Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Approximate Approximations from Scattered Data

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submitted: 29 Sep 2005

No. 1058
Berlin 2005

1991 *Mathematics Subject Classification.* 41A30, 65D15, 41A63, 41A25.

Key words and phrases. scattered data quasi-interpolation, cubature of integral operators, multivariate approximation, error estimates .

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Abstract

The aim of this paper is to extend the approximate quasi-interpolation on a uniform grid by dilated shifts of a smooth and rapidly decaying function on a uniform grid to scattered data quasi-interpolation. It is shown that high order approximation of smooth functions up to some prescribed accuracy is possible, if the basis functions, which are centered at the scattered nodes, are multiplied by suitable polynomials such that their sum is an approximate partition of unity. For Gaussian functions we propose a method to construct the approximate partition of unity and describe the application of the new quasi-interpolation approach to the cubature of multi-dimensional integral operators.

1 Introduction

The approximation of multivariate functions from scattered data is an important theme in numerical mathematics. One of the methods to attack this problem is quasi-interpolation. One takes values $u(\mathbf{x_j})$ of a function u on a set of nodes $\{\mathbf{x_j}\}$ and constructs an approximant of u by linear combinations

$$\sum u(\mathbf{x_j})\eta_{\mathbf{j}}(\mathbf{x})\,,$$

where $\eta_{\mathbf{j}}(\mathbf{x})$ is a set of basis functions. Using quasi-interpolation there is no need to solve large algebraic systems. The approximation properties of quasi-interpolants in the case that $\mathbf{x_j}$ are the nodes of a uniform grid are well-understood. For example, the quasi-interpolant

$$\sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\varphi\Big(\frac{\mathbf{x}-h\mathbf{j}}{h}\Big) \tag{1.1}$$

can be studied via the theory of principal shift-invariant spaces, which has been developed in several articles by de Boor, DeVore and Ron (see *e.g.* [2], [3]). Here φ is supposed to be a compactly supported or rapidly decaying function. Based on the Strang-Fix condition for φ, which is equivalent to polynomial reproduction, convergence and approximation orders for several classes of basis functions were obtained (see also Schaback/Wu [20], Jetter/Zhou [7]). Scattered data quasi-interpolation by functions, which reproduce polynomials, has been studied by Buhmann, Dyn, Levin in [1] and Dyn, Ron in [4] (see also [24] for further references).

In order to extend the quasi-interpolation (1.1) to general classes of approximating functions, another concept of approximation procedures, called *Approximate Approximations*, was proposed in [9] and [10]. These procedures have the common feature, that they are accurate without being convergent in a rigorous sense. Consider, for example, the quasi-interpolant on the uniform grid

$$\mathcal{M}\,u(\mathbf{x}) = D^{-n/2}\sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\,\eta\Big(\frac{\mathbf{x}-h\mathbf{j}}{h\sqrt{D}}\Big)\,, \tag{1.2}$$

where η is sufficiently smooth and of rapid decay, h and D are two positive parameters. It was shown that if $\mathcal{F}\eta - 1$ has a zero of order N at the origin ($\mathcal{F}\eta$ denotes the Fourier transform of η), then $\mathcal{M}\,u$ approximates u pointwise

$$|\mathcal{M}u(\mathbf{x}) - u(\mathbf{x})| \leqslant c_{N,\eta}\,(h\sqrt{D})^N \sup_{\mathbb{R}^n}|\nabla_N u| + \varepsilon\,|\nabla_{N-1}u(\mathbf{x})| \tag{1.3}$$

with a constant $c_{N,\eta}$ not depending on u, h, and D, and ε can be made arbitrarily small if D is sufficiently large (see [13], [14]). In general, there is no convergence of the *approximate quasi-interpolant* $\mathcal{M}u(\mathbf{x})$ to $u(\mathbf{x})$ as $h \to 0$. However, one can fix D such that up to any prescribed accuracy $\mathcal{M}u$ approximates u with order $O(h^N)$. The lack of convergence as $h \to 0$, which is even not perceptible in numerical computations for appropriately chosen D, is compensated by a greater flexibility in the choice of approximating functions η. In applications, this flexibility enables one to obtain simple and accurate formulae for values of various integral and pseudo-differential operators of mathematical physics (see [12], [15], [17] and the review paper [21]) and to develop explicit semi-analytic time-marching algorithms for initial boundary value problems for linear and non linear evolution equations ([11], [8]).

The approximate quasi-interpolation approach was extended to nonuniform grids up to now in two directions. The case that the set of nodes is a smooth image of a uniform grid have been studied in [16]. It was shown that formulae similar to (1.2) preserve the basic properties of approximate quasi-interpolation. A similar result for quasi-interpolation on piecewise uniform grids was obtained in [6].

It is the purpose of the present paper to generalize the method of approximate quasi-interpolation to functions with values given on a rather general grid. We start with a simple quasi-interpolant for a set of nodes close to a uniform grid of size h in the sense, that for some positive constant κ and any $\mathbf{j} \in \mathbb{Z}^n$ there exists at least one node $\mathbf{x_j}$ with $|\mathbf{x_j} - h\mathbf{j}| < \kappa h$. Then under some additional assumption on the nodes we construct a quasi-interpolant with gridded centers

$$\mathbb{M}u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} \Lambda_{\mathbf{j}}(u)\eta\Big(\frac{\mathbf{x} - h\,\mathbf{j}}{h\sqrt{D}}\Big) .$$

Here $\Lambda_{\mathbf{j}}$ are linear functionals of the data at a finite number of nodes around $\mathbf{x_j}$. It can be shown that estimate (1.3) remains true for $\mathbb{M}u$ under the same assumptions on the function η.

In order to treat more general distributions of the nodes $\mathbf{x_j}$ we modify the approximating functions. More precisely, we consider approximations of the form

$$Mu(\mathbf{x}) = \sum_{\mathbf{j}} u(\mathbf{x_j})\mathcal{P}_{\mathbf{j}}(\mathbf{x})\eta\Big(\frac{\mathbf{x} - \mathbf{x_j}}{h_{\mathbf{j}}}\Big) \tag{1.4}$$

with some polynomials $\mathcal{P}_{\mathbf{j}}$. We show that one can achieve the approximation of u with arbitrary order N up to a small saturation error, as long as an "approximate partition of unity" $\Big\{\widetilde{\mathcal{P}}_{\mathbf{j}}(\mathbf{x})\}\eta\Big(\frac{\mathbf{x} - \mathbf{x_j}}{h_{\mathbf{j}}}\Big)\Big\}$ with other polynomials $\widetilde{\mathcal{P}}_{\mathbf{j}}$ exists. Here we mean that for any $\varepsilon > 0$ one can find polynomials such that

$$\Big|\sum_{\mathbf{j}} \widetilde{\mathcal{P}}_{\mathbf{j}}(\mathbf{x})\eta\Big(\frac{\mathbf{x} - \mathbf{x_j}}{h_{\mathbf{j}}}\Big) - 1\Big| < \varepsilon\,.$$

Then one can choose the polynomials $\mathcal{P}_{\mathbf{j}}$ in (1.4) such that

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C \sup_{\mathbf{j}} h_{\mathbf{j}}^N \,\|\nabla_N u\|_{L_\infty} + \varepsilon\,|u(\mathbf{x})|\,.$$

This estimate is valid as long as

$$\sum_{\mathbf{j}} \eta\Big(\frac{\mathbf{x} - \mathbf{x_j}}{h_{\mathbf{j}}}\Big) \geq c > 0$$

and η is sufficiently smooth and of rapid decay, but is not subjected to additional requirements as the Strang-Fix condition. Moreover, we propose a method to construct the polynomials such that the series

$$\sum_{\mathbf{j}} \widetilde{\mathcal{P}}_{\mathbf{j}}(\mathbf{x})\, \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_j|^2/h_{\mathbf{j}}^2 D}$$

approximates the constant function 1 up to an arbitrary prescribed accuracy. This method does not require solving a large system of linear equations. Instead, in order to obtain the local representation of the partition of unity, one has to solve a small number of approximation problems, which are reduced to linear systems of moderate size.

By a suitable choice of η it is possible to obtain explicit semi-analytic or other efficient approximation formulae for multi-dimensional integral and pseudo-differential operators which are based on the quasi-interpolant (1.4). So the cubature of those integrals, which is one of the applications of the approximate quasi-interpolation on uniform grids, can be carried over to the case when the integral operators are applied to functions given at scattered nodes.

We give a simple example of formula (1.4). Let $\{x_i\}$ be a sequence of points on $\mathbb{R}$ such that $0 < x_{i+1} - x_i \leqslant 1$. Consider a sequence of functions ζ_j on $\mathbb{R}$ supported by a fixed neighborhood of the origin. Suppose that the sequence $\{\zeta_j(x - x_j)\}$ forms an approximate partition of unity on $\mathbb{R}$,

$$|1 - \sum_j \zeta_j(x - x_j)| < \varepsilon \ .$$

One can easily see that the quasi-interpolant

$$M_h u(x) = \sum_j u(hx_j) \left(\frac{x_{j+1} - x/h}{x_{j+1} - x_j} \zeta_j\Big(\frac{x}{h} - x_j\Big) + \frac{x/h - x_{j-1}}{x_j - x_{j-1}} \zeta_{j-1}\Big(\frac{x}{h} - x_{j-1}\Big) \right)$$

satisfies

$$|M_h u(x) - u(x)| \leqslant c\, h^2 \, \|u''\|_{L_\infty(\mathbb{R})} + \varepsilon \, |u(x)| \, ,$$

where the constant c depends on the functions ζ_j.

The outline of the paper is as follows. In Section 2 we consider an extension of the approximate quasi-interpolation to scattered nodes close to a uniform grid. We construct the quasi-interpolant $\mathbb{M}u$ with gridded centers and coefficients depending on scattered data and obtain approximation estimates. Further the results of some numerical experiments are presented which confirm the predicted approximation orders. In Section 3 we show that an approximate partition of unity can be obtained from a given system of rapidly decaying approximating functions if these functions are multiplied by polynomials. Using the approximate partition of unity, one can construct approximate quasi-interpolants of high order approximation rate up to some prescribed saturation error. This is the topic of Section 4. Section 5 contains an application to the cubature of convolution integral operators. A construction of the approximate partition of unity for the case of Gaussians and some numerical examples are given in Section 6.

2 Quasi-interpolants with gridded centers

Here we give a simple extension of the quasi-interpolation operator on uniform grids (1.2) to a quasi-interpolant, which uses the values $u(\mathbf{x_j})$ on a set of scattered nodes $\mathbf{X} = \{\mathbf{x_j}\} \subset \mathbb{R}^n$ if it is close to a uniform grid. Precisely we suppose

Condition 2.1 There exist $h > 0$ and $\kappa_1 > 0$ such that for any $\mathbf{j} \in \mathbb{Z}^n$ the ball $B(h\mathbf{j}, h\kappa_1)$ centered at $h\mathbf{j}$ with radius $h\kappa_1$ contains nodes of $\mathbf{X}$.

Definition 2.1 Let $\mathbf{x_j} \in \mathbf{X}$. The collection of $m_N = \dfrac{(N-1+n)!}{n!(N-1)!} - 1$ nodes $\mathbf{x_k} \in \mathbf{X}$ will be called the *star* of $\mathbf{x_j}$ and denoted by $\mathrm{st}\,(\mathbf{x_j})$ if the Vandermonde matrix

$$V_{\mathbf{j},h} = \Big\{ \Big(\frac{\mathbf{x_k} - \mathbf{x_j}}{h}\Big)^{\boldsymbol{\alpha}} \Big\}, \ |\boldsymbol{\alpha}| = 1, ..., N-1, \tag{2.1}$$

is not singular. The union of the node $\mathbf{x_j}$ and its star is denoted by $\mathrm{ST}\,(\mathbf{x_j}) = \mathbf{x_j} \cup \mathrm{st}\,(\mathbf{x_j})$.

Condition 2.2 Assume Condition 2.1 and denote by $\widetilde{\mathbf{x}}_{\mathbf{j}} \in \mathbf{X}$ the node closest to $h\,\mathbf{j}$. There exists $\kappa_2 > 0$ such that for any $\mathbf{j} \in \mathbb{Z}^n$

(a) $\mathrm{st}\,(\widetilde{\mathbf{x}}_{\mathbf{j}}) \subset B(\widetilde{\mathbf{x}}_{\mathbf{j}}, h\,\kappa_2)$ with $|\det V_{\mathbf{j},h}| \geqslant c > 0$;

(b) $\bigcup_{\mathbf{j}\in\mathbb{Z}^n} \mathrm{ST}\,(\widetilde{\mathbf{x}}_{\mathbf{j}}) = \mathbf{X}$.

2.1 Error estimate

To formulate our first result we denote by $\{b^{(\mathbf{j})}_{\boldsymbol{\alpha},\mathbf{k}}\}$, $\mathbf{x}_{\mathbf{k}} \in \mathrm{st}\,(\widetilde{\mathbf{x}}_{\mathbf{j}})$, $|\boldsymbol{\alpha}| = 1, ..., N-1$, the elements of the inverse matrix of $V_{\mathbf{j},h}$ and define the functional

$$\Lambda_{\mathbf{j}}(u) = u(\widetilde{\mathbf{x}}_{\mathbf{j}})\Big(1 - \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \Big(\mathbf{j} - \frac{\widetilde{\mathbf{x}}_{\mathbf{j}}}{h}\Big)^{\boldsymbol{\alpha}} \sum_{\mathbf{x}_{\mathbf{k}}\in \mathrm{st}\,(\widetilde{\mathbf{x}}_{\mathbf{j}})} b^{(\mathbf{j})}_{\boldsymbol{\alpha},\mathbf{k}}\Big) + \sum_{\mathbf{x}_{\mathbf{k}}\in \mathrm{st}\,(\widetilde{\mathbf{x}}_{\mathbf{j}})} u(\mathbf{x}_{\mathbf{k}}) \sum_{|\boldsymbol{\alpha}|=1}^{N-1} b^{(\mathbf{j})}_{\boldsymbol{\alpha},\mathbf{k}} \Big(\mathbf{j} - \frac{\widetilde{\mathbf{x}}_{\mathbf{j}}}{h}\Big)^{\boldsymbol{\alpha}}.$$

Theorem 2.1 *Suppose that for some $K > n$ and the smallest integer $n_0 > n/2$ the function $\eta(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^n$, satisfies the conditions $(1+|\mathbf{x}|)^K|\partial^{\boldsymbol{\beta}}\eta(\mathbf{x})| \le C_{\boldsymbol{\beta}}$ for all $0 \le |\boldsymbol{\beta}| \le n_0$, and $\partial^{\boldsymbol{\alpha}}(\mathcal{F}\eta - 1)(\mathbf{0}) = 0$, $0 \le |\boldsymbol{\alpha}| < N$. If the set of nodes $\mathbf{X}$ satisfies Conditions* 2.1 *and* 2.2, *then for any $\varepsilon > 0$ there exists D such that the quasi-interpolant*

$$\mathbb{M}\,u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} \Lambda_{\mathbf{j}}(u)\eta\Big(\frac{\mathbf{x} - h\,\mathbf{j}}{h\sqrt{D}}\Big) \tag{2.2}$$

approximates any $u \in W^N_\infty(\mathbb{R}^n)$ with

$$|\mathbb{M}u(\mathbf{x}) - u(\mathbf{x})| \leqslant c_{N,\eta,D}\; h^N \sup_{\mathbb{R}^n} |\nabla_N u| + \varepsilon \sum_{k=0}^{N-1} |\nabla_k u(\mathbf{x})|(\sqrt{D}h)^k\,, \tag{2.3}$$

where $c_{N,\eta,D}$ does not depend on u and h.

Proof. For given $u \in W^N_\infty(\mathbb{R}^n)$ we consider the quasi-interpolant (1.2) on the uniform grid $\{h\mathbf{j}\}$

$$\mathcal{M}\,u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} u(h\mathbf{j})\,\eta\Big(\frac{\mathbf{x} - h\mathbf{j}}{h\sqrt{D}}\Big)\,,$$

with h given by Condition 2.1. It was proved in [16] that under the decay and moment conditions on η, formulated in the statement of the theorem, $\mathcal{M}\,u$ can be represented as

$$\mathcal{M}\,u(\mathbf{x}) = u(\mathbf{x}) + \sum_{|\boldsymbol{\alpha}|=\mathbf{0}}^{N-1} \Big(\frac{\sqrt{D}h}{2\pi i}\Big)^{|\boldsymbol{\alpha}|} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x})}{\boldsymbol{\alpha}!} \sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{D}\mathbf{k})\,\mathrm{e}^{2\pi i(\mathbf{k},\mathbf{x})} + U_N(\mathbf{x})$$

with a function U_N bounded by

$$|U_N(\mathbf{x})| \le C_1\; (\sqrt{D}h)^N \sup_{\mathbb{R}^n} |\nabla_N u|$$

and a constant C_1 depending only on η. Moreover, the sequences $\{\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{D}\,\cdot)\} \in l_1(\mathbb{Z}^n)$ and

$$\sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}}\mathcal{F}\eta(\sqrt{D}\mathbf{k})| \to 0 \quad \text{as} \quad D \to \infty\,.$$

Hence, we can find D such that $\mathcal{M}u$ satisfies the inequality

$$|\mathcal{M}u(\mathbf{x}) - u(\mathbf{x})| \leqslant |U_N| + \varepsilon \sum_{k=0}^{N-1} |\nabla_k u(\mathbf{x})| (\sqrt{D}h)^k \,.$$

It remains to estimate $|\mathcal{M}u - \mathbb{M}u|$. Recall the Taylor expansion of u around $\mathbf{y} \in \mathbb{R}^n$

$$u(\mathbf{x}) = \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{y})}{\boldsymbol{\alpha}!} (\mathbf{x} - \mathbf{y})^{\boldsymbol{\alpha}} + R_N(\mathbf{y}, \mathbf{x}) \tag{2.4}$$

with the remainder satisfying

$$|R_N(\mathbf{y}, \mathbf{x})| \leqslant c_N |\mathbf{x} - \mathbf{y}|^N \sup_{B(\mathbf{y}, |\mathbf{x}-\mathbf{y}|)} |\nabla_N u| \,. \tag{2.5}$$

For $\mathbf{j} \in \mathbb{Z}^n$ we choose $\widetilde{\mathbf{x}}_{\mathbf{j}} \in \mathbf{X}$ and use (2.4) with $\mathbf{y} = \widetilde{\mathbf{x}}_{\mathbf{j}}$. We split

$$\mathcal{M}u(\mathbf{x}) = M^{(1)} u(\mathbf{x}) + D^{-n/2} \sum_{\mathbf{j} \in \mathbb{Z}^n} R_N(\widetilde{\mathbf{x}}_{\mathbf{j}}, h\mathbf{j}) \, \eta\Big(\frac{\mathbf{x} - h\,\mathbf{j}}{h\sqrt{D}}\Big)$$

with

$$M^{(1)} u(\mathbf{x}) = D^{-n/2} \sum_{\mathbf{j} \in \mathbb{Z}^n} \sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_{\mathbf{j}})}{\boldsymbol{\alpha}!} (h\mathbf{j} - \widetilde{\mathbf{x}}_{\mathbf{j}})^{\boldsymbol{\alpha}} \, \eta\Big(\frac{\mathbf{x} - h\,\mathbf{j}}{h\sqrt{D}}\Big) \,. \tag{2.6}$$

Because of $|h\mathbf{j} - \widetilde{\mathbf{x}}_{\mathbf{j}}| \leq \kappa_1 h$ for any $\mathbf{j}$ we derive from (2.5)

$$|M^{(1)} u(\mathbf{x}) - \mathcal{M}u(\mathbf{x})| \leqslant c_N \, (\kappa_1 h)^N D^{-n/2} \sum_{\mathbf{j} \in \mathbb{Z}^n} \Big| \eta\Big(\frac{\mathbf{x} - h\,\mathbf{j}}{h\sqrt{D}}\Big) \Big| \sup_{B(\mathbf{x}, h\kappa_1)} |\nabla_N u| \,. \tag{2.7}$$

The next step is to approximate $\partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_{\mathbf{j}})$, $1 \leqslant |\boldsymbol{\alpha}| < N$, by a linear combination of $u(\mathbf{x}_{\mathbf{k}})$, $\mathbf{x}_{\mathbf{k}} \in \mathrm{st}\,(\widetilde{\mathbf{x}}_{\mathbf{j}})$. Let $\{a_{\boldsymbol{\alpha}}^{(\mathbf{j})}\}_{1 \leqslant |\boldsymbol{\alpha}| \leq N-1}$ be the unique solution of the linear system with m_N unknowns

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{a_{\boldsymbol{\alpha}}^{(\mathbf{j})}}{\boldsymbol{\alpha}!} (\mathbf{x}_{\mathbf{k}} - \widetilde{\mathbf{x}}_{\mathbf{j}})^{\boldsymbol{\alpha}} = u(\mathbf{x}_{\mathbf{k}}) - u(\widetilde{\mathbf{x}}_{\mathbf{j}}) \,, \quad \mathbf{x}_{\mathbf{k}} \in \mathrm{st}\,(\widetilde{\mathbf{x}}_{\mathbf{j}}) \,. \tag{2.8}$$

From (2.4) and (2.8) follows that

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{h^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} (a_{\boldsymbol{\alpha}}^{(\mathbf{j})} - \partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_{\mathbf{j}})) \left(\frac{\mathbf{x}_{\mathbf{k}} - \widetilde{\mathbf{x}}_{\mathbf{j}}}{h} \right)^{\boldsymbol{\alpha}} = R_N(\widetilde{\mathbf{x}}_{\mathbf{j}}, \mathbf{x}_{\mathbf{k}}) \,.$$

By Condition 2.2(a) the norms of $V_{\mathbf{j},h}^{-1}$ are bounded uniformly in $\mathbf{j}$, this leads together with (2.5) to the inequality

$$\frac{|a_{\boldsymbol{\alpha}}^{(\mathbf{j})} - \partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_{\mathbf{j}})|}{\boldsymbol{\alpha}!} \leqslant C_2 \, h^{N - |\boldsymbol{\alpha}|} \sup_{B(\widetilde{\mathbf{x}}_{\mathbf{j}}, h\kappa_2)} |\nabla_N u|. \tag{2.9}$$

Hence, if we replace the derivatives $\partial^{\boldsymbol{\alpha}} u(\widetilde{\mathbf{x}}_{\mathbf{j}})$ in (2.6) by $a_{\boldsymbol{\alpha}}^{(\mathbf{j})}$, then we get the sum

$$D^{-n/2} \sum_{\mathbf{j} \in \mathbb{Z}^n} \Big(u(\widetilde{\mathbf{x}}_{\mathbf{j}}) + \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{a_{\boldsymbol{\alpha}}^{(\mathbf{j})}}{\boldsymbol{\alpha}!} (h\mathbf{j} - \widetilde{\mathbf{x}}_{\mathbf{j}})^{\boldsymbol{\alpha}} \Big) \, \eta\Big(\frac{\mathbf{x} - h\,\mathbf{j}}{h\sqrt{D}}\Big),$$

which in view of

$$a_{\boldsymbol{\alpha}}^{(\mathbf{j})} = \frac{\boldsymbol{\alpha}!}{h^{|\boldsymbol{\alpha}|}} \sum_{\mathbf{x_k} \in \mathrm{st}\,(\widetilde{\mathbf{x}}_\mathbf{j})} b_{\boldsymbol{\alpha},\mathbf{k}}^{(\mathbf{j})} \left(u(\mathbf{x_k}) - u(\widetilde{\mathbf{x}}_\mathbf{j})\right)$$

coincides with (2.2). Moreover,

$$|\mathbb{M}\, u(\mathbf{x}) - M^{(1)}u(\mathbf{x})| \leqslant C_2 h^N \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \kappa_1^{|\boldsymbol{\alpha}|}\, D^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} \Big|\eta\Big(\frac{\mathbf{x}-h\,\mathbf{j}}{h\sqrt{D}}\Big)\Big| \sup_{B(\mathbf{x},h\kappa_2)} |\nabla_N u|\,. \tag{2.10}$$

Now the inequality

$$\sup_{\mathbb{R}^n} D^{-n/2} \sum_{\mathbf{j}\in\mathbb{Z}^n} \Big|\eta\Big(\frac{\mathbf{x}-\mathbf{j}}{\sqrt{D}}\Big)\Big| \le C_3$$

for all $D \geq D_0 > 0$ implies that (2.7) and (2.10) lead to

$$|\mathcal{M}u(\mathbf{x}) - \mathbb{M}\, u(\mathbf{x})| \le C_4 h^N \sup_{B(\mathbf{x},h\kappa_2)} |\nabla_N u|\,,$$

which proves (2.3). ■

2.2 Numerical Experiments with Quasi-interpolants

The behavior of the quasi-interpolant $\mathbb{M}u$ was tested by one- and two-dimensional experiments. In all cases the scattered grid is chosen such that any ball $B(h\mathbf{j}, h/2)$, $\mathbf{j} \in \mathbb{Z}^n$, contains one randomly chosen node $\mathbf{x_j}$, thus $\widetilde{\mathbf{x}}_\mathbf{j} = \mathbf{x_j}$. All the computations were carried out with MATHEMATICA®.

The following figures show the graph of $\mathbb{M}u - u$ for different smooth functions u using basis functions for second (Fig. 1) and fourth (Fig. 2) order of approximation with $h = 1/32$ (dashed line) and $h = 1/64$ (solid line).

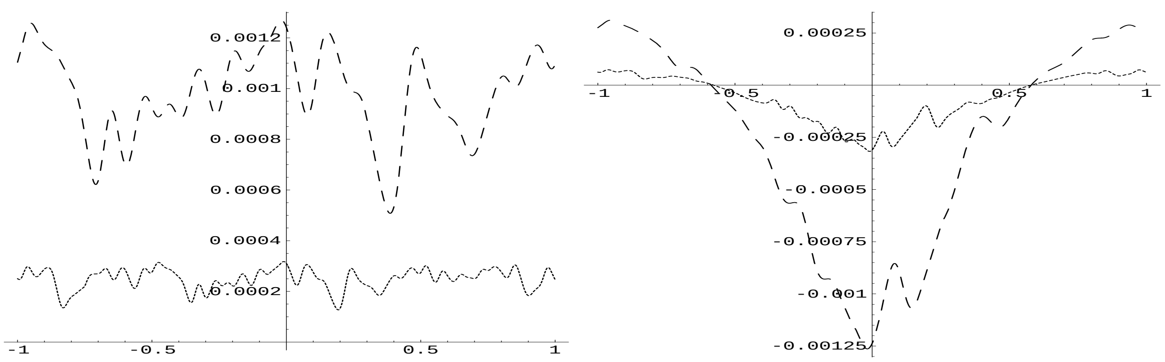

Figure 1: The graphs of $\mathbb{M}\, u(x) - u(x)$ with $D = 2$, $N = 2$, $\mathrm{st}\,(x_j) = \{x_{j+1}\}$, when $u(x) = x^2$ (on the left) and $u(x) = 1/(1+x^2)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

In Fig. 3 the difference $\mathbb{M}u - u$ is plotted for the function $u(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{-1}$, $\mathbf{x} \in \mathbb{R}^2$. In formula (2.2) we use $\eta(\mathbf{x}) = \pi^{-1}\mathrm{e}^{-|\mathbf{x}|^2}$, $D = 2$, for which $N = 2$, and therefore the star $\mathrm{st}\,(\mathbf{x_j})$ contains 2 nodes. We have chosen $\mathrm{st}\,(\mathbf{x}_{j_1,j_2}) = \{\mathbf{x}_{j_1+1,j_2}, \mathbf{x}_{j_1,j_2+1}\}$. In Table 1 we give some computed values of $\mathbb{M}\, u(\mathbf{0}) - u(\mathbf{0})$ for $D = 2$ and $D = 4$ with different h, which confirm h^2-convergence of the two-dimensional quasi-interpolant.

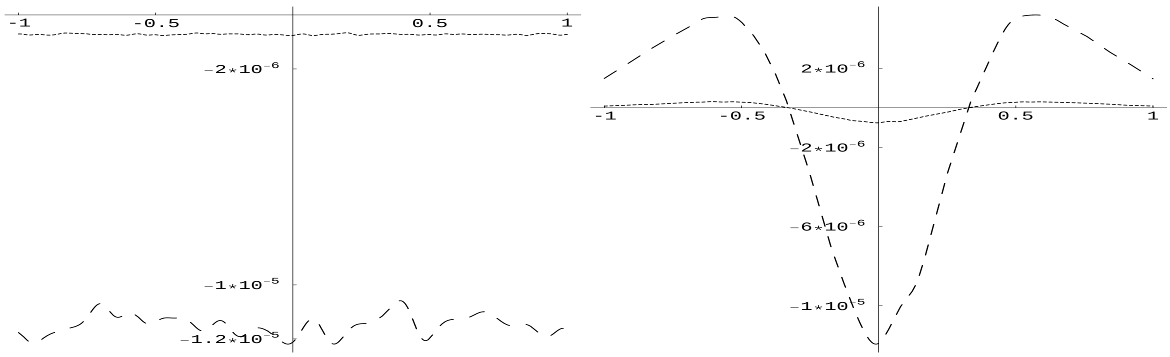

Figure 2: The graphs of $\mathbb{M}\,u(x) - u(x)$ with $D = 4$, $N = 4$, $\operatorname{st}(x_j) = \{x_{j-2}, x_{j-1}, x_{j+1}\}$, when $u(x) = x^4$ (on the left) and $u(x) = 1/(1 + x^2)$. Dashed and solid lines correspond to $h = 1/32$ and $h = 1/64$.

h	$D = 2$	$D = 4$
2^{-4}	$-6.2 \cdot 10^{-3}$	$-1.3 \cdot 10^{-2}$
2^{-5}	$-1.6 \cdot 10^{-3}$	$-3.3 \cdot 10^{-3}$
2^{-6}	$-3.9 \cdot 10^{-4}$	$-8.3 \cdot 10^{-4}$
2^{-7}	$-9.8 \cdot 10^{-5}$	$-2.1 \cdot 10^{-4}$
2^{-8}	$-2.4 \cdot 10^{-5}$	$-5.2 \cdot 10^{-5}$

Table 1: Values of $\mathbb{M}\,u(\mathbf{0}) - u(\mathbf{0})$

3 Approximate partition of unity

In the following two sections we consider irregularly distributed nodes. First we show that an approximate partition of unity can be obtained from a given system of approximating functions centered at the scattered nodes if these functions are multiplied by polynomials. We are mainly interested in rapidly decaying basis functions which are supported on the whole space. But we start with the simpler case of compactly supported basis functions.

3.1 Basis functions with compact support

Lemma 3.1 *Let $\{B(\mathbf{x_j}, h_\mathbf{j})\}_{\mathbf{j}\geqslant 0}$ be an open locally finite covering of $\mathbb{R}^n$ by balls centered in $\mathbf{x_j}$ and radii $h_\mathbf{j}$. Suppose that the multiplicity of this covering does not exceed a positive constant μ_n and that there are positive constants c_1 and c_2 satisfying*

$$c_1 h_\mathbf{m} \leqslant h_\mathbf{j} \leqslant c_2 h_\mathbf{m} \tag{3.1}$$

provided the balls $B(\mathbf{x_j}, h_\mathbf{j})$ and $B(\mathbf{x_m}, h_\mathbf{m})$ have common points. Furthermore, let $\{\eta_\mathbf{j}\}$ be a bounded sequence of continuous functions on $\mathbb{R}^n$ such that $\operatorname{supp} \eta_\mathbf{j} \subset B(\mathbf{x_j}, h_\mathbf{j})$. We assume that the functions $\mathbb{R}^n \ni \mathbf{y} \to \eta_\mathbf{j}(h_\mathbf{j}\,\mathbf{y})$ are continuous uniformly with respect to $\mathbf{j}$ and

$$s(\mathbf{x}) := \sum_\mathbf{j} \eta_\mathbf{j}(\mathbf{x}) \geqslant c \quad \text{on } \mathbb{R}^n \tag{3.2}$$

where c is a positive constant. Then for any $\varepsilon > 0$ there exists a sequence of polynomials $\{\mathcal{P}_\mathbf{j}\}$ with the following properties:

(i) *the degrees of all $\mathcal{P}_\mathbf{j}$ are bounded (they depend on the least majorant of the continuity modulae of $\eta_\mathbf{j}$ and the constants ε, c, c_1, c_2, μ_n);*

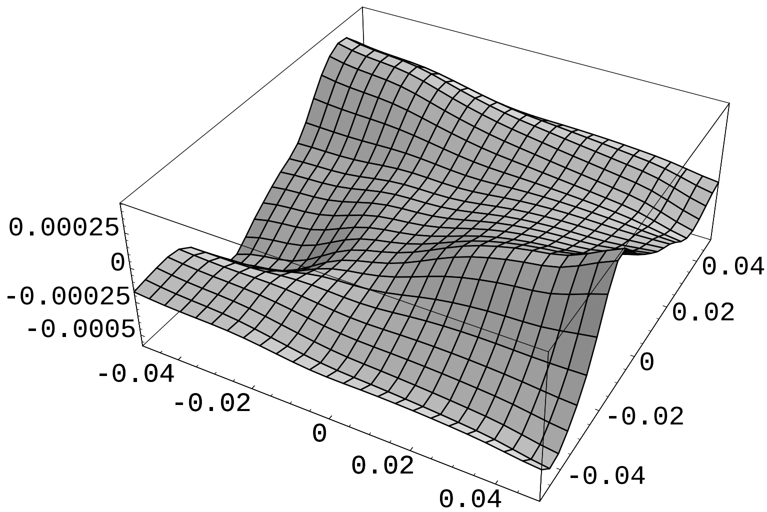

Figure 3: The graph of $\mathbb{M}u - u$ with $D = 2$, $N = 2$, $h = 1/128$, when $u(\mathbf{x}) = 1/(1 + |\mathbf{x}|^2)$, $\mathbf{x} \in \mathbb{R}^2$.

(ii) *there is such a constant* c_0 *that* $|\mathcal{P}_{\mathbf{j}}| < c_0$ *on* $B(\mathbf{x_j}, h_{\mathbf{j}})$*;*

(iii) *the function*

$$\Theta := \sum_{\mathbf{j}} \mathcal{P}_{\mathbf{j}}\, \eta_{\mathbf{j}} \tag{3.3}$$

satisfies

$$|\Theta(\mathbf{x}) - 1| < \varepsilon \quad \textit{for all } \mathbf{x} \in \mathbb{R}^n\,. \tag{3.4}$$

Proof. Since the functions $B(\mathbf{x_j}, 1) \ni \mathbf{y} \to s(h_{\mathbf{j}}\, \mathbf{y})$ are continuous uniformly with respect to $\mathbf{j}$, for an arbitrary positive δ there exist polynomials $\mathcal{P}_{\mathbf{j}}$ subject to

$$\Big| \mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})} \Big| < \delta \ \text{ on } \ B(\mathbf{x_j}, h_{\mathbf{j}})\,,$$

and the degree of $\mathcal{P}_{\mathbf{j}}$, $\deg \mathcal{P}_{\mathbf{j}}$, is independent of $\mathbf{j}$. Letting $\delta = \varepsilon\, (\mu_n \|\eta\|_{L_\infty})^{-1}$ we obtain

$$\Big| \eta_{\mathbf{j}}(\mathbf{x}) \Big(\mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})} \Big) \Big| \le \frac{\varepsilon}{\mu_n}\,. \tag{3.5}$$

Then

$$\sup_{\mathbb{R}^n} \sum_{\mathbf{j}} \Big| \eta_{\mathbf{j}}(\mathbf{x}) \Big(\mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})} \Big) \Big| \le \varepsilon\,, \tag{3.6}$$

since at most μ_n terms of this sum are different from zero. But

$$\sum_{\mathbf{j}} \eta_{\mathbf{j}}(\mathbf{x}) \Big(\mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})} \Big) = \sum_{\mathbf{j}} \eta_{\mathbf{j}}(\mathbf{x}) \mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})} \sum_{\mathbf{j}} \eta_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{j}} \eta_{\mathbf{j}}(\mathbf{x}) \mathcal{P}_{\mathbf{j}}(\mathbf{x}) - 1\,,$$

which proves (3.4). ∎

Remark 3.1 Let the functions $\{\eta_{\mathbf{j}}\}$ in Lemma 3.1 satisfy the additional hypothesis $\eta_{\mathbf{j}} \in C^k(\mathbb{R}^n)$. Then one can find a sequence of polynomials $\{\mathcal{P}_{\mathbf{j}}\}$ of degrees $L_{\mathbf{j}}$ such that

$$\sup_{B(\mathbf{x}_{\mathbf{j}},h_{\mathbf{j}})} \left|\mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\right| \leqslant C(k)\, \frac{h_{\mathbf{j}}^k}{L_{\mathbf{j}}^k} \sup_{B(\mathbf{x}_{\mathbf{j}},h_{\mathbf{j}})} |\nabla_k s(\mathbf{x})|$$

(see, *e.g.*, [18]). This shows that it suffices to take polynomials $\mathcal{P}_{\mathbf{j}}$ with $\deg \mathcal{P}_{\mathbf{j}} > c(k)\, \varepsilon^{-1/k}$ in order to achieve the error ε in (3.6).

3.2 Basis functions with noncompact support

Here we consider approximating functions supported on the whole $\mathbb{R}^n$. We suppose that the functions $\eta_{\mathbf{j}}$ are scaled translates

$$\eta_{\mathbf{j}}(\mathbf{x}) = \eta\Big(\frac{\mathbf{x} - \mathbf{x}_{\mathbf{j}}}{h_{\mathbf{j}}}\Big)$$

of a sufficiently smooth function η with rapid decay.

Lemma 3.2 *For any $\varepsilon > 0$ there exists L_ε and polynomials $\{\mathcal{P}_{\mathbf{j}}\}$ of degree $\deg \mathcal{P}_{\mathbf{j}} \le L_\varepsilon$ such that the function Θ defined by* (3.3) *satisfies* (3.4) *under the following assumptions on η, the nodes $\{\mathbf{x}_{\mathbf{j}}\}$ and the scaling parameters $\{h_{\mathbf{j}}\}$:*

1. *There exists $K > 0$ such that*
$$c_K := \Big\| \sum_{\mathbf{j}} \big(1 + h_{\mathbf{j}}^{-1}|\cdot - \mathbf{x}_{\mathbf{j}}|\big)^{-K} \Big\|_{L_\infty} < \infty\,. \tag{3.7}$$

2. *There exists $p > 0$ such that*
$$\Big\| \big(1 + |\cdot|\big)^K \mathrm{e}^{p^2|\cdot|^2}\, \eta \Big\|_{L_\infty}, \; \Big\| \big(1 + |\cdot|\big)^K \nabla\eta \Big\|_{L_\infty} \le c_p < \infty\,. \tag{3.8}$$

3. *There exists $C \ge 1$ such that all indices* $\mathbf{j}$, $\mathbf{m}$
$$\frac{h_{\mathbf{j}}}{h_{\mathbf{m}}} \le C\,. \tag{3.9}$$

4. (3.2) *is valid.*

Proof. From (3.7) and (3.8) the sum $s(\mathbf{x}) = \sum_{\mathbf{j}} \eta_{\mathbf{j}}(\mathbf{x})$ converges absolutely for any $\mathbf{x}$ to a positive, smooth and bounded function s. Suppose that we have shown that for any $\varepsilon > 0$ and all indices $\mathbf{j}$ there exist polynomials $\mathcal{P}_{\mathbf{j}}$ such that

$$\Big|\eta_{\mathbf{j}}(\mathbf{x})\Big(\mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\Big)\Big| \le \frac{\varepsilon}{c_K}\, \big(1 + {h_{\mathbf{j}}}^{-1}|\mathbf{x} - \mathbf{x}_{\mathbf{j}}|\big)^{-K}, \tag{3.10}$$

(c_K is defined in (3.7)) and $\deg \mathcal{P}_{\mathbf{j}} \le L_\varepsilon$. Then

$$\sup_{\mathbb{R}^n} \sum_{\mathbf{j}} \Big|\eta_{\mathbf{j}}(\mathbf{x})\Big(\mathcal{P}_{\mathbf{j}}(\mathbf{x}) - \frac{1}{s(\mathbf{x})}\Big)\Big| \le \varepsilon\,,$$

and as in the proof of Lemma 3.1 we conclude

$$\sup_{\mathbb{R}^n} \Big| \sum_{\mathbf{j}} \eta_{\mathbf{j}}(\mathbf{x}) \mathcal{P}_{\mathbf{j}}(\mathbf{x}) - 1\Big| \le \varepsilon\,.$$

To establish (3.10) we use a result on weighted polynomial approximation from [5], which will be stated in a simplified form. Introduce the weight $w_p(\mathbf{x}) = \mathrm{e}^{-p^2|\mathbf{x}|^2}$, $p > 0$, and consider the best weighted polynomial approximation of a function g given on $\mathbb{R}^n$

$$E_N(g)_{w_p,\infty} := \inf_{\mathcal{P}\in\Pi_N} \sup_{\mathbb{R}^n} |w_p(\mathbf{x})(g(\mathbf{x}) - \mathcal{P}(\mathbf{x}))|\,,$$

where Π_N denotes the set of polynomials, which are of degree at most N in each variable $x_1,\dots,x_n$. Then for $g \in W^1_\infty(\mathbb{R}^n)$

$$E_N(g)_{w_p,\infty} \le C\,\frac{\|\nabla g\|_{L_\infty}}{N^{1/2}}\,. \tag{3.11}$$

Let us fix an index $\mathbf{j}$ and make the change of variables $\mathbf{y} = {h_\mathbf{j}}^{-1}(\mathbf{x} - \mathbf{x_j})$. Then (3.10) is proved if we show that there exists a polynomial $\mathcal{P}_\mathbf{j}$ such that for all $\mathbf{y} \in \mathbb{R}^n$

$$\Big|\eta(\mathbf{y})\Big(\mathcal{P}_\mathbf{j}(\mathbf{y}) - \frac{1}{\tilde{s}(\mathbf{y})}\Big)\Big| \le \frac{\varepsilon}{c_K}\big(1 + |\mathbf{y}|\big)^{-K} \tag{3.12}$$

with $\tilde{s}(\mathbf{y}) = s(h_\mathbf{j}\mathbf{y} + \mathbf{x_j})$. Since $\tilde{s}^{-1} \in W^1_\infty(\mathbb{R}^n)$ according to (3.11) we can find a polynomial $\mathcal{P}_\mathbf{j}$ satisfying

$$\sup_{\mathbb{R}^n}\Big|\mathcal{P}_\mathbf{j}(\mathbf{y}) - \frac{1}{\tilde{s}(\mathbf{y})}\Big|\mathrm{e}^{-p^2|\mathbf{y}|^2} < \frac{\varepsilon}{c_p\,c_K}$$

with the constant c_p in the decay condition (3.8). Now (3.12) follows immediately from

$$|\eta(\mathbf{y})|\big(1 + |\mathbf{y}|\big)^K \le c_p\,\mathrm{e}^{-p^2|\mathbf{y}|^2}\,.$$

From (3.11) we see that $\deg\mathcal{P}_\mathbf{j}$ depends on the norm of the gradient

$$\sup_{\mathbb{R}^n}\Big|\nabla\frac{1}{\tilde{s}(\mathbf{y})}\Big| = h_\mathbf{j}\sup_{\mathbb{R}^n}\Big|\nabla\frac{1}{s(\mathbf{x})}\Big| \le \sup_{\mathbb{R}^n}\frac{1}{(s(\mathbf{x}))^2}\sum_{\mathbf{m}}\frac{h_\mathbf{j}}{h_\mathbf{m}}\Big|\nabla\eta\Big(\frac{\mathbf{x} - \mathbf{x_m}}{h_\mathbf{m}}\Big)\Big|\,,$$

which is bounded uniformly in $\mathbf{j}$ in view of (3.2), (3.8), and (3.9). ■

4 Quasi-interpolants of a general form

In this section we study the approximation of functions $u \in W^N_\infty(\mathbb{R}^n)$ by the quasi-interpolant (1.4). We will show that within the class of generating functions of the form polynomial times compactly supported or rapidly decaying generating function it suffices to have an approximate partition of unity in order to construct approximate quasi-interpolants of high order accuracy up to some prescribed saturation error.

Let us assume the following hypothesis concerning the grid $\{\mathbf{x_j}\}$:

Condition 4.1 For any $\mathbf{x_j}$ there exists a ball $B(\mathbf{x_j}, h_\mathbf{j})$ which contains m_N nodes $\mathbf{x_k} \in \mathrm{st}\,(\mathbf{x_j})$ with

$$|\det V_{\mathbf{j},h_\mathbf{j}}| = \Big|\det\Big\{\Big(\frac{\mathbf{x_k} - \mathbf{x_j}}{h_\mathbf{j}}\Big)^{\boldsymbol{\alpha}}\Big\}_{|\boldsymbol{\alpha}|=1}^{N-1}\Big| \geqslant c\,, \tag{4.1}$$

(see Definition 2.1), with $c > 0$ not depending on $\mathbf{x_j}$.

4.1 Compactly supported basis functions

Theorem 4.1 *Suppose that the function system* $\{\eta_{\mathbf{j}}\}$ *satisfies the conditions of Lemma* 3.1*, let* $u \in W_\infty^N(\mathbb{R}^n)$ *and* $\varepsilon > 0$ *arbitrary. There exist polynomials* $\mathcal{P}_{\mathbf{j},\mathbf{k}}$*, independent on* u*, whose degrees are uniformly bounded, such that the quasi-interpolant*

$$Mu(\mathbf{x}) = \sum_{\mathbf{k}} u(\mathbf{x_k}) \sum_{\mathrm{ST}\,(\mathbf{x_j})\ni \mathbf{x_k}} \mathcal{P}_{\mathbf{j},\mathbf{k}}(\mathbf{x})\eta_{\mathbf{j}}(\mathbf{x}) \tag{4.2}$$

satisfies the estimate

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C h_{\mathbf{m}}^N \sup_{B(\mathbf{x_m},\lambda\, h_{\mathbf{m}})} |\nabla_N\, u| \, + \varepsilon\, |u(\mathbf{x})|, \tag{4.3}$$

where $\mathbf{x_m}$ *is an arbitrary node and* $\mathbf{x}$ *is any point of the ball* $B(\mathbf{x_m}, h_{\mathbf{m}})$*. By* λ *we denote a constant greater than* 1 *which depends on* c_1 *and* c_2 *in* (3.1)*. The constant* C *does not depend on* $h_{\mathbf{m}}$*,* $\mathbf{m}$ *and* ε*.*

Proof. For given ε we choose polynomials $\mathcal{P}_{\mathbf{j}}(\mathbf{x})$ such that the function (3.3) satisfies

$$|\Theta(\mathbf{x}) - 1| < \varepsilon \quad \text{for all } \mathbf{x} \in \mathbb{R}^n \,,$$

and introduce the auxiliary quasi-interpolant

$$M^{(1)}u(\mathbf{x}) = \sum_{\mathbf{j}} \Big(\sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x_j})}{\boldsymbol{\alpha}!} (\mathbf{x} - \mathbf{x_j})^{\boldsymbol{\alpha}} \Big) \mathcal{P}_{\mathbf{j}}(\mathbf{x}) \eta_{\mathbf{j}}(\mathbf{x}) \,. \tag{4.4}$$

Using the Taylor expansion (2.4) with $\mathbf{y} = \mathbf{x_j}$ we write $M^{(1)}u(\mathbf{x})$ as

$$M^{(1)}u(\mathbf{x}) = u(\mathbf{x})\Theta(\mathbf{x}) - \sum_{\mathbf{j}} R_N(\mathbf{x_j}, \mathbf{x}) \mathcal{P}_{\mathbf{j}}(\mathbf{x}) \eta_{\mathbf{j}}(\mathbf{x}) \,,$$

which gives

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant \sum_{\mathbf{j}} |R_N(\mathbf{x_j}, \mathbf{x}) \mathcal{P}_{\mathbf{j}}(\mathbf{x}) \eta_{\mathbf{j}}(\mathbf{x})| + |u(\mathbf{x})|\, |\Theta(\mathbf{x}) - 1| \,.$$

This, together with the estimate for the remainder (2.5), shows that for $\mathbf{x} \in B(\mathbf{x_m}, h_{\mathbf{m}})$

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant C_1 h_{\mathbf{m}}^N \sup_{B(\mathbf{x_m},\lambda\, h_{\mathbf{m}})} |\nabla_N\, u| + \varepsilon |u(\mathbf{x})| \,, \tag{4.5}$$

where the ball $B(\mathbf{x_m}, \lambda\, h_{\mathbf{m}})$ contains all balls $B(\mathbf{x_j},\, h_{\mathbf{j}})$ such that $B(\mathbf{x_j},\, h_{\mathbf{j}})$ and $B(\mathbf{x_m},\, h_{\mathbf{m}})$ intersect.

Similar to the proof of Theorem 2.1 we approximate in $M^{(1)}u$ the values of the derivatives $\partial^{\boldsymbol{\alpha}} u(\mathbf{x_j})$ by a linear combination of $u(\mathbf{x_k})$, where $\mathbf{x_k} \in \mathrm{st}\,(\mathbf{x_j})$. The solution of

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{a_{\boldsymbol{\alpha}}^{(\mathbf{j})}}{\boldsymbol{\alpha}!} (\mathbf{x_k} - \mathbf{x_j})^{\boldsymbol{\alpha}} = u(\mathbf{x_k}) - u(\mathbf{x_j}) \,, \quad \mathbf{x_k} \in \mathrm{st}\,(\mathbf{x_j}) \,,$$

is given by

$$a_{\boldsymbol{\alpha}}^{(\mathbf{j})} = \frac{\boldsymbol{\alpha}!}{h_{\mathbf{j}}^{|\boldsymbol{\alpha}|}} \sum_{\mathbf{x_k} \in \mathrm{st}\,(\mathbf{x_j})} b_{\boldsymbol{\alpha},\mathbf{k}}^{(\mathbf{j})} \, (u(\mathbf{x_k}) - u(\mathbf{x_j})),$$

where $\{b^{(\mathbf{j})}_{\boldsymbol{\alpha},\mathbf{k}}\}$ are the elements of the inverse of $V_{\mathbf{j},h_{\mathbf{j}}}$. Replacing the derivatives $\{\partial^{\boldsymbol{\alpha}} u(\mathbf{x_j})\}$ in (4.4) by $\{a^{(\mathbf{j})}_{\boldsymbol{\alpha}}\}$ gives the quasi-interpolant

$$\begin{aligned} Mu(\mathbf{x}) = \sum_{\mathbf{j}} \Big\{ u(\mathbf{x_j}) \Big(1 - \sum_{\mathbf{x_k}\in \mathrm{st}\,(\mathbf{x_j})} \sum_{|\boldsymbol{\alpha}|=1}^{N-1} b^{(\mathbf{j})}_{\boldsymbol{\alpha},\mathbf{k}} \Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}} \Big)^{\boldsymbol{\alpha}} \Big) \\ + \sum_{\mathbf{x_k}\in \mathrm{st}\,(\mathbf{x_j})} u(\mathbf{x_k}) \sum_{|\boldsymbol{\alpha}|=1}^{N-1} b^{(\mathbf{j})}_{\boldsymbol{\alpha},\mathbf{k}} \Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}} \Big)^{\boldsymbol{\alpha}} \Big\} \mathcal{P}_{\mathbf{j}}(\mathbf{x}) \eta_{\mathbf{j}}(\mathbf{x}) \\ = \sum_{\mathbf{j}} \sum_{\mathbf{x_k}\in \mathrm{ST}\,(\mathbf{x_j})} u(\mathbf{x_k})\, \mathcal{P}_{\mathbf{j},\mathbf{k}}(\mathbf{x})\, \eta_{\mathbf{j}}(\mathbf{x}) \end{aligned}$$

which can be rewritten as the quasi-interpolant (4.2). By (2.4) we obtain again

$$\sum_{|\boldsymbol{\alpha}|=1}^{N-1} \frac{h_{\mathbf{j}}^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} (a^{(\mathbf{j})}_{\boldsymbol{\alpha}} - \partial^{\boldsymbol{\alpha}} u(\mathbf{x_j})) \Big(\frac{\mathbf{x_k}-\mathbf{x_j}}{h_{\mathbf{j}}} \Big)^{\boldsymbol{\alpha}} = R_N(\mathbf{x_j}, \mathbf{x_k})\,,$$

hence the boundedness of $\|V^{-1}_{\mathbf{j},h_{\mathbf{j}}}\|$ from Condition 4.1 and the estimate of the remainder (2.5) imply

$$|a^{(\mathbf{j})}_{\boldsymbol{\alpha}} - \partial^{\boldsymbol{\alpha}} u(\mathbf{x_j})| \leqslant \boldsymbol{\alpha}!\, C_2\, h_{\mathbf{j}}^{N-|\boldsymbol{\alpha}|} \sup_{B(\mathbf{x_j},h_{\mathbf{j}})} |\nabla_N u|.$$

Therefore we obtain the inequality

$$|Mu(\mathbf{x}) - M^{(1)}u(\mathbf{x})| \leqslant C_2 \sum_{\mathbf{j}} h_{\mathbf{j}}^N \sup_{B(\mathbf{x_j},h_{\mathbf{j}})} |\nabla_N u| \sum_{|\boldsymbol{\alpha}|=1}^{N-1} \Big| \frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}} \Big|^{|\boldsymbol{\alpha}|} |\mathcal{P}_{\mathbf{j}}(\mathbf{x}) \eta_{\mathbf{j}}(\mathbf{x})|$$

and, for any $\mathbf{x} \in B(\mathbf{x_m}, h_{\mathbf{m}})$,

$$|Mu(\mathbf{x}) - M^{(1)}u(\mathbf{x})| \leqslant C_3\, h_{\mathbf{m}}^N \sup_{B(\mathbf{x_m},\lambda\, h_{\mathbf{m}})} |\nabla_N u|.$$

This inequality and (4.5) lead to (4.3). ■

4.2 Quasi-interpolants with noncompactly supported basis functions

Theorem 4.2 *Suppose that additionally to the conditions of Lemma 3.2 the inequality*

$$\Big\| \sum_{\mathbf{j}} \big(1 + h_{\mathbf{j}}^{-1} |\cdot - \mathbf{x_j}|\big)^{N-K} \Big\|_{L_\infty} < \infty \tag{4.6}$$

is fulfilled, let $u \in W^N_\infty(\mathbb{R}^n)$ and $\varepsilon > 0$ arbitrary. There exist polynomials $\mathcal{P}_{\mathbf{j},\mathbf{k}}$, independent on u, whose degrees are uniformly bounded, such that the quasi-interpolant

$$Mu(\mathbf{x}) = \sum_{\mathbf{k}} u(\mathbf{x_k}) \sum_{\mathrm{ST}\,(\mathbf{x_j})\ni \mathbf{x_k}} \mathcal{P}_{\mathbf{j},\mathbf{k}} \Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}} \Big) \eta \Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}} \Big) \tag{4.7}$$

satisfies the estimate

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C \sup_{\mathbf{m}} h_{\mathbf{m}}^N\, \|\nabla_N\, u\|_{L_\infty} + \varepsilon\, |u(\mathbf{x})|\,. \tag{4.8}$$

The constant C does not depend on u and ε.

Proof. Analogously to (4.4) we introduce the quasi-interpolant

$$M^{(1)}u(\mathbf{x}) = \sum_{\mathbf{j}} \Big(\sum_{|\boldsymbol{\alpha}|=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x_j})}{\boldsymbol{\alpha}!} (\mathbf{x}-\mathbf{x_j})^{\boldsymbol{\alpha}} \Big) \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big) \eta\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big)$$

and obtain the estimate

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant \sum_{\mathbf{j}} \Big| R_N(\mathbf{x_j},\mathbf{x}) \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big) \eta\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big)\Big| + |u(\mathbf{x})|\, |\Theta(\mathbf{x}) - 1| \,.$$

From (3.12) we have

$$\Big|\mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big) \eta\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big)\Big| \le \frac{1}{c}\Big|\eta\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big)\Big| + \frac{\varepsilon}{c_K}\Big(1 + \frac{|\mathbf{x}-\mathbf{x_j}|}{h_{\mathbf{j}}}\Big)^{-K}$$

with the lower bound c of $s(\mathbf{x})$ (see (3.2)). Together with (3.8) and (2.5) this provides

$$\Big| R_N(\mathbf{x_j},\mathbf{x}) \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big) \eta\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big)\Big|$$
$$\le c_N h_{\mathbf{j}}^N \|\nabla_N u\|_{L_\infty} \Big|\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big|^N \Big(\frac{c_p}{c}\, \mathrm{e}^{-p^2|\mathbf{x}-\mathbf{x_j}|^2/h_{\mathbf{j}}^2} + \frac{\varepsilon}{c_K}\Big) \Big(1 + \Big|\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big|\Big)^{-K}$$

resulting in

$$|M^{(1)}u(\mathbf{x}) - u(\mathbf{x})| \leqslant |u(\mathbf{x})|\, |\Theta(\mathbf{x}) - 1| + c_N \|\nabla_N u\|_{L_\infty}$$
$$\times \Big(\frac{c_p}{c} \big\|\mathrm{e}^{-p^2|\mathbf{x}|^2} |\mathbf{x}|^N\big\|_{L_\infty} \sum_{\mathbf{j}} h_{\mathbf{j}}^N \Big(1 + \Big|\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big|\Big)^{-K} + \frac{\varepsilon}{c_K} \sum_{\mathbf{j}} h_{\mathbf{j}}^N \Big(1 + \Big|\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big|\Big)^{N-K} \Big).$$

Now we can proceed as in the proof of Theorem 4.1. ■

Remark 4.1 Let for fixed $\mathbf{x}$ the parameter $\kappa_{\mathbf{x}}$ be chosen such that

$$\sum_{|\mathbf{x_j}-\mathbf{x}|>\kappa_{\mathbf{x}}} \mathrm{e}^{-p^2|\mathbf{x}-\mathbf{x_j}|^2/h_{\mathbf{j}}^2} \Big|\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big|^N \Big(1 + \Big|\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big|\Big)^{-K} < \varepsilon \,.$$

Then the estimate (4.8) can be sharpened to

$$|Mu(\mathbf{x}) - u(\mathbf{x})| \leqslant C \max_{|\mathbf{x_j}-\mathbf{x}|\le\kappa_{\mathbf{x}}} h_{\mathbf{j}}^N \sup_{B(\mathbf{x},\kappa_{\mathbf{x}})} |\nabla_N u| + \varepsilon\, (|u(\mathbf{x})| + \|\nabla_N u\|_{L_\infty}) \,.$$

5 Application to the computation of integral operators

Here we discuss a direct application of the quasi-interpolation formula (4.7) for the important example $\eta(\mathbf{x}) = \mathrm{e}^{-|\mathbf{x}|^2}$. Suppose that the density of the integral operator with radial kernel

$$\mathcal{K}u(\mathbf{x}) = \int_{\mathbb{R}^n} g(|\mathbf{x}-\mathbf{y}|) u(\mathbf{y})\, d\mathbf{y} \tag{5.1}$$

is approximated by the quasi-interpolant

$$Mu(\mathbf{x}) = \sum_{\mathbf{j}} \sum_{\mathbf{x_k}\in \mathrm{ST}\,(\mathbf{x_j})} u(\mathbf{x_k})\, \mathcal{P}_{\mathbf{j},\mathbf{k}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_{\mathbf{j}}}\Big)\, \mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/h_{\mathbf{j}}^2} \,. \tag{5.2}$$

Using the following lemma it is easy to derive cubature formulae for (5.1).

Lemma 5.1 *For any* $\mathcal{P}(\mathbf{x}) = \sum\limits_{|\boldsymbol{\beta}|=0}^{L} c_{\boldsymbol{\beta}}\, \mathbf{x}^{\boldsymbol{\beta}}$ *one can write* $\mathcal{P}(\mathbf{x})\mathrm{e}^{-|\mathbf{x}|^2} = \sum\limits_{|\boldsymbol{\beta}|=0}^{L} c_{\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\, \mathrm{e}^{-|\mathbf{x}|^2}$ *with the polynomial* $\mathcal{S}_{\boldsymbol{\beta}}(\mathbf{t})$ *being defined by*

$$\mathcal{S}_{\boldsymbol{\beta}}(\mathbf{t}) = \Big(\frac{1}{2\,i}\Big)^{|\boldsymbol{\beta}|} H_{\boldsymbol{\beta}}\Big(\frac{\mathbf{t}}{2\,i}\Big)\,, \tag{5.3}$$

where $H_{\boldsymbol{\beta}}$ *denotes the Hermite polynomial of* n *variables* $H_{\boldsymbol{\beta}}(\mathbf{t}) = \mathrm{e}^{|\mathbf{t}|^2}(-\partial_{\mathbf{t}})^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{t}|^2}$.

Proof. We are looking for the polynomial $\mathcal{S}_{\boldsymbol{\beta}}(\mathbf{t})$ defined by the relation

$$\mathbf{x}^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{x}|^2} = \mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}|^2}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{5.4}$$

Since

$$\mathcal{F}(\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}|^2})(\lambda) = \pi^{n/2}\mathrm{e}^{-\pi^2|\lambda|^2}\mathcal{S}_{\boldsymbol{\beta}}(2\pi i\lambda)$$

and

$$\mathcal{F}(\mathbf{x}^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{x}|^2})(\lambda) = \pi^{n/2}\Big(-\frac{\partial_\lambda}{2\pi i}\Big)^{\boldsymbol{\beta}}\mathrm{e}^{-\pi^2|\lambda|^2}$$

we obtain (5.3). ∎

In view of Lemma 5.1 we can write $\mathcal{P}_{\mathbf{j},\mathbf{k}}(\mathbf{x})\,\mathrm{e}^{-|\mathbf{x}|^2} = \mathcal{T}_{\mathbf{j},\mathbf{k}}(\partial_{\mathbf{x}})\,\mathrm{e}^{-|\mathbf{x}|^2}$ with some polynomials $\mathcal{T}_{\mathbf{j},\mathbf{k}}(\mathbf{x})$. Then (5.2) can be rewritten as

$$Mu(\mathbf{x}) = \sum_{\mathbf{j}} \sum_{\mathbf{x}_\mathbf{k}\in \mathrm{ST}\,(\mathbf{x}_\mathbf{j})} u(\mathbf{x}_\mathbf{k})\, \mathcal{T}_{\mathbf{j},\mathbf{k}}(-h_\mathbf{j}\,\partial_{\mathbf{x}_\mathbf{j}})\, \mathrm{e}^{-|\mathbf{x}-\mathbf{x}_\mathbf{j}|^2/h_\mathbf{j}^2}\,.$$

The cubature formula for the integral $\mathcal{K}u$ is obtained by replacing u by its quasi-interpolant Mu

$$\tilde{\mathcal{K}}u(\mathbf{x}) = \mathcal{K}Mu(\mathbf{x}) = \sum_{\mathbf{j}} \sum_{\mathbf{x}_\mathbf{k}\in \mathrm{ST}\,(\mathbf{x}_\mathbf{j})} u(\mathbf{x}_\mathbf{k})\, \mathcal{T}_{\mathbf{j},\mathbf{k}}(-h_\mathbf{j}\,\partial_{\mathbf{x}_\mathbf{j}})\, h_\mathbf{j}^n \int_{\mathbb{R}^n} g(h_\mathbf{j}|\mathbf{z}|)\,\mathrm{e}^{-|\mathbf{z}+\mathbf{t}_\mathbf{j}|^2}\,d\mathbf{z}\,, \tag{5.5}$$

where $\mathbf{t}_\mathbf{j} = (\mathbf{x}-\mathbf{x}_\mathbf{j})/h_\mathbf{j}$. By introducing spherical coordinates in $\mathbb{R}^n$ we obtain

$$\int_{\mathbb{R}^n} g(h_\mathbf{j}|\mathbf{z}|)\mathrm{e}^{-|\mathbf{z}+\mathbf{t}_\mathbf{j}|^2}d\mathbf{z} = \mathrm{e}^{-|\mathbf{t}_\mathbf{j}|^2}\int_0^\infty \varrho^{n-1} g(h_\mathbf{j}\varrho)\,\mathrm{e}^{-\varrho^2}d\,\varrho \int_{S^{n-1}} \mathrm{e}^{-2\varrho|\mathbf{t}_\mathbf{j}|\cos(\omega_{\mathbf{t}_\mathbf{j}},\omega)}\,d\sigma_\omega\,,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. The integral over S^{n-1} can be represented by means of the modified Bessel functions of the first kind I_n in the following way

$$\int_{S^{n-1}} \mathrm{e}^{-2\varrho|\mathbf{t}_\mathbf{j}|\cos(\omega_{\mathbf{t}_\mathbf{j}},\omega)}d\sigma_\omega = \frac{2\,\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})}\int_0^\pi \mathrm{e}^{-2\varrho|\mathbf{t}_\mathbf{j}|\cos\vartheta}(\sin\vartheta)^{n-2}\,d\vartheta = 2\pi^{n/2}(\varrho\,|\mathbf{t}_\mathbf{j}|)^{1-n/2}I_{\frac{n-2}{2}}(2\varrho|\mathbf{t}_\mathbf{j}|)$$

(see [22, p.154] and [23, p.79]). If we denote by

$$\mathcal{L}(r) = 2\,\pi^{n/2}\,r^{1-n/2}\mathrm{e}^{-r^2}\int_0^\infty \varrho^{n/2}\,\mathrm{e}^{-\varrho^2}g(h_\mathbf{j}\,\varrho)\,I_{(n-2)/2}(2\varrho\,r)\,d\varrho\,,$$

then (5.5) leads to the following cubature formula for the integral Ku

$$\tilde{\mathcal{K}}u(\mathbf{x}) = \sum_{\mathbf{j}} h_\mathbf{j}^n \sum_{\mathbf{x}_\mathbf{k}\in \mathrm{ST}\,(\mathbf{x}_\mathbf{j})} u(\mathbf{x}_\mathbf{k})\, \mathcal{T}_{\mathbf{j},\mathbf{k}}(-h_\mathbf{j}\,\partial_{\mathbf{x}_\mathbf{j}})\, \mathcal{L}\Big(\frac{|\mathbf{x}-\mathbf{x}_\mathbf{j}|}{h_\mathbf{j}}\Big)\,.$$

6 Construction of the Θ-function with Gaussians

In this section we propose a method to construct the approximate partition of unity for the basis functions

$$\eta_{\mathbf{j}}(\mathbf{x}) = (\pi D)^{-n/2} \mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/h_{\mathbf{j}}^2 D}$$

if the set of nodes $\{\mathbf{x_j}\}$ satisfy Condition 2.1 piecewise with different grid sizes h_j.

6.1 Scattered nodes close to a piecewise uniform grid

Let us explain the assumption on the nodes: Suppose that a subset of nodes $\mathbf{x_j} \in J_0$ satisfies Condition 2.1 with $h = h_1$. The remaining nodes $\mathbf{x_k} \in \mathbf{X} \setminus J_0$ lie in a bounded domain $\Omega \subset \mathbb{R}^n$ and satisfy Condition 2.1 with $h = h_2 = H h_1$ for some small H. To keep good local properties of quasi-interpolants one wants to approximate the data at these nodes by functions of the form polynomial times $\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/h_2^2 D}$, whereas outside Ω quasi-interpolants with functions of the form polynomial times $\mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/h_1^2 D}$ should be used.

Our aim is, to develop a simple method to construct polynomials $\mathcal{P}_{\mathbf{j}}$ such that

$$\Theta(\mathbf{x}) = (\pi D)^{-n/2} \Big(\sum_{\mathbf{x_j} \in J_1} \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_1\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/h_1^2 D} + \sum_{\mathbf{x_k} \in J_2} \mathcal{P}_{\mathbf{k}}\Big(\frac{\mathbf{x}-\mathbf{x_k}}{h_2\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/h_2^2 D} \Big) \tag{6.1}$$

is almost the constant function 1. Here J_2 denotes the set of nodes $\mathbf{x_k} \in \Omega$ and $J_1 = \{\mathbf{x_j}\} \setminus J_2$ the remaining nodes.

First we derive a piecewise uniform grid on $\mathbb{R}^n$ which is associated to the splitting of the set of scattered nodes into J_1 and J_2. We start with Poisson's summation formula for Gaussians

$$(\pi D)^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D} = \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathrm{e}^{-\pi^2 D|\mathbf{k}|^2} \mathrm{e}^{2\pi i(\mathbf{x},\mathbf{k})/h_1},$$

which shows that

$$\Big|1 - (\pi D)^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D}\Big| \le C_1\, \mathrm{e}^{-\pi^2 D}$$

with some constant C_1 depending only on the space dimension.

Thus for any $\varepsilon > 0$ there exists $D > 0$ such that the function system $\{\mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D}\}_{\mathbf{m}\in\mathbb{Z}^n}$ forms an approximate partition of unity with accuracy ε. We can represent any of these functions very accurately by a linear combination of dilated Gaussians due to the equation (see [15])

$$\begin{aligned}\mathrm{e}^{-|\mathbf{x}|^2/D_1} = \Big(\frac{D_1}{\pi D(D_1 - h^2 D)}\Big)^{n/2} & \sum_{\mathbf{m}\in\mathbb{Z}^n} \mathrm{e}^{-h^2|\mathbf{m}|^2/(D_1-h^2D)}\, \mathrm{e}^{-|\mathbf{x}-h\mathbf{m}|^2/h^2 D} \\ - \mathrm{e}^{-|\mathbf{x}|^2/D_1} & \sum_{\mathbf{k}\in\mathbb{Z}^n\setminus\{\mathbf{0}\}} \mathrm{e}^{2\pi i(D_1-h^2D)(\mathbf{x},\mathbf{k})/hD_1}\, \mathrm{e}^{-\pi^2 D(D_1-h^2D)|\mathbf{k}|^2/D_1},\end{aligned} \tag{6.2}$$

which is valid for any $D_1 > h^2 D > 0$. Applied to our setting with $h = h_2$ and $D_1 = h_1^2 D$ we obtain the approximate refinement relation

$$\Big|\mathrm{e}^{-|\mathbf{x}|^2/h_1^2 D} - \sum_{\mathbf{k}\in\mathbb{Z}^n} a_{\mathbf{k}}\, \mathrm{e}^{-|\mathbf{x}-h_2\mathbf{k}|^2/h_2^2 D}\Big| \le C_2\, \mathrm{e}^{-|\mathbf{x}|^2/h_1^2 D}\, \mathrm{e}^{-\pi^2 D(1-H^2)} \tag{6.3}$$

(because by assumption $h_2 = H h_1$) with the coefficients

$$a_{\mathbf{k}} = \big(\pi D(1-H^2)\big)^{-n/2} \mathrm{e}^{-H^2|\mathbf{k}|^2/(1-H^2)D}.$$

Again, the constant C_2 depends only on the space dimension. Define by $S \in \mathbb{Z}^n$ the minimal index set such that

$$\sum_{\mathbf{k}\in\mathbb{Z}^n\setminus S} a_{\mathbf{k}} < \mathrm{e}^{-\pi^2 D(1-H^2)} .$$

Then it is clear from (6.3) that for any disjoint Z_1 and Z_2 with $Z_1 \cup Z_2 = \mathbb{Z}^n$

$$\Big|1 - (\pi D)^{-n/2}\Big(\sum_{\mathbf{m}\in Z_1} \mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}|^2/h_1^2 D} + \sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S} a_{\mathbf{k}}\, \mathrm{e}^{-|\mathbf{x}-h_1\mathbf{m}-h_2\mathbf{k}|^2/h_2^2 D}\Big)\Big| \le C_3\, \mathrm{e}^{-\pi^2 D(1-H^2)} . \tag{6.4}$$

Condition 6.1 Denote $Z_2 = \{\mathbf{m} \in \mathbb{Z}^n : h_1\mathbf{m} + h_2\mathbf{k} \in \Omega \text{ for all } \mathbf{k} \in S\}$. The constant κ_1 of Condition 2.1 and the domain Ω are such that for all nodes $\mathbf{x}_{\mathbf{k}} \in \Omega$, i.e. the nodes belonging to J_2, one can find $\mathbf{m} \in Z_2$, $\mathbf{k} \in S$ with $|\mathbf{x}_{\mathbf{k}} - h_1\mathbf{m} - h_2\mathbf{k}| < \kappa_1 h_2$.

Setting $Z_1 = \mathbb{Z}^n \setminus Z_2$ we connect the index sets Z_1, Z_2 with the splitting of the scattered nodes into J_1, J_2. By this way we construct an approximate partition of unity using Gaussians with the "large" scaling factor h_1 centered at the uniform grid $G_1 := \{h_1\mathbf{m}\}_{\mathbf{m}\in Z_1}$ outside Ω and using Gaussians with scaling factor h_2 and the centers $G_2 := \{h_1\mathbf{m} + h_2\mathbf{k}\}_{\mathbf{m}\in Z_2,\mathbf{k}\in S}$ in Ω.

It is obvious, that the above definition of piecewise quasi-uniformly distributed scattered nodes and the construction of an associated approximate partition of unity on piecewise uniform grids can be extended to finitely many scaling factors h_j. Since there will be no difference for the subsequent considerations we will restrict to the two-scale case.

From (6.4) we see that for any $\varepsilon > 0$, and given h_1 and h_2 there exists $D > 0$ such that the linear combination

$$(\pi D)^{-n/2}\Big(\sum_{g_1\in G_1} \mathrm{e}^{-|\mathbf{x}-g_1|^2/h_1^2 D} + \sum_{g_2\in G_2} \tilde{a}_{g_2}\, \mathrm{e}^{-|\mathbf{x}-g_2|^2/h_2^2 D}\Big) \tag{6.5}$$

with $\tilde{a}_{g_2} = a_{\mathbf{k}}$ for $g_2 = h_1\mathbf{m}+h_2\mathbf{k}$, $\mathbf{m} \in Z_2, \mathbf{k} \in S$, approximates the constant function 1 with an error less than $\varepsilon/2$. The idea of constructing the Θ-function (6.1) is to choose for each $g_1 \in G_1$ and $g_2 \in G_2$ finite sets of nodes $\Sigma(g_1) \subset J_1$ and $\Sigma(g_2) \subset J_2$, respectively, and to determine polynomials $\mathcal{P}_{\mathbf{j},g_\ell}$ such that

$$\sum_{\mathbf{x_j}\in\Sigma(g_\ell)} \mathcal{P}_{\mathbf{j},g_\ell}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_\ell\sqrt{D}}\Big)\mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/h_\ell^2 D} \quad \text{approximate} \quad \mathrm{e}^{-|\mathbf{x}-g_\ell|^2/h_\ell^2 D} , \ \ell = 1, 2 .$$

If the L_∞-error of the sums over g_ℓ can be controlled, then we get

$$\sum_{g_1\in G_1} \mathrm{e}^{-|\mathbf{x}-g_1|^2/h_1^2 D} \asymp \sum_{\mathbf{x_j}\in J_1} \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x_j}}{h_1\sqrt{D}}\Big)\mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/h_1^2 D}$$

with the polynomials

$$\mathcal{P}_{\mathbf{j}} = \sum_{g_1\in G(\mathbf{x_j})} \mathcal{P}_{\mathbf{j},g_1} \tag{6.6}$$

and

$$\sum_{g_2\in G_2} \tilde{a}_{g_2}\mathrm{e}^{-|\mathbf{x}-g_2|^2/h_2^2 D} \asymp \sum_{\mathbf{x_k}\in J_2} \mathcal{P}_{\mathbf{k}}\Big(\frac{\mathbf{x}-\mathbf{x_k}}{h_2\sqrt{D}}\Big)\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/h_2^2 D}$$

with the polynomials

$$\mathcal{P}_{\mathbf{k}} = \sum_{g_2\in G(\mathbf{x_k})} \tilde{a}_{g_2}\mathcal{P}_{\mathbf{k},g_2} , \tag{6.7}$$

where we denote $G(\mathbf{x_j}) = \{g : \mathbf{x_j} \in \Sigma(g)\}$. Note that we have to choose the subsets $\Sigma(g_\ell)$ such that the sets $G(\mathbf{x_j}) \subset G_\ell$ are finite and nonempty for any node $\mathbf{x_j} \in J_\ell$. Additionally, one has to choose these sets such that for some $\kappa_1 > 0$ and any $g_\ell \in G_\ell$ the ball $B(g_\ell, \kappa_1 h_\ell)$ contains at least one node $\mathbf{x_j} \in J_\ell$. This is always possible, since Conditions 2.1 resp. 6.1 are valid.

The proposed construction method of Θ does not require solving a large algebraic system. Instead, to obtain the local representation of Θ one has to solve a small number of approximation problems, which are reduced in the next sections to linear systems of moderate size.

After this preparation we write Θ as

$$\begin{aligned}\Theta(\mathbf{x}) =& (\pi\, D)^{-n/2} \sum_{g_1 \in G_1} \mathrm{e}^{-|\mathbf{x}-g_1|^2/h_1^2 D} + \sum_{g_1 \in G_1} \omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big) \\ &+ (\pi\, D)^{-n/2} \sum_{g_2 \in G_2} \tilde{a}_{g_2}\, \mathrm{e}^{-|\mathbf{x}-g_2|^2/h_2^2 D} + \sum_{g_2 \in G_2} \tilde{a}_{g_2} \omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big)\,,\end{aligned}$$

where

$$\omega_{g_\ell}(\mathbf{y}) = (\pi\, D)^{-n/2} \Big\{ \sum_{h_\ell \mathbf{y_j} \in \Sigma(g_\ell)} \mathcal{P}_{\mathbf{j}, g_\ell}\Big(\frac{\mathbf{y}-\mathbf{y_j}}{\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D} - \mathrm{e}^{-|\mathbf{y}-g_\ell/h_\ell|^2/D} \Big\} \tag{6.8}$$

with $\mathbf{y_j} = \mathbf{x_j}/h_\ell$, $\mathbf{x_j} \in J_\ell$. Hence for sufficiently large D

$$\big|\Theta(\mathbf{x}) - 1\big| < \frac{\varepsilon}{2} + \sum_{g_1 \in G_1} \Big|\omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big)\Big| + \sum_{g_2 \in G_2} \tilde{a}_{g_2} \Big|\omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big)\Big|\,. \tag{6.9}$$

6.2 Construction of Polynomials

Let us introduce

$$\omega(\mathbf{y}) := (\pi\, D)^{-n/2} \Big\{ \sum_{\mathbf{y_j} \in \Sigma} \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{y}-\mathbf{y_j}}{\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D} - \mathrm{e}^{-|\mathbf{y}|^2/D} \Big\}, \tag{6.10}$$

where Σ is some finite point set in $\mathbb{R}^n$. We will describe a method for constructing polynomials $\mathcal{P}_{\mathbf{j}}$ such that $\mathrm{e}^{\rho|\mathbf{y}|^2} |\omega(\mathbf{y})|$ for some $\rho > 0$ becomes small. In what follows we use the representation

$$\mathcal{P}_{\mathbf{j}}(\mathbf{x}) = \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}}\, \mathbf{x}^{\boldsymbol{\beta}}\,.$$

Hence by Lemma 5.1

$$\mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{y}-\mathbf{y_j}}{\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D} = \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D} \partial_{\mathbf{y}})\, \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D}\,.$$

and ω can be written as

$$\omega(\mathbf{y}) = (\pi D)^{-n/2} \Big(\sum_{\mathbf{y_j} \in \Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D} \partial_{\mathbf{y}})\, \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D} - \mathrm{e}^{-|\mathbf{y}|^2/D} \Big). \tag{6.11}$$

To estimate the L_∞-norm of ω we represent this function as convolution.

Lemma 6.1 *Let $\mathcal{P}(\mathbf{t})$ be a polynomial and let $0 < D_0 < D$. Then*

$$\mathcal{P}(\partial_\xi)\, \mathrm{e}^{-|\xi-\mathbf{x}|^2/D} = c_1\, \mathrm{e}^{-|\xi|^2/(D-D_0)} * \mathcal{P}(\partial_\xi)\, \mathrm{e}^{-|\xi-\mathbf{x}|^2/D_0}\,,$$

where $*$ stands for the convolution operator and

$$c_1 = \Big(\frac{D}{\pi D_0(D-D_0)}\Big)^{n/2}.$$

Proof. From

$$\mathrm{e}^{-|\xi-\mathbf{x}|^2/D} = c_1 \int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\xi-\mathbf{t}|^2/(D-D_0)}\, \mathrm{e}^{-|\mathbf{t}-\mathbf{x}|^2/D_0}\, d\mathbf{t}$$

we obtain

$$\mathcal{P}(\partial_\xi)\mathrm{e}^{-|\xi-\mathbf{x}|^2/D} = \mathcal{P}(-\partial_\mathbf{x})\mathrm{e}^{-|\xi-\mathbf{x}|^2/D} = c_1 \int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\xi-\mathbf{t}|^2/(D-D_0)} \mathcal{P}(\partial_\mathbf{t})\mathrm{e}^{-|\mathbf{t}-\mathbf{x}|^2/D_0}\, d\mathbf{t}\,. \qquad \blacksquare$$

Using Lemma 6.1 and (6.11) we write ω as

$$\omega(\mathbf{y}) = \frac{c_1}{(\pi D)^{n/2}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{y}-\mathbf{t}|^2/(D-D_0)} \Big(\sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_\mathbf{t})\mathrm{e}^{-|\mathbf{t}-\mathbf{y_j}|^2/D_0} - \mathrm{e}^{-|\mathbf{t}|^2/D_0}\Big)\, d\mathbf{t} \tag{6.12}$$

and, by Cauchy's inequality, we obtain

$$||\omega||_{L_\infty} \leqslant c_2 \Big\| \sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_\mathbf{t})\mathrm{e}^{-|\mathbf{t}-\mathbf{y_j}|^2/D_0} - \mathrm{e}^{-|\mathbf{t}|^2/D_0} \Big\|_{L^2}, \tag{6.13}$$

where

$$c_2 = (\pi D_0)^{-n/2}(2\pi(D-D_0))^{-n/4}.$$

If we define polynomials $T_{\boldsymbol{\beta}}$ by

$$T_{\boldsymbol{\beta}}(\mathbf{x}) = \mathrm{e}^{|\mathbf{x}|^2/D_0} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_\mathbf{x})\mathrm{e}^{-|\mathbf{x}|^2/D_0}\,, \tag{6.14}$$

then

$$||\omega||_{L_\infty} \leqslant c_2 \Big\| \mathrm{e}^{-|\,\cdot\,|^2/D_0} - \sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} T_{\boldsymbol{\beta}}(\,\cdot\, - \mathbf{y_j})\mathrm{e}^{-|\,\cdot\,-\mathbf{y_j}|^2/D_0} \Big\|_{L^2}.$$

An estimate for the sum of $|\omega_{g_\ell}|$ can be derived from

Lemma 6.2 *Let $0 < D_0 < D$ and denote $\rho = \dfrac{D-D_0}{(D-D_0)^2 + DD_0}$. Then the estimate*

$$\sup_{\mathbb{R}^n} |\omega(\mathbf{y})|\, \mathrm{e}^{\rho|\mathbf{y}|^2} \leqslant c_3 \sqrt{Q(\mathbf{c})} \tag{6.15}$$

is valid, where for $\mathbf{c} = \{c_{\mathbf{j},\boldsymbol{\beta}}\}$ the quadratic form $Q(\mathbf{c})$ is defined by

$$Q(\mathbf{c}) = \int\limits_{\mathbb{R}^n} \mathrm{e}^{2(D-D_0)|\mathbf{t}|^2/DD_0} \Big(\mathrm{e}^{-|\mathbf{t}|^2/D_0} - \sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} T_{\boldsymbol{\beta}}(\mathbf{t}-\mathbf{y_j})\, \mathrm{e}^{-|\mathbf{t}-\mathbf{y_j}|^2/D_0} \Big)^2 d\mathbf{t} \tag{6.16}$$

and

$$c_3 = \frac{D^{n/4}}{(2\pi^3 D_0(D-D_0)((D-D_0)^2 + DD_0))^{n/4}}\,.$$

Proof. Starting with (6.12), using (6.14) and

$$|\mathbf{x}-\mathbf{t}|^2 = \Big|\sqrt{a}\mathbf{x} - \frac{\mathbf{t}}{\sqrt{a}}\Big|^2 + (1-a)|\mathbf{x}|^2 + \frac{a-1}{a}|\mathbf{t}|^2$$

for $a > 0$, we derive the representation

$$\omega(\mathbf{y}) = \frac{c_1}{(\pi D)^{n/2}}\, \mathrm{e}^{-(1-a)|\mathbf{y}|^2/(D-D_0)} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}-a\mathbf{x}|^2/a(D-D_0)}\, \mathrm{e}^{(1-a)|\mathbf{t}|^2/a(D-D_0)}$$

$$\times \Big(\sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} T_{\boldsymbol{\beta}}(\mathbf{t}-\mathbf{y_j}) \mathrm{e}^{-|\mathbf{t}-\mathbf{y_j}|^2/D_0} - \mathrm{e}^{-|\mathbf{t}|^2/D_0} \Big)\, d\mathbf{t}\,.$$

Then Cauchy's inequality leads to

$$\Big|\omega(\mathbf{y})\, \mathrm{e}^{(1-a)|\mathbf{y}|^2/(D-D_0)}\Big|$$

$$\le c_3 \Big(\int\limits_{\mathbb{R}^n} \mathrm{e}^{2(1-a)|\mathbf{t}|^2/a(D-D_0)} \Big(\sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} T_{\boldsymbol{\beta}}(\mathbf{t}-\mathbf{y_j}) e^{-|\mathbf{t}-\mathbf{y_j}|^2/D_0} - \mathrm{e}^{-|\mathbf{t}|^2/D_0} \Big)^2 d\mathbf{t} \Big)^{1/2} \tag{6.17}$$

with

$$c_3 = (\pi D_0)^{-n/2} \Big(\frac{a}{2\pi(D-D_0)} \Big)^{n/4}.$$

If we choose the parameter a such that

$$\frac{(1-a)|\mathbf{t}|^2}{a(D-D_0)} - \frac{|\mathbf{t}|^2}{D_0} = -\frac{|\mathbf{t}|^2}{D}, \quad \text{i.e.} \quad a = \frac{D\, D_0}{(D-D_0)^2 + D\, D_0},$$

then the right hand side of (6.17) takes the form (6.16). ■

Next we estimate

$$r := \min_{\mathbf{c}} Q(\mathbf{c})\,. \tag{6.18}$$

Using (6.14), after elementary calculations one obtains

$$Q(\mathbf{c}) = \int\limits_{\mathbb{R}^n} \Big(\mathrm{e}^{(D-D_0)|\mathbf{t}|^2/DD_0} \Big(\sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{S}_{\boldsymbol{\beta}}(\sqrt{D}\partial_\mathbf{t})\, \mathrm{e}^{-|\mathbf{t}-\mathbf{y_j}|^2/D_0} - \mathrm{e}^{-|\mathbf{t}|^2/D_0} \Big) \Big)^2 d\mathbf{t}$$

$$= \Big(\frac{\pi D}{2}\Big)^{n/2} \Big(1 - 2 \sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{C}_{\boldsymbol{\beta},0}(\mathbf{y_j},\mathbf{0}) + \sum_{\mathbf{y_j},\mathbf{y_k}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} \sum_{|\boldsymbol{\gamma}|=0}^{L_\mathbf{k}} c_{\mathbf{j},\boldsymbol{\beta}} c_{\mathbf{k},\boldsymbol{\gamma}} \mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{y_j},\mathbf{y_k}) \Big)$$

with

$$\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) := \mathcal{S}_{\boldsymbol{\beta}}(-\sqrt{D}\partial_\mathbf{x}) \mathcal{S}_{\boldsymbol{\gamma}}(-\sqrt{D}\partial_\mathbf{y})\, \mathrm{e}^{(D-D_0)(|\mathbf{x}|^2+|\mathbf{y}|^2)/D_0^2}\, \mathrm{e}^{-D|\mathbf{x}-\mathbf{y}|^2/2D_0^2}.$$

The minimum of $Q(\mathbf{c})$ is attained by the solution $\mathbf{c} = \{c_{\mathbf{j},\boldsymbol{\beta}}\}$ of the linear system

$$\sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_\mathbf{j}} \mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{y_j},\mathbf{y_k}) c_{\mathbf{j},\boldsymbol{\beta}} = \mathcal{C}_{\mathbf{0},\boldsymbol{\gamma}}(\mathbf{0},\mathbf{y_k}), \quad \mathbf{y_k}\in\Sigma,\ 0 \le |\boldsymbol{\gamma}| \le L_\mathbf{k}\,. \tag{6.19}$$

Then by Lemma 6.2 the sum

$$\sum_{\mathbf{y_j}\in\Sigma} \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{y}-\mathbf{y_j}}{\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D} = \sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}} \Big(\frac{\mathbf{y}-\mathbf{y_j}}{\sqrt{D}}\Big)^{\boldsymbol{\beta}} \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D} \tag{6.20}$$

approximates $\mathrm{e}^{-|\mathbf{y}|^2/D}$ with

$$(\pi D)^{-n/2}\Big|\mathrm{e}^{-|\mathbf{y}|^2/D} - \sum_{\mathbf{y_j}\in\Sigma} \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{y}-\mathbf{y_j}}{\sqrt{D}}\Big) \mathrm{e}^{-|\mathbf{y}-\mathbf{y_j}|^2/D}\Big| \le c_3\, \mathrm{e}^{-\rho|\mathbf{y}|^2} r^{1/2} .$$

In the next section we show that (6.19) has a unique solution and give an estimate of r.

6.3 Existence and estimates

Let us give another representation of the quadratic form $Q(\mathbf{c})$ defined by (6.16). Introduce the transformed points

$$\mathbf{t_j} = \frac{D}{D_0}\mathbf{y_j}\,, \quad \mathbf{y_j} \in \Sigma\,,$$

then, because of

$$\frac{D-D_0}{D\,D_0}|\mathbf{t}|^2 - \frac{1}{D_0}|\mathbf{t}-\mathbf{y_j}|^2 = -\frac{1}{D}|\mathbf{t}-\mathbf{t_j}|^2 + \frac{D-D_0}{D_0^2}\,|\mathbf{y_j}|^2$$

$Q(\mathbf{c})$ can be written as

$$Q(\mathbf{c}) = \int\limits_{\mathbb{R}^n} \Big(\mathrm{e}^{-|\mathbf{t}|^2/D} - \sum_{\mathbf{y_j}\in\Sigma} \mathrm{e}^{(D-D_0)|\mathbf{y_j}|^2/D_0^2} \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}} T_{\boldsymbol{\beta}}(\mathbf{t}-\mathbf{y_j})\, \mathrm{e}^{-|\mathbf{t}-\mathbf{t_j}|^2/D}\Big)^2 d\mathbf{t}\,. \tag{6.21}$$

Since $T_{\boldsymbol{\beta}}$ are polynomials of degree $\boldsymbol{\beta}$, the minimum problem for $Q(\mathbf{c})$ is equivalent to finding the best L_2-approximation

$$\min_{d_{\mathbf{j},\boldsymbol{\beta}}} \int\limits_{\mathbb{R}^n} \Big(\mathrm{e}^{-|\mathbf{t}|^2/D} - \sum_{\mathbf{y_j}\in\Sigma} \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} d_{\mathbf{j},\boldsymbol{\beta}} (\mathbf{t}-\mathbf{t_j})^{\boldsymbol{\beta}}\, \mathrm{e}^{-|\mathbf{t}-\mathbf{t_j}|^2/D}\Big)^2 d\mathbf{t}\,.$$

Lemma 6.3 *Let $\{\mathbf{x_j}\}$ a finite collection of nodes. For all $L_{\mathbf{j}} \ge 0$ the polynomials $\mathcal{P}_{\mathbf{j}}$ of degree $L_{\mathbf{j}}$, which minimize*

$$\Big\|\mathrm{e}^{-|\,\cdot\,|^2} - \sum_{\mathbf{j}} \mathcal{P}_{\mathbf{j}}(\,\cdot\, - \mathbf{x_j})\, \mathrm{e}^{-|\,\cdot\,-\mathbf{x_j}|^2}\Big\|_{L_2}\,, \tag{6.22}$$

are uniquely determined.

Proof. The application of Lemma 5.1 gives for $\mathcal{P}_{\mathbf{j}}(\mathbf{x}) = \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}}\, \mathbf{x}^{\boldsymbol{\beta}}$

$$\Big\|\mathrm{e}^{-|\,\cdot\,|^2} - \sum_{\mathbf{j}} \mathcal{P}_{\mathbf{j}}(\,\cdot\, - \mathbf{x_j})\mathrm{e}^{-|\,\cdot\,-\mathbf{x_j}|^2}\Big\|_{L_2}^2 = \int\limits_{\mathbb{R}^n} \Big(\mathrm{e}^{-|\mathbf{x}|^2} - \sum_{\mathbf{j}} \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}}\, \mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\, \mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2}\Big)^2 d\mathbf{x}$$

$$= \Big(\frac{\pi}{2}\Big)^{n/2}\Big(1 - 2\sum_{\mathbf{j}} \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}} \mathcal{B}_{\boldsymbol{\beta},0}(\mathbf{x_j},0) + \sum_{\mathbf{j},\mathbf{k}} \sum_{|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|=0}^{L_{\mathbf{j}},L_{\mathbf{k}}} c_{\mathbf{j},\boldsymbol{\beta}} c_{\mathbf{k},\boldsymbol{\gamma}} \mathcal{B}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x_j},\mathbf{x_k})\Big),$$

where we use the notation

$$\mathcal{B}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \mathcal{S}_{\boldsymbol{\beta}}(-\partial_{\mathbf{x}})\,\mathcal{S}_{\boldsymbol{\gamma}}(-\partial_{\mathbf{y}})\,\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/2}\,.$$

The coefficients $\{\mathbf{c}_{\mathbf{j},\boldsymbol{\beta}}\}$ are chosen to minimize (6.22), that is the vector $\{\mathbf{c}_{\mathbf{j},\boldsymbol{\beta}}\}$ is a solution of the linear system

$$\sum_{\mathbf{j}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}}\,\mathcal{B}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x}_{\mathbf{j}},\mathbf{x}_{\mathbf{k}}) = \mathcal{B}_{0,\boldsymbol{\gamma}}(0,\mathbf{x}_{\mathbf{k}}). \tag{6.23}$$

To show that the matrix of this system is positive definite we use the representation

$$\mathrm{e}^{-|\mathbf{x}-\mathbf{y}|^2/2} = (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2/2}\mathrm{e}^{i(\mathbf{t},\mathbf{x})}\mathrm{e}^{-i(\mathbf{t},\mathbf{y})}\,d\mathbf{t}\,,$$

which implies

$$\mathcal{B}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} \mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\overline{\mathcal{S}_{\boldsymbol{\gamma}}(-i\mathbf{t})}\,\mathrm{e}^{-|\mathbf{t}|^2/2}\mathrm{e}^{i(\mathbf{t},\mathbf{x})}\mathrm{e}^{-i(\mathbf{t},\mathbf{y})}\,d\mathbf{t}\,.$$

Let $\{v_{\mathbf{j},\boldsymbol{\beta}}\}$ be a constant vector and consider the sesquilinear form

$$\begin{aligned}
&\sum_{\mathbf{j},\mathbf{k}}\sum_{|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|=0}^{L_{\mathbf{j}},L_{\mathbf{k}}} \mathcal{B}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x}_{\mathbf{j}},\mathbf{x}_{\mathbf{k}})\,v_{\mathbf{j},\boldsymbol{\beta}}\,\overline{v_{\mathbf{k},\boldsymbol{\gamma}}}\\
&\quad= (2\pi)^{-n/2}\sum_{\mathbf{j},\mathbf{k}}\sum_{|\boldsymbol{\beta}|,|\boldsymbol{\gamma}|=0}^{L_{\mathbf{j}},L_{\mathbf{k}}} v_{\mathbf{j},\boldsymbol{\beta}}\,\overline{v_{\mathbf{k},\boldsymbol{\gamma}}}\int\limits_{\mathbb{R}^n} \mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\overline{\mathcal{S}_{\boldsymbol{\gamma}}(-i\mathbf{t})}\,\mathrm{e}^{-|\mathbf{t}|^2/2}\mathrm{e}^{i(\mathbf{t},\mathbf{x}_{\mathbf{j}}-\mathbf{x}_{\mathbf{k}})}\,d\mathbf{t}\\
&\quad= (2\pi)^{-n/2}\int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2/2}\Big|\sum_{\mathbf{j}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} v_{\mathbf{j},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_{\mathbf{j}})}\Big|^2\,d\mathbf{t} \geq 0\,.
\end{aligned}$$

The change of integration and summation is valid because the integrand is absolutely integrable and the sums are finite. We have to show that the inequality is strict when $\{v_{\mathbf{j},\boldsymbol{\beta}}\} \neq 0$. This is equivalent to show that

$$\sigma(\mathbf{t}) = \sum_{\mathbf{j}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} v_{\mathbf{j},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(-i\mathbf{t})\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_{\mathbf{j}})} = 0$$

identically only if all components $v_{\mathbf{j},\boldsymbol{\beta}} = 0$ for all $\mathbf{j}$ and $\boldsymbol{\beta}$. To this end similar to [19, Lemma 3.1] we introduce the function

$$\begin{aligned}
f_\varepsilon(\mathbf{x}) :=& \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\varepsilon^2|\mathbf{t}|^2/4}\,\sigma(\mathbf{t})\,\mathrm{e}^{-i(\mathbf{t},\mathbf{x})}\,d\mathbf{t} = \sum_{\mathbf{j}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} v_{\mathbf{j},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\int\limits_{\mathbb{R}^n} \mathrm{e}^{-\varepsilon^2|\mathbf{t}|^2/4}\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_{\mathbf{j}}-\mathbf{x})}\,d\mathbf{t}\\
=&\,\varepsilon^{-n}\sum_{\mathbf{j}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} v_{\mathbf{j},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2/4}\,\mathrm{e}^{i(\mathbf{t},\mathbf{x}_{\mathbf{j}}-\mathbf{x})/\varepsilon}\,d\mathbf{t}\\
=&\Big(\frac{4\pi}{\varepsilon^2}\Big)^{n/2}\sum_{\mathbf{j}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} v_{\mathbf{j},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x}_{\mathbf{j}}|^2/\varepsilon^2}\,.
\end{aligned}$$

Let us fix $\mathbf{k}$ and consider $f_\varepsilon(\mathbf{x})$ for $|\mathbf{x}-\mathbf{x_k}| < \varepsilon$ for sufficiently small $\varepsilon > 0$. We have $\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x_j}|^2/\varepsilon^2} \to 0$ as $\varepsilon \to 0$ and

$$\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{k}}} v_{\mathbf{k},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/\varepsilon^2} = \sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{k}}} v_{\mathbf{k},\boldsymbol{\beta}}\mathcal{S}_{\boldsymbol{\beta}}(\varepsilon^{-1}\partial_{\mathbf{t}})\mathrm{e}^{-|\mathbf{t}|^2}\Big|_{\mathbf{t}=(\mathbf{x}-\mathbf{x_k})/\varepsilon}.$$

Because of $f_\varepsilon(\mathbf{x}) = 0$ for all $\varepsilon > 0$ there exist ε_0 such that

$$\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{k}}} v_{\mathbf{k},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/\varepsilon^2} = 0 \tag{6.24}$$

for $\varepsilon \le \varepsilon_0$. On the other hand,

$$\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{k}}} v_{\mathbf{k},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/\varepsilon^2} = \varepsilon^{-2L_{\mathbf{k}}}\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2/\varepsilon^2}\,\Pi_{2L_{\mathbf{k}}}(\varepsilon)\,,$$

where $\Pi_{2L_{\mathbf{k}}}(\varepsilon)$ is a polynomial of degree $2L_{\mathbf{k}}$ in ε with coefficients depending on $|\mathbf{x}-\mathbf{x_k}|$. Therefore (6.24) holds for any $\varepsilon > 0$, in particular

$$\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{k}}} v_{\mathbf{k},\boldsymbol{\beta}}\,\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2} = 0\,.$$

Since by (5.4)

$$\mathcal{S}_{\boldsymbol{\beta}}(\partial_{\mathbf{x}})\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2} = (\mathbf{x}-\mathbf{x_k})^{\boldsymbol{\beta}}\mathrm{e}^{-|\mathbf{x}-\mathbf{x_k}|^2}\,,$$

we conclude $v_{\mathbf{k},\boldsymbol{\beta}} = 0$ for all $\boldsymbol{\beta}$. ■

Let now for given Σ and degrees $L_{\mathbf{j}}$ the coefficient vector $\mathbf{c} = \{c_{\mathbf{j},\boldsymbol{\beta}}\}$ be a unique solution of the linear system (6.19). To estimate $r = Q(\mathbf{c})$ we denote by $\mathbf{y}_{\boldsymbol{\mu}} \in \Sigma$ the point closest to $\mathbf{0}$ and by $L_{\boldsymbol{\mu}}$ the degree of the polynomial $\mathcal{P}_{\boldsymbol{\mu}}$.

Lemma 6.4 *The minimal value of* (6.18) *can be estimated by*

$$r \le \Big(\frac{\pi}{2}\Big)^{n/2}\frac{D^{L_{\boldsymbol{\mu}}+1+n/2}\,|\mathbf{y}_{\boldsymbol{\mu}}|^{2(L_{\boldsymbol{\mu}}+1)}}{D_0^{2(L_{\boldsymbol{\mu}}+1)}(L_{\boldsymbol{\mu}}+1)!}\,.$$

Proof. It follows from the representation (6.21) that

$$\begin{aligned} r &= \int\limits_{\mathbb{R}^n}\Big(\sum_{\mathbf{y_j}\in\Sigma}\mathrm{e}^{(D-D_0)|\mathbf{y_j}|^2/D_0^2}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mathbf{j}}} c_{\mathbf{j},\boldsymbol{\beta}}T_{\boldsymbol{\beta}}(\mathbf{t}-\mathbf{y_j})\,\mathrm{e}^{-|\mathbf{t}-\mathbf{t_j}|^2/D} - \mathrm{e}^{-|\mathbf{t}|^2/D}\Big)^2 d\mathbf{t} \\ &\le \min_{\mathcal{P}\in\Pi_{L_{\mu}}}\int\limits_{\mathbb{R}^n}\Big(\mathcal{P}(\mathbf{t})\mathrm{e}^{-|\mathbf{t}-\mathbf{t}_{\mu}|^2/D} - \mathrm{e}^{-|\mathbf{t}|^2/D}\Big)^2 d\mathbf{t} \\ &= \Big(\frac{D}{2}\Big)^{n/2}\min_{\mathcal{P}\in\Pi_{L_{\mu}}}\int\limits_{\mathbb{R}^n}\mathrm{e}^{-|\mathbf{t}|^2}\Big(\mathcal{P}(\mathbf{t}) - \mathrm{e}^{-|\mathbf{z}_{\mu}|^2}\mathrm{e}^{-\sqrt{2}(\mathbf{t},\mathbf{z}_{\mu})}\Big)^2 d\mathbf{t} \end{aligned}$$

with $\mathbf{t}_{\boldsymbol{\mu}} = D\mathbf{y}_{\boldsymbol{\mu}}/D_0$, $\mathbf{z}_{\boldsymbol{\mu}} = \sqrt{D}\mathbf{y}_{\boldsymbol{\mu}}/D_0$, and $\Pi_{L_{\boldsymbol{\mu}}}$ denotes the set of polynomials of degree $L_{\boldsymbol{\mu}}$. The minimum is attained when

$$\mathcal{P}(\mathbf{t}) = \frac{1}{\sqrt{2^{|\boldsymbol{\beta}|}\boldsymbol{\beta}!\pi^{n/2}}}\sum_{|\boldsymbol{\beta}|=0}^{L_{\mu}} a_{\boldsymbol{\beta}}H_{\boldsymbol{\beta}}(\mathbf{t})$$

with the coefficients

$$a_{\boldsymbol\beta} = \frac{\mathrm{e}^{-|\mathbf{z}_{\boldsymbol\mu}|^2}}{\sqrt{2^{|\boldsymbol\beta|}\,\boldsymbol\beta!\,\pi^{n/2}}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-|\mathbf{t}|^2} H_{\boldsymbol\beta}(\mathbf{t}) \mathrm{e}^{-\sqrt{2}(\mathbf{t},\mathbf{z}_{\boldsymbol\mu})}\, d\mathbf{t} = \frac{\mathrm{e}^{-|\mathbf{z}_{\boldsymbol\mu}|^2}}{\sqrt{2^{|\boldsymbol\beta|}\boldsymbol\beta!\pi^{n/2}}} \int\limits_{\mathbb{R}^n} \mathrm{e}^{-\sqrt{2}(\mathbf{t},\mathbf{z}_{\boldsymbol\mu})} (-\partial_{\mathbf{t}})^{\boldsymbol\beta} \mathrm{e}^{-|\mathbf{t}|^2}\, d\mathbf{t}\,.$$

Integrating by parts, we obtain

$$a_{\boldsymbol\beta} = \pi^{n/4} \frac{(-1)^{|\boldsymbol\beta|} \mathbf{z}_{\boldsymbol\mu}^{\boldsymbol\beta}}{\sqrt{\boldsymbol\beta!}}\, \mathrm{e}^{-|\mathbf{z}_{\boldsymbol\mu}|^2/2}\,,$$

which together with

$$\sum_{|\boldsymbol\beta|=L_{\boldsymbol\mu}+1}^{\infty} \frac{\mathbf{z}_{\boldsymbol\mu}^{2\boldsymbol\beta}}{\boldsymbol\beta!} = \sum_{s=L_{\boldsymbol\mu}+1}^{\infty} \frac{|\mathbf{z}_{\boldsymbol\mu}|^{2s}}{s!} \leqslant \frac{|\mathbf{z}_{\boldsymbol\mu}|^{2(L_{\boldsymbol\mu}+1)}}{(L_{\boldsymbol\mu}+1)!}\, \mathrm{e}^{|\mathbf{z}_{\boldsymbol\mu}|^2}$$

leads to

$$r \le \Big(\frac{D}{2}\Big)^{n/2} \sum_{|\boldsymbol\beta|=L_{\boldsymbol\mu}+1}^{\infty} |a_{\boldsymbol\beta}|^2 = \pi^{n/2} \Big(\frac{D}{2}\Big)^{n/2} \frac{|\mathbf{z}_{\boldsymbol\mu}|^{2(L_{\boldsymbol\mu}+1)}}{(L_{\boldsymbol\mu}+1)!}\,. \qquad \blacksquare$$

6.4 Approximate partition of unity with Gaussians

Now we are in position to prove the main result of this section. Suppose that the nodes $\{\mathbf{x}_{\mathbf{j}}\}$ are as described in subsection 6.1 and let $G_1 \cup G_2$ be the associated piecewise uniform grid with stepsizes h_1 and h_2. Assign to each grid point $g_\ell \in G_\ell$, $\ell = 1, 2$, a finite set of nodes $\Sigma(g_\ell)$, fix a common degree L for all polynomials $\mathcal{P}_{\mathbf{j}}$ in (6.1) and solve the linear system

$$\sum_{\mathbf{x}_{\mathbf{j}} \in \Sigma(g_\ell)} \sum_{|\boldsymbol\beta|=0}^{L} \mathcal{C}_{\boldsymbol\beta,\boldsymbol\gamma}\Big(\frac{\mathbf{x}_{\mathbf{j}} - g_\ell}{h_\ell}, \frac{\mathbf{x}_{\mathbf{k}} - g_\ell}{h_\ell}\Big) c_{\mathbf{j},\boldsymbol\beta}(g_\ell) = \mathcal{C}_{\mathbf{0},\boldsymbol\gamma}\Big(\mathbf{0}, \frac{\mathbf{x}_{\mathbf{k}} - g_\ell}{h_\ell}\Big) \tag{6.25}$$

for all $\mathbf{x}_{\mathbf{k}} \in \Sigma(g_\ell)$ and $0 \le |\boldsymbol\gamma| \le L$ with

$$\mathcal{C}_{\boldsymbol\beta,\boldsymbol\gamma}(\mathbf{x}, \mathbf{y}) = \mathcal{S}_{\boldsymbol\beta}(-\sqrt{D}\partial_{\mathbf{x}}) \mathcal{S}_{\boldsymbol\gamma}(-\sqrt{D}\partial_{\mathbf{y}})\, \mathrm{e}^{(D-D_0)(|\mathbf{x}|^2+|\mathbf{y}|^2)/D_0^2}\, \mathrm{e}^{-D|\mathbf{x}-\mathbf{y}|^2/2D_0^2}\,,$$

$D_0 < D$ is some arbitrary positive number. Following (6.20) define the polynomials

$$\begin{aligned} \mathcal{P}_{\mathbf{j}}\Big(\frac{\mathbf{x}-\mathbf{x}_{\mathbf{j}}}{h_1\sqrt{D}}\Big) &= \sum_{g_1 \in G(\mathbf{x}_{\mathbf{j}})} \sum_{|\boldsymbol\beta|=0}^{L} c_{\mathbf{j},\boldsymbol\beta}(g_1) \Big(\frac{\mathbf{x}-\mathbf{x}_{\mathbf{j}}}{h_1\sqrt{D}}\Big)^{\boldsymbol\beta}, \qquad \mathbf{x}_{\mathbf{j}} \in J_1, \\ \mathcal{P}_{\mathbf{k}}\Big(\frac{\mathbf{x}-\mathbf{x}_{\mathbf{k}}}{h_2\sqrt{D}}\Big) &= \sum_{g_2 \in G(\mathbf{x}_{\mathbf{k}})} \sum_{|\boldsymbol\beta|=0}^{L} \tilde{a}_{g_2} c_{\mathbf{k},\boldsymbol\beta}(g_2) \Big(\frac{\mathbf{x}-\mathbf{x}_{\mathbf{k}}}{h_2\sqrt{D}}\Big)^{\boldsymbol\beta}, \qquad \mathbf{x}_{\mathbf{k}} \in J_2, \end{aligned} \tag{6.26}$$

Theorem 6.1 *Under Conditions* 2.1 *and* 6.1 *on the scattered nodes* $\{\mathbf{x}_{\mathbf{j}}\}$ *for any* $\varepsilon > 0$ *there exist* $D > 0$ *and* L *such that the function* (6.1) *is an approximate partition of unity satisfying*

$$|\Theta(\mathbf{x}) - 1| < \varepsilon \quad \text{for all } \mathbf{x} \in \mathbb{R}^n\,,$$

if the polynomials $\{\mathcal{P}_{\mathbf{j}}\}$ *of degree* L *are generated via* (6.26) *by the solutions* $\{c_{\mathbf{j},\boldsymbol\beta}(g_\ell)\}$ *of the linear systems* (6.25) *for all* $g_\ell \in G_1 \cup G_2$.

Proof. From (6.9) we have to show that

$$\sup_{\mathbb{R}^n}\Big(\sum_{g_1\in G_1}\Big|\omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big)\Big| + \sum_{g_2\in G_2}\tilde a_{g_2}\Big|\omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big)\Big|\Big) \le \frac{\varepsilon}{2} \tag{6.27}$$

if L is sufficiently large. We start with estimating the first sum

$$\sum_{g_1\in G_1}\Big|\omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big)\Big|,$$

where $g_1 = h_1\mathbf{m}$, $\mathbf{m}\in Z_1\subset\mathbb{Z}^n$. Using (6.10) we can write

$$\omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big) = \omega\big(\frac{\mathbf{x}}{h_1}-\mathbf{m}\big),$$

where the points $\mathbf{y_j}$ in (6.10) are given by $\mathbf{y_j} = \mathbf{x_j}/h_1 - \mathbf{m}$, $\mathbf{x_j}\in\Sigma(g_1)$. By Lemmas 6.2 and 6.4 we have

$$\sum_{g_1\in G_1}\Big|\omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big)\Big| \le c_3\Big(\frac{\pi}{2}\Big)^{n/4}\sum_{\mathbf{m}\in Z_1}\mathrm{e}^{-\rho|\mathbf{x}/h_1-\mathbf{m}|^2}\frac{D^{(L_{\boldsymbol{\mu}_\mathbf{m}}+1+n/2)/2}}{D_0^{L_{\boldsymbol{\mu}_\mathbf{m}}+1}\sqrt{(L_{\boldsymbol{\mu}_\mathbf{m}}+1)!}}\Big|\frac{\mathbf{x}_{\boldsymbol{\mu}_\mathbf{m}}}{h_1}-\mathbf{m}\Big|^{L_{\boldsymbol{\mu}_\mathbf{m}}+1}$$

where $\mathbf{x}_{\boldsymbol{\mu}_\mathbf{m}}\in\Sigma(g_1)$ is the node closest to $g_1 = h_1\mathbf{m}$ and $L_{\boldsymbol{\mu}_\mathbf{m}}$ is the degree of the polynomial $\mathcal{P}_{\boldsymbol{\mu}_\mathbf{m},g_1}$. Since $|\mathbf{x}_{\boldsymbol{\mu}_\mathbf{m}} - h_1\mathbf{m}|\le\kappa_1 h_1$ by Condition 2.1 and $L_{\boldsymbol{\mu}_\mathbf{m}} = L$ for all $\boldsymbol{\mu}_\mathbf{m}$ we conclude that

$$\sum_{g_1\in G_1}\Big|\omega_{g_1}\big(\frac{\mathbf{x}}{h_1}\big)\Big| \le c_3\Big(\frac{\pi}{2}\Big)^{n/4}\frac{D^{(L+1+n/2)/2}\,\kappa_1^{L+1}}{D_0^{L+1}\sqrt{(L+1)!}}\sup_{\mathbb{R}^n}\sum_{\mathbf{m}\in Z_1}\mathrm{e}^{-\rho|\mathbf{x}/h_1-\mathbf{m}|^2}. \tag{6.28}$$

From

$$\rho = \frac{D-D_0}{(D-D_0)^2+DD_0}\in(0,D)\quad\text{for any fixed}\quad D_0\in(0,D),$$

we see, that for fixed D and D_0

$$\sum_{g_1\in G_1}\Big|\omega_{g_1}\big(\frac{\mathbf{x}}{h}\big)\Big| \to 0 \quad\text{if}\quad L\to\infty. \tag{6.29}$$

We turn to

$$\sum_{g_2\in G_2}\tilde a_{g_2}\Big|\omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big)\Big|$$

with $g_2 = h_1\mathbf{m}+h_2\mathbf{k}$, $\mathbf{m}\in Z_2, \mathbf{k}\in S$. Using (6.10) we have

$$\omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big) = \omega\big(\frac{\mathbf{x}-\mathbf{m}h_1}{h_2}-\mathbf{k}\big),$$

and the points $\mathbf{y_j}$ in (6.10) are given by $\mathbf{y_j} = (\mathbf{x_j}-\mathbf{m}h_1)/h_2-\mathbf{k}$ with $\mathbf{x_j}\in\Sigma(g_2)$. Hence

$$\sum_{g_2\in G_2}\tilde a_{g_2}\Big|\omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big)\Big| = \sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S}a_\mathbf{k}\Big|\omega\big(\frac{\mathbf{x}-\mathbf{m}h_1}{h_2}-\mathbf{k}\big)\Big|$$

$$\le c_3\Big(\frac{\pi}{2}\Big)^{n/4}\sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S}a_\mathbf{k}\mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2}\frac{D^{(L_{\boldsymbol{\mu}_\mathbf{k}}+1+n/2)/2}}{D_0^{L_{\boldsymbol{\mu}_\mathbf{k}}+1}\sqrt{(L_{\boldsymbol{\mu}_\mathbf{k}}+1)!}}\Big|\frac{\mathbf{x}_{\boldsymbol{\mu}_\mathbf{k}}-\mathbf{m}h_1}{h_2}-\mathbf{k}\Big|^{L_{\boldsymbol{\mu}_\mathbf{k}}+1}.$$

Here $\mathbf{x}_{\boldsymbol{\mu}_\mathbf{k}} \in \Sigma(g_2)$ is the node closest to $g_2 = h_1\mathbf{m} + h_2\mathbf{k}$ and $L_{\boldsymbol{\mu}_\mathbf{k}}$ is the degree of the polynomial $\mathcal{P}_{\boldsymbol{\mu}_\mathbf{k},g_2}$. By Condition 6.1 for fixed D and D_0

$$\frac{D^{(L+1+n/2)/2}}{D_0^{L+1}\sqrt{(L+1)!}}\Big|\frac{\mathbf{x}_{\boldsymbol{\mu}_\mathbf{k}} - \mathbf{m}h_1}{h_2} - \mathbf{k}\Big|^{L+1} \le \delta(L) \to 0 \quad \text{if} \quad L \to \infty$$

uniformly for all $g_2 \in G_2$. Hence we obtain

$$\sum_{g_2\in G_2} \tilde{a}_{g_2}\Big|\omega_{g_2}\big(\frac{\mathbf{x}}{h_2}\big)\Big| \le C_1\delta(L) \sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S} a_\mathbf{k} \mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2} \tag{6.30}$$

because of $L_{\boldsymbol{\mu}_\mathbf{k}} = L$ for all $\boldsymbol{\mu}_\mathbf{k}$. The sum

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} a_\mathbf{k} \mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2} = \Big(\frac{h_1^2}{\pi D(h_1^2-h_2^2)}\Big)^{n/2} \sum_{\mathbf{k}\in\mathbb{Z}^n} \mathrm{e}^{-h_2^2|\mathbf{k}|^2/(h_1^2-h_2^2)D}\, \mathrm{e}^{-\rho|\mathbf{x}-\mathbf{m}h_1-h_2\mathbf{k}|^2/h_2^2}$$

can be easily estimated by using equation (6.2). Setting

$$(h_1^2 - h_2^2)D = h_1^2 D_1 - h_2^2/\rho$$

we derive

$$D_1 = D + \frac{h_2^2}{h_1^2}\Big(\frac{1}{\rho} - D\Big) = D + H^2\,\frac{D_0^2}{D-D_0}\,.$$

and after some algebra

$$\sum_{\mathbf{k}\in\mathbb{Z}^n} \mathrm{e}^{-h_2^2|\mathbf{k}|^2/(h_1^2-h_2^2)D}\, \mathrm{e}^{-\rho|\mathbf{x}-h_2\mathbf{k}|^2/h_2^2}$$
$$= \Big(\frac{\pi D(1-H^2)}{\rho D_1}\Big)^{n/2} \mathrm{e}^{-|\mathbf{x}|^2/h_1^2 D_1}\Big(1 + O(\mathrm{e}^{-\pi^2 D^2(1-H^2)/D_1})\Big).$$

Therefore we obtain

$$\sup_{\mathbb{R}^n} \sum_{\mathbf{m}\in Z_2}\sum_{\mathbf{k}\in S} a_\mathbf{k}\mathrm{e}^{-\rho|(\mathbf{x}-\mathbf{m}h_1)/h_2-\mathbf{k}|^2} \le C_2 \sup_{\mathbb{R}^n} \sum_{\mathbf{m}\in Z_2} \mathrm{e}^{-|\mathbf{x}-\mathbf{m}h_1|^2/h_1^2 D_1} \le C_3$$

with some constant C_3 depending on D, D_0 and the space dimension n. Now (6.27) follows immediately from (6.29) and (6.30). ■

Remark 6.1 It can be seen from (6.28) that in principle the parameter D_0 can be any value of the interval $(0, D)$ not too close the its end points. In numerical experiments we have not seen any significant dependence on this parameter. The choice $D_0 = D/2$ might be advantageous because the differential expressions $\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y})$ simplify to

$$\mathcal{C}_{\boldsymbol{\beta},\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \mathcal{S}_{\boldsymbol{\beta}}(-\sqrt{D}\partial_\mathbf{x})\mathcal{S}_{\boldsymbol{\gamma}}(-\sqrt{D}\partial_\mathbf{y})\,\mathrm{e}^{4(\mathbf{x},\mathbf{y})/D}\,.$$

6.5 Numerical Experiments

We have tested the construction (6.26, 6.25) for a quasi-uniform distribution of nodes on $\mathbb{R}$ with the parameters $D = 2$, $D_0 = 3/2$, $h = 1$, $\kappa_1 = 1/2$. To see the dependence of the approximation error from the number of nodes in $\Sigma(m)$, $m \in \mathbb{Z}$, and the degree of polynomials we provide graphs of the difference to 1 for the following cases :

- $\Sigma(m)$ consists of 1 point, $L = 1, 2$ (Fig. 4) and $L = 3, 4$ (Fig. 5);
- $\Sigma(m)$ consists of 3 points, $L = 1, 2$ (Fig. 6) and $L = 3, 4$ (Fig. 7);
- $\Sigma(m)$ consists of 5 points, $L = 1, 2$ (Fig. 8) and $L = 3, 4$ (Fig. 9).

As expected, the approximation becomes better with increasing degree L and more points in the subsets $\Sigma(m)$. The use of only one node in $\Sigma(m)$ reduces the approximation error by a factor 10^{-1} if L increases by 1. The cases of 3 and 5 points indicate, that enlarging the degree L of the polynomials by 1 gives a factor 10^{-2} for the approximation error.

One should notice, that the plotted total error consists of two parts. Using (6.26, 6.25) we approximate the θ-function

$$(2\pi)^{-1/2} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-(x-m)^2/2} = 1 + 2 \sum_{j=1}^{\infty} \mathrm{e}^{-2\pi^2 j^2} \cos 2\pi j x \,. \tag{6.31}$$

Hence, the plotted total error is the sum of the difference between (6.1) and (6.31) and the function

$$2 \sum_{j=1}^{\infty} \mathrm{e}^{-2\pi^2 j^2} \cos 2\pi j x \,, \tag{6.32}$$

which is the saturation term obtained on the uniform grid. The error plots in Figure 7 for $L = 4$ and in Figure 9 show that the total error is already majorized by (6.32), which is shown by dashed lines.

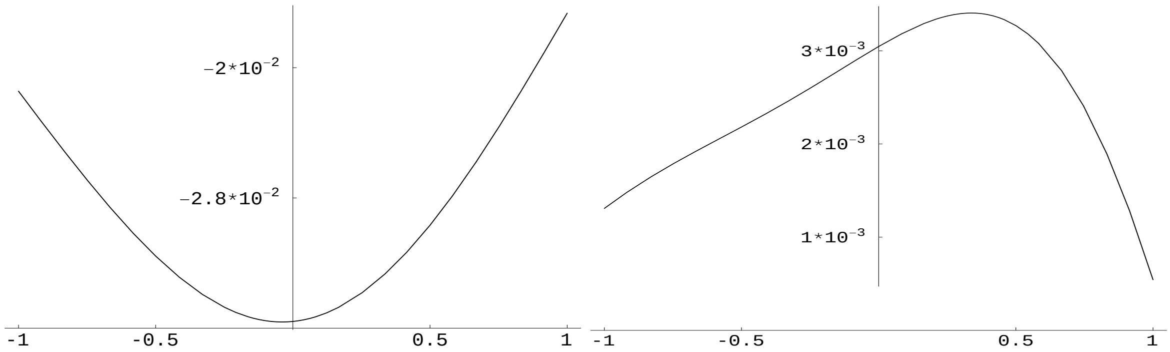

Figure 4: The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 1 point, $L = 1$ (on the left) and $L = 2$ (on the right).

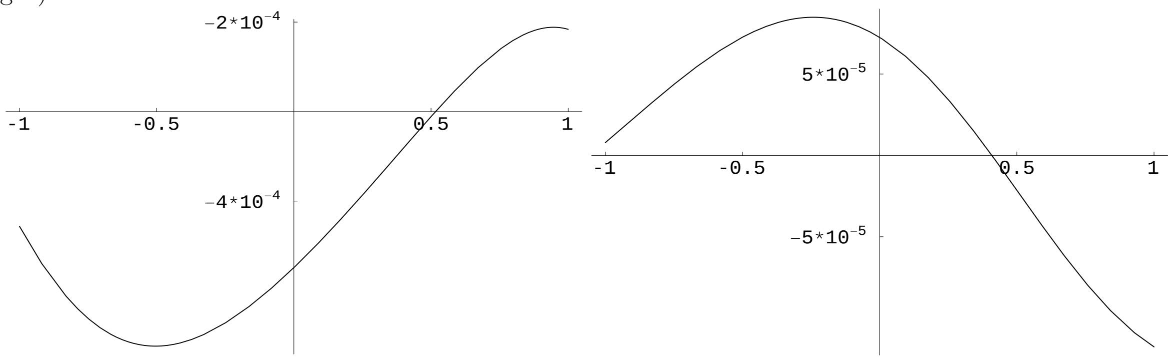

Figure 5: The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 1 point, $L = 3$ (on the left) and $L = 4$ (on the right).

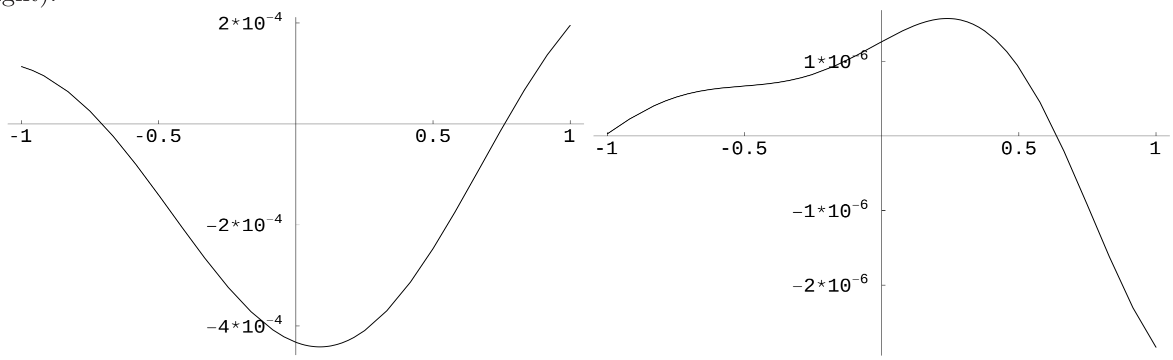

Figure 6: The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 3 points, $L = 1$ (on the left) and $L = 2$ (on the right).

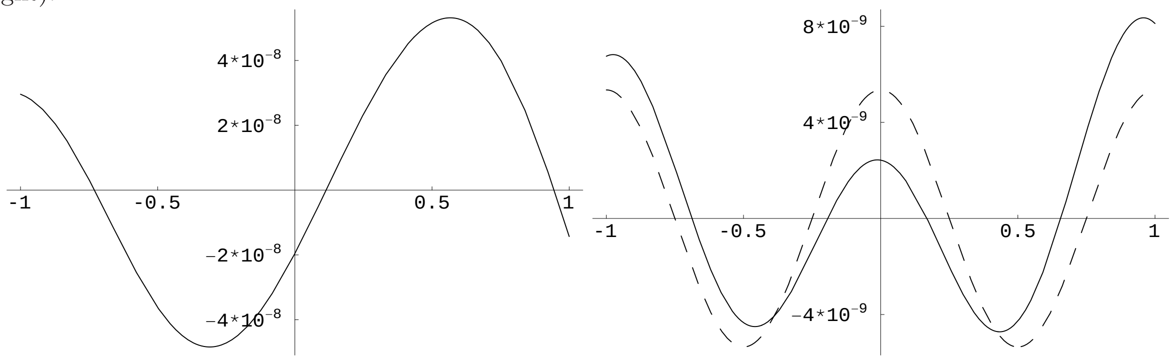

Figure 7: The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 3 points, $L = 3$ (on the left) and $L = 4$ (on the right). The saturation term (6.32) is depicted by dashed lines.

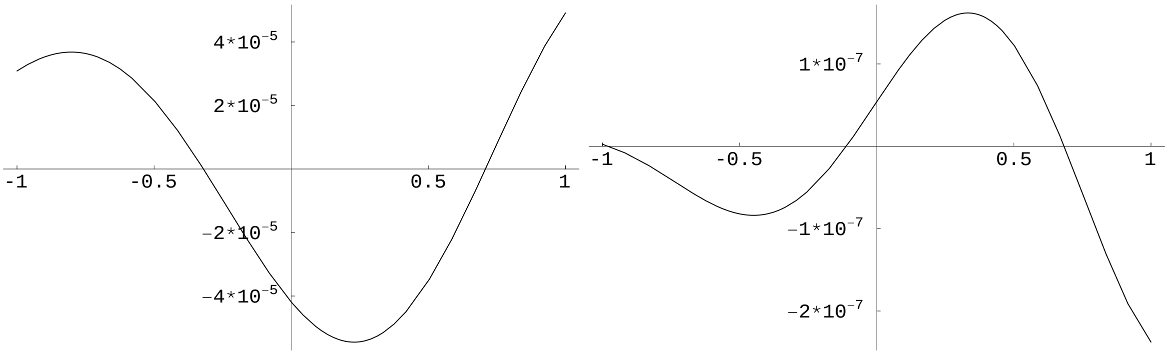

Figure 8: The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 5 points, $L = 1$ (on the left) and $L = 2$ (on the right).

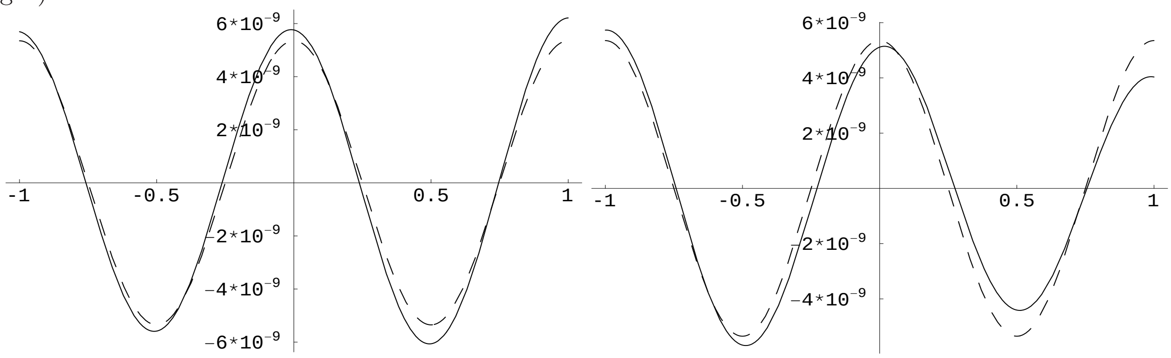

Figure 9: The graph of $\Theta(\mathbf{x}) - 1$ when $\Sigma(m)$ consists of 5 points, $L = 3$ (on the left) and $L = 4$ (on the right). The saturation term (6.32) is depicted by dashed lines.

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