Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

The critical Galton–Watson process without further power moments

Sergei Nagaev[*, 1], Vitali Wachtel[†, 2]

submitted: 01. November 2005

[1] Sobolev Institute of Mathematics
Siberian Branch
of the Russian Academy of Sciences
4 Acad. Koptyug avenue
630090 Novosibirsk
Russia
Email: nagaev@math.nsc.ru

[2] Weierstrass Institute for
Applied Analysis and Stochastics
Mohrenstrasse 39
10117 Berlin
Germany
Email: vakhtel@wias-berlin.de

No. 1062
Berlin 2005

2000 *Mathematics Subject Classification.* 60J80, 60F05.

Key words and phrases. Critical Galton–Watson process, conditional limit theorem, slowly varying function, functional normalization .

*) Supported in part by the INTAS grant 03-51-5018
†) Supported in part by the German Science Foundation.

.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract. In this paper we prove a conditional limit theorem for a critical Galton – Watson branching process $\{Z_n; n \geq 0\}$ with offspring generating function $s+(1-s)L((1-s)^{-1})$, where $L(x)$ is slowly varying. In contrast to a well-known theorem of Slack (1968, 1972) we use a functional normalization, which gives an exponential limit. We give also an alternative proof of Sze's (1976) result on the asymptotic behavior of the nonextinction probability

1. Introduction, statement of results, and discussion

Let $Z = \{Z_n; n \geq 0\}$ be a critical Galton – Watson process initiated by a single particle.The main purpose of this note is to study processes with an offspring generating function $f(s)$ satisfying the condition

$$f(s) = s + (1-s)L((1-s)^{-1}) \text{ for some slowly varying } L(x). \tag{1.1}$$

Note that $L(x) \to 0$ as $x \to \infty$by the assumed criticality of our process.

Evidently, $\mathbf{E}Z_n^{1+\delta} = \infty$for every $\delta > 0$, provided that (1.1) holds. For critical branching processes with this property there areonly a few papers. Zubkov [12] proved limit theorems for the distance to thecommon nearest ancestor under some additional restrictions on the function $L(x)$. In Sze's paper [11] the asymptotic behaviorof the nonextinction probability $Q_n := \mathbf{P}(Z_n > 0)$ was studied. Bondarenko and Topchii [2] obtained lower and upper bounds for theexpectation of the maximum $M_n := \max_{k \leq n} Z_k$ under the condition $\mathbf{E}Z_1 \log^\beta(1 + Z_1) < \infty$ for some $\beta > 0$.

We begin with the following general result on critical Galton – Watson processes which was proven by Slack [9, 10].

Theorem 1 (Slack)**.** *For a critical Galton – Watson process the following four assertions are equivalent.*

- *(a) The sequence of distributions $F_n(x) := \mathbf{P}(Q_n Z_n < x | Z_n > 0)$ converges weakly to some nondegenerate limit;*
- *(b) $f(s) = s + (1-s)^{1+\alpha} L((1-s)^{-1})$ for some $\alpha > 0$;*
- *(c) There exists a slowly varying function $L^*(x)$ such that $Q_n = n^{-1/\alpha} L^*(n)$ for some $\alpha > 0$;*
- *(d) Laplace transform of the limit of the sequence F_n is $\lambda \mapsto 1 - \lambda(1+\lambda^\alpha)^{-1/\alpha}$ for some $\alpha > 0$.*

Therefore, the sequence $F_n(x)$ cannot have a nondegenerate limit if representation (1.1) holds. In other words, the normalization with the nonextinction probability doesn't work in the present case, and we need to find an alternative way to normalize the branching process Z_n.

For a general offspring generating function $f(s)$ we set

$$H(x) := x\Big(f(1-x^{-1}) - 1 + x^{-1}\Big),\ x \geq 1 \tag{1.2}$$

and

$$V(y) := \int_0^{1-1/y} \frac{ds}{f(s) - s} = \int_1^y \frac{dx}{xH(x)},\ y \geq 1. \tag{1.3}$$

Note that $H(x) \equiv L(x)$ if (1.1) holds.

The following conditional limit theorem is our main result.

Theorem 2. *Assume that $f(s)$ satisfies (1.1). Then for all $x > 0$,*

$$\lim_{n\to\infty} \mathbf{P}\Big(H(Q_n^{-1})V(Z_n) < x \Big| Z_n > 0\Big) = 1 - e^{-x}. \tag{1.4}$$

Nonextinction probabilities are in a sense natural norming constants for critical branching processes, sincealways

$$\mathbf{E}\{Q_n Z_n | Z_n > 0\} \equiv 1.$$

But under condition (1.1) the expectation overnormalizes Z_n:

Corollary 1. *Under the assumptions of Theorem 2,*

$$\lim_{n\to\infty} \mathbf{P}(Q_n Z_n < x | Z_n > 0) = 1 \tag{1.5}$$

for every $x > 0$.

It is well-known that for supercritical Galton – Watson processes the normalization with the expectation leads to a nondegenerate limit if and only if $\mathbf{E}Z_1 \log Z_1 < \infty$. Further, if $\mathbf{E}Z_1 \log Z_1 = \infty$, then one can find a sequence $c_n > 0$ such that $c_n Z_n$ converges almost surely. Consequently, in this irregular case, a linear normalization is possible. In contrast to the supercritical case, it follows from (1.4) that there is no linear normalization for Z_n satisfying (1.1).

Apparently Darling [4] was first who used the functional normalization for proving limit theorems. In [4] he studied the limit behavior of a sum of independent identically distributed random variables with slowly varying right tails. Concerning branching processes, this type of normalization was used usually if the expectation of the number of the offsprings is infinite. The first contribution to this area was made also by Darling [5]. He has shown that under some additional assumptions on $f(s)$ there exists $\gamma \in (0,1)$ such that the sequence $\mathbf{P}\Big(\gamma^n \log(1+Z_n) < x\Big)$ converges to a proper distribution function $\Psi(x)$. Hadson and Seneta [7] give sufficient conditions for the weak convergence of $\gamma^n L(Z_n)$ for some slowly varying function $L(x)$ and some $\gamma \in (0,1)$. Barbour and Schuh [1] proved that for every Galton – Watson process with infinite mean there exists a norming function $U(x)$ such that $e^{-n}U(Z_n)$ converges almost surely to some nondegenerate random variable.

The functional normalization $V(x)$ in Theorem 2 is individual: For processes with different offspring generating functions we have different normalizations. In order to compare the limiting behavior of Z_n for different functions $L(x)$ in (1.1), we must reduce individual normalizations to a common one. Below we give some examples of the reduction to the logarithmic normalizing function. In each example we have a limit theorem of the following form: There exist a centering sequence A_n and a norming sequence B_n such that

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{\log Z_n - A_n}{B_n} < x \Big| Z_n > 0\Big) = F(x), \tag{1.6}$$

where $F(x)$ is a distribution function.

Example 1. Assume that

$$L(x) = \beta^{-1}(\log^{1-\beta} x)\exp\{-\log^\beta x\}(1+o(1)) \text{ as } x \to \infty, \tag{1.7}$$

where $\beta \in (0,1)$. Then, recalling that $H(x) \equiv L(x)$ under assumption (1.1) and using definition (1.3) of $V(x)$, we have

$$V(y) = \int_1^y \frac{dx}{xL(x)} = \exp\{\log^\beta y\}(1+o(1)) \text{ as } y \to \infty. \tag{1.8}$$

Because of continuity of the limiting distribution in (1.4), we may replace H and V by their asymptotic equivalents given in (1.7) and (1.8) respectively. Thus,

$$\lim_{n\to\infty} \mathbf{P}\Big(\beta^{-1}(\log^{1-\beta} Q_n^{-1})\exp\{\log^\beta Z_n - \log^\beta Q_n^{-1}\} < x \Big| Z_n > 0\Big) = 1 - e^{-x}$$

underassumption (1.7). Substituting $x = \beta^{-1}e^y$ and taking logarithm, we get

$$\lim_{n\to\infty} \mathbf{P}\Big(\log^\beta Z_n - \log^\beta Q_n^{-1} + (1-\beta)\log\log Q_n^{-1} < y \Big| Z_n > 0\Big) = 1 - \exp\Big(-\frac{e^y}{\beta}\Big).$$

Therefore,

$$\lim_{n\to\infty} \mathbf{P}\Big(\log Z_n < (b_n+y)^{1/\beta}\Big| Z_n>0\Big) = 1-\exp\Big(-\frac{e^y}{\beta}\Big), \tag{1.9}$$

where

$$b_n := \log^\beta Q_n^{-1} - (1-\beta)\log\log Q_n^{-1}. \tag{1.10}$$

Noting that

$$(b_n+y)^{1/\beta} = b_n^{1/\beta} + \frac{y}{\beta} b_n^{1/\beta-1}(1+o(1)) \text{ as } n\to\infty,$$

and taking into account the continuity of the right-hand side in (1.9), we conclude that

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{\log Z_n - b_n^{1/\beta}}{b_n^{1/\beta-1}} < \frac{y}{\beta}\Big| Z_n>0\Big) = 1-\exp\Big(-\frac{e^y}{\beta}\Big). \tag{1.11}$$

The next equalities follow from definition (1.10) of b_n,

$$b_n^{1/\beta} = \log Q_n^{-1} - (\beta^{-1}-1)(\log^{1-\beta} Q_n^{-1})\log\log Q_n^{-1} + o(\log^{1-\beta} Q_n^{-1}),$$

$$b_n^{1/\beta-1} = \log^{1-\beta} Q_n^{-1}(1+o(1)) \text{ as } n\to\infty.$$

Substituting these expressions for $b_n^{1/\beta}$ and $b_n^{1/\beta-1}$ into (1.11) and dropping the $o(1)$-term under the $\mathbf{P}$-symbol, we observe that (1.6) holds with $F(x) := 1-\exp(-e^x/\beta)$ and

$$A_n := \log Q_n^{-1} - (\beta^{-1}-1)(\log^{1-\beta} Q_n^{-1})\log\log Q_n^{-1}, \quad B_n := \log^{1-\beta} Q_n^{-1}.$$

Noting that $\log Z_n - A_n = \log Z_n - \log Q_n^{-1} - B_n(\beta^{-1}-1)\log\log Q_n^{-1}$, we get from (1.6) the relation

$$\log Z_n - \log Q_n^{-1} \sim -(\beta^{-1}-1)\log\log Q_n^{-1} \tag{1.12}$$

on the set $\{Z_n>0\}$.

In the next two examples the process $\log Z_n$ converges without centering, i.e. $A_n\equiv 0$ in (1.6).

Example 2. If $L(x)\sim \log^{-\beta} x$ as $x\to\infty$ for some $\beta>0$, then

$$V(y) = (\beta+1)^{-1}\log^{\beta+1} x(1+o(1)).$$

As we have already mentioned, we may insert the asymptotic equivalents of H and V into (1.4). We thus have

$$\lim_{n\to\infty} \mathbf{P}\Big(\frac{\log^{\beta+1} Z_n}{(\beta+1)\log^\beta Q_n^{-1}} < x\Big| Z_n>0\Big) = 1-e^{-x},$$

which is equivalent to

$$\lim_{n\to\infty} \mathbf{P}\Big(\log Z_n < x\log^{\beta/(\beta+1)} Q_n^{-1}\Big| Z_n>0\Big) = 1-\exp\Big(-\frac{x^{\beta+1}}{\beta+1}\Big). \tag{1.13}$$

Roughly speaking, here $\log Z_n$ grows as $\log^{\beta/(\beta+1)} Q_n^{-1}$. This is slower than for the process from the previous example, since there $\log Z_n\sim\log Q_n^{-1}$ by (1.12).

Example 3. Let $\log_{(1)} x := \log x$ and, for all $k\ge 1$, define recursively $\log_{(k+1)} x := \log(\log_{(k)} x)$. Suppose that $L(x)\sim(\log_{(k)} x)^{-1}$ for some $k\ge 3$.For this choice of $L(x)$,

$$V(y) = \int_0^{\log y}\frac{dx}{L(e^x)} = (\log y)\log_{(k-1)} y(1+o(1)) \text{ as } y\to\infty.$$

Hence, by Theorem 2,

$$\lim_{n\to\infty} \mathbf{P}(\log Z_n \log_{(k-1)} Z_n < x\log_{(k)} Q_n^{-1} | Z_n>0) = 1-e^{-x}. \tag{1.14}$$

Taking logarithm, we obtain

$$\lim_{n\to\infty} \mathbf{P}(\log_{(2)} Z_n + \log_{(k)} Z_n < \log_{(k+1)} Q_n^{-1} + \log x | Z_n>0) = 1-e^{-x}.$$

Since $Z_n \to \infty$ on the set $\{Z_n > 0\}$, and $\log_{(k+1)} Q_n^{-1} \to \infty$ as $n \to \infty$, we infer that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\Big(\Big|\frac{\log_{(2)} Z_n}{\log_{(k+1)} Q_n^{-1}} - 1\Big| > \varepsilon \Big| Z_n > 0\Big) = 0.$$

Therefore,for every $k \geq 3$,

$$\lim_{n\to\infty} \mathbf{P}\Big(\Big|\frac{\log_{(k-1)} Z_n}{\log_{(2k-1)} Q_n^{-1}} - 1\Big| > \varepsilon \Big| Z_n > 0\Big) = 0.$$

This allows to replace $\log_{(k)} Z_n$ in (1.14) by $\log_{(2k-1)} Q_n^{-1}$. As a result, we get

$$\lim_{n\to\infty} \mathbf{P}\Big(\log Z_n < x\frac{\log_{(k)} Q_n^{-1}}{\log_{(2k-1)} Q_n^{-1}}\Big| Z_n > 0\Big) = 1 - e^{-x}. \tag{1.15}$$

This example shows that the process $\log Z_n$ can grow witn an arbitrarily small speed.

Next we turn again to the situation which is described in Theorem 1. Assume that $f(s) = s + (1-s)^{1+\alpha}L((1-s)^{-1})$, where $\alpha \in (0,1]$ and $L(x)$ is a slowly varying function. Thenby definitions (1.2) and (1.3),

$$H(x) = x^{-\alpha}L(x) \tag{1.16}$$

and

$$V(y) = \int_1^y \frac{dx}{x^{1-\alpha}L(x)} = \frac{y^\alpha}{\alpha L(y)}(1 + o(1)) \text{ as } y \to \infty. \tag{1.17}$$

Since $V(y)$ increases,

$$\mathbf{P}(Q_n Z_n < x | Z_n > 0) = \mathbf{P}(V(Z_n) < V(xQ_n^{-1}) | Z_n > 0).$$

Multiplying both parts by $H(Q_n^{-1})$ and taking into account (1.16) and(1.17), we arrive at the identity

$$\mathbf{P}(Q_n Z_n < x | Z_n > 0) = \mathbf{P}(H(Q_n^{-1})V(Z_n) < \alpha^{-1}x^\alpha + \varepsilon_n(x) | Z_n > 0),$$

where $\varepsilon_n(x) \to 0$ as $n \to \infty$. Combining this equality with Theorem 1, we conclude thatthe sequence of distributions $\mathbf{P}\Big(H(Q_n^{-1})V(Z_n) < x | Z_n > 0\Big)$ converges weakly to some nondegenerate limit. Thus, we can combine Theorems 1 and 2 in the following result: *If*

$$f(s) = s + (1-s)^{1+\alpha}L((1-s)^{-1}) \tag{1.18}$$

for some $0 \leq \alpha \leq 1$ *and some slowly varying* $L(x)$*, then*

$$\lim_{n\to\infty} \mathbf{P}\Big(H(Q_n^{-1})V(Z_n) < x | Z_n > 0\Big) = F^{(\alpha)}(x), \tag{1.19}$$

where $F^{(\alpha)}(x)$*is a distribution function.*

The rest of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 2. Section 3 contains an alternative proof of Sze's result on the asymptotic behavior of the nonextinction probability and some remarks related to this.

2. Proof of the main result

In this section we prove Theorem 2 and Corollary 1.

2.1. **Auxiliary results.** An essential step in our method is to connect the weak convergence of the functional normalized sequence $V(Z_n)$ with the convergence of Laplace transforms of Z_n.

Lemma 1. *Let $V(x)$ be a continuous, increasing, slowly varying function. The inverse function of $V(x)$ we denote by $G(x)$. If there exist a continuous function $\varphi(x)$ and a sequence $a_n > 0$ such that for all $x > 0$,*

$$\lim_{n\to\infty} \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\} = \varphi(x) \tag{2.1}$$

then for all $x > 0$,

$$\lim_{n\to\infty} \mathbf{P}\Big(a_n^{-1}V(Z_n) < x\Big|Z_n > 0\Big) = \varphi(x). \tag{2.2}$$

Proof. One can easily verify that for all $x, \varepsilon > 0$ and an arbitrary sequence $\{a_n\}$ the following estimates hold:

$$\begin{aligned}\mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\} &\le \mathbf{P}\Big(Z_n < G(a_n(x+\varepsilon))\Big|Z_n > 0\Big) \\ &\quad + \exp\Big(-\frac{G(a_n(x+\varepsilon))}{G(a_n x)}\Big), \\ \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\} &\ge \mathbf{P}\Big(Z_n < G(a_n(x-\varepsilon))\Big|Z_n > 0\Big) \times \\ &\quad \exp\Big(-\frac{G(a_n(x-\varepsilon))}{G(a_n x)}\Big).\end{aligned}$$

Since $V(y)$ is increasing and slowly varying, then by Theorem 1.11 of [8],

$$\lim_{x\to\infty} \frac{G(x)}{G(cx)} = 0 \tag{2.3}$$

for every constant $c > 1$. Thus,for any given $\varepsilon > 0$ and all $x > 0$,

$$\liminf_{n\to\infty} \mathbf{P}\Big(Z_n < G(a_n(x-\varepsilon))\Big|Z_n > 0\Big) \le \limsup_{n\to\infty} \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\},$$

$$\liminf_{n\to\infty} \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\} \le \limsup_{n\to\infty} \mathbf{P}\Big(Z_n < G(a_n(x+\varepsilon))\Big|Z_n > 0\Big),$$

or, equivalently,

$$\liminf_{n\to\infty} \mathbf{P}\Big(a_n^{-1}V(Z_n) < x-\varepsilon\Big|Z_n > 0\Big) \le \limsup_{n\to\infty} \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\}, \tag{2.4}$$

$$\liminf_{n\to\infty} \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\} \le \limsup_{n\to\infty} \mathbf{P}\Big(a_n^{-1}V(Z_n) < x+\varepsilon\Big|Z_n > 0\Big). \tag{2.5}$$

If (2.1) holds, then (2.4) and (2.5) imply

$$\liminf_{n\to\infty} \mathbf{P}\Big(a_n^{-1}V(Z_n) < x\Big|Z_n > 0\Big) \le \varphi(x+\varepsilon)$$

and

$$\limsup_{n\to\infty} \mathbf{P}\Big(a_n^{-1}V(Z_n) < x\Big|Z_n > 0\Big) \ge \varphi(x-\varepsilon).$$

Letting $\varepsilon \to 0$ and taking into account the continuity of $\varphi(x)$, we get (2.2). $\square$

Remark 1. This proof of Lemma 1 is in the spirit of Lemma 1 from [7], but essentially simpler.

Lemma 2. *Let the sequence $\{y_k;\ k \geq 0\}$ be recursively defined by*

$$y_{k+1} := y_k - y_k l(y_k),\ \ y_0 \in (0,1],$$

where $l(y)$ is a monotone incresing function, $l(y) < 1$ for all y, and $\lim_{y \downarrow 0} l(y) = 0$. Then

$$\lim_{k\to\infty} \frac{y_{k+1}}{y_k} = 1. \tag{2.6}$$

Furthermore, if

$$\limsup_{k\to\infty} \frac{y_{k+2} - y_{k+1}}{y_{k+1} - y_k} = \limsup_{k\to\infty} \frac{y_{k+1} l(y_{k+1})}{y_k l(y_k)} < \infty, \tag{2.7}$$

then there exists a constant $C > 0$ such that for all $n \geq 0$ and $j \geq 1$,

$$j < \int_{y_{n+j}}^{y_n} \frac{dy}{yl(y)} < j + C \sum_{n}^{n+j-1} l(y_k). \tag{2.8}$$

Corollary 2. *If $(n+j) \to \infty$, then*

$$j^{-1} \int_{y_{n+j}}^{y_n} \frac{dy}{yl(y)} \to 1.$$

Proof. Since y_n decreases, the limit $y^* := \lim_{n\to\infty} y_n$ exists, and y^* is theroot of the equation $y = y(1 - l(y))$. But under the condition $l(y) < 1$ the latter equation has the unique solution $y = 0$. Thus, $y^* = 0$, i.e. the sequence y_n converges to zero.Therefore,

$$\frac{y_{n+1}}{y_n} = 1 - l(y_n) \to 1 \text{ as } n \to \infty,$$

i.e. (2.6) is proved.

Obviously,

$$\frac{y_k - y_{k+1}}{y_k l(y_k)} \equiv 1. \tag{2.9}$$

This, together with the monotonicity of $yl(y)$, yields

$$j = \sum_{k=n}^{n+j-1} \frac{y_k - y_{k+1}}{y_k l(y_k)} < \int_{y_{n+j}}^{y_n} \frac{dy}{yl(y)},$$

finishing the proof of the lower bound in (2.8).

Using again the monotonicity of $yl(y)$, we get

$$\sum_{k=n}^{n+j-1} \frac{y_k - y_{k+1}}{y_{k+1} l(y_{k+1})} > \int_{y_{n+j}}^{y_n} \frac{dy}{yl(y)}.$$

On the other hand,

$$\sum_{k=n}^{n+j-1} \frac{y_k - y_{k+1}}{y_{k+1} l(y_{k+1})} - \sum_{k=n}^{n+j-1} \frac{y_k - y_{k+1}}{y_k l(y_k)} < \sum_{k=n}^{n+j-1} \frac{y_k l^2(y_k)}{y_{k+1} l(y_{k+1})}, \tag{2.10}$$

since by the monotonicity of $l(y)$,

$$\begin{aligned} \frac{y_k - y_{k+1}}{y_{k+1} l(y_{k+1})} - \frac{y_k - y_{k+1}}{y_k l(y_k)} &< \frac{1}{l(y_{k+1})} \frac{(y_k - y_{k+1})^2}{y_k y_{k+1}} \\ &= \frac{y_k l^2(y_k)}{y_{k+1} l(y_{k+1})} \frac{(y_k - y_{k+1})^2}{(y_k l(y_k))^2} = \frac{y_k l^2(y_k)}{y_{k+1} l(y_{k+1})}. \end{aligned}$$

By (2.7), there exists a finite constant C such that

$$\frac{y_k l^2(y_k)}{y_{k+1} l(y_{k+1})} < C l(y_k).$$

Applying this bound to the right-hand side in (2.10),

$$\sum_{k=n}^{n+j-1} \frac{y_k - y_{k+1}}{y_{k+1} l(y_{k+1})} < \sum_{k=n}^{n+j-1} \frac{y_k - y_{k+1}}{y_k l(y_k)} + C \sum_{k=n}^{n+j-1} l(y_k).$$

Using (2.9), we see that the first sum at the right-hand side equals j. This proves the upper boundin (2.8). □

Set

$$W(x) := \int_x^1 \frac{dy}{y l(y)} \quad 0 < x < 1, \tag{2.11}$$

where $l(y)$ is from Lemma 2.

Lemma 3. *Let $l(y)$ and $\{y_k\}$ be defined as in Lemma 2. Assume also that $l(y)$ slowly varies at zero. Let the sequence b_n be decreasing and satisfying the condition*

$$W(b_n) = a_n x(1 + o(1)) \text{ as } n \to \infty, \tag{2.12}$$

where $x \in (0, \infty)$, $a_n := 1/l(y_n)$, and put $k_n := \min\{k : y_k < b_n\}$. Then

$$\lim_{n\to\infty} \frac{y_{n+k_n}}{y_n} = e^{-x}. \tag{2.13}$$

Proof. First of all, we note that b_n converges to zero as $n \to \infty$. Indeed, since y_n converges to zero and $\lim_{y\to 0} l(y) = 0$, the sequence a_n tends to infinity. According to condition (2.12), $b_n \to 0$ as $n \to \infty$. Hence, k_n goes to infinity.

Note also that (2.7) holds for every slowly varying function $l(x)$. Consequently, Lemma 2 holds under the conditions of Lemma 3. Recalling that $k_n \to \infty$ and using Corollary 2 with $n = 0$ and $j = k_n$, we have

$$W(y_{k_n}) \sim k_n \text{ as } n \to \infty. \tag{2.14}$$

Using the definition of k_n and the monotonicity of b_n gives

$$W(y_{k_{n-1}}) < W(b_n) \le W(y_{k_n}).$$

Moreover, by (2.8),

$$0 < W(y_{k_n}) - W(y_{k_n - 1}) = \int_{y_{k_n}}^{y_{k_n-1}} \frac{dy}{y l(y)} < C l(y_{k_n}) = o(1) \text{ as } n \to \infty.$$

Hence,

$$W(y_{k_n}) \sim W(b_n) \text{ as } n \to \infty. \tag{2.15}$$

Combining (2.14), (2.15), and (2.12), we have finally

$$k_n \sim W(y_{k_n}) \sim W(b_n) \sim a_n x \text{ as } n \to \infty. \tag{2.16}$$

Since $l(y)$ increases,

$$\int_{y_{n+k_n}}^{y_n} \frac{dy}{y l(y)} > \frac{1}{l(y_n)} \log \frac{y_n}{y_{n+k_n}}.$$

On the other hand, by Lemma 2,

$$\int_{y_{n+k_n}}^{y_n} \frac{dy}{y l(y)} \sim k_n \text{ as } n \to \infty.$$

As a result, we have

$$\frac{1}{l(y_n)} \log \frac{y_n}{y_{n+k_n}} < k_n(1 + o(1)).$$

Taking into account (2.16) and recalling that $a_n = 1/l(y_n)$, we obtain, for every $\varepsilon \in (0,1)$ and all large n, the inequality

$$x > (1-\varepsilon)\log\frac{y_n}{y_{n+k_n}}$$

whenever

$$y_n < e^{x/(1-\varepsilon)} y_{n+k_n}.$$

Since $l(x)$ is slowly varying, it follows from the previous inequality that

$$\frac{l(y)}{l(y_n)} \to 1 \text{ as } n \to \infty$$

uniformly for $y \in [y_n, y_{n+k_n}]$. This, together with Lemma 2, gives

$$k_n \sim \int_{y_{n+k_n}}^{y_n} \frac{dy}{yl(y)} \sim \frac{1}{l(y_n)} \log\frac{y_n}{y_{n+k_n}} \quad \text{as } n \to \infty.$$

On the other hand, it follows from (2.16) that $k_n \sim a_n x$. Hence, recalling that $a_n = 1/l(y_n)$, we get

$$\lim_{n\to\infty} \log\frac{y_n}{y_{n+k_n}} = x.$$

This completes the proof. □

2.2. **Proof of Theorem 2.** The function $V(x)$, defined in (1.3), satifies the conditions of Lemma 1. Thus, to prove the theorem it suffices to show that (2.1) holds with $a_n = [H(1/Q_n)]^{-1}$ and $\varphi(x) = 1 - e^{-x}$.

For a critical Galton – Watson process the sequence Q_n satisfies the recursion equation

$$Q_{k+1} = 1 - f(1-Q_k) = Q_k(1 - H(1/Q_k)), \tag{2.17}$$

i.e. the sequence$\{Q_k\}$ coincides with $\{y_k\}$ defined in Lemma 2 for $l(x) := H(1/x)$. Furthermore,one can easily verify that the function $(f(s)-s)/(1-s)$is decreasing. Recalling definition (1.2) of $H(x)$, we see that $l(x)$ is also decreasing. By assumption (1.1), $H(x) = L(x)$ varies slowly at infinity. Therefore, $l(x)$ varies slowly at zero. We note finally that $l(x) \le l(1) = p_0 < 1$ for $x \in [0,1]$. Summarizing, we conclude that $l(x) = H(1/x)$ satisfies all conditions of Lemma 2.

Let

$$s_n = s_n(x) = \exp\Big\{-\frac{1}{G(a_n x)}\Big\}.$$

Evidently, s_n increases and $(1-s_n)^{-1} \sim G(a_n x)$ as $n \to \infty$. Hence,

$$V\Big(\frac{1}{1-s_n}\Big) \sim a_n x \text{ as } n \to \infty. \tag{2.18}$$

Noting that $V(1/x) = W(x)$for all $x \in [0,1]$, we can rewrite (2.18) in the following form:

$$W(1-s_n) \sim a_n x \text{ as } n \to \infty.$$

Consequently, all conditions of Lemma 3 are fulfilled with $l(x) = H(1/x)$, $y_n = Q_n$ and $b_n = 1 - s_n$. Therefore,

$$\lim_{n\to\infty} \frac{Q_{n+k_n}}{Q_n} = e^{-x}, \tag{2.19}$$

where

$$k_n := \min\{k : Q_k < 1 - s_n\}. \tag{2.20}$$

From this definition of k_n it follows that $f_{k_n-1}(0) \le s_n < f_{k_n}(0)$. Thus,

$$f_{n+k_n-1} \le f_n(s_n) < f_{n+k_n}(0). \tag{2.21}$$

Using the inequality

$$1 - f(f_j(0)) = \int_{f_j(0)}^{1} f'(y)dy > f'(f_j(0))(1 - f_j(0)),$$

we get

$$f'(f_j(0))(1 - f_j(0)) < 1 - f_{j+1}(0) < 1 - f_j(0) \tag{2.22}$$

for every critical Galton – Watson process. Since $\lim_{j\to\infty} f'(f_j(0)) = 1$, we conclude from (2.22) that $(1 - f_{j-1}(0)) \sim (1 - f_j(0))$ as $j \to \infty$. Combining this with (2.21) yields

$$1 - f_n(s_n) \sim Q_{n+k_n} \text{ as } n \to \infty. \tag{2.23}$$

From this relation and (2.19) we find that

$$\lim_{n\to\infty} \mathbf{E}\Big\{\exp\Big(-\frac{Z_n}{G(a_n x)}\Big)\Big|Z_n > 0\Big\} = \lim_{n\to\infty}\Big(1 - \frac{1 - f_n(s_n)}{1 - f_n(0)}\Big) = 1 - e^{-x}.$$

Thus, the theorem is proved.

Remark 2. The reduction of $(1-f_n(s_n))$ to $(1-f_{n+k_n}(0))$ with a proper k_n, which realized in the proof of Theorem 2, was proposed in [9]; see also [3]. If the asymptotic behavior of the nonextinction probability Q_n is known, we can derive immediately thecorresponding limit theorem (see Theorems 1 and 2 in [3]). Assume, for example,that $Q_n \sim n^{-1/\alpha}$ as $n \to \infty$ for some $\alpha \in (0, 1]$. Letting $s_n = 1 - xQ_n$ and recalling definition (2.20) of k_n, we see that $k_n \sim n/x^\alpha$ as $n \to \infty$. Therefore, by (2.23),

$$1 - f_n(s_n) \sim Q_{n+k_n} \sim (1 + x^{-\alpha})^{-1/\alpha} Q_n \text{ as } n \to \infty.$$

Finally,

$$\lim_{n\to\infty} \mathbf{E}\{s_n^{Z_n}|Z_n > 0\} = 1 - (1 + x^{-\alpha})^{-1/\alpha},$$

i.e. we have (a) and (d) of Theorem 1.

In contrast to paper [3], our way to prove a limit theorem for Z_n doesn't require any information on Q_n. Instead of this we consider the ratio Q_{n+k_n}/Q_n, which is in the spirit of [9].

Remark 3. An asymptotic expression for $1 - f_n(s)$ can be found under the additional condition

$$L(x) = o(\log^{-1} x). \tag{2.24}$$

By Theorem 1 of [11], this condition is sufficient for the validity of relation

$$1 - f_n(s) \sim \Big[G\Big(n + V((1 - s)^{-1})\Big)\Big]^{-1} \text{ as } n \to \infty.$$

In particular,

$$Q_n \sim [G(n)]^{-1} \text{ as } n \to \infty. \tag{2.25}$$

Using the method described in Remark 2, we can derive our Theorem 2 from (2.25). Note that assumption (2.24) is superfluous.

2.3. **Proof of Corollary 1.** Obviously,

$$\mathbf{P}(Q_n Z_n < x|Z_n > 0) = \mathbf{P}\Big(H(Q_n^{-1})V(Z_n) < H(Q_n^{-1})V(xQ_n^{-1})\Big|Z_n > 0\Big). \tag{2.26}$$

It follows from definition (1.3) of $V(x)$ that for arbitrary $\varepsilon \in (0, 1)$,

$$V(y) \geq \int_{\varepsilon y}^{y} \frac{dx}{xH(x)} = \frac{\log \varepsilon^{-1}}{H(y)}(1 + o(1)) \text{ as } y \to \infty. \tag{2.27}$$

It means that $H(y)V(y) \to \infty$ as $y \to \infty$. Since $H(x)(= L(x))$ is slowly varying,

$$L(Q_n^{-1})V(xQ_n^{-1}) \sim L(xQ_n^{-1})V(xQ_n^{-1}) \text{ as } n \to \infty. \tag{2.28}$$

Combining (2.27) and (2.28), we conclude that $L(Q_n^{-1})V(xQ_n^{-1}) \to \infty$ as $n \to \infty$. This relation, together with (2.26) and (1.4), proves the corollary.

3. On the nonextinction probability

In Subsection 3.1 we give sufficient conditions when the sequence y_j, defined in Lemma 2, is asymptotically, as $j \to \infty$, equivalent to $W^{-1}(j)$, see Lemma 4 below. An application of this result to the sequence Q_n gives us the asymptotic behavior of Q_n. In Subsection 3.2 we discuss the influence of the function $L(x)$ of (1.1) on the nonextinction probability.

3.1. **On the inversion problem for the function $W(x)$.** Let $\{y_j\}$ be the sequence defined in Lemma 2. It follows from (2.11) and Corollary 2 with $n = 0$ that

$$W(y_j) = j + \psi_j \tag{3.1}$$

and $\psi_j = o(j)$ as $j \to \infty$. Hence,

$$y_j = W^{-1}(j + \psi_j).$$

Fix $\alpha > 0$ and let $l(x) = x^\alpha$ in (2.11). Then $W(x) = \alpha^{-1}(x^{-\alpha} - 1)$, and, consequently,

$$W^{-1}(x + o(x)) \sim W^{-1}(x) \text{ as } x \to \infty. \tag{3.2}$$

In particular, $y_j \sim W^{-1}(j)$ as $j \to \infty$. But if $W(x)$ is defined by a slowly varying $l(x)$, then (3.2) is not true in general. However, if $l(x)$ goes to zero sufficiently fast, then $W^{-1}(j + \psi_j) \sim W^{-1}(j)$, as the next lemma shows.

Lemma 4. *Assume that (2.7) holds and $l(x) = o(\log^{-1} x)$. Then, as $j \to \infty$,*

$$y_j = W^{-1}(j)(1 + o(1)).$$

Proof. Putting $n = 0$ in (2.8) and taking into account the definition (2.11) of $W(x)$, we have

$$j < W(y_j) < j + C \sum_{k=0}^{j-1} l(y_k). \tag{3.3}$$

It follows from the left inequality and monotonicity of $W(x)$ that $y_j < W^{-1}(j)$. Since $l(x)$ increases, this bound implies

$$\sum_{k=0}^{j-1} l(y_k) < \sum_{k=0}^{j-1} l(W^{-1}(k)).$$

Noting that $l(W^{-1}(x))$ decreases, one has, for every $k \geq 1$, the bound

$$l(W^{-1}(k)) \leq \int_{k-1}^{k} l(W^{-1}(x))dx.$$

Therefore,

$$\sum_{k=1}^{j-1} l(W^{-1}(k)) \leq \int_0^j l(W^{-1}(x))dx.$$

Substituting $x = W(y)$ in the last integral and taking into account the equality

$$W'(x) = -\frac{1}{xl(x)}, \tag{3.4}$$

one can easily verify that

$$\int_0^j l(W^{-1}(x))dx = \int_1^{W^{-1}(j)} l(y)W'(y)dy = -\int_1^{W^{-1}(j)} y^{-1}dy = -\log W^{-1}(j).$$

Thus,

$$\sum_{k=1}^{j-1} l(W^{-1}(k)) \le -\log W^{-1}(j).$$

Substituting this bound into (3.3), we get, for some $C < \infty$, the inequality

$$j < W(y_j) < j - C \log W^{-1}(n),$$

or, equivalently,

$$W^{-1}(j - C \log W^{-1}(n)) < y_j < W^{-1}(j). \tag{3.5}$$

To show that these bounds for y_j are asymptotically equivalent, we consider the difference

$$\log W^{-1}(j) - \log W^{-1}(j - C \log W^{-1}(n)).$$

Evidently, this difference does not exceed

$$C \sup_{x \in [n, n - C \log W^{-1}(n)]} (\log W^{-1})'(x) \, \log W^{-1}(n) < \infty. \tag{3.6}$$

Applying (3.4), we get

$$(\log W^{-1})'(x) = \frac{(W^{-1})'(x)}{W^{-1}(x)} = \frac{1}{W^{-1}(x) W'(W^{-1}(x))} = -l(W^{-1}(x)).$$

Since $l(W^{-1}(x))$ is decreasing, the left-hand side of (3.6) equals

$$-Cl(W^{-1}(n)) \log W^{-1}(n).$$

It means that

$$\lim_{j \to \infty} \log W^{-1}(j) - \log W^{-1}(j - C \log W^{-1}(n)) = 0$$

if $l(x) = o(\log^{-1} x)$ as $x \to 0$. The statement of lemma follows from (3.5) and the last relation. $\square$

Now we use Lemma 4 to determine the asymptotic behavior of the nonextinction probability of Z_n.

By (2.17),

$$Q_{n+1} = Q_n - Q_n H(1/Q_n).$$

As we showed in the proof of Theorem 2, the function $l(x) := H(1/x)$ satisfies the conditions of Lemma 2. Thus, in order to apply Lemma 4 to the sequence Q_n, we need only to check condition (2.7) with $H(1/x)$. Since

$$Q_n - Q_{n+1} = f_{n+1}(0) - f_n(0) = f(f_n(0)) - f(f_{n-1}(0))$$

and $f'(s)$ increases,

$$f'(f_{n-1}(0))(Q_{n-1} - Q_n) < Q_n - Q_{n+1} < (Q_{n-1} - Q_n).$$

It follows from these bounds that

$$\lim_{n \to \infty} \frac{Q_n - Q_{n+1}}{Q_{n-1} - Q_n} = f'(1) = 1,$$

i.e. (2.7) holds. Hence,

$$Q_n \sim W^{-1}(n) \text{ as } n \to \infty,$$

provided that $H(1/x) = o(\log^{-1} x)$ as $x \to 0$. This result was obtained in [11] (see Theorem 1 and Corollary 2 there) by using another method.

We conclude this subsection with an example, which shows that the condition $l(x) = o(\log^{-1} x)$ is close to being necessary for the validity of the statement of Lemma 4.

Example 4. Assume that $l(x) = \log^{-\alpha} x^{-1}$ for some $\alpha \in (0, 1]$. In this case,

$$y_{n+1} = y_n\Big(1 - \log^{-\alpha} \frac{1}{y_n}\Big) \text{ for } y_0 < e^{-1}.$$

Taking logarithm at both sides, we arrive at the equality

$$\log \frac{1}{y_{n+1}} = \log \frac{1}{y_n} - \log\Big(1 - \log^{-\alpha} \frac{1}{y_n}\Big).$$

Using Taylor expansion for $\log(1 - x)$, we get

$$\log \frac{1}{y_{n+1}} = \log \frac{1}{y_n} + \sum_{j=1}^{\infty} \frac{1}{j} \log^{-j\alpha} \frac{1}{y_n}.$$

Hence,

$$\begin{aligned}\log^{\alpha+1} \frac{1}{y_{n+1}} &= \log^{\alpha+1} \frac{1}{y_n}\Big[1 + \log^{-\alpha-1} \frac{1}{y_n} + \frac{1}{2}\log^{-2\alpha-1} \frac{1}{y_n} + O\Big(\log^{-3\alpha-1} \frac{1}{y_n}\Big)\Big]^{\alpha+1} \\ &= \log^{\alpha+1} \frac{1}{y_n} + (\alpha+1) + \frac{(\alpha+1)}{2} \log^{-\alpha} \frac{1}{y_n} + O\Big(\log^{-2\alpha} \frac{1}{y_n}\Big).\end{aligned}$$

Letting $x_n := \log^{\alpha+1} y_n^{-1}$, we have

$$x_{n+1} = x_n + (\alpha + 1) + \frac{\alpha+1}{2} x_n^{-\alpha/(\alpha+1)} + O(x_n^{-2\alpha/(\alpha+1)}).$$

Hence,

$$x_n = x_0 + (\alpha+1)n + \sum_{j=0}^{n-1} \frac{\alpha+1}{2} x_j^{-\alpha/(\alpha+1)} + O\Big(\sum_{j=0}^{n-1} x_j^{-2\alpha/(\alpha+1)}\Big). \tag{3.7}$$

Clearly, $x_n = (\alpha + 1)n + o(n)$ as $n \to \infty$. Therefore,

$$x_j^{-\alpha/(\alpha+1)} = (\alpha+1)^{-\alpha/(\alpha+1)} j^{-\alpha/(\alpha+1)}(1 + o(1)) \text{ as } j \to \infty.$$

Summing over $j \in [0, n)$, we get

$$\sum_{j=0}^{n-1} \frac{\alpha+1}{2} x_j^{-\alpha/(\alpha+1)} = c(\alpha) n^{1/(\alpha+1)}(1 + o(1)) \text{ as } n \to \infty,$$

where $c(\alpha) = 2^{-1}(\alpha+1)^{-\alpha/(\alpha+1)}$. Substituting this equality into (3.7), we have

$$x_n = (\alpha+1)n + c(\alpha)n^{1/(\alpha+1)}(1 + o(1)) \text{ as } n \to \infty.$$

Going back to y_n,

$$y_n = \exp\big\{-[(\alpha+1)n]^{1/(\alpha+1)} - c'(\alpha)n^{(1-\alpha)/(1+\alpha)}(1 + o(1))\big\} \text{ as } n \to \infty, \tag{3.8}$$

where $c'(\alpha) = 2^{-1}(\alpha+1)^{-2\alpha/(\alpha+1)}$.

On the other hand, $W(x) = (\alpha+1)^{-1} \log^{\alpha+1} x^{-1}$ if $l(x) = \log^{-\alpha} x^{-1}$. Thus,

$$W^{-1}(n) = \exp\big\{-[(\alpha+1)n]^{1/(\alpha+1)}\big\}. \tag{3.9}$$

Comparing (3.8) and (3.9), we see that y_n and $W^{-1}(n)$ are not asymptotically equivalent, for all $\alpha \in (0, 1]$.

3.2. **On the connection between the asymptotics of $L(x)$ and of Q_n.** To use (2.25) we need to determine the asymptotic behavior of $G(x) = V^{-1}(x)$. But this is not easy, because of the slow variation of $V(x)$. We will demonstrate it with the following example. Assume, that

$$V(x) = a\log^\theta x + b\log^{\theta-\beta} x + o(\log^{\theta-1} x), \tag{3.10}$$

where a and b are positive, $\theta > 1$ and $\beta \in (0, 1]$. To find the asymptotic behavior of the function $G(x)$, we consider the equation

$$az^\theta + bz^{\theta-\beta} + g(z) = x, \tag{3.11}$$

where $g(z) = o(z^{\theta-1})$. It is easily seen, that $z = (x/a)^{1/\theta}(1 + o(1))$ as $x \to \infty$. Letting $z = (x/a)^{1/\theta}(1 + \delta(x))$ in (3.11), we have

$$x\big(1 + \theta\delta(x) + O(\delta^2(x))\big) + b(x/a)^{1-\beta/\theta}\big(1 + (\theta - \beta)\delta(x) + O(\delta^2(x))\big) + o(x^{1-1/\theta}) = x.$$

Therefore,as $x \to \infty$,

$$\delta(x) = -\frac{b}{a\theta}(x/a)^{-\beta/\theta}(1 + o(1))$$

and

$$z = (x/a)^{1/\theta} - \frac{b}{a\theta}(x/a)^{(1-\beta)/\theta}(1 + o(1)).$$

This means that under (3.10),

$$G(x) = \exp\Big\{(x/a)^{1/\theta} - \frac{b}{a\theta}(x/a)^{(1-\beta)/\theta}(1 + o(1))\Big\} \text{ as } x \to \infty. \tag{3.12}$$

Therefore, in order to find the asymptotics of $G(x)$, it is not enough to know only themain term of the asymptotics of $V(x)$. Consequently, if $Z^{(i)}$, $i = 1, 2$, are Galton-Watson processes satisfying (1.1) with slowly varying functions $L^{(i)}$ and $L^{(1)}(x) \sim L^{(2)}(x)$, then it may happen that $Q_n^{(1)}$ and $Q_n^{(2)}$ are *not* asymptotically equivalent.

Acknowledgement. This work was finished during the first author's visit to the Weierstrass Institute in 2005. Hospitality is gratefully acknowledged. We would like to thank Klaus Fleischmann for remarks which led to a better presentation of the results.

References

[1] Barbour, A. D. and Schuh, H.-J. (1977). On the asymptotic behavior of branching processes with infinite mean. *Adv. Appl. Probab.* **9,** 681-723.

[2] Bondarenko, E. M. and Topchii, V. A. (2001). Estimates for expectation of the maximum of a critical Galton – Watson process on a finite interval. *Siberian Math. J.* **42,** 209-216.

[3] Borovkov, K. A. (1988). A method for the proof of limit theorems for branching processes. *Teor. Verojatn. i Primenen.* **33,** 115-123.

[4] Darling, D. A.(1952). The influence of the maximum term in the addition of independent random variables. *Trans. Amer. Math. Soc.*, **73,** 95–107.

[5] Darling, D. A. (1970). The Galton – Watson process with infinite mean. *J. Appl. Probab.* **7,** 455-456.

[6] Feller, W. (1966) *An introduction to probability theory and its applications*, Vol.2 Wiley and Sons.

[7] Hudson, I. L. and Seneta, E. (1977). A note on simple branching processes with infinite mean. *J. Appl. Probab.* **14,** 836-842.

[8] Seneta, E (1976). *Regularly varying functions.* Lecture Notes in Mathematics **508**, Springer.

[9] Slack, R. S. (1968). A branching process with mean one and possibly infinite variance. *Z. Warhscheinlichkeitstheor. und Verw. Geb.* **9,** 139-145.

[10] Slack, R. S. (1972). Further notes on branching process with mean one. *Z. Warhscheinlichkeitstheor. und Verw. Geb.* **25,** 31-38.

[11] Sze, M. (1976). Markov processes associated with critical Galton – Watson processes with application to extinction probability. *Adv. Appl. Probab.* **8,** 278-295.

[12] Zubkov, A. M. (1975). Limit distributions of the distance to the nearest common ancestor. *Teor. Verojatn. i Primenen.* **20,** 614-623.