u},\tilde{v}\in\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{H}} = \{(f\circ\theta)^{\lambda}(v\circ\theta);\, v\in\mathcal{H}\}$, from (2.4) and (3.6) we get

$$a(u,v;\lambda) = \frac{1}{|\det L|}\frac{1}{\log 2}\int\limits_{S^1\cap L(G)}\int\limits_{f(\theta(\vartheta))}^{2f(\theta(\vartheta))}\big(\tilde{b}_2\,\tilde{u}'\,\overline{\tilde{v}'} - \lambda\tilde{b}_1(\tilde{u}\,\overline{\tilde{v}'} - \tilde{u}'\,\overline{\tilde{v}}) - \lambda^2\tilde{b}_0\,\tilde{u}\overline{\tilde{v}}\big)\,\frac{d\rho}{\rho}\,d\vartheta$$

with the above mentioned coefficients $\tilde{b}_j$. Because of

$$\int\limits_{f(\theta(\vartheta))}^{2f(\theta(\vartheta))}\frac{d\rho}{\rho} = \log 2 = \int\limits_1^2\frac{d\rho}{\rho}\,,$$

we therefore obtain

$$\begin{aligned}|\det L|\,a(u,v;\lambda) &= \frac{1}{\log 2}\int\limits_{\{1<\rho<2\}\cap L(G)}\rho^{-2}\big(\tilde{b}_2\,\tilde{u}'\,\overline{\tilde{v}'} - \lambda\tilde{b}_1(\tilde{u}\overline{\tilde{v}'} - \tilde{u}'\,\overline{\tilde{v}}) - \lambda^2\tilde{b}_0\,\tilde{u}\overline{\tilde{v}}\big)\,d\mathrm{y}\\ &= \tilde{a}(\tilde{u},\tilde{v};\lambda)\,.\end{aligned}$$

□

Now Lemma 2.9 follows from

Corollary 3.2. *For any constant, symmetric, and positive definite matrix $\hat{\mu}$, the eigenvalue λ° with minimal real part of the homogeneous Dirichlet problem $u(\theta_j) = 0$, $j = 0,1$, or of the homogeneous Neumann problem* (2.10) *for the differential equation* (2.7) *is real and satisfies $\lambda^\circ > 1/2$ if $\theta_1 - \theta_0 \in (\pi, 2\pi)$, and $\lambda^\circ = 1$ if $\theta_1 - \theta_0 \in (0,\pi]$.*

Proof. Choose the matrix L such that $L\hat{\mu}L^T = I$. Then, by Lemma 3.1, the eigenvalues of the Dirichlet or Neumann problem for (2.7) coincide with those of the Sturm-Liouville problem

$$u'' + \lambda^2 u = 0\,,$$

with $u \in H_0^1(S^1\cap L(G))$ or $u\in H^1(S^1\cap L(G))$, respectively, where $S^1\cap G = \Sigma = (\theta_0,\theta_1)$. The positive eigenvalues of this classical problem are the singular exponents of solutions to the Laplace equation with homogeneous Dirichlet or Neumann conditions on the infinite angle $L(G)$. It is well known that $\lambda^\circ = \min(1,\pi/\omega)$, where ω is the opening angle of $L(G)$. Since $\omega - \pi$ and $(\theta_1-\theta_0)-\pi$ have obviously the same sign, the assertion follows. □

4 Boundary bimaterial angles

Here we prove Lemma 2.10, where no monotonicity conditions on the matrices $\hat{\mu}_j$ are imposed. Since the sesquilinear form (2.2) is invariant under orthogonal transformations, one can rotate the coordinate system such that the intersection of the material interface with the x-plane lies on the positive x_1-axis. Then $\hat{\mu}$ transforms to a matrix A which takes constant values $A_\pm = (a_{jk}^\pm)_{j,k=1}^2$ on the sectors $\mathcal{K}_+ = \{r>0, 0<\theta<\alpha_+\}$, $\mathcal{K}_- = \{r>0, -\alpha_- < \theta < 0\}$ with $\alpha_+ + \alpha_- < 2\pi$.

4.1 Transformation

As proposed by Il'in in [12], in $\mathcal{K}_+$ and $\mathcal{K}_-$ two different linear transformations are performed with the matrices

$$L_\pm = \begin{pmatrix} 1 & -a^\pm_{21}/a^\pm_{22} \\ 0 & a_\pm/a^\pm_{22} \end{pmatrix}, \qquad \text{where we denote } a_\pm \stackrel{\text{def}}{=} \sqrt{\det A_\pm}\,.$$

Then in view of $a^\pm_{12} = a^\pm_{21}$

$$L_\pm A_\pm (L_\pm)^T = \frac{\det A_\pm}{a^\pm_{22}} I = a_\pm \det L_\pm\, I \tag{4.1}$$

with the 2×2 identity matrix I. Hence, the study of the eigenvalues of the operator pencil $\Pi_{\hat{\mu}}$, i.e. of nontrivial solutions $u \in \mathcal{H}$ of the equations

$$a(u, v; \lambda) = \frac{1}{\log 2} \int\limits_{\{1<r<2\}\cap G} A\, \nabla_{\mathrm{x}}\, r^\lambda u(\theta) \cdot \nabla_{\mathrm{x}}\, r^{-\lambda} \overline{v(\theta)}\, d\mathrm{x} = 0\,, \ \forall v \in \mathcal{H}\,,$$

can be reduced to the isotropic case. Here $\mathcal{H} = H^1_0(-\alpha_-, \alpha_+)$ for the D problem and $\mathcal{H} = H^1(-\alpha_-, \alpha_+)$ for the N problem.

To justify this reduction we first consider the images $\widetilde{\mathcal{K}}_\pm = L_\pm(\mathcal{K}_\pm)$. Since the x_1-axis is invariant for both transformations, the sectors $\widetilde{\mathcal{K}}_\pm$ have again the common side $\{x_1 > 0, x_2 = 0\}$. Let us denote by $\widetilde{\alpha}_\pm$ the opening angles of the sectors $\widetilde{\mathcal{K}}_\pm$, respectively. Since $a^\pm_{22} > 0$, they can be given by

$$\begin{aligned} \widetilde{\alpha}_+ &= \arg((a^+_{22} \cos\alpha_+ - a^+_{21} \sin\alpha_+) + i a_+ \sin\alpha_+)\,, \\ \widetilde{\alpha}_- &= \arg((a^-_{22} \cos\alpha_- + a^-_{21} \sin\alpha_-) + i a_- \sin\alpha_-)\,, \end{aligned} \tag{4.2}$$

implying

$$\text{either} \quad \alpha, \widetilde{\alpha} \in (0, \pi) \quad \text{or} \quad \alpha = \widetilde{\alpha} = \pi \quad \text{or} \quad \alpha, \widetilde{\alpha} \in (\pi, 2\pi)\,, \tag{4.3}$$

where α stands for α_+ or α_-. Hence, that if $\alpha_\pm \le \pi$, then $\widetilde{\alpha}_\pm \le \pi$ and $\widetilde{\alpha}_+ + \widetilde{\alpha}_- < 2\pi$. Otherwise, if the opening of one of the sectors $\mathcal{K}_\pm$ is greater than π, then possibly $\widetilde{\alpha}_+ + \widetilde{\alpha}_- \ge 2\pi$.

The linear transformations $L_\pm$ generate by

$$\theta = \frac{\mathrm{x}}{|\mathrm{x}|} \to \vartheta_\pm(\theta) = \frac{L_\pm \mathrm{x}}{|L_\pm \mathrm{x}|}$$

smooth one-to-one functions $\vartheta_+ : [0, \alpha_+] \to [0, \widetilde{\alpha}_+]$ and $\vartheta_- : [-\alpha_-, 0] \to [-\widetilde{\alpha}_-, 0]$, which form via

$$\vartheta(\theta) = \begin{cases} \vartheta_-(\theta)\,, & \theta \le 0\,, \\ \vartheta_+(\theta)\,, & \theta \ge 0\,, \end{cases}$$

a Lipschitz isomorphism $\vartheta : [-\alpha_-, \alpha_+] \to [-\widetilde{\alpha}_-, \widetilde{\alpha}_+]$. Thus the composition with $\theta = \vartheta^{-1}$ realizes an isomorphism between the Sobolev spaces $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{H}} = H^1_0(-\widetilde{\alpha}_-, \widetilde{\alpha}_+)$ for the D problem and $\widetilde{\mathcal{H}} = H^1(-\widetilde{\alpha}_-, \widetilde{\alpha}_+)$ for the N problem. For the following we denote by $\Sigma_\pm$ the intersection of $\mathcal{K}_\pm$ with the annulus $\{1 < r < 2\}$, i.e.

$$\Sigma_+ = \{r(\cos\theta, \sin\theta) : 1 < r < 2,\ 0 < \theta < \alpha_+\}\,,\ \Sigma_- = \{r(\cos\theta, \sin\theta) : 1 < r < 2,\ -\alpha_- < \theta < 0\}$$

and by (ρ, ϑ) the polar coordinates $\rho(\cos\vartheta, \sin\vartheta) = (y_1, y_2)$. Then

$$\rho^2 = \frac{r^2}{a_{22}^\pm}\big(a_{22}^\pm \cos^2\theta - 2a_{21}^\pm \sin\theta\cos\theta + a_{11}^\pm \sin^2\theta\big)$$

leads to the representations

$$\begin{aligned} L_+(\Sigma_+) &= \big\{(\rho,\vartheta) : 0 \le \vartheta < \widetilde{\alpha}_+,\ f_+(\vartheta) < \rho < 2f_+(\vartheta)\big\}\,, \\ L_-(\Sigma_-) &= \big\{(\rho,\vartheta) : -\widetilde{\alpha}_- < \vartheta < 0,\ f_-(\vartheta) < \rho < 2f_-(\vartheta)\big\} \end{aligned}$$

with the smooth functions

$$f_\pm(\vartheta) = \Big(\frac{a_{22}^\pm \cos^2\theta_\pm(\vartheta) - 2a_{21}^\pm \sin\theta_\pm(\vartheta)\cos\theta_\pm(\vartheta) + a_{11}^\pm \sin^2\theta_\pm(\vartheta)}{a_{22}^\pm}\Big)^{1/2}.$$

Since $f_-(0) = f_+(0)$, the union $\overline{L_-(\Sigma_-)} \cup \overline{L_+(\Sigma_+)}$ has a piecewise smooth boundary and, in general, corner points at $(1,0)$ and $(2,0)$, but with angles different from 0 and 2π.

Lemma 4.1. *The complex number λ is an eigenvalue of the pencil $\Pi_{\hat\mu}(\lambda)$ associated with the D problem (N problem) for the bimaterial angle if and only if there exists $u \in H_0^1(-\widetilde{\alpha}_-, \widetilde{\alpha}_+)$ $(u \in H^1(-\widetilde{\alpha}_-, \widetilde{\alpha}_+))$ such that*

$$\int_{-\widetilde{\alpha}_-}^{\widetilde{\alpha}_+} \sqrt{\det A}\,\big(u'(\vartheta)\,\overline{v'(\vartheta)} - \lambda^2 u(\vartheta)\overline{v(\vartheta)}\big)\,d\vartheta = 0$$

for all $v \in H_0^1(-\widetilde{\alpha}_-, \widetilde{\alpha}_+)$ $(v \in H^1(-\widetilde{\alpha}_-, \widetilde{\alpha}_+))$.

Proof. Let $u, v \in \mathcal{H}$. Because of (4.1) the substitution $\mathrm{y} = L_\pm \mathrm{x}$ for $\mathrm{x} \in \Sigma_\pm$ leads to

$$\begin{aligned} J &\stackrel{\text{def}}{=} \frac{1}{\log 2}\int_\Sigma A\,\nabla_{\mathrm{x}}\, r^\lambda u(\theta)\cdot\nabla_{\mathrm{x}}\, r^{-\lambda}\overline{v(\theta)}\,d\mathrm{x} \\ &= \frac{a_+}{\log 2}\int_{L_+(\Sigma_+)} \nabla_{\mathrm{y}}\,\rho^\lambda\tilde u(\vartheta)\cdot\nabla_{\mathrm{y}}\,\rho^{-\lambda}\overline{\tilde v(\vartheta)}\,d\mathrm{y} + \frac{a_-}{\log 2}\int_{L_-(\Sigma_-)} \nabla_y\,\rho^\lambda\tilde u(\vartheta)\cdot\nabla_{\mathrm{y}}\,\rho^{-\lambda}\overline{\tilde v(\vartheta)}\,d\mathrm{y} \end{aligned}$$

with $\tilde u(\vartheta) = (f_\pm(\vartheta))^{-\lambda}u(\theta(\vartheta))$, $\tilde v(\vartheta) = (f_\pm(\vartheta))^{\lambda}v(\theta(\vartheta))$ for $\vartheta \gtrless 0$. Since $f = f_\pm(\vartheta)$ for $\vartheta \gtrless 0$ belongs to $H^1(-\widetilde{\alpha}_-, \widetilde{\alpha}_+)$, it follows that $\tilde u, \tilde v \in \widetilde{\mathcal{H}}$. Moreover, from

$$\nabla_{\mathrm{y}}\,\rho^\lambda\tilde u(\vartheta)\cdot\nabla_{\mathrm{y}}\,\rho^{-\lambda}\overline{\tilde v(\vartheta)} = \rho^{-2}\big(\tilde u'(\vartheta)\,\overline{\tilde v'(\vartheta)} - \lambda^2\tilde u(\vartheta)\overline{\tilde v(\vartheta)}\big)$$

we obtain

$$J = \frac{a_+}{\log 2}\int_0^{\widetilde{\alpha}_+}\int_{f_+(\vartheta)}^{2f_+(\vartheta)} \big(\tilde u'\,\overline{\tilde v'} - \lambda^2\tilde u\overline{\tilde v}\big)\,\frac{d\rho}{\rho}\,d\vartheta + \frac{a_-}{\log 2}\int_{-\widetilde{\alpha}_-}^{0}\int_{f_-(\vartheta)}^{2f_-(\vartheta)} \big(\tilde u'\,\overline{\tilde v'} - \lambda^2\tilde u\overline{\tilde v}\big)\,\frac{d\rho}{\rho}\,d\vartheta\,. \qquad \square$$

4.2 Proof of Lemma 2.10

Due to Lemma 4.1 the eigenvalues of the operator pencil $\Pi_{\hat\mu}$ coincide with the eigenvalues of the following one-dimensional problem: Find nontrivial $u \in H^1(-\widetilde\alpha_-, \widetilde\alpha_+)$ satisfying

$$u'' + \lambda^2 u = 0\,, \quad u(-0) = u(+0)\;,\; a_- u'(-0) = a_+ u'(+0)\,, \tag{4.4}$$

with the corresponding boundary conditions

$$u(-\widetilde\alpha_-) = u(\widetilde\alpha_+) = 0 \qquad \text{or} \qquad u'(-\widetilde\alpha_-) = u'(\widetilde\alpha_+) = 0\,. \tag{4.5}$$

Recall that $a_\pm = \sqrt{\det A} = \sqrt{\det \hat\mu}$. The eigenvalues of (4.4) have been investigated in [28] and [24] for different boundary conditions. In particular, for the Dirichlet or Neumann conditions (4.5) the sharp lower bounds

$$\begin{aligned} \lambda^\circ &> \frac{\pi}{2\max(\widetilde\alpha_+, \widetilde\alpha_-)} && \text{for} \quad \widetilde\alpha_+ \neq \widetilde\alpha_-\,, \\ \lambda^\circ &= \frac{\pi}{2\alpha} && \text{if} \quad \widetilde\alpha_+ = \widetilde\alpha_- = \alpha \end{aligned} \tag{4.6}$$

have been obtained. Since the opening angles of the sectors $\widetilde{\mathcal{K}}_\pm = L(\mathcal{K}_\pm)$ satisfy $0 < \widetilde\alpha_\pm < 2\pi$, the relation (4.6) implies $\operatorname{Re}\lambda^\circ > 1/4$ for any $\alpha_\pm$ and positive definite $A_\pm$.

Furthermore, in view of (4.3) condition **A3**(i) implies $\widetilde\alpha_\pm \le \pi$ and $\widetilde\alpha_+ + \widetilde\alpha_- < 2\pi$, and hence $\operatorname{Re}\lambda^\circ > 1/2$. To prove the last assertion of Lemma 2.10 we suppose that $\alpha_+ > \pi$, hence $\widetilde\alpha_+ > \pi$. It follows from (4.6) that $\operatorname{Re}\lambda^\circ > 1/3$ if $\widetilde\alpha_+ \le 3\pi/2$, which in view of (4.2) and $\sin\alpha_+ < 0$ leads to the sufficient condition

$$a_{21}^+ \sin\alpha_+ - a_{22}^+ \cos\alpha_+ \ge 0\,.$$

To write the condition in a form which is invariant under rotations we note that

$$a_{21}^+ \sin\alpha_+ - a_{22}^+ \cos\alpha_+ = (A_+\nu_1, \nu_2)\,,$$

where ν_j, $j = 1, 2$, are the interior normals to the two sides of $\mathcal{K}_+$. □

5 Interior bimaterial angles

Here we provide a more detailed study of operator pencils corresponding to interior bimaterial angles. Compared to Lemma 2.8 we obtain the slightly better bound $\operatorname{Re}\lambda^\circ > 1/2$ if the two material matrices are comparable. Furthermore, we give an example that even for right angles and two diagonal coefficient matrices $\operatorname{Re}\lambda^\circ$ can be arbitrarily small.

5.1 Characteristic equation

By Lemma 3.1, it suffices to consider the periodic eigenvalue problem (2.7, 2.8) for the special case that the differential operator (2.1) has the form

$$\nabla\cdot\hat\mu\nabla = \begin{cases} \partial_{x_1}^2 + t^2\partial_{x_2}^2\,, & \mathrm{x} \in \mathcal{K}_+ \overset{\text{def}}{=} \{\mathrm{x} : \arg \mathrm{x} \in (\gamma, \delta)\}\,, \\ a(\partial_{x_1}^2 + \partial_{x_2}^2)\,, & \mathrm{x} \in \mathcal{K}_- \overset{\text{def}}{=} \mathbb{R}^2 \setminus \overline{\mathcal{K}_+} \end{cases} \tag{5.1}$$

with $t \geq 1$ and $a > 0$, where $-\gamma \geq 0$, $0 < \delta$ with $\delta - \gamma \in (0, 2\pi)$. The corresponding linear transformation L can be constructed in the following way: The matrix $\hat{\mu}$ takes the two values $\hat{\mu}_j$ on the sectors $\mathcal{K}_j$, $j = 1, 2$. First one can diagonalize the symmetric matrix $\hat{\mu}_1$ by rotation. If the diagonal elements of the resulting matrix are not equal, then one of the axes is stretched so that these entries become equal. This results in the differential operator $-a_0\Delta$ with some $a_0 > 0$, whereas in the other sector we obtain the operator $-\nabla^T\tilde{\hat{\mu}}_2\nabla$ with a transformed, but symmetric and positive definite matrix $\tilde{\hat{\mu}}_2$. By another rotation, which obviously does not change $a_0\Delta$, one makes the matrix $\tilde{\hat{\mu}}_2$ diagonal with entries $a_1, a_2 > 0$. If $a_1 \leq a_2$, then the scaling with a_1^{-1} gives the desired form. Otherwise, one has to interchange the y_1- and y_2-axes and to scale with a_2^{-1}.

The characteristic equation for $\Pi(\lambda)$ associated with (5.1) was obtained in Il'in [13]: Introduce in the complex plane the transformation $T : x_1 + ix_2 \to tx_1 + ix_2$, which stretches (because of $t \geq 1$) the unit circle to the ellipse with the two main radii t and 1. Then

$$Te^{i\delta} = K_\delta e^{iF(\delta)}$$

with

$$K_\delta = |t\cos\delta + i\sin\delta| = \sqrt{t^2\cos^2\delta + \sin^2\delta}\,, \quad F(\delta) = \arg(t\cos\delta + i\sin\delta)\,. \tag{5.2}$$

The application of T in $\mathcal{K}_+$ transforms the differential operator (5.1) to $t^2\Delta$. This allows to take $e^{\pm i\lambda\theta}$ as elementary solutions of the differential equations (2.7) in $S^1 \cap T(\mathcal{K}_+)$ as well as in $S^1 \cap \mathcal{K}_-$. Matching the boundary values at δ, $F(\delta)$ and γ, $F(\gamma)$ in accordance with the transmission conditions (2.8) then gives, after some algebraic manipulations, the equation

$$4at\cosh(\lambda\log|\zeta|) = (a+t)^2\cos\lambda(2\pi - \beta + \phi) - (a-t)^2\cos\lambda(2\pi - \beta - \phi)\,, \tag{5.3}$$

where $\beta = \delta - \gamma$ denotes the angle of $\mathcal{K}_+$ and the complex number ζ is defined by

$$\zeta \overset{\text{def}}{=} \frac{t\cos\delta + i\sin\delta}{t\cos\gamma + i\sin\gamma} = \frac{K_\delta}{K_\gamma}\,e^{i\phi}\,, \tag{5.4}$$

i.e., $\phi = \arg\zeta$ is the opening angle of $T(\mathcal{K}_+)$.

Consider the characteristic equation (5.3) for two well known special cases. If the interface between $\mathcal{K}_+$ and $\mathcal{K}_-$ is a straight line, then $\beta = \phi = \pi$ and $|\zeta| = 1$. So (5.3) has the simple form

$$4at = (a+t)^2\cos 2\lambda\pi - (a-t)^2$$

and therefore only the roots $\lambda \in \mathbb{Z}$.

The case $t = 1$ corresponds to the transmission problem for the two operators Δ in $\mathcal{K}_+$ and $a\Delta$ in $\mathcal{K}_-$. Here $\zeta = e^{i\beta}$ and equation (5.3) can be transformed to

$$(a+1)^2\sin^2\lambda\pi = (a-1)^2\sin^2\lambda(\pi - \beta)\,. \tag{5.5}$$

It was shown in [6, Lemma 6.2] that for any positive $a \neq 1$ the solution λ° of (5.5) with minimal positive real part is real and satisfies

$$\lambda^\circ > \min\left(\frac{\pi}{\beta}, \frac{\pi}{2\pi - \beta}\right) > 1/2\,.$$

Hence it remains to study equation (5.3) for $\beta = \delta - \gamma \neq \pi$ and $t > 1$, which is always supposed in the following.

Remark 5.1. There exist simple configurations for which (5.3) has solutions with arbitrarily small positive real part. Let, for example, $t = a$, $\delta = k\pi/2$, $k \in \mathbb{Z}$, and $\beta = \pi/2$. Then $\mathcal{K}_+$ coincides with a quarter plane and is therefore invariant under the transformation T, i.e., $\phi = \beta$. Moreover, either $K_\delta = t$, $K_\gamma = 1$, or $K_\delta = 1$, $K_\gamma = t$, and equation (5.3) takes the form

$$\cosh(\lambda \log t) = \cos 2\pi\lambda \,. \tag{5.6}$$

Since

$$\cos 2\pi\lambda - \cos(i\lambda \log t) = -2 \sin \frac{\lambda}{2}(2\pi + i \log t) \sin \frac{\lambda}{2}(2\pi - i \log t)$$

(5.6) is satisfied if

$$\frac{\lambda}{2}\,(2\pi \pm i \log t) = 2k\pi \,,\ k \in \mathbb{Z} \,,$$

which implies that the eigenvalue of the operator pencil $\Pi(\lambda)$ with minimal positive real part is

$$\lambda^\circ = \frac{8\pi^2}{4\pi^2 + \log^2 t} \pm i \frac{4\pi \log t}{4\pi^2 + \log^2 t} \,.$$

For the analysis of equation (5.3), we need some simple relations involving the function F defined in (5.2). From

$$Te^{i\delta} = t \cos\delta + i \sin\delta$$

we have that

$$\cos F(\delta) = \frac{t \cos\delta}{K_\delta} \,,\ \sin F(\delta) = \frac{\sin\delta}{K_\delta} \,, \tag{5.7}$$

which implies

$$\sin(F(\delta) \pm F(\gamma)) = \frac{t \sin(\delta \pm \gamma)}{K_\delta K_\gamma}$$

for any values of δ and γ. In particular, the opening angles of $\mathcal{K}_+$ and $T(\mathcal{K}_+)$ satisfy

$$\sin\phi = \frac{t}{K_\delta K_\gamma} \sin\beta \,, \tag{5.8}$$

which gives, because of $\beta \neq \pi$,

$$\cos\frac{\beta}{2} \cos\frac{\phi}{2} > 0 \,. \tag{5.9}$$

5.2 Analysis of equation (5.3)

In this subsection we prove

Lemma 5.2. *The characteristic equation* (5.3) *has no roots in the strip* $0 < \operatorname{Re}\lambda \leq 1/2$ *for any* $\delta \neq \gamma$ *if the coefficients of the differential operator* (5.1) *satisfy* $a \in (0, 1] \cup [t^2, \infty)$.

Because of

$$a(x_1^2 + x_2^2) = (\hat{\mu}_1 L^T x, L^T x) \quad \text{and} \quad x_1^2 + t^2 x_2^2 = (\hat{\mu}_2 L^T x, L^T x) \,,$$

we then obtain the equivalent form of Lemma 2.11:

Corollary 5.3. *If* $\hat{\mu}_1 \geq \hat{\mu}_2$ *or* $\hat{\mu}_1 \leq \hat{\mu}_2$, *then* $\operatorname{Re}\lambda^\circ > 1/2$ *for any interior bimaterial angle.*

5.2.1 Preparation

Rewrite (5.3) in the form

$$\sinh^2\frac{\lambda}{2}\log|\zeta| + \frac{(a+t)^2}{4at}\sin^2\frac{\lambda}{2}(2\pi-\beta+\phi) - \frac{(a-t)^2}{4at}\sin^2\frac{\lambda}{2}(2\pi-\beta-\phi) = 0$$

and note that

$$\frac{(a+t)^2}{4at} = \frac{1}{2}\Big(\cosh\Big(\log\frac{a}{t}\Big)+1\Big), \quad \frac{(a-t)^2}{4at} = \frac{1}{2}\Big(\cosh\Big(\log\frac{a}{t}\Big)-1\Big).$$

Using the notation

$$C \stackrel{\text{def}}{=} \cosh\Big(\log\frac{a}{t}\Big), \tag{5.10}$$

$$f(\lambda) \stackrel{\text{def}}{=} (C+1)\sin^2\frac{\lambda}{2}(2\pi-\beta+\phi) - (C-1)\sin^2\frac{\lambda}{2}(2\pi-\beta-\phi), \tag{5.11}$$

$$g(\lambda) \stackrel{\text{def}}{=} 2\sinh^2\frac{\lambda}{2}\log|\zeta|, \tag{5.12}$$

one has therefore to determine for which factors C the function

$$f+g = (C+1)\sin^2\frac{\lambda}{2}(2\pi-\beta+\phi) - (C-1)\sin^2\frac{\lambda}{2}(2\pi-\beta-\phi) + 2\sinh^2\frac{\lambda}{2}\log|\zeta| \tag{5.13}$$

has no roots in the strip $0 < \operatorname{Re}\lambda \le 1/2$. Since the functions f and g are even, we will study the roots of $f+g$ in the extended strip $|\operatorname{Re}\lambda| \le 1/2$ using Rouché's Theorem.

5.2.2 Roots

Lemma 5.4. *For any $C \ge 1$, the function f has two roots in the strip $|\operatorname{Re}\lambda| \le 1/2$, and $f+g$ has a double root at $\lambda = 0$.*

Proof. It is obvious that $\lambda = 0$ is a root of multiplicity 2 for both f and $f+g$. To show that $f(\lambda)$ has no other roots satisfying $|\operatorname{Re}\lambda| \le 1/2$, we use the relation

$$|\sin(\xi+i\eta)|^2 = \sin^2\xi + \sinh^2\eta = \cosh^2\eta - \cos^2\xi = \frac{1}{2}\big(\cosh 2\eta - \cos 2\xi\big), \tag{5.14}$$

which gives

$$\Big|(C+1)\sin^2\frac{\lambda}{2}(2\pi-\beta+\phi)\Big| = (C+1)\Big(\sin^2\frac{\mu}{2}(2\pi-\beta+\phi) + \sinh^2\frac{\nu}{2}(2\pi-\beta+\phi)\Big),$$

$$\Big|(C-1)\sin^2\frac{\lambda}{2}(2\pi-\beta-\phi)\Big| = (C-1)\Big(\sin^2\frac{\mu}{2}(2\pi-\beta-\phi) + \sinh^2\frac{\nu}{2}(2\pi-\beta-\phi)\Big),$$

with $\lambda = \mu + i\nu$. Since

$$\sin^2\frac{\mu}{2}(2\pi-\beta+\phi) - \sin^2\frac{\mu}{2}(2\pi-\beta-\phi) = \sin\mu(2\pi-\beta)\sin\mu\phi,$$

the relations $2\pi-\beta, \phi \in (0, 2\pi)$ and $|\mu| \le 1/2$ imply that

$$\sin^2\frac{\mu}{2}(2\pi-\beta+\phi) \ge \sin^2\frac{\mu}{2}(2\pi-\beta-\phi),$$

with equality only for $\mu = 0$. Furthermore, because of $2\pi - \beta + \phi > \left|2\pi - \beta - \phi\right|$,

$$\sinh^2 \frac{\nu}{2}(2\pi - \beta + \phi) \geq \sinh^2 \frac{\nu}{2}(2\pi - \beta - \phi)$$

for all ν with equality only if $\nu = 0$. Thus

$$(C+1)\left|\sin^2 \frac{\lambda}{2}(2\pi - \beta + \phi)\right| > (C-1)\left|\sin^2 \frac{\lambda}{2}(2\pi - \beta - \phi)\right|$$

for all $\lambda \neq 0$ with $|\operatorname{Re}\lambda| \leq 1/2$. Consequently, by (5.11)

$$|f(\lambda)| \geq (C+1)\left|\sin^2 \frac{\lambda}{2}(2\pi - \beta + \phi)\right| - (C-1)\left|\sin^2 \frac{\lambda}{2}(2\pi - \beta - \phi)\right|, \tag{5.15}$$

which shows that $f(\lambda) = 0$ and $|\operatorname{Re}\lambda| \leq 1/2$ imply $\lambda = 0$. □

Remark 5.5. Using (5.14) one can rewrite

$$\left|\sin^2 \frac{\lambda}{2}(2\pi - \beta + \phi)\right| - \left|\sin^2 \frac{\lambda}{2}(2\pi - \beta - \phi)\right| = \sin\mu(2\pi - \beta)\sin\mu\phi + \sinh\nu(2\pi - \beta)\sinh\nu\phi$$

and

$$\left|\sin^2 \frac{\lambda}{2}(2\pi - \beta + \phi)\right| + \left|\sin^2 \frac{\lambda}{2}(2\pi - \beta - \phi)\right| = \cosh\nu(2\pi - \beta)\cosh\nu\phi - \cos\mu(2\pi - \beta)\cos\mu\phi\,,$$

which transforms inequality (5.15) to

$$\begin{aligned} |f(\lambda)| \geq C\big(&\sin\mu(2\pi - \beta)\sin\mu\phi + \sinh\nu(2\pi - \beta)\sinh\nu\phi\big) \\ &+ \big(\cosh\nu(2\pi - \beta)\cosh\nu\phi - \cos\mu(2\pi - \beta)\cos\mu\phi\big)\,. \end{aligned} \tag{5.16}$$

5.2.3 Estimate on the boundary

Now we arrive at the problem to determine for which C the inequality

$$|g(\lambda)| < |f(\lambda)|\,,\ \lambda \in \partial R\,, \tag{5.17}$$

is valid, where $R = (-1/2, 1/2) \times (-b, b)$ with arbitrarily large b. Then, by Rouché's Theorem, $f + g$ and f have the same number of roots with $|\operatorname{Re}\lambda| \leq 1/2$. Hence, if (5.17) holds, then by Lemma 5.4 the characteristic equation (5.3) does not have other roots than $\lambda = 0$ in this strip.

Lemma 5.6. *For any fixed $t \geq 1$ and $C \geq 1$, there exists b_0 such that*

$$|g(\mu + i\nu)| < |f(\mu + i\nu)| \quad \textit{for all} \quad |\mu| \leq 1/2 \quad \textit{and} \quad |\nu| > b_0\,.$$

Proof. Since $1 \leq K_\delta, K_\gamma \leq t$ by (5.2) and

$$|g(\mu + i\nu)| = 2\left|\sinh^2 \frac{\mu + i\nu}{2}\log|\zeta|\right| = \cosh(\mu\log|\zeta|) - \cos(\nu\log|\zeta|)\,,$$

the function g is bounded by

$$|g(\lambda)| \leq \cosh\frac{\log t}{2} + 1 = 2\cosh^2\frac{\log t}{4}$$

if $|\operatorname{Re}\lambda| \le 1/2$. On the other hand, from (5.16)

$$|f(\lambda)| \ge C \sinh \nu(2\pi-\beta) \sinh \nu\phi + \cosh \nu(2\pi-\beta) \cosh \nu\phi - 1\,.$$

The inequality $\max(2\pi-\beta, \phi) > \pi$, which follows from (5.8), implies the lower bound

$$|f(\lambda)| \ge \cosh \nu\pi - 1 = 2\sinh^2 \frac{\nu\pi}{2}\,.$$

This proves the assertion with $b_0 = \dfrac{2}{\pi}\operatorname{Arsh}\cosh \dfrac{\log t}{4}$. □

It remains to determine C such that

$$|g(\lambda)| < |f(\lambda)| \quad \text{for all} \quad \lambda = \mu + i\nu \quad \text{with} \quad |\mu| = 1/2\,.$$

In this case, the right hand side of (5.16) takes the form

$$\begin{aligned} &C\Big(\sin\frac{2\pi-\beta}{2}\sin\frac{\phi}{2} + \sinh \nu(2\pi-\beta)\sinh \nu\phi\Big) + \cosh \nu(2\pi-\beta)\cosh\nu\phi - \cos\frac{2\pi-\beta}{2}\cos\frac{\phi}{2} \\ &= C\sin\frac{\beta}{2}\sin\frac{\phi}{2} + \cos\frac{\beta}{2}\cos\frac{\phi}{2} + C\sinh\nu(2\pi-\beta)\sinh\nu\phi + \cosh 2\nu(2\pi-\beta)\cosh\nu\phi\,. \end{aligned}$$

Therefore C should be chosen such that

$$|g(\lambda)| = 2\sinh^2\frac{1}{4}\log|\zeta| + 2\sin^2\frac{\nu}{2}\log|\zeta| = \cosh\frac{1}{2}\log|\zeta| - \cos\nu\log|\zeta|$$

satisfies

$$|g(\lambda)| < C\sin\frac{\beta}{2}\sin\frac{\phi}{2} + \cos\frac{\beta}{2}\cos\frac{\phi}{2} + C\sinh\nu(2\pi-\beta)\sinh\nu\phi + \cosh 2\nu(2\pi-\beta)\cosh\nu\phi\,.$$

For any $\nu \in \mathbb{R}$ and $C \ge 0$,

$$\big|\cos(\nu\log|\zeta|)\big| \le 1 \le C\sinh\nu(2\pi-\beta)\sinh\nu\phi + \cosh 2\nu(2\pi-\beta)\cosh\nu\phi\,.$$

Thus one has to find C satisfying the inequality

$$\cosh\frac{1}{2}\log|\zeta| < C\sin\frac{\beta}{2}\sin\frac{\phi}{2} + \cos\frac{\beta}{2}\cos\frac{\phi}{2}\,.$$

Lemma 5.7. *For all $t > 1$, δ and γ with $\delta - \gamma \ne \pi$,*

$$\cosh\frac{1}{2}\log|\zeta| < \frac{t^2+1}{2t}\sin\frac{\beta}{2}\sin\frac{\phi}{2} + \cos\frac{\beta}{2}\cos\frac{\phi}{2}\,. \tag{5.18}$$

Proof. The proof is based on simple geometry. Recall that

$$\zeta = \frac{t\cos\delta + i\sin\delta}{t\cos\gamma + i\sin\gamma} = \frac{K_\delta}{K_\gamma}\,e^{i\phi}\,,$$

which gives

$$\cosh\frac{1}{2}\log|\zeta| = \frac{1}{2}\left(\sqrt{\frac{K_\delta}{K_\gamma}} + \sqrt{\frac{K_\gamma}{K_\delta}}\right) = \frac{K_\delta + K_\gamma}{2\sqrt{K_\delta K_\gamma}}\,, \tag{5.19}$$

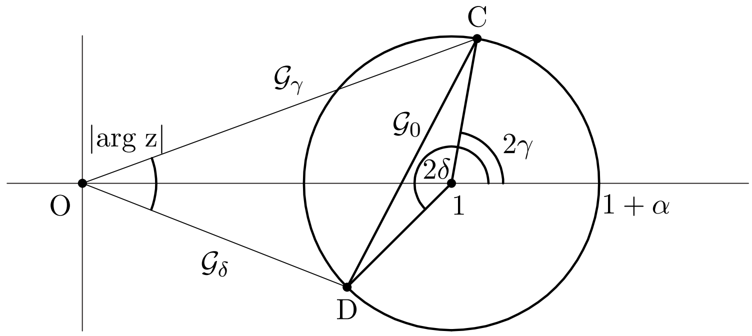

Figure 1: Argument of z

and introduce

$$z \overset{\text{def}}{=} e^{i\beta}\,\overline{\zeta} = e^{i(\delta-\gamma)}\,\frac{t\cos\delta - i\sin\delta}{t\cos\gamma - i\sin\gamma} = \frac{1+\alpha e^{2i\delta}}{1+\alpha e^{2i\gamma}} \quad \text{with} \quad \alpha = \frac{t-1}{t+1}\,. \tag{5.20}$$

Figure 1 depicts the numerator $1+\alpha e^{2i\delta}$ and the denominator $1+\alpha e^{2i\gamma}$ of z, and $|\arg z|$ which is the angle at O of the triangle ODC. Its sides have the lengths

$$\mathcal{G}_\delta = |\alpha e^{2i\delta}+1|\,,\ \mathcal{G}_\gamma = |\alpha e^{2i\gamma}+1|\,, \quad \text{and} \quad \mathcal{G}_0 = 2\alpha|\sin(\delta-\gamma)|\,,$$

hence by the half angle formula for plane triangles,

$$\cos^2\frac{\arg z}{2} = \frac{(\mathcal{G}_\delta+\mathcal{G}_\gamma+\mathcal{G}_0)(\mathcal{G}_\delta+\mathcal{G}_\gamma-\mathcal{G}_0)}{4\mathcal{G}_\delta\mathcal{G}_\gamma} = \frac{(\mathcal{G}_\delta+\mathcal{G}_\gamma)^2 - 4\alpha^2\sin^2\beta}{4\mathcal{G}_\delta\mathcal{G}_\gamma}\,. \tag{5.21}$$

Note that this relation is also valid in the exceptional case $\arg z = 0$, i.e., if $\arg(\alpha e^{2i\delta}+1) = \arg(\alpha e^{2i\gamma}+1)$. Since $\arg z = \beta - \arg\zeta = \beta - \phi$ and, on recalling (5.2),

$$K_\delta = \frac{t+1}{2}|\alpha e^{2i\delta}+1| = \frac{t+1}{2}\,\mathcal{G}_\delta\,,$$

the half angle formula (5.21) can be written as

$$\cos^2\frac{\beta-\phi}{2} = \frac{(K_\delta+K_\gamma)^2 - (t-1)^2\sin^2\beta}{4K_\delta K_\gamma}\,. \tag{5.22}$$

Finally, by using (5.19) and

$$\frac{\sin\beta}{K_\delta K_\gamma} = \frac{\sin\phi}{t}\,,$$

which follows from (5.8), elementary algebra transforms (5.22) to the equality

$$\begin{aligned}\cosh^2\frac{1}{2}\log|\zeta| &= \cos^2\frac{\beta-\phi}{2} + \frac{(t-1)^2}{4t}\sin\beta\sin\phi \\ &= \Big(\frac{t^2+1}{2t}\sin\frac{\beta}{2}\sin\frac{\phi}{2} + \cos\frac{\beta}{2}\cos\frac{\phi}{2}\Big)^2 - \frac{(t^2-1)^2}{4t^2}\sin^2\frac{\beta}{2}\sin^2\frac{\phi}{2}\,.\end{aligned}$$

Because of (5.9), the sum within the first brackets on the right hand side is positive. Therefore the assumption $t > 1$ implies (5.18). $\square$

Since

$$\frac{t^2+1}{2t} = \cosh(\log t)\,,$$

we conclude from Lemmas 5.6 and 5.7:

Corollary 5.8. *If $t > 1$ and $C \geq \cosh(\log t)$, then $|g(\lambda)| < |f(\lambda)|$ for all λ with $|\operatorname{Re}\lambda| = 1/2$.*

Hence, the proof of Lemma 5.2 is completed by noting (5.10) and that

$$C = \cosh\left(\log \frac{a}{t}\right) \geq \cosh(\log t)$$

if and only if either $\frac{a}{t} \geq t$ or $\frac{t}{a} \geq t$.

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