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Pointwise asymptotic convergence of solutions for a phase separation model

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Abstract

A new technique, combining the global energy and entropy balance equations with the local stability theory for dynamical systems, is used for proving that every solution to a non-smooth temperature-driven phase separation model with conserved energy converges pointwise in space to an equilibrium as time tends to infinity. Three main features are observed: the limit temperature is uniform in space, there exists a partition of the physical body into at most three constant limit phases, and the phase separation process has a hysteresis-like character.

Introduction

This paper deals with the asymptotic behavior of solutions for a phase separation model which involves the subdifferential of an indicator function. Before precisely stating our mathematical results and giving their proof, let us briefly recall some related results in the literature.

Given a nonlinear evolution equation, once we establish the global existence and uniqueness of a solution, a central issue is to study its asymptotic behavior for large times. As pointed out in [25], this study can be divided into two categories. The first category includes the investigations of a single orbit starting from a given initial datum. In particular, a relevant question is whether the solution converges to an equilibrium as time goes to infinity. The second category of problems is related to all orbits starting from any bounded set of initial data, with the intention to see, for instance, whether this family of orbits will eventually converge to an invariant compact set, which is usually called a global attractor. In the context of classical phase-field equations, which were first studied by Caginalp [7], we refer to [9], [1], [24] for results in the first category, and to [3], [4], and [5] for the second category. For other types of phase-field models, we mention, e. g., [20], [2], [10], [12], [21], [23] for the first category, and, e. g., [8], [19], [13] for the second category.

In this paper, we stay within the first category, and prove that for any given initial datum, the solution converges to an equilibrium as time tends to infinity. Our problem is new in several respects. First, it involves the subdifferential of an indicator function. It turns out that the usual Łojasiewicz-Simon approach suitable for analytic nonlinearities thus seems difficult to apply here, also because the limit asymptotic state may be discontinuous. Secondly, in our problem the temperature satisfies the Neumann boundary condition, hence the steady temperature is not uniquely determined as in the Dirichlet boundary condition case (cf. a similar situation in [20], [24], [23]). In other words, we have to prove convergence simultaneously for both the temperature and the order parameter.

In order to overcome the corresponding mathematical difficulties due to these new features, we propose a new technique, combining the first and the second principle of thermodynamics with a local phase dynamics argument, to describe the pointwise convergence of the order parameter trajectories towards the natural singular values, with an a priori unknown temperature equilibrium. As a model example, we consider the following phase-field system for the state variables θ (the *absolute temperature*) and χ (an *order parameter* characterizing the physical phase – see the comments after Fig.1 below).

$$\partial_t(c_V\theta + \Lambda(\chi)) - \kappa\Delta\theta \;=\; 0\,, \tag{1.1}$$

$$\mu\partial_t\chi + \lambda(\chi) + \partial I_{[0,1]}(\chi) \;\ni\; \frac{L}{\theta_c}\,\theta\,, \tag{1.2}$$

$$\Lambda(\chi) \;=\; L\chi + \alpha\chi(1-\chi)\,, \quad \lambda(\chi) \;=\; \Lambda'(\chi) \;=\; L + \alpha - 2\alpha\chi \tag{1.3}$$

for $(x,t) \in \Omega \times (0,\infty)$, where $\Omega \subset \mathbb{R}^N$ with $N \ge 1$, representing the physical body, is an open bounded domain with Lipschitzian boundary, $I_{[0,1]}$ is the indicator function of the interval $[0,1]$ (that is, $I_{[0,1]}(\chi) = 0$ if $\chi \in [0,1]$, $I_{[0,1]}(\chi) = +\infty$ if $\chi \notin [0,1]$), and $\partial I_{[0,1]}$ is its (maximal monotone) subdifferential. The specific heat c_V, heat conductivity κ, latent heat L, phase relaxation coefficient μ, mean phase transition temperature θ_c, and undercooling/overheating parameter $\alpha < L$ are assumed to be positive constants. We will see that the exact shape (1.3) of Λ enables us to simplify some formulas in Section 4. The argument however remains valid if Λ is any *strictly concave increasing function* in $C^2([0,1])$.

We couple the above system with the homogeneous Neumann boundary condition

$$\frac{\partial\theta}{\partial n} \;=\; 0 \qquad \text{on } \partial\Omega \times (0,\infty) \tag{1.4}$$

and initial conditions

$$\theta(x,0) \;=\; \theta^0(x)\,, \quad \chi(x,0) \;=\; \chi^0(x)\,, \tag{1.5}$$

with given functions

$$\left.\begin{array}{ll} \theta^0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)\,, & \operatorname{inf\,ess}\{\theta^0(x)\,;\, x \in \Omega\} > 0\,, \\ \chi^0 \in L^\infty(\Omega)\,, & \chi^0(x) \in [0,1] \ \text{ a. e.} \end{array}\right\} \tag{1.6}$$

The *free energy density* F corresponding to (1.1)–(1.3) is of the form

$$\begin{aligned} F(\theta,\chi) \;&=\; c_V\theta(1-\log\theta) + \Lambda(\chi) + I_{[0,1]}(\chi) - \frac{L\theta}{\theta_c}\chi \\ &=\; c_V\theta(1-\log\theta) + L\chi\left(1 - \frac{\theta}{\theta_c}\right) + \alpha\chi(1-\chi) + I_{[0,1]}(\chi)\,, \end{aligned} \tag{1.7}$$

hence it is of double obstacle type with respect to χ as in [22, Sect. VII.3] with two local minima at $\chi = 0$ and $\chi = 1$ in the temperature range

$$1 - \frac{\alpha}{L} \;<\; \frac{\theta}{\theta_c} \;<\; 1 + \frac{\alpha}{L}\,. \tag{1.8}$$

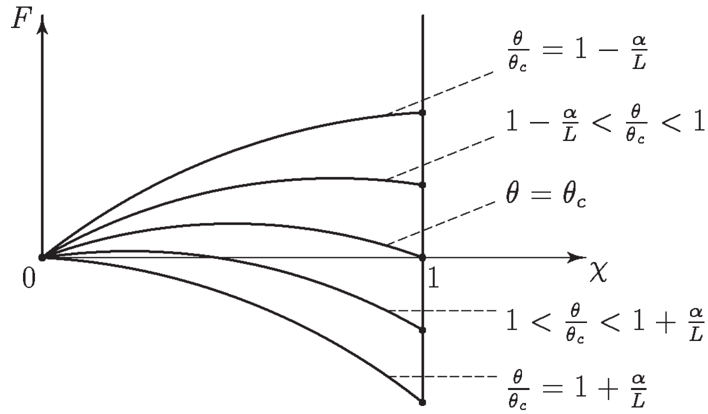

Figure 1: The "phase component" of the free energy at different temperatures.

Beyond this interval, only one local minimum persists, namely $\chi = 1$ for high temperatures, and $\chi = 0$ for low temperatures. The values of χ outside $[0, 1]$ are not accessible due to the term $I_{[0,1]}(\chi)$ in the free energy. Figure 1 shows the shape of $F(\theta, \chi)$ for different values of θ, not accounting for the purely caloric component $c_V\theta(1 - \log\theta)$, which only produces vertical shifts in the diagram.

The order parameter χ can thus be interpreted as a characterization of the phase: the body Ω is in high temperature phase at point x and time t if $\chi(x, t) = 1$, and in low temperature phase if $\chi(x, t) = 0$, while the intermediate values of χ correspond to a mixture of both. Intuitively, the mixtures can be expected to be unstable because of the concave character of the free energy in the open interval $(0, 1)$.

Similarly to the general scheme in [6], we associate with the free energy density F given by (1.7) the densities of *internal energy* U and *entropy* S in the form

$$U(\theta, \chi) = c_V\theta + \Lambda(\chi) + I_{[0,1]}(\chi), \tag{1.9}$$

$$S(\theta, \chi) = c_V \log\theta + \frac{L}{\theta_c}\chi. \tag{1.10}$$

Using the identity

$$\partial_t I_{[0,1]}(\chi(t)) = \xi(t)\,\partial_t\chi(t) = 0 \quad \text{a.e.}, \tag{1.11}$$

which holds for every absolutely continuous function χ and every measurable selection $\xi(t) \in \partial I_{[0,1]}(\chi(t))$, we may interpret Eq. (1.1) as the energy balance

$$\partial_t U + \operatorname{div} \mathbf{q} = 0 \tag{1.12}$$

with Fourier heat flux $\mathbf{q} = -\kappa\nabla\theta$. Eq. (1.2) describes the phase relaxation dynamics similarly as in [22, Sect. V.1]. Using (1.11), we easily check that every solution of (1.1)–(1.6) with the properties stated in Theorem 2.1 below ($\theta > 0$, in particular) satisfies the entropy balance equation

$$\partial_t S + \operatorname{div} \frac{\mathbf{q}}{\theta} = \frac{\mu}{\theta}(\partial_t\chi)^2 + \frac{\kappa|\nabla\theta|^2}{\theta^2}, \tag{1.13}$$

where the entropy production term on the right-hand side is non-negative in agreement with the Second principle of thermodynamics.

The main result of this paper is Theorem 2.2 below on the convergence of (θ, χ) towards an equilibrium $(\theta_\infty, \chi_\infty)$ as $t \to \infty$. In particular, if θ_∞ is within the interval given by (1.8), the range of χ_∞ consists of three points at most: the two pure phases $\chi = 0$ and $\chi = 1$, and possibly one intermediate phase.

We obtain the convergence result from the energy conservation principle in cases where phase transition can only take place in the mixture (Steps (i)–(iii) of the proof). Otherwise, in order to get a possible mass exchange between the pure phases under control, we take also the entropy balance into account (Steps (iv)–(viii)).

In Section 2 we state the main result. Section 3 is devoted to a uniform estimate of the difference between the local temperature and the mean temperature using a semigroup argument, and the convergence of the solution towards an equilibrium is proved in Section 4.

2 Main result

The exact values of the physical constants in (1.1)–(1.3) are not relevant for the qualitative behaviour of the solution. We therefore assume that

$$c_V = \kappa = L = \mu = \theta_c = 1\,, \quad 0 < \alpha < 1\,, \tag{2.1}$$

$$|\Omega| = 1\,, \quad \text{where } |\cdot| \text{ denotes the Lebesgue measure in } \mathbb{R}^N. \tag{2.2}$$

In other words, system (1.1)–(1.3) now reads

$$\partial_t(\theta + \Lambda(\chi)) - \Delta\theta \;=\; 0\,, \tag{2.3}$$

$$\partial_t\chi + \lambda(\chi) + \partial I_{[0,1]}(\chi) \;\ni\; \theta\,, \tag{2.4}$$

$$\Lambda(\chi) \;=\; \chi + \alpha\chi(1-\chi)\,, \quad \lambda(\chi) \;=\; 1 + \alpha - 2\alpha\chi\,. \tag{2.5}$$

This is a special case of the system

$$\partial_t(\theta + F_1[w]) - \Delta\theta \;=\; 0\,, \tag{2.6}$$

$$\mu(\theta)\,\partial_t w + f_1[w] + \theta f_2[w] \;=\; 0 \tag{2.7}$$

with hysteresis operators f_1, f_2, F_1, which was investigated in [16]. Indeed, (2.3)–(2.5) can be transformed into (2.6)–(2.7) by introducing an auxiliary function

$$w(x,t) \;=\; \int_0^t (\theta(x,\tau) - \lambda(\chi(x,\tau))\,d\tau\,. \tag{2.8}$$

Then the inclusion

$$\partial_t\chi + \partial I_{[0,1]}(\chi) \;\ni\; \partial_t w\,, \quad \chi(0) = \chi^0 \tag{2.9}$$

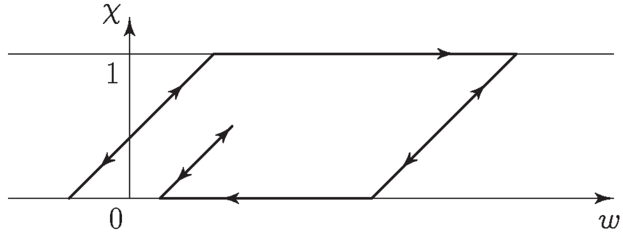

Figure 2: A diagram of the stop $\chi = \mathfrak{s}[\chi^0, w]$ with $\chi^0 = 0$.

defines the so-called *stop operator* $\chi = \mathfrak{s}[\chi^0, w]$ (see Figure 2), and we obtain (2.6)–(2.7) with $F_1[w] = \Lambda(\mathfrak{s}[\chi^0, w])$, $f_1[w] = \lambda(\mathfrak{s}[\chi^0, w])$, $f_2[w] = -1$, $\mu(\theta) = 1$, see [15] for details.

The main result in [16] was Theorem 2.1, which reads (with respect to the present notation) as follows.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitzian boundary, and let θ^0, χ^0 satisfying (1.6) be given. Then system (2.3)–(2.5), (1.4)–(1.5) admits a unique global solution $(\theta, \chi) \in [L^\infty(\Omega \times (0,\infty))]^2$ such that $\partial_t \chi \in L^\infty(\Omega \times (0,\infty))$, $\partial_t \theta, \Delta\theta \in L^2(\Omega \times (0,\infty))$, $\theta(x,t) > 0, \chi(x,t) \in [0,1]$ a. e. in $\Omega \times (0,\infty)$, and the function*

$$V(t) := \int_\Omega \left(|\nabla\theta|^2 + |\partial_t \chi|^2\right)(x,t)\, dx \tag{2.10}$$

has the property

$$\int_0^\infty V(t)\, dt < \infty\,, \quad \operatorname*{Var}_{[0,\infty)} V^2 < \infty\,, \quad \lim_{t\to\infty} \operatorname*{sup\,ess}_{s>t} V(s) = 0\,. \tag{2.11}$$

Note that $V(t)$ may be discontinuous. This makes the proof of the convergence of $V(t)$ towards zero technically complicated, and special dissipation properties of hysteresis operators have to be taken into account.

The total energy $\mathcal{E}(t)$ and entropy $\mathcal{S}(t)$ are given by the respective formulas

$$\mathcal{E}(t) = \int_\Omega (\theta + \Lambda(\chi))(x,t)\, dx \tag{2.12}$$

$$\mathcal{S}(t) = \int_\Omega (\log\theta + \chi)(x,t)\, dx\,. \tag{2.13}$$

Integrating (2.3) and (1.13) over Ω and using the boundary condition (1.4) we obtain

$$\mathcal{E}(t) = \mathcal{E}(0) =: \mathcal{E}_0 \tag{2.14}$$

$$\dot{\mathcal{S}}(t) \geq 0 \text{ a. e.} \tag{2.15}$$

We further have $\log\theta \leq \theta$, $\chi \leq \Lambda(\chi)$ for all admissible arguments, hence $\mathcal{S}(t)$ is a bounded non-decreasing function, and there exists $\mathcal{S}_\infty \leq \mathcal{E}_0$ such that

$$\mathcal{S}(t) \nearrow \mathcal{S}_\infty \quad \text{as} \quad t \to \infty\,. \tag{2.16}$$

The above balance principles for $\mathcal{E}(t)$ and $\mathcal{S}(t)$ will play a crucial role in Section 4 in the proof of the following main result of this paper.

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1, assume that $\Omega \subset \mathbb{R}^N$ is of class C^2 if $N \geq 4$. Then there exist a constant $\theta_\infty > 0$ and a function $\chi_\infty \in L^\infty(\Omega)$ such that the solution to (2.3)–(2.5), (1.4)–(1.5) has the properties*

$$\lim_{t\to\infty} \sup_{x\in\Omega} |\theta(x,t) - \theta_\infty| = 0, \tag{2.17}$$

$$\lim_{t\to\infty} \chi(x,t) = \chi_\infty(x) \qquad a.\,e., \tag{2.18}$$

$$\theta_\infty \in \lambda(\chi_\infty(x)) + \partial I_{[0,1]}(\chi_\infty(x)) \qquad a.\,e. \tag{2.19}$$

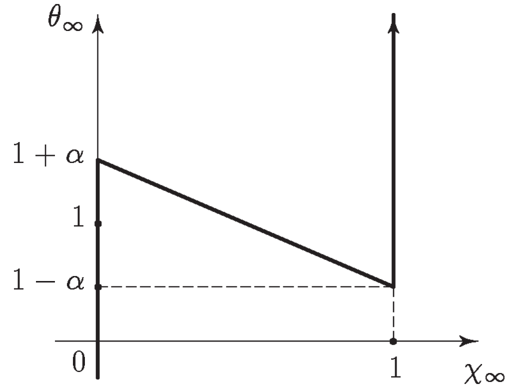

Figure 3: A diagram of the equilibrium set (2.19).

In other words, condition (2.19) means that $\chi_\infty(x) = 1$ a. e. if $\theta_\infty > 1+\alpha$, and $\chi_\infty(x) = 0$ a. e. if $\theta_\infty < 1-\alpha$. For $1-\alpha \leq \theta_\infty \leq 1+\alpha$ (see Figure 3), the domain Ω is decomposed into $\Omega = A_\infty \cup B_\infty \cup C_\infty$, with $\chi_\infty(x) = 0$ for $x \in A_\infty$, $\chi_\infty(x) = 1$ for $x \in C_\infty$, and $\chi_\infty(x) = (1+\alpha-\theta_\infty)/(2\alpha)$ for $x \in B_\infty$. In fact, we will see in the proof, in particular in Lemma 4.1 below, that the intermediate value of χ between 0 and 1 is unstable with respect to small perturbations and is actually unlikely to persist for $t \to \infty$ except for some particular cases, like for instance:

- $\theta^0(x) = \bar{\theta} =$ const., $\bar{\theta} \in \lambda(\chi^0(x)) + \partial I_{[0,1]}(\chi^0(x))$ a. e. Then the solution remains constant in time $\theta(x,t) = \theta^0(x)$, $\chi(x,t) = \chi^0(x)$ independently of the distribution of $\chi^0(x)$ (time-independent solutions).

- $\theta^0(x) = \bar{\theta} =$ const., $\chi^0(x) = \bar{\chi} =$ const. If $\mathcal{E}_0$ is such that
$$1+\alpha < \mathcal{E}_0 < 2-\alpha, \tag{2.20}$$
then the function $\Gamma(\chi) = \mathcal{E}_0 - \lambda(\chi) - \Lambda(\chi)$ has only one null point χ_∞ in $(0,1)$, $\Gamma(0) > 0$, $\Gamma(1) < 0$, hence the solution $\chi^*(t)$ of the differential equation
$$\dot{\chi}^*(t) = \Gamma(\chi^*(t)), \quad \chi^*(0) = \bar{\chi}, \tag{2.21}$$
stays in $(0,1)$ for all $t > 0$, $\lim_{t\to\infty} \chi^*(t) = \chi_\infty$, and $\chi(x,t) = \chi^*(t)$, $\theta(x,t) = \mathcal{E}_0 - \Lambda(\chi^*(t))$ is a solution to (2.3)–(2.5), (1.4)–(1.5) which entirely lies in the unstable region (space-independent solutions).

The two above examples seem to be quite isolated, and we make the following conjecture.

Conjecture. For a generic set of initial data (for example a set of second Baire's category like in [14, Remark 5.3]), the Lebesgue measure of the set B_∞ is zero.

3 Space variation of the temperature

We define for $t \ge 0$ the mean temperature

$$\theta_\Omega(t) \;=\; \int_\Omega \theta(x,t)\,dx\,. \tag{3.1}$$

Note that we use Hypothesis (2.2) here and in the sequel. The function θ_Ω is positive, bounded, and from (2.10), (2.12), (2.14), and Hölder's inequality it follows that

$$|\dot\theta_\Omega(t)| \;=\; \left|\int_\Omega \lambda(\chi(x,t))\,\partial_t\chi(x,t)\,dx\right| \;\le\; (1+\alpha)\,V^{1/2}(t)\,, \tag{3.2}$$

hence, by Theorem 2.1,

$$\int_0^\infty |\dot\theta_\Omega(t)|^2 dt \;<\; \infty\,, \qquad \lim_{t\to\infty} \operatorname*{sup\,ess}_{s>t} |\dot\theta_\Omega(s)| \;=\; 0\,. \tag{3.3}$$

This is not enough to conclude that $\theta_\Omega(t)$ converges to a limit as $t \to \infty$, and we need further estimates. We stay in the framework of the usual spaces $L^p(\Omega), L^p(\Omega;\mathbb{R}^N)$ with $1 \le p \le \infty$, and denote by $|\cdot|_p$ the standard norm in both these spaces. We also introduce the closed subspaces $L^p_0(\Omega)$ of $L^p(\Omega)$ consisting of zero mean functions

$$L^p_0(\Omega) \;=\; \left\{ v \in L^p(\Omega)\,;\; \int_\Omega v(x)\,dx = 0 \right\}\,. \tag{3.4}$$

We define in $\Omega \times (0,\infty)$ auxiliary functions

$$v(x,t) \;=\; \theta(x,t) - \theta_\Omega(t)\,, \tag{3.5}$$

$$f(x,t) \;=\; -\partial_t(\Lambda(\chi(x,t)) + \theta_\Omega(t))\,, \tag{3.6}$$

$$v^0(x) \;=\; \theta^0(x) - \int_\Omega \theta^0(y)\,dy\,. \tag{3.7}$$

Then v is the solution of the problem

$$\partial_t v - \Delta v \;=\; f(x,t) \qquad \text{in} \;\; \Omega \times (0,\infty)\,, \tag{3.8}$$

$$\frac{\partial v}{\partial n} \;=\; 0 \qquad \text{on} \;\; \partial\Omega \times (0,\infty)\,, \tag{3.9}$$

$$\int_\Omega v(x,t)\,dx \;=\; 0 \qquad \text{for a.e.}\; t > 0\,, \tag{3.10}$$

$$v(x,0) \;=\; v^0(x) \qquad \text{in} \;\; \Omega\,. \tag{3.11}$$

For the reader's convenience, we recall the following estimate for linear parabolic equations as a special case of the general theory explained in [17]. In all what follows, we denote by $C_1, C_2, \ldots$ positive constants independent of t.

Proposition 3.1. *Let $r > p \geq 2$ be given such that $1/r > 1/p - 1/N$, and assume that Ω is of class C^2 if $p > 2$. Let v be the solution to (3.8)–(3.11) with some $f \in L^\infty(0,\infty\,;\, L^p_0(\Omega))$. Then there exists $C > 0$ independent of t such that*

$$|\nabla v(t)|_r \;\leq\; C \quad \textit{for } t \geq 1\,.$$

Proof. Consider the semigroup $T_p(t)$ in $L^p_0(\Omega)$ for $1 < p < \infty$ generated by the Laplace operator with the homogeneous Neumann boundary conditions. The operator Δ is sectorial for $p = 2$ by [17, Proposition 2.1.11] and for $p > 2$ by [17, Proposition 3.1.3], hence $T_p(t)$ is analytic. The solution v is given by the Duhamel formula

$$v(t) \;=\; T_p(t)\, v_0 + \int_0^t T_p(t-s)\, f(s)\, ds\,. \tag{3.12}$$

By [17, Propositions 2.1.1, 2.2.15], there exist positive constants C_1 and ϱ such that for every $\xi \in L^p_0(\Omega)$ and $t > 0$ we have

$$|\Delta T_p(t)\,\xi|_p \;\leq\; \frac{C_1}{t}\, \mathrm{e}^{-\varrho t} |\xi|_p\,, \tag{3.13}$$

$$|\nabla T_p(t)\,\xi|_p \;\leq\; \frac{C_1}{\sqrt{t}}\, \mathrm{e}^{-\varrho t} |\xi|_p\,. \tag{3.14}$$

Set $\eta = N(1/p - 1/r) \in (0,1)$. By the Gagliardo-Nirenberg inequality (see [18], [11]), there exists $C_2 > 0$ such that every function $w \in W^{2,p}(\Omega) \cap L^p_0(\Omega)$ satisfying homogeneous Neumann boundary conditions on $\partial\Omega$ fulfils the following inequality:

$$|\nabla w|_r \;\leq\; C_2 \left(|\nabla w|_p^{1-\eta}\, |\Delta w|_p^\eta + |\nabla w|_p \right)\,. \tag{3.15}$$

From (3.13)–(3.15) we thus obtain for all $\xi \in L^p_0(\Omega)$ and $t > 0$ that

$$|\nabla T_p(t)\,\xi|_r \;\leq\; C_3\, (t^{-(1+\eta)/2} + t^{-1/2})\, \mathrm{e}^{-\varrho t} |\xi|_p\,. \tag{3.16}$$

Using (3.12)–(3.16) we conclude for $t \geq 1$ (the argument works for $t \searrow 0+$ only if v_0 has the corresponding regularity!) that

$$\begin{aligned} |\nabla v(t)|_r \;&\leq\; |\nabla T_p(t)\, v_0|_r + \int_0^t |\nabla T_p(t-s)\, f(s)|_r\, ds \\ &\leq\; C_4 \left(|v_0|_p + \int_0^t \left((t-s)^{-(1+\eta)/2} + (t-s)^{-1/2}\right) \mathrm{e}^{-\varrho(t-s)}\, |f(s)|_p\, ds \right) \\ &\leq\; C_5 \left(1 + \int_0^t \left(s^{-(1+\eta)/2} + s^{-1/2}\right) \mathrm{e}^{-\varrho s}\, ds \right) \;\leq\; C_6\,, \end{aligned} \tag{3.17}$$

and the proof is complete. ∎

Corollary 3.2. *Under the hypotheses of Theorem 2.2, we have*

$$\lim_{t\to\infty} \sup_{x\in\Omega} |\theta(x,t) - \theta_\Omega(t)| \;=\; 0\,.$$

Proof. By Theorem 2.1, the function f given by (3.6) belongs to $L^\infty(\Omega\times(0,\infty)) \cap L^2(\Omega\times(0,\infty))$. For $N \le 3$ we put $p=2$, $q=4$, $r=5$, for $N\ge 4$ we fix arbitrary $p \in (N/2, N)$ and $r > q > N$ such that $1/r > 1/p - 1/N$. For $t \ge 1$ we have by Proposition 3.1 that $|\nabla\theta(t)|_r \le C_7$. Using the interpolation inequality

$$|\nabla\theta(t)|_q \;\le\; |\nabla\theta(t)|_r^\gamma \, |\nabla\theta(t)|_2^{1-\gamma} \quad \text{with } \gamma = \frac{q-2}{r-2}\,, \tag{3.18}$$

and Theorem 2.1, which guarantees that $|\nabla\theta(t)|_2 \to 0$ as $t\to\infty$, we obtain that $\lim_{t\to\infty} |\nabla\theta(t)|_q = 0$. The assertion follows from the continuous embedding of $W^{1,q}(\Omega)$ into $C(\bar\Omega)$ and the fact that the function $\theta(\cdot,t) - \theta_\Omega(t)$ has zero mean value on Ω. ■

Remark 3.3. In the case $p=2$, we can prove the inequalities (3.13)–(3.14) directly by considering e. g. the Fourier expansions into the orthonormal system of eigenfunctions of the operator $-\Delta$ in $L^2_0(\Omega)$ with the Neumann boundary conditions.

4 Convergence

In this section, we prove that $\theta_\Omega(t)$ converges to a limit θ_∞ as $t\to\infty$ and show that this implies the pointwise convergence of $\chi(x,t)$ as well. We treat Eq. (2.4) for each $x\in\Omega$ as a one-dimensional dynamical system of the form

$$\left.\begin{aligned} &\dot\chi(t) + \lambda(\chi(t)) + \partial I_{[0,1]}(\chi(t)) \;\ni\; \theta(t)\,,\\ &\chi(0) = \chi^0 \in [0,1] \end{aligned}\right\} \tag{4.1}$$

with a given function $\theta \in L^\infty(0,\infty)$, and derive the following crucial estimate.

Lemma 4.1. *Let θ, χ satisfy Eq. (4.1), and let there exist $\varepsilon > 0$ and $t_0 \ge 0$ such that*

$$\theta(t) \;\ge\; \lambda(\chi(t_0)) + \varepsilon \quad \textit{for a. e. } t \in \left(t_0, t_0 + \frac{1}{\varepsilon}\right). \tag{4.2}$$

Then there exists $t_1 \in [t_0, t_0 + \frac{1}{\varepsilon}]$ such that $\chi(t) < 1$ for $t\in[t_0,t_1)$, $\chi(t) = 1$ for $t \in [t_1, t_0+\frac{1}{\varepsilon}]$. Similarly, if

$$\theta(t) \;\le\; \lambda(\chi(t_0)) - \varepsilon \quad \textit{for a. e. } t \in \left(t_0, t_0 + \frac{1}{\varepsilon}\right), \tag{4.3}$$

then there exists $t_1 \in [t_0, t_0 + \frac{1}{\varepsilon}]$ such that $\chi(t) > 0$ for $t\in[t_0,t_1)$, $\chi(t) = 0$ for $t \in [t_1, t_0+\frac{1}{\varepsilon}]$.

Proof. It can easily be shown that Eq. (4.1) has a unique absolutely continuous solution χ (see e.g. [15]). Rewriting the identity (1.11) in the form

$$\dot{\chi}(t)(\dot{\chi}(t) + \lambda(\chi(t)) - \theta(t)) \;=\; 0 \quad \text{a.e.} \tag{4.4}$$

and using the fact that both $\lambda(\chi(t))$ and $\theta(t)$ are bounded, we see that χ belongs to $W^{1,\infty}(0,\infty)$.

Assume first that (4.2) holds, and that there exists $t_1 \in [t_0, t_0+\frac{1}{\varepsilon})$ such that $\chi(t_1) = 1$. The function λ is decreasing, hence for a.e. $t \in (t_1, t_0 + \frac{1}{\varepsilon})$ we have

$$\theta(t) - \lambda(1) \geq \theta(t) - \lambda(\chi(t_0)) \geq \varepsilon\,.$$

In particular,

$$\theta(t) - \lambda(1) \;\in\; \partial I_{[0,1]}(1)\,,$$

hence the constant function $\chi(t) \equiv 1$ is the unique solution of (4.1) in $[t_1, t_0 + \frac{1}{\varepsilon}]$.

If $\chi(t) < 1$ in $[t_0, t_0 + \frac{1}{\varepsilon})$, we define $y(t)$ as the solution of the equation

$$\dot{y}(t) = (\lambda(\chi(t_0)) - \lambda(y(t)))^+ + \theta(t) - \lambda(\chi(t_0))\,, \quad y(t_0) = \chi(t_0)\,, \tag{4.5}$$

where $(\cdot)^+$ denotes the positive part. By hypothesis, we have $\dot{y}(t) \geq \varepsilon$ in $(t_0, t_0 + \frac{1}{\varepsilon})$, hence y is increasing in $(t_0, t_0 + \frac{1}{\varepsilon})$ and $y(t_0 + \frac{1}{\varepsilon}) \geq 1$. Set $t_2 = \sup\{t \in (t_0, t_0 + \frac{1}{\varepsilon})\,;\, y(t) < 1\}$. As λ is decreasing, we obtain $\dot{y}(t) = -\lambda(y(t)) + \theta(t)$ in (t_0, t_2), hence $y(t) = \chi(t)$ in $[t_0, t_2]$. We have in particular $y(t_2) = \chi(t_2) = 1$ and $t_2 = t_0 + \frac{1}{\varepsilon}$.

The case (4.3) is analogous. We replace (4.5) by

$$\dot{y}(t) = -(\lambda(\chi(t_0)) - \lambda(y(t)))^- + \theta(t) - \lambda(\chi(t_0))\,, \quad y(t_0) = \chi(t_0)\,. \tag{4.6}$$

and argue as above. ■

The rest of the paper is devoted to a local analysis of a fixed solution to (2.3)–(2.5), (1.4)–(1.5) with properties as in Theorem 2.1. We will assume that χ^0 is defined for all $x \in \Omega$, so that $\chi(x,t)$ is also defined for all $x \in \Omega$ as the pointwise solution to (2.4).

Lemma 4.2. *Let θ_Ω be as in (3.1), and for $t \geq 0$ set*

$$\left.\begin{aligned} A(t) &= \{x \in \Omega\,;\, \chi(x,t) = 0\}\,, \\ B(t) &= \{x \in \Omega\,;\, 0 < \chi(x,t) < 1\}\,, \\ C(t) &= \{x \in \Omega\,;\, \chi(x,t) = 1\}\,. \end{aligned}\right\} \tag{4.7}$$

If

$$\liminf_{t\to\infty} \theta_\Omega(t) \;>\; 1 - \alpha\,, \tag{4.8}$$

then there exists $\hat{t} > 0$ such that for $t_1 > t_0 \geq \hat{t}$ we have $C(t_1) \supset C(t_0)$. Moreover, putting $C_\infty = \bigcup_{t\geq\hat{t}} C(t)$, we have

$$\limsup_{t\to\infty} \theta_\Omega(t) - \lambda(\chi(x,t)) \;\leq\; 0 \quad \forall x \in \Omega \setminus C_\infty\,. \tag{4.9}$$

Similarly, if

$$\limsup_{t\to\infty} \theta_\Omega(t) \;<\; 1+\alpha\,, \tag{4.10}$$

then there exists $\hat t > 0$ *such that for* $t_1 > t_0 \ge \hat t$ *we have* $A(t_1) \supset A(t_0)$. *Moreover, putting* $A_\infty = \bigcup_{t\ge\hat t} A(t)$, *we have*

$$\liminf_{t\to\infty} \theta_\Omega(t) - \lambda(\chi(x,t)) \;\ge\; 0 \quad \forall x\in\Omega\setminus A_\infty\,. \tag{4.11}$$

Proof. We assume first that (4.8) holds, fix $\varepsilon > 0$ sufficiently small, and find $\hat t > 0$ such that for (almost all) $t > \hat t$ we have

$$\left.\begin{array}{rcl} \theta_\Omega(t) & \ge & 1-\alpha+2\varepsilon\,, \\ |\dot\theta_\Omega(t)| & < & \varepsilon^2/2\,, \\ |\theta(x,t)-\theta_\Omega(t)| & < & \varepsilon/2 \qquad \text{for all } x\in\Omega\,. \end{array}\right\} \tag{4.12}$$

Let $t_0 \ge \hat t$ and $x \in C(t_0)$ be arbitrary. We have for $t \ge t_0$ that

$$\theta(x,t) - \lambda(\chi(x,t_0)) = \theta(x,t) - \lambda(1) \ge 1-\alpha+2\varepsilon - |\theta(x,t)-\theta_\Omega(t)| - (1-\alpha) \ge \varepsilon\,,$$

hence, by Lemma 4.1, $\chi(x,t) = 1$ for all $t \in [t_0, t_0+\frac{1}{\varepsilon}]$. We can continue by induction and obtain $C(t_1) \supset C(t_0)$ for all $t_1 \ge t_0$.

Let now $x \in \Omega\setminus C_\infty$ be fixed, and consider the difference

$$\beta(x,t) \;:=\; \theta_\Omega(t) - \lambda(\chi(x,t)) \quad \text{ for } \; t\ge\hat t\,. \tag{4.13}$$

Assume for contradiction that there exists $\delta > 0$ and a sequence $t_n \to \infty$ such that $\beta(x,t_n) \ge 2\delta$ for all n. We find n_0 sufficiently large such that $|\dot\theta_\Omega(t)| < \delta^2/2$, $|\theta(x,t) - \theta_\Omega(t)| < \delta/2$ for (almost all) $t > t_{n_0}$, and obtain from Lemma 4.1 that $\chi(x, t_{n_0} + \frac{1}{\delta}) = 1$ in contradiction with the choice of $x \in \Omega\setminus C_\infty$, and (4.9) follows. The case (4.10) is similar. ■

Remark 4.3. We see from the above Lemma that χ can switch from 1 to 0 only if the value of θ is below $1-\alpha$, and from 0 to 1 only if θ is above $1+\alpha$. The phase separation process thus exhibits a *hysteresis-like behavior.*

We conclude the proof with a case distinction in eight consecutive steps. Steps (i)–(iii) deal with the situation where no mass exchange between the pure phases $\chi = 0$ and $\chi = 1$ takes place after a finite time, and the convergence towards the equilibrium then follows from the energy balance (2.14) alone. Mass exchange can only occur when the temperature oscillates around the critical values $1\pm\alpha$, and in the corresponding Steps (iv)–(viii) we also use the entropy balance (2.16) to prove the stabilization result.

Step (i) *Assume that there exists a sequence* $t_n \to \infty$ *such that* $\hat\theta = \lim_{n\to\infty} \theta_\Omega(t_n) > 1+\alpha$.

In view of (3.3) and Corollary 3.2, we may assume, passing possibly to a subsequence, that

$$\left.\begin{aligned} |\theta_\Omega(t_n) - \hat\theta| &< \tfrac{1}{n}\,, \\ |\dot\theta_\Omega(t)| &< \tfrac{1}{2n^2} \qquad \text{for a. e. } t > t_n\,, \\ |\theta(x,t) - \theta_\Omega(t)| &< \tfrac{1}{2n} \qquad \text{in } \Omega\times(t_n,\infty)\,. \end{aligned}\right\} \tag{4.14}$$

We fix n_0 such that $\theta_\Omega(t_n) \ge 1+\alpha+\frac{2}{n}$ for $n \ge n_0$. For $x\in\Omega$ and $t \in [t_n, t_n+n]$ we then have

$$\begin{aligned} \theta(x,t) - \lambda(\chi(x,t_n)) &\ge \theta(x,t) - \theta_\Omega(t) + \theta_\Omega(t) - \theta_\Omega(t_n) + \theta_\Omega(t_n) - \lambda(0) \\ &\ge -\frac{1}{2n} - (t-t_n)\frac{1}{2n^2} + \frac{2}{n} \ \ge\ \frac{1}{n}\,, \end{aligned}$$

hence, by Lemma 4.1, we have $\chi(x, t_n+n) = 1$ for all $x\in\Omega$ and $n\ge n_0$. Using (2.14) we obtain that

$$\mathcal{E}_0 = \mathcal{E}(t_n+n) = \theta_\Omega(t_n+n) + 1 \qquad \forall n \ge n_0\,, \tag{4.15}$$

where $|\theta_\Omega(t_n+n) - \hat\theta| < 3/(2n)$ by (4.14). Letting $n\to\infty$ thus yields that

$$\mathcal{E}_0 = \hat\theta + 1\,. \tag{4.16}$$

On the other hand, for all $t\ge 0$ we have $\mathcal{E}_0 = \mathcal{E}(t) \le \theta_\Omega(t) + 1$, hence $\theta_\Omega(t) \ge \hat\theta$ for all $t \ge 0$. In particular, we have $\chi(x,t) = 1$ and $\theta_\Omega(t) = \hat\theta =: \theta_\infty$ for all $x\in\Omega$ and $t \ge t_{n_0} + n_0$. In this case, we see that stabilization of χ and of the mean temperature occurs in finite time, and $\theta(x,t)$ converges to θ_∞ exponentially as solution of the linear homogeneous heat equation.

Step (ii) *Assume that there exists a sequence* $t_n\to\infty$ *such that* $\hat\theta = \lim_{n\to\infty}\theta_\Omega(t_n) < 1-\alpha$.

We argue in the same way as in Step (i). Note only that we obtain $\chi(x,t) = 0$ for all $x\in\Omega$ and all t sufficiently large, hence $\mathcal{E}_0 = \theta_\infty$ as a counterpart of (4.16). Since the initial energy is positive, we necessarily have $\theta_\infty > 0$.

Step (iii) *Assume that* $1-\alpha < \liminf_{t\to\infty}\theta_\Omega(t) \le \limsup_{t\to\infty}\theta_\Omega(t) < 1+\alpha$.

Let $\tilde\theta \in (1-\alpha, 1+\alpha)$ be any element of the ω-limit set of $\theta_\Omega(t)$, and let $t_n \nearrow \infty$ be such that $\theta_\Omega(t_n) \to \tilde\theta$. With the notation of Lemma 4.2 we have for all n that

$$\mathcal{E}_0 \ = \ \theta_\Omega(t_n) + \int_{B(t_n)} \Lambda(\chi(x,t_n))\,dx + |C(t_n)|\,. \tag{4.17}$$

Set

$$B_\infty \ = \ \Omega\setminus(A_\infty\cup C_\infty) = \bigcup_{\hat t>0}\bigcap_{t\ge\hat t} B(t)\,.$$

For $x \in B_\infty$ we have by Lemma 4.2 that

$$\lim_{n\to\infty} 1+\alpha-2\alpha\,\chi(x,t_n)-\theta_\Omega(t_n) \;=\; 0\,, \tag{4.18}$$

hence

$$\lim_{n\to\infty} \chi(x,t_n) \;=\; \frac{1}{2\alpha}(1+\alpha-\tilde\theta)\,. \tag{4.19}$$

Using the formula

$$\Lambda\left(\frac{1}{2\alpha}(1+\alpha-\tilde\theta)\right) \;=\; \frac{1}{4\alpha}((1+\alpha)^2-\tilde\theta^2)\,, \tag{4.20}$$

we obtain, after passing to the limit in (4.17), that

$$\mathcal{E}_0 \;=\; \tilde\theta + \frac{|B_\infty|}{4\alpha}((1+\alpha)^2-\tilde\theta^2)+|C_\infty|\,. \tag{4.21}$$

The values of $\mathcal{E}_0, |B_\infty|, |C_\infty|$ are independent of the choice of the sequence $t_n \nearrow \infty$. Hence, (4.21) is an equation for $\tilde\theta$ which admits at most two solutions. Since the ω-limit set of $\theta_\Omega(t)$ is connected, we necessarily have $\lim_{t\to\infty}\theta_\Omega(t)=\theta_\infty$, where θ_∞ is a solution of (4.21). We then have $\lim_{t\to\infty}\chi(x,t)=0$ for $x\in A_\infty$, $\lim_{t\to\infty}\chi(x,t)=1$ for $x\in C_\infty$, $\lim_{t\to\infty}\chi(x,t)=\frac{1}{2\alpha}(1+\alpha-\theta_\infty)$ for $x\in B_\infty$.

Step (iv) *Assume that there exists a sequence $t_n \nearrow \infty$ such that $\lim_{n\to\infty}\theta_\Omega(t_n)=1+\alpha$.*

Passing to a subsequence, if necessary, we may assume that (cf. (4.14))

$$\left.\begin{aligned}
|\theta_\Omega(t_n)-(1+\alpha)| &< \tfrac{1}{n}\,, && \\
|\dot\theta_\Omega(t)| &< \tfrac{1}{2n^2} && \text{for a.e. } t>t_n\,,\\
|\theta(x,t)-\theta_\Omega(t)| &< \tfrac{1}{2n} && \text{in } \Omega\times(t_n,\infty)\,,
\end{aligned}\right\} \tag{4.22}$$

and define the sets

$$\left.\begin{aligned}
A_n &= \left\{x\in\Omega\,;\ \chi(x,t_n)<\tfrac{2}{\alpha n}\right\}\,,\\
B_n &= \left\{x\in\Omega\,;\ \tfrac{2}{\alpha n}\le\chi(x,t_n)<1-\tfrac{1}{\sqrt{n}}\right\}\,,\\
C_n &= \left\{x\in\Omega\,;\ \chi(x,t_n)\ge 1-\tfrac{1}{\sqrt{n}}\right\}\,.
\end{aligned}\right\} \tag{4.23}$$

For $x\in B_n\cup C_n$ and $t\in[t_n,t_n+n]$ we have

$$\theta(x,t)-\lambda(\chi(x,t_n)) \;\ge\; 1+\alpha-\frac{2}{n}-\lambda(\chi(x,t_n)) \;=\; 2\alpha\chi(x,t_n)-\frac{2}{n} \;\ge\; \frac{2}{n}\,,$$

and from Lemma 4.1 we obtain

$$\chi(x,t_n+n) \;=\; 1 \quad \forall x\in B_n\cup C_n\,. \tag{4.24}$$

We now compare the internal energies at times $t = t_n$ and $t = t_n + n$. We have

$$\begin{aligned}
\mathcal{E}_0 &= \mathcal{E}(t_n) = \theta_\Omega(t_n) + \int_{A_n \cup B_n \cup C_n} \Lambda(\chi(x, t_n))\, dx \qquad (4.25) \\
&\leq \theta_\Omega(t_n) + |A_n|\, \Lambda\left(\frac{2}{\alpha n}\right) + |B_n|\, \Lambda\left(1 - \frac{1}{\sqrt{n}}\right) + |C_n|\,,
\end{aligned}$$

and from (4.24) it follows that

$$\begin{aligned}
\mathcal{E}_0 &= \mathcal{E}(t_n + n) = \theta_\Omega(t_n + n) + \int_{A_n \cup B_n \cup C_n} \Lambda(\chi(x, t_n + n))\, dx \qquad (4.26) \\
&\geq \theta_\Omega(t_n + n) + |B_n| + |C_n|\,,
\end{aligned}$$

hence

$$|B_n| \left(1 - \Lambda\left(1 - \frac{1}{\sqrt{n}}\right)\right) \leq |A_n|\, \Lambda\left(\frac{2}{\alpha n}\right) + \theta_\Omega(t_n) - \theta_\Omega(t_n + n) \leq \Lambda\left(\frac{2}{\alpha n}\right) + \frac{1}{2n}\,. \qquad (4.27)$$

Using the inequalities

$$\Lambda(\chi) \leq (1 + \alpha)\, \chi\,, \quad 1 - \Lambda(\chi) \geq (1 - \alpha)\, (1 - \chi)\,, \qquad (4.28)$$

we obtain from (4.27) that

$$\frac{1 - \alpha}{\sqrt{n}} |B_n| \leq \frac{1}{n} \left(\frac{2(1 + \alpha)}{\alpha} + \frac{1}{2}\right)\,, \qquad (4.29)$$

hence $\lim_{n\to\infty} |B_n| = 0$. Letting $n \to \infty$ in (4.25), we see that the limit $c^* := \lim_{n\to\infty} |C_n|$ exists and satisfies the identity

$$\mathcal{E}_0 = 1 + \alpha + c^*\,. \qquad (4.30)$$

Using the fact that $\theta(x, t_n)$ converge uniformly to $1 + \alpha$ as $n \to \infty$, we also obtain

$$\begin{aligned}
\mathcal{S}_\infty &= \lim_{n\to\infty} \mathcal{S}(t_n) = \lim_{n\to\infty} \int_\Omega \log \theta(x, t_n)\, dx + \int_{A_n \cup B_n \cup C_n} \chi(x, t_n)\, dx \qquad (4.31) \\
&= \log(1 + \alpha) + c^*\,.
\end{aligned}$$

Step (v) *Assume that there exists a sequence* $t'_n \nearrow \infty$ *such that* $\lim_{n\to\infty} \theta_\Omega(t'_n) = 1 - \alpha$.

As in Step (iv), we assume that

$$\left.\begin{aligned}
|\theta_\Omega(t'_n) - (1 - \alpha)| &< \tfrac{1}{n}\,, \\
|\dot\theta_\Omega(t)| &< \tfrac{1}{2n^2} && \text{for a.e. } t > t'_n\,, \\
|\theta(x, t) - \theta_\Omega(t)| &< \tfrac{1}{2n} && \text{in } \Omega \times (t'_n, \infty)\,,
\end{aligned}\right\} \qquad (4.32)$$

and define the sets

$$\left.\begin{aligned} A'_n &= \left\{x \in \Omega\,;\ \chi(x,t'_n) < \tfrac{1}{\sqrt{n}}\right\}, \\ B'_n &= \left\{x \in \Omega\,;\ \tfrac{1}{\sqrt{n}} \le \chi(x,t'_n) < 1 - \tfrac{2}{\alpha n}\right\}, \\ C'_n &= \left\{x \in \Omega\,;\ \chi(x,t'_n) \ge 1 - \tfrac{2}{\alpha n}\right\}. \end{aligned}\right\} \tag{4.33}$$

For $x \in A'_n \cup B'_n$ and $t \in [t'_n, t'_n + n]$ we have

$$\theta(x,t) - \lambda(\chi(x,t'_n)) \le 1 - \alpha + \frac{2}{n} - \lambda(\chi(x,t'_n)) = 2\alpha(\chi(x,t'_n) - 1) + \frac{2}{n} \le -\frac{2}{n},$$

and Lemma 4.1 yields

$$\chi(x, t'_n + n) = 0 \quad \forall x \in A'_n \cup B'_n\,. \tag{4.34}$$

We continue as in Step (iv) and obtain

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}(t'_n) = \theta_\Omega(t'_n) + \int_{A'_n \cup B'_n \cup C'_n} \Lambda(\chi(x,t'_n))\,dx \\ &\ge \theta_\Omega(t'_n) + |B'_n|\,\Lambda\left(\frac{1}{\sqrt{n}}\right) + |C'_n|\,\Lambda\left(1 - \frac{2}{\alpha n}\right), \end{aligned} \tag{4.35}$$

and from (4.34) it follows that

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}(t'_n + n) \le \theta_\Omega(t'_n + n) + \int_{A'_n \cup B'_n \cup C'_n} \Lambda(\chi(x,t'_n + n))\,dx \\ &\le \theta_\Omega(t'_n + n) + |C'_n|, \end{aligned} \tag{4.36}$$

hence, by virtue of (4.35)–(4.36) and the inequality $\Lambda(\chi) \ge \chi$, we obtain that

$$\frac{1}{\sqrt{n}}\,|B'_n| \le \frac{2}{\alpha n}\,|C'_n| + \theta_\Omega(t'_n + n) - \theta_\Omega(t'_n) \le \frac{2}{\alpha n} + \frac{1}{2n}, \tag{4.37}$$

hence $\lim_{n\to\infty} |B'_n| = 0$. Letting $n \to \infty$ in (4.35), we see that the limit $c_* := \lim_{n\to\infty} |C'_n|$ exists and satisfies the identities

$$\mathcal{E}_0 = 1 - \alpha + c_*, \tag{4.38}$$

$$\begin{aligned} \mathcal{S}_\infty &= \lim_{n\to\infty} \mathcal{S}(t'_n) = \lim_{n\to\infty} \int_\Omega \log\theta(x,t'_n)\,dx + \int_{A'_n \cup B'_n \cup C'_n} \chi(x,t'_n)\,dx \\ &= \log(1-\alpha) + c_*\,. \end{aligned} \tag{4.39}$$

Note that the situation in Steps (iv) and (v) is different from Step (iii) in the sense that there is a priori no inclusion between the sets C_{n_1} and C_{n_2} or between C'_{n_1} and C'_{n_2} for $n_1 < n_2$; only their Lebesgue measures converge.

Step (vi) *Assume that* $\limsup_{t\to\infty} \theta_\Omega(t) = 1 + \alpha$, $\liminf_{t\to\infty} \theta_\Omega(t) = 1 - \alpha$.

The hypotheses of both Step (iv) and Step (v) are fulfilled, hence there exist $c_* > c^* \geq 0$ such that

$$\mathcal{E}_0 = 1 - \alpha + c_* = 1 + \alpha + c^*, \tag{4.40}$$

$$\mathcal{S}_\infty = \log(1-\alpha) + c_* = \log(1+\alpha) + c^*. \tag{4.41}$$

We have used the fact that, due to (2.16), the limit value $\mathcal{S}_\infty$ of the entropy is independent of how t converges to infinity. From (4.40)–(4.41) we obtain the equation $\log(1+\alpha) - \log(1-\alpha) = 2\alpha$ which only holds if $\alpha = 0$, so that this case never occurs.

Step (vii) *Assume that* $\limsup_{t\to\infty} \theta_\Omega(t) = 1+\alpha$, $\liminf_{t\to\infty} \theta_\Omega(t) = \underline{\theta} > 1-\alpha$.

With the notation of Lemma 4.2 put

$$D_\infty = \Omega \setminus C_\infty. \tag{4.42}$$

Let t_n be as in (4.22). We have $\lim_{n\to\infty} \chi(x,t_n) = 1$ for $x \in C_\infty$ by definition, and $\lim_{n\to\infty} \chi(x,t_n) = 0$ for $x \in D_\infty$ by (4.9). From (4.30)–(4.31) it follows that

$$\mathcal{E}_0 = 1 + \alpha + |C_\infty|, \quad \mathcal{S}_\infty = \log(1+\alpha) + |C_\infty|. \tag{4.43}$$

We now consider any $\hat{\theta} \in [\underline{\theta}, 1+\alpha]$, and find a sequence $t''_n \nearrow \infty$ such that

$$\left.\begin{array}{rcll} |\theta_\Omega(t''_n) - \hat{\theta}| & < & \frac{1}{n}, & \\ |\dot{\theta}_\Omega(t)| & < & \frac{1}{2n^2} & \text{for a.e. } t > t''_n, \\ |\theta(x,t) - \theta_\Omega(t)| & < & \frac{1}{2n} & \text{in } \Omega \times (t''_n, \infty), \\ |C_\infty \setminus C(t''_n)| & < & \frac{1}{n} & \end{array}\right\} \tag{4.44}$$

for all $n \in \mathbb{N}$. As in previous steps, we define the sets

$$\left.\begin{array}{rcl} A''_n & = & \left\{x \in D_\infty \,;\, \chi(x,t''_n) < \frac{1}{\sqrt{n}}\right\}, \\ B''_n & = & \left\{x \in D_\infty \,;\, \frac{1}{\sqrt{n}} \leq \chi(x,t''_n) < \frac{1}{2\alpha}\left(1+\alpha-\hat{\theta}-\frac{3}{n}\right)\right\}, \\ C''_n & = & \left\{x \in D_\infty \,;\, \chi(x,t''_n) \geq \frac{1}{2\alpha}\left(1+\alpha-\hat{\theta}-\frac{3}{n}\right)\right\}. \end{array}\right\} \tag{4.45}$$

For $x \in A''_n \cup B''_n$ we have $\hat{\theta} + 3/n < \lambda(\chi(x,t''_n))$, hence $\theta(x,t) - \lambda(\chi(x,t''_n)) \leq -1/n$ for $t \in [t''_n, t''_n + n]$. From Lemma 4.1 we conclude that

$$\chi(x, t''_n + n) = 0 \quad \forall x \in A''_n \cup B''_n. \tag{4.46}$$

For $x \in C''_n$ and $\tau \in [t''_n, t''_n + n]$ set

$$\delta_n(x,\tau) = \hat{\theta} - \lambda(\chi(x,\tau)), \tag{4.47}$$

and assume that $\delta_n(x,\tau) > 4/n$ for some $(x,\tau) \in C''_n \times [t''_n, t''_n + n]$. Then $\theta(x,t) - \lambda(\chi(x,\tau)) > 1/n$ for $t \in [\tau, \tau + n]$, and Lemma 4.1 yields that $\chi(x, \tau + n) = 1$ in contradiction with the hypothesis $C''_n \subset D_\infty$. We thus have $\delta_n(x,\tau) < 4/n$, that is,

$$\chi(x,\tau) \ \le\ \frac{1}{2\alpha}\left(1+\alpha-\hat\theta+\frac{4}{n}\right) \quad \forall (x,\tau) \in C''_n \times [t''_n, t''_n + n]. \tag{4.48}$$

We now compare again the energies and entropies at times t''_n and $t''_n + n$. By (4.46)–(4.48) we have

$$\begin{aligned}
\mathcal{E}_0 \ &=\ \mathcal{E}(t''_n) \ =\ \theta_\Omega(t''_n) + \int_{A''_n \cup B''_n \cup C''_n \cup C_\infty} \Lambda(\chi(x,t''_n))\,dx \qquad (4.49)\\
&\ge\ \theta_\Omega(t''_n) + |B''_n|\,\Lambda\left(\frac{1}{\sqrt{n}}\right) + |C''_n|\,\Lambda\left(\frac{1}{2\alpha}\left(1+\alpha-\hat\theta-\frac{3}{n}\right)\right) + |C(t''_n)|\,,\\
\mathcal{E}_0 \ &=\ \mathcal{E}(t''_n + n) \ \le\ \theta_\Omega(t''_n + n) + \int_{A''_n \cup B''_n \cup C''_n \cup C_\infty} \Lambda(\chi(x,t''_n + n))\,dx \qquad (4.50)\\
&\le\ \theta_\Omega(t''_n + n) + |C''_n|\,\Lambda\left(\frac{1}{2\alpha}\left(1+\alpha-\hat\theta+\frac{4}{n}\right)\right) + |C_\infty|\,.
\end{aligned}$$

As in Step (v), we conclude that $\lim_{n\to\infty} |B''_n| = 0$, $\lim_{n\to\infty} |C''_n| =: c_*$, and

$$\mathcal{E}_0 \ =\ \hat\theta + c_*\,\Lambda\left(\frac{1}{2\alpha}\left(1+\alpha-\hat\theta\right)\right) + |C_\infty| \ =\ \hat\theta + \frac{c_*}{4\alpha}\left((1+\alpha)^2 - \hat\theta^2\right) + |C_\infty|\,. \tag{4.51}$$

We further have

$$\mathcal{S}(t''_n) \ =\ \int_\Omega \log\theta(x,t''_n)\,dx + \int_{A''_n \cup B''_n \cup C''_n \cup C_\infty} \chi(x,t''_n)\,dx\,, \tag{4.52}$$

which for $n \to \infty$ yields

$$\mathcal{S}_\infty \ =\ \log\hat\theta + \frac{c_*}{2\alpha}\left(1+\alpha-\hat\theta\right) + |C_\infty|\,. \tag{4.53}$$

We now combine (4.43) with (4.51) and (4.53), and obtain

$$\left.\begin{aligned}
1+\alpha-\hat\theta \ &=\ \tfrac{c_*}{4\alpha}\left((1+\alpha)^2-\hat\theta^2\right)\\
\log(1+\alpha) - \log\hat\theta \ &=\ \tfrac{c_*}{2\alpha}\left(1+\alpha-\hat\theta\right)\,.
\end{aligned}\right\} \tag{4.54}$$

We either have $c_* = 0$ and $\hat\theta = 1+\alpha$, or $c_* > 0$ and

$$\log(1+\alpha) - \log\hat\theta \ =\ 2\,\frac{1+\alpha-\hat\theta}{1+\alpha+\hat\theta}\,. \tag{4.55}$$

The unique solution of Eq. (4.55) is again $\hat\theta = 1+\alpha$, hence $\lim_{t\to\infty}\theta_\Omega(t) = 1+\alpha$.

Step (viii) *Assume that* $\liminf_{t\to\infty}\theta_\Omega(t) = 1-\alpha$, $\limsup_{t\to\infty}\theta_\Omega(t) = \bar\theta < 1+\alpha$.

Referring to Lemma 4.2, put

$$G_\infty = \Omega \setminus A_\infty . \tag{4.56}$$

For t'_n as in (4.32), we have $\lim_{n\to\infty} \chi(x,t'_n) = 0$ for $x \in A_\infty$ by definition of A_∞, and $\lim_{n\to\infty} \chi(x,t'_n) = 1$ for $x \in G_\infty$ by (4.11). Formulas (4.38)–(4.39) then yield

$$\mathcal{E}_0 = 1-\alpha+|G_\infty|, \quad \mathcal{S}_\infty = \log(1-\alpha)+|G_\infty| . \tag{4.57}$$

We continue as in Step (vii) choosing any $\hat\theta \in [1-\alpha, \bar\theta]$ and a suitable sequence $t'''_n \nearrow \infty$ with properties analogous to (4.44), and define the sets

$$\left.\begin{aligned} A'''_n &= \left\{x \in G_\infty \,;\, \chi(x,t'''_n) < \tfrac{1}{2\alpha}\left(1+\alpha-\hat\theta+\tfrac{3}{n}\right)\right\} , \\ B'''_n &= \left\{x \in G_\infty \,;\, \tfrac{1}{2\alpha}\left(1+\alpha-\hat\theta+\tfrac{3}{n}\right) \le \chi(x,t'''_n) < 1-\tfrac{1}{\sqrt{n}}\right\} , \\ C'''_n &= \left\{x \in G_\infty \,;\, \chi(x,t'''_n) \ge 1-\tfrac{1}{\sqrt{n}}\right\} . \end{aligned}\right\} \tag{4.58}$$

By Lemma 4.1, we have again $\chi(x, t'''_n+n) = 1$ for all $x \in B'''_n \cup C'''_n$, and

$$\chi(x,\tau) \ \ge\ \frac{1}{2\alpha}\left(1+\alpha-\hat\theta-\frac{4}{n}\right) \quad \forall (x,\tau) \in A'''_n \times [t'''_n, t'''_n+n] . \tag{4.59}$$

The energy balance now reads

$$\begin{aligned} \mathcal{E}_0 &= \mathcal{E}(t'''_n) = \theta_\Omega(t'''_n) + \int_{A'''_n\cup B'''_n\cup C'''_n\cup A_\infty} \Lambda(\chi(x,t'''_n))\,dx \qquad (4.60)\\ &\le \theta_\Omega(t'''_n) + |A'''_n|\,\Lambda\left(\frac{1}{2\alpha}\left(1+\alpha-\hat\theta+\frac{3}{n}\right)\right) + |B'''_n|\,\Lambda\left(1-\frac{1}{\sqrt{n}}\right) \\ &\quad +|C'''_n| + |A_\infty \setminus A(t'''_n)| , \\ \mathcal{E}_0 &= \mathcal{E}(t'''_n+n) = \theta_\Omega(t'''_n+n) + \int_{A'''_n\cup B'''_n\cup C'''_n\cup A_\infty} \Lambda(\chi(x,t'''_n+n))\,dx \qquad (4.61)\\ &\ge \theta_\Omega(t'''_n+n) + |A'''_n|\,\Lambda\left(\frac{1}{2\alpha}\left(1+\alpha-\hat\theta-\frac{4}{n}\right)\right) + |B'''_n| + |C'''_n| , \end{aligned}$$

hence $|B'''_n| \to 0$ as $n \to \infty$. Selecting a subsequence, if necessary, we obtain $|A'''_n| \to a^{**}$, $|C'''_n| \to c^{**}$, $a^{**}+c^{**} = |G_\infty|$, and

$$\begin{aligned} \mathcal{E}_0 &= \lim_{n\to\infty} \theta_\Omega(t'''_n) + \int_{A'''_n\cup B'''_n\cup C'''_n\cup A_\infty} \Lambda(\chi(x,t'''_n))\,dx \qquad (4.62)\\ &= \hat\theta + \frac{a^{**}}{4\alpha}\left((1+\alpha)^2 - \hat\theta^2\right) + c^{**} , \end{aligned}$$

$$\begin{aligned} \mathcal{S}_\infty &= \lim_{n\to\infty} \int_\Omega \log\theta(x,t'''_n)\,dx + \int_{A'''_n\cup B'''_n\cup C'''_n\cup A_\infty} \chi(x,t'''_n)\,dx \qquad (4.63)\\ &= \log\hat\theta + \frac{a^{**}}{2\alpha}\left(1+\alpha-\hat\theta\right) + c^{**} . \end{aligned}$$

Combining (4.62)–(4.63) with (4.57), we have

$$\left.\begin{aligned} 1-\alpha+a^{**}-\hat\theta &= \tfrac{a^{**}}{4\alpha}\left((1+\alpha)^2-\hat\theta^2\right) \\ \log(1-\alpha)+a^{**}-\log\hat\theta &= \tfrac{a^{**}}{2\alpha}\left(1+\alpha-\hat\theta\right), \end{aligned}\right\} \tag{4.64}$$

that is,

$$\left.\begin{aligned} 1-\alpha-\hat\theta &= \tfrac{a^{**}}{4\alpha}\left((1-\alpha)^2-\hat\theta^2\right) \\ \log(1-\alpha)-\log\hat\theta &= \tfrac{a^{**}}{2\alpha}\left(1-\alpha-\hat\theta\right), \end{aligned}\right\} \tag{4.65}$$

and we argue as in Step (vii) to conclude that $\bar\theta = 1-\alpha$.

This enables us to finish the proof of Theorem 2.2. From Steps (i)–(viii) it follows that there exists $\theta_\infty > 0$ with the desired properties. The pointwise convergence of $\chi(x,t)$ as $t\to\infty$ has already been proved in the cases where $\theta_\infty \neq 1\pm\alpha$. For $\theta_\infty = 1+\alpha$ we go back to Step (vii) with $\hat\theta = 1+\alpha$, and from (4.42), (4.46), and (4.48) we obtain that $\chi(x,t)\to 1$ for $x\in C_\infty$ and $\chi(x,t)\to 0$ for $x\in D_\infty$. The case $\theta_\infty = 1-\alpha$ is similar.

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