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Zeno product formula revisited

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Abstract

We introduce a new product formula which combines an orthogonal projection with a complex function of a non-negative operator. Under certain assumptions on the complex function the strong convergence of the product formula is shown. Under more restrictive assumptions even operator-norm convergence is verified. The mentioned formula can be used to describe Zeno dynamics in the situation when the usual non-decay measurement is replaced by a particular generalized observables in the sense of Davies.

1 Introduction

Product formulæ are a traditional tool in various branches of mathematics; their use dates back to the time of Sophus Lie. Such formulæ are often of the form

$$\operatorname*{s-lim}_{n\to\infty}\left(e^{-itA/n}e^{-itB/n}\right)^n = e^{-itC}, \quad C := A + B, \quad t \in \mathbb{R}, \tag{1.1}$$

where A and B are bounded operators on some separable Hilbert space $\mathfrak{H}$ and s-lim stands for the strong operator topology. A natural generalization to unbounded self-adjoint operators A and B is due to Trotter [23, 24] who showed that the limit exists and is equal to e^{-itC}, $t \in \mathbb{R}$, if the operator C,

$$Cf := Af + Bf, \quad f \in \mathrm{dom}(C) := \mathrm{dom}(A) \cap \mathrm{dom}(B),$$

is essentially self-adjoint. In [16, 17] Kato focused his interest to a formula of the type

$$\operatorname*{s-lim}_{n\to\infty}\left(f(tA/n)g(tB/n)\right)^n \text{ and } \operatorname*{s-lim}_{n\to\infty}\left(g(tB/n)^{1/2}f(tA/n)g(tB/n)^{1/2}\right)^n \tag{1.2}$$

where A, B are non-negative self-adjoint operators and f, g are now so-called Kato functions. Recall that a Borel measurable function $f(\cdot) : [0,\infty) \longrightarrow \mathbb{R}$ is usually called a Kato function if the conditions

$$0 \le f(x) \le 1, \quad f(0) = 1, \quad f'(+0) = -1,$$

are satisfied. Under these conditions he was able to show that the limits (1.2) exist and are equal to e^{-tC}, $t \in \mathbb{R}$, where C is the form sum of A and B. Notice $f(x) = e^{-x}$ is a Kato-functions which yields the well-known Trotter-Kato product formulæ

$$\operatorname*{s-lim}_{n\to\infty}\left(e^{-tA/n}e^{-tB/n}\right)^n = \operatorname*{s-lim}_{n\to\infty}\left(e^{-tB/2n}e^{-tA/n}e^{-tB/2n}\right)^n = e^{-tC}, \quad t \ge 0.$$

Both formulas (1.1) and (1.2) are very useful and admit applications to functional integration, quantum statistical physics and other parts of physics – see, e.g., [8, Chap. V],

[25] and references therein. The last decade brought a progress in understanding of the convergence properties of such formulæ in operator-norm and trace-class topology, for a review of these results we refer to the monograph [25].

In the last few years we have witnessed a surge of interest to another type of product formulæ, namely

$$\text{s-}\lim_{n\to\infty}\left(Pe^{-itH/n}P\right)^n \tag{1.3}$$

where H is a self-adjoint operator on some separable Hilbert space $\mathfrak{H}$ and P is a orthogonal projection on some closed subspace $\mathfrak{h} \subseteq \mathfrak{H}$. Formulæ of such type are motivated by the "quantum Zeno effect" (QZE). We call them therefore Zeno product formulæ. The fact that the limit, if it exists, may be unitary on $\mathfrak{h}$ is a venerable problem known already to Alan Turing and formulated in the usual decay context of quantum mechanics for the first time by Beskow and Nilsson [2]: frequent measurements can slow down a decay of an unstable system, or even fully stop it in the limit of infinite measurement frequency. The effect was analyzed mathematically by Friedman [12] but became popular only after the authors of [21] invented the above stated name. Recent interest is motivated mainly by the fact that now the effect is within experimental reach; an up-to-date bibliography can be found, e.g., in [10] or [22]. The physical interpretation of this formula can be given in the context of particle decay, cf. [8, Chap. II]. The unstable system is characterized by a projection P to a subspace $\mathfrak{h}$ of the state Hilbert space $\mathfrak{H}$ of a larger, closed system, the dynamics of which is governed by a self-adjoint Hamiltonian H. Repeating the non-decay measurement experiment with the period t/n, we can describe the time evolution over the interval $[0,t]$ of a state originally in the subspace $P\mathfrak{H}$ by the interlaced product $(Pe^{-itH/n}P)^n$; the question is how this operator will behave as $n \to \infty$.

In [21, Theorem 1] it was shown that if the limit (1.3) exists and there is a conjugation J commuting with P and H, then the Zeno product formula defines a unitary group on the subspace $\mathfrak{h}$. Another simple example shows that this result is not valid generally: the limit (1.3) may exist without defining a unitary group. Let $\mathfrak{H} = L^2(\mathbb{R})$ and H be the momentum operator, i.e $H = -i\partial_x$ and let $P = P_{[a,b]}$ be the orthogonal projection on some subspace $\mathfrak{h} = L^2([a,b])$, $[a,b] \subseteq \mathbb{R}$. A straightforward calculation shows that

$$P_{[a,b]}e^{-isH}P_{[a,b]} = P_{[a,b]}P_{[a+s,b+s]}e^{-isH}P_{[a,b]}, \quad s \in \mathbb{R}.$$

Therefore, we get

$$T(t) := \text{s-}\lim_{n\to\infty}\left(Pe^{-itH/n}P\right)^n = Pe^{-itH} \upharpoonright \mathfrak{h}, \quad t \in \mathbb{R}, \tag{1.4}$$

which is neither unitary nor it satisfies the group property but defines a contraction semi-group for $t \geq 0$. This example is covered by the following more general one. Let H be the minimal self-adjoint dilation of a maximal dissipative operator K defined on the subspace $\mathfrak{h}$. Since by definition of self-adjoint dilations, cf. [11],

$$Pe^{-isH}P := e^{-isK} = Pe^{-isH} \upharpoonright \mathfrak{h}, \quad s \geq 0,$$

we find

$$\text{s-}\lim_{n\to\infty}\left(Pe^{-itH/n}P\right)^n = e^{-itK}, \quad t \geq 0.$$

Now on the strong convergence in the product formula is considered only on $\mathfrak{h}$. Further examples can be found in [20]. However, in all of them the non-unitarity of the limit is related to the fact that H is not semibounded. So we restrict ourself in the following to the case that H is semi-bounded from below; it is clear that without loss of generality we may assume that H is non-negative.

It has to be stressed that the last mentioned assumption does not ensure the existence of the limit (1.3). Indeed, if $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is not dense, then it can happen that the Zeno product formula does not converge, cf. [8, Rem. 2.4.9] or [20].

With these facts in mind we assume in the following that H is a non-negative self-adjoint operator such that $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is dense in $\mathfrak{h}$. Under these assumptions we claim that a "natural" candidate for the limit of the Zeno product formula (1.3) is the unitary group e^{-itK}, $t \in \mathbb{R}$, on $\mathfrak{h}$, generated by the non-negative self-adjoint operator K associated with the closed sesquilinear form $\mathfrak{k}$,

$$\mathfrak{k}(f,g) := (\sqrt{H}f, \sqrt{H}g), \quad f,g \in \mathrm{dom}(\mathfrak{k}) = \mathrm{dom}(\sqrt{H}) \cap \mathfrak{h} \subseteq \mathfrak{h}. \tag{1.5}$$

The claim rests upon the paper [9, Theorem 2.1] where it is shown that

$$\lim_{n\to\infty} \int_0^T \| \left(Pe^{-itH/n}P\right)^n f - e^{-itK} f\|^2 dt = 0, \quad \text{for each } f \in \mathfrak{h} \text{ and } T > 0 \tag{1.6}$$

holds. This result yields the existence of the limit of the Zeno product formula for almost all t in the strong operator topology, along a subsequence $\{n'\}$ of natural numbers[1].

The reason behind this weaker result is that the exponential function involved in the interlaced product gives rise to oscillations which are not easy to deal with. One of the main ingredients in the present paper is a simple observation that one can avoid the mentioned problem when $\phi(x) = e^{-ix}$ is replaced by functions with an imaginary part of constant sign. In analogy with the Kato class of the product formula (1.2) it seems to be useful to introduce a class of admissible functions.

Defintion 1.1 *We call a Borel measurable function $\phi(\cdot) : [0,\infty) \longrightarrow \mathbb{C}$ admissible if the conditions*

$$|\phi(x)| \le 1, \quad x \in [0,\infty), \qquad \phi(0) = 1, \quad \text{and} \quad \phi'(+0) = -i, \tag{1.7}$$

are satisfied.

Typical examples are

$$\phi(x) = (1 + ix/k)^{-k}, \quad k = 1,2,\ldots, \quad \text{and} \quad \phi(x) = e^{-ix}, \quad x \in [0,\infty). \tag{1.8}$$

The main goal of this paper is to prove the following result.

Theorem 1.2 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$ such that $P : \mathfrak{H} \longrightarrow \mathfrak{h}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{h}$. If $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is dense in $\mathfrak{h}$ and ϕ is admissible function which obeys*

$$\Im\mathrm{m}\,(\phi(x)) \le 0, \quad x \in [0,\infty), \tag{1.9}$$

[1] This fact was omitted in the first version of the paper from which the claim was reproduced in the review [22]. A complete proof of this claim is known at present only in the case claim when P is finite-dimensional, cf. [9].

then for any $t_0 > 0$ *one has*

$$\text{s-}\lim_{n\to\infty} \left(P\phi(tH/n)P\right)^n = e^{-itK}\,, \tag{1.10}$$

uniformly in $t \in [0, t_0]$*, where the generator* K *is defined by (1.5) and the strong convergence is meant on* $\mathfrak{h}$*.*

One may consider formulæ of type (1.10) as modified Zeno product formulæ. Examples of admissible functions obeying (1.9) are

$$\phi(x) = (1 + ix)^{-1} \quad \text{and} \quad \phi(x) = (1 + ix/2)^{-2}, \quad x \in [0, \infty).$$

Unfortunately, not all admissible function do satisfy the condition (1.9). Indeed, the functions $\phi(x) = (1 + ix/3)^{-3}$ and $\phi(x) = e^{-ix}$, $x \in [0, \infty)$, are admissible but do not obey (1.9). In particular this yields that the convergence problem for the original Zeno product formula (1.3) is not solved by Theorem 1.2 and remains open.

However, Theorem (1.2) suggests the following regularizing procedure. We set

$$\Delta_\phi := \{x \in [0, \infty) : \Im\mathrm{m}\,(\phi(x)) \le 0\}.$$

By (1.7) the set Δ_ϕ contains a neighbourhood of zero. If the subset $\Delta \subseteq \Delta_\phi$ contains also a neighbourhood of zero, then

$$\phi_\Delta(x) := \phi(x)\chi_\Delta(x), \quad x \in [0, \infty),$$

defines an admissible function obeying $\Im\mathrm{m}\,(\phi_\Delta(x)) \le 0$, $x \in [0, \infty)$. By Theorem 1.2 we obtain that for any $t_0 > 0$ one has

$$\text{s-}\lim_{n\to\infty} \left(P\phi_\Delta(tH/n)P\right)^n = e^{-itK}$$

uniformly in $t \in [0, t_0]$. Applying this procedure to $\phi(x) = e^{-ix}$, $x \in [0, \infty)$, one has to choose a subset $\Delta \subseteq \Delta_\phi := \cup_{m=0}^{\infty}[2m\pi, (2m + 1)\pi]$ containing a neighbourhood of zero. From $\phi(x) = e^{-ix}$ one can construct a "cutoff" admissible function $\phi_\Delta(x) := e^{-ix}\chi_\Delta(x)$, $x \in [0, \infty)$, obeying (1.9). In particular for $\Delta = [0, \pi)$, the function

$$\phi_\Delta(x) = e^{-ix}\chi_\Delta(x), \quad x \in [0, \infty),$$

is admissible and obeys (1.9). This leads immediately to the following corollary.

Corollary 1.3 *If the assumptions of Theorem 1.2 are satisfied, then for any* $t_0 > 0$ *one has*

$$\text{s-}\lim_{n\to\infty} \left(P(I + itH/n)^{-1}P\right)^n = \text{s-}\lim_{n\to\infty} \left(P(I + itH/2n)^{-2}P\right)^n = e^{-itK}\,, \tag{1.11}$$

and

$$\text{s-}\lim_{n\to\infty} \left(PE_H([0, \pi n/t))e^{-itH/n}P\right)^n = e^{-itK} \tag{1.12}$$

uniformly in $t \in [0, t_0]$ *where* $E_H(\cdot)$ *is the spectral measure of* H*, i.e.* $H = \int_{[0,\infty)} \lambda \, dE_H(\lambda)$*.*

The ideas to replace the unitary group e^{-itH} by a resolvent, cf. (1.11), or to employ a spectral cut-off together with e^{-itH}, cf. (1.12), are not new: they were used to derive a modification of the unitary Lie-Trotter formula in [14] and [18, 19], respectively, both for the form sum of two non-negative self-adjoint operators. See also [3].

Finally, let us note that formula (1.12) admits a physical interpretation in the context of the Zeno effect. To this end we note that the combination of the energy filtering and non-decay measurement following immediately one after another, see (1.12), can be regarded as *a single generalized measurement.* In fact, a product of two, in general non-commuting[2] projections represents the simplest non-trivial example of generalized observables[3] introduced by Davies which are realized as positive maps of the respective space of density matrices [6, Sec. 2.1]. Thus formula (1.12) corresponds to a modified Zeno situation with such generalized measurements, which depend on n and tend to the standard non-decay yes-no experiment as $n \to \infty$.

Let us describe briefly the contents of the paper. Section 2 is completely devoted to the proof of Theorem 1.2. In Section 3 we handle the general case of admissible functions under the stronger assumption $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$. We show that under this assumption the modified Zeno product formula converges to e^{-itK} for any admissible function not necessary satisfying the additional condition (1.9). In particular, one has

$$\text{s-lim}\left(Pe^{-itH/n}P\right)^n = e^{-itK}, \tag{1.13}$$

uniformly in $t \in [0, t_0]$ for any $t_0 > 0$. Moreover, we shall demonstrate there that under stronger assumptions, unfortunately too restrictive from the viewpoint of physical applications, even the operator-norm convergence can be obtained. We finish the paper with a conjecture which takes into account the results of [9] and the present paper.

2 Proof of Theorem 1.2

We set

$$F(\tau) := P\phi(\tau H)P : \mathfrak{h} \longrightarrow \mathfrak{h}, \quad \tau \geq 0, \tag{2.1}$$

and

$$S(\tau) := \frac{I_{\mathfrak{h}} - F(\tau)}{\tau} : \mathfrak{h} \longrightarrow \mathfrak{h}, \quad \tau > 0, \tag{2.2}$$

where $I_{\mathfrak{h}}$ is the identity operator in the subspace $\mathfrak{h}$. In the following for an operator X in $\mathfrak{H}$ we use the notation PXP for the operator $PXP := PX \restriction \mathfrak{h} : \mathfrak{h} \longrightarrow \mathfrak{h}$ as well as for its extension by zero in $\mathfrak{h}^{\perp}$. Let us assume that

$$\mathrm{dom}(T) := \mathrm{dom}(\sqrt{H}) \cap \mathfrak{h} \tag{2.3}$$

is dense in $\mathfrak{h}$. We define a linear operator $T : \mathfrak{h} \longrightarrow \mathfrak{H}$ by

$$Tf := \sqrt{H}f, \quad f \in \mathrm{dom}(T). \tag{2.4}$$

[2]We are primarily interested, of course, in the nontrivial case when the P does not commute with H, and thus also with the spectral projections $E_H([0, \pi n/t))$.

[3]Since the spectral projections involved commute with the evolution operator, one can also replace the product $PE_H([0, \pi t/n))$ in our formulæ by $E_H([0, \pi t/n))PE_H([0, \pi t/n))$. Such generalized observables represented by symmetrized projection products have been recently studied as *almost sharp quantum effects* – cf. [1].

Since $\sqrt{H}$ is closed and $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ is dense the operator T is closed and its domain $\mathrm{dom}(T)$ is dense in $\mathfrak{h}$. Then $T^*T : \mathfrak{h} \longrightarrow \mathfrak{h}$ is a self-adjoint operator which is identical with K defined by (1.5), i.e.

$$K := T^*T : \mathfrak{h} \longrightarrow \mathfrak{h} \tag{2.5}$$

which defines a non-negative self-adjoint operator in $\mathfrak{h}$.

Further, let us represent the function ϕ as

$$\phi(x) = \psi(x) - i\omega(x), \quad x \in [0,\infty),$$

where $\psi, \omega : [0,\infty) \longrightarrow \mathbb{R}$ are real-valued, Borel measurable functions obeying

$$|\psi(x)| \le 1, \quad \psi(0) = 1, \quad \psi'(+0) = 0 \tag{2.6}$$

and

$$0 \le \omega(x) \le 1, \quad \omega(0) = 0, \quad \omega'(+0) = 1.$$

Setting

$$\varphi(x) := 1 - \omega(x), \quad x \in [0,\infty),$$

one has

$$0 \le \varphi(x) \le 1, \quad \varphi(0) = 1, \quad \varphi'(+0) = -1, \tag{2.7}$$

which shows that φ is a Kato function. In terms of ψ, φ the function ϕ admits the representation

$$\phi(x) = \psi(x) - i(1 - \varphi(x)), \quad x \in [0,\infty).$$

We set

$$\begin{aligned} p_-(x) &:= \begin{cases} 1, & x = 0, \\ \inf_{s\in(0,x]}(1-\varphi(s))/s, & x > 0, \end{cases} \quad \text{and} \\ p_+(x) &:= \begin{cases} 1, & x = 0, \\ \sup_{s\in(0,x]}(1-\varphi(s))/s, & x > 0. \end{cases} \end{aligned} \tag{2.8}$$

Both functions are bounded on $[0,\infty)$ and obey

$$0 \le p_-(x) \le 1 \le p_+(x) < \infty, \quad x \in [0,\infty). \tag{2.9}$$

The function p_- is decreasing, i.e. $p_-(x) \ge p_-(y)$, $0 \le x \le y$, and p_+ is increasing, i.e. $p_-(x) \le p_-(y)$, $0 \le x \le y$. We define the sesquilinear forms

$$\mathfrak{k}^-_\tau(f,g) := (p_-(\tau H)\sqrt{H}f, \sqrt{H}g), \quad f,g \in \mathrm{dom}(\mathfrak{k}^-_\tau) := \mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}, \quad \tau \ge 0,$$

and

$$\mathfrak{k}^+_\tau(f,g) := (p_+(\tau H)\sqrt{H}f, \sqrt{H}g), \quad f,g \in \mathrm{dom}(\mathfrak{k}^+_\tau) := \mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}, \quad \tau \ge 0.$$

Notice that for $\tau = 0$ one has $\mathfrak{k}^-_0 = \mathfrak{k}^+_0 = \mathfrak{k}$ where the sesquilinear form $\mathfrak{k}$ is defined by (1.5). Obviously, both forms $\mathfrak{k}^\pm_\tau$ are non-negative for each $\tau \ge 0$. Moreover, the form $\mathfrak{k}^-_\tau$ is closable for each $\tau > 0$ and its closure is a bounded form on $\mathfrak{h}$ while the form $\mathfrak{k}^+_\tau$ is already closed for each $\tau \ge 0$. By $K^\pm_\tau$ we denote the associated non-negative self-adjoint operators on $\mathfrak{h}$. We note that $K^\pm_0 = K$. By (2.9) we get

$$\mathfrak{k}^-_\tau(f,f) \le \mathfrak{k}(f,f) \le \mathfrak{k}^+_\tau(f,f), \quad f \in \mathrm{dom}(\mathfrak{k}^-_\tau) = \mathrm{dom}(\mathfrak{k}) = \mathrm{dom}(\mathfrak{k}^+_\tau), \quad \tau \ge 0,$$

which yields

$$K_\tau^- \le K \le K_\tau^+, \quad \tau \ge 0.$$

Since p_- is decreasing the family $\{K_\tau^-\}_{\tau\ge 0}$ is increasing as $\tau \downarrow 0$. Further, from (2.8) one gets that $\text{s-lim}_{\tau\to+0}\, p_-(\tau H) = I_{\mathfrak{H}}$. Since $\mathfrak{k}_\tau^- \le \mathfrak{k}$ and

$$\lim_{\tau\to+0} \mathfrak{k}_\tau^-(f,g) = \lim_{\tau\to+0}(p_-(\tau H)\sqrt{H}f, \sqrt{H}g) = \mathfrak{k}(f,g), \quad f,g \in \text{dom}(\mathfrak{k}_\tau^-) = \text{dom}(\mathfrak{k}),$$

we obtain from Theorem VIII.3.13 of [15] that

$$\text{s-}\lim_{\tau\to+0}(I_{\mathfrak{h}} + K_\tau^-)^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{2.10}$$

Further, since p_+ is increasing the family $\{K_\tau^+\}_{\tau\ge 0}$ is decreasing as $\tau \downarrow 0$. By $\text{s-lim}_{\tau\to+0}\, p_+(\tau H) = I_{\mathfrak{H}}$ we find

$$\lim_{\tau\to+0} \mathfrak{k}_\tau^+(f,g) = \lim_{\tau\to+0}(p_+(\tau H)\sqrt{H}f, \sqrt{H}g) = \mathfrak{k}(f,g), \quad f,g \in \text{dom}(\mathfrak{k}_\tau^+) = \text{dom}(\mathfrak{k}).$$

Since $\mathfrak{k}$ is closed we obtain from Theorem VIII.3.11 of [15] that

$$\text{s-}\lim_{\tau\to+0}(I_{\mathfrak{h}} + K_\tau^+)^{-1} = (I_{\mathfrak{h}} + K)^{-1}. \tag{2.11}$$

Lemma 2.1 *Let $\{X(\tau)\}_{\tau>0}$, $\{Y(\tau)\}_{\tau>0}$, and $\{A(\tau)\}_{\tau>0}$ be families of bounded non-negative self-adjoint operators in $\mathfrak{h}$ such that the condition*

$$0 \le X(\tau) \le A(\tau) \le Y(\tau)\,, \quad \tau > 0\,,$$

is satisfied. If $s-\lim_{\tau\to 0} X(\tau) = s-\lim_{\tau\to 0} Y(\tau) = A$, where A is a bounded self-adjoint operator in $\mathfrak{h}$, then $s-\lim_{\tau\to 0} A(\tau) = A$.

Proof. Since for each $f \in \mathfrak{h}$ we have

$$(X(\tau)f,f) \le (A(\tau)f,f) \le (Y(\tau)f,f)\,, \quad \tau > 0\,,$$

we get $\lim_{\tau\to 0}(A(\tau)f,f) = (Af,f)$, $f \in \mathfrak{h}$, or $w-\lim_{\tau\to 0} A(\tau) = A$. Hence

$$w - \lim_{\tau\to 0}(Y(\tau) - A(\tau)) = 0.$$

Since $Y(\tau) - A(\tau) \ge 0$, $\tau > 0$, we find

$$\text{s-}\lim_{\tau\to 0}(Y(\tau) - A(\tau))^{1/2} = 0$$

which yields $\text{s-lim}_{\tau\to 0}(Y(\tau) - A(\tau)) = 0$. Hence $\text{s-lim}_{\tau\to 0} A(\tau) = A$. □

From (2.2) we obtain

$$S(\tau) = \frac{1}{\tau}P(I_{\mathfrak{H}} - \psi(\tau H))P + i\frac{1}{\tau}P(I_{\mathfrak{H}} - \varphi(\tau H))P, \quad \tau > 0. \tag{2.12}$$

Let

$$L_0(\tau) := \frac{1}{\tau}P(I_{\mathfrak{H}} - \varphi(\tau H))P : \mathfrak{h} \longrightarrow \mathfrak{h}, \quad \tau > 0. \tag{2.13}$$

Lemma 2.2 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$. If* $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ *is dense in $\mathfrak{h}$ and φ is a Kato function, then we have*

$$\text{s-}\lim_{\tau\to 0}\,(I_\mathfrak{h} + L_0(\tau))^{-1} = (I_\mathfrak{h} + K)^{-1} \tag{2.14}$$

Proof. Since

$$\mathfrak{k}_\tau^-(f,f) \le \left(\frac{I_\mathfrak{H} - \varphi(\tau H)}{\tau} f, f\right) \le \mathfrak{k}_\tau^+(f,f), \quad f \in \mathrm{dom}(\sqrt{H}) \cap \mathfrak{h},$$

we find

$$K_\tau^- \le P\frac{I_\mathfrak{H} - \varphi(\tau H)}{\tau} P \le K_\tau^+, \quad \tau > 0.$$

Hence

$$X(\tau) := (I_\mathfrak{h} + K_\tau^+)^{-1} \le \left(I_\mathfrak{h} + P\frac{I_\mathfrak{H} - \varphi(\tau H)}{\tau}P\right)^{-1} \le (I_\mathfrak{h} + K_\tau^-)^{-1} =: Y(\tau),$$

$\tau > 0$. Taking into account (2.10),(2.11) and applying Lemma 2.1 we prove (2.14). □

We set

$$L(\tau) := \frac{1}{\tau}P(I - \psi(\tau H))P + \frac{1}{\tau}P(I - \varphi(\tau H))P : \mathfrak{h} \longrightarrow \mathfrak{h}, \quad \tau > 0.$$

Lemma 2.3 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$. If* $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ *is dense in $\mathfrak{h}$, the real-valued Borel measurable function ψ obeys* (2.6) *and φ is a Kato function, then*

$$\text{s-}\lim_{\tau\to 0}(I_\mathfrak{h} + L(\tau))^{-1} = (I_\mathfrak{h} + K)^{-1}. \tag{2.15}$$

Proof. Let

$$\zeta(x) := \frac{\psi(2x) + \varphi(2x)}{2}, \quad x \in [0,\infty).$$

Notice that ζ is a Kato function. Setting

$$\widetilde{L}_0(\tau) := \frac{1}{\tau}P(I_\mathfrak{H} - \zeta(\tau H))P, \quad \tau > 0,$$

we obtain from Lemma 2.2 that $\text{s-}\lim_{\tau\to 0} = (I_\mathfrak{h} + \widetilde{L}_0(\tau))^{-1} = (I_\mathfrak{h} + K)^{-1}$. By $L(2\tau) = \widetilde{L}_0(\tau)$ we prove (2.15). □

We set

$$M(\tau) := (I_\mathfrak{h} + L_0(\tau))^{-1/2} P\frac{I - \psi(\tau H)}{\tau} P(I_\mathfrak{h} + L_0(\tau))^{-1/2}, \quad \tau > 0.$$

Lemma 2.4 *Let H be a non-negative self-adjoint operator in $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$. If* $\mathrm{dom}(\sqrt{H}) \cap \mathfrak{h}$ *is dense in $\mathfrak{h}$, the real-valued, Borel measurable functions ψ obeys* (2.6) *and φ is a Kato function, then we have*

$$\text{s-}\lim_{\tau\to 0}\,(I_\mathfrak{h} + M(\tau))^{-1} = I_\mathfrak{h}. \tag{2.16}$$

Proof. A straightforward computation proves the representation

$$(I_{\mathfrak{h}} + L(\tau))^{-1} = (I_{\mathfrak{h}} + L_0(\tau))^{-1/2} (I_{\mathfrak{h}} + M(\tau))^{-1} (I_{\mathfrak{h}} + L_0(\tau))^{-1/2}, \quad \tau > 0.$$

By (2.14) and (2.15) we get

$$w - \lim_{\tau \to 0} (I_{\mathfrak{h}} + M(\tau))^{-1} = I_{\mathfrak{h}}$$

which yields

$$\text{s-}\lim_{\tau \to 0} \left(I_{\mathfrak{h}} - (I_{\mathfrak{h}} + M(\tau))^{-1} \right)^{1/2} = 0.$$

Hence

$$\text{s-}\lim_{\tau \to 0} \left(I_{\mathfrak{h}} - (I_{\mathfrak{h}} + M(\tau))^{-1} \right) = 0$$

which proves (2.16). □

From (2.16) one gets

$$\text{s-}\lim_{\tau \to 0} (iI_{\mathfrak{h}} + M(\tau))^{-1} = -iI_{\mathfrak{h}}.$$

Hence

$$\text{s-}\lim_{\tau \to 0} (I_{\mathfrak{h}} + L_0(\tau))^{-1/2} (iI_{\mathfrak{h}} + M(\tau))^{-1} (I_{\mathfrak{h}} + L_0(\tau))^{-1/2} = -i(I_{\mathfrak{h}} + K)^{-1}.$$

or

$$\text{s-}\lim_{\tau \to 0} \left(iI_{\mathfrak{h}} + \frac{1}{\tau} P(I_{\mathfrak{h}} - \psi(\tau H))P + iL_0(\tau) \right)^{-1} = (iI_{\mathfrak{h}} + iK)^{-1}.$$

Using (2.12) and(2.13) we obtain

$$\text{s-}\lim_{\tau \to 0} (iI_{\mathfrak{h}} + S(\tau))^{-1} = (iI_{\mathfrak{h}} + iK)^{-1}$$

which yields

$$\text{s-}\lim_{\tau \to 0} (I_{\mathfrak{h}} + S(\tau))^{-1} = (I_{\mathfrak{h}} + iK)^{-1}$$

We finish the proof of Theorem 1.2 applying Chernoff's theorem [5] or Lemma 3.29 of [7].

3 Arbitrary admissible functions

Theorem 1.2 needs the additional assumption (1.9) and it is unclear whether this assumption can be dropped. In the following we are going to show that under stronger assumptions on the domain of $\sqrt{H}$ the condition (1.9) is indeed not necessary.

Theorem 3.1 *Let H be a non-negative self-adjoint operator on $\mathfrak{H}$ and let $\mathfrak{h}$ be a closed subspace of $\mathfrak{H}$ such that $P : \mathfrak{H} \longrightarrow \mathfrak{h}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{h}$. If $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$ and ϕ is admissible, then*

$$\text{s-}\lim_{n \to \infty} (P\phi(tH/n)P)^n = e^{-itK} \tag{3.1}$$

uniformly in $t \in [0, t_0]$ for any $t_0 > 0$ where K is defined by (1.5).

Proof. We note that $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$ implies that $T = \sqrt{H}P$ is a bounded operator, and consequently, $K = T^*T$ is also bounded. We may employ the representation

$$\left(\frac{I_{\mathfrak{H}} - \phi(\tau H)}{\tau} f, g\right) = \left(p(\tau H)\sqrt{H}f, \sqrt{H}g\right), \quad \tau > 0, \tag{3.2}$$

for $f \in \mathrm{dom}(H)$ and $g \in \mathrm{dom}(\sqrt{H})$ where

$$p(x) := \begin{cases} i, & x = 0 \\ (1 - \phi(x))/x, & x > 0 \end{cases} .$$

Since $C_p := \sup_{x\in[0,\infty)} |p(x)| < \infty$ by (1.7) one gets $\|p(\tau H)\|_{\mathcal{B}(\mathfrak{H})} \le C_p$, $\tau > 0$. Hence the equality (3.2) extends to $f, g \in \mathrm{dom}(\sqrt{H})$, in particular, to $f, g \in \mathfrak{h}$. This leads to the representation

$$(I_{\mathfrak{h}} - F(\tau))f = T^* p(\tau H)Tf\,, \quad \tau > 0\,, \quad f \in \mathfrak{h}\,, \tag{3.3}$$

or

$$S(\tau)f - iKf = T^*\left(p(\tau H) - iI_{\mathfrak{H}}\right)Tf\,, \quad \tau > 0\,, \quad f \in \mathfrak{h}\,. \tag{3.4}$$

By assumption (1.7) we find $\text{s-lim}_{\tau\to 0}\, p(\tau H) = iI_{\mathfrak{H}}$ which yields $\text{s-lim}_{\tau\to 0}\, S(\tau) = iK$. In this way we obtain the relation

$$\text{s-}\lim_{\tau\to 0}(I_{\mathfrak{h}} + S(\tau))^{-1} = (I_{\mathfrak{h}} + iK)^{-1},$$

and using Chernoff's theorem [5] one more time we have proved (3.1). □

It turns out that the convergence (3.1) can be improved to operator-norm convergence under some stronger assumption.

Corollary 3.2 *Let the assumptions of Theorem 3.1 be satisfied. One has*

$$\lim_{n\to\infty} \left\|(P\phi(tH/n)P)^n - e^{-itK}\right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{3.5}$$

uniformly in $t \in [0, t_0]$ *for any* $t_0 > 0$ *if in addition*

(i) the operator T *is compact or*

(ii) there is $\alpha > 0$ *such that* $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H^{1+\alpha}})$ *and* $C_\alpha := \sup_{x\in(0,\infty)} |p_\alpha(x)| < \infty$ *where*

$$p_\alpha(x) := \begin{cases} 0, & x = 0 \\ (p(x) - i)/x^\alpha, & x > 0 \end{cases} .$$

Proof. From (3.4) and the compactness of T we find

$$\lim_{\tau\to 0} \|S(\tau) - iK\|_{\mathcal{B}(\mathfrak{h})} = 0. \tag{3.6}$$

If $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H^{1+\alpha}})$ for some $\alpha > 0$, then we set $T_\alpha := \sqrt{H^{1+\alpha}}P$ and $K_\alpha = T_\alpha^* T_\alpha$. Notice that T_α is a bounded operator. From (3.4) we obtain the representation

$$S(\tau) - iK = \tau^\alpha\, T_\alpha^*\, p_\alpha(\tau H)\, T_\alpha\,, \quad \tau > 0.$$

Hence we find the estimate

$$\|S(\tau) - iK\|_{\mathcal{B}(\mathfrak{h})} \le \tau^\alpha C_\alpha \|K_\alpha\|_{\mathcal{B}(\mathfrak{h})}, \quad \tau > 0,$$

which yields (3.6). Using the representation

$$e^{-itK} - e^{-tS(t/n)} = \int_0^t e^{-(t-s)S(t/n)}(S(t/n) - iK)e^{-isK} ds$$

we get the estimate

$$\left\| e^{-itK} - e^{-tS(t/n)} \right\|_{\mathcal{B}(\mathfrak{h})} \le t\|S(t/n) - iK\|_{\mathcal{B}(\mathfrak{h})}, \quad t \ge 0.$$

Using (3.6) we find

$$\lim_{n\to\infty} \left\| e^{-tS(t/n)} - e^{-itK} \right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{3.7}$$

holds for any $t > 0$, uniformly in $t \in [0, t_0]$. We shall combine it with the telescopic estimate

$$\begin{aligned} &\left\| F(t/n)^n - e^{-itK} \right\|_{\mathcal{B}(\mathfrak{h})} \le \\ &\le \left\| F(t/n)^n - e^{-tS(t/n)} \right\|_{\mathcal{B}(\mathfrak{h})} + \left\| e^{-tS(t/n)} - e^{-itK} \right\|_{\mathcal{B}(\mathfrak{h})}, \end{aligned} \tag{3.8}$$

where the first term can be treated as in Lemma 2 of [4], see also [7, Lemma 3.27],

$$\left\| \left(F(t/n)^n - e^{-tS(t/n)} \right) f \right\| \le \sqrt{n} \left\| \left(F(t/n) - I_{\mathfrak{h}} \right) f \right\|, \quad f \in \mathfrak{h}. \tag{3.9}$$

Using the representation (3.3) with $\tau = t/n$, we can estimate the right-hand side of (3.9) by

$$\left\| \left(F(t/n) - I_{\mathfrak{h}} \right) f \right\| \le \frac{t}{n} \left\| T^* p(tH/n) T f \right\|, \quad f \in \mathfrak{h}, \quad t > 0.$$

Since $\|p(\tau H)\|_{\mathcal{B}(\mathfrak{H})} \le C_p$, $\tau > 0$, we find

$$\left\| \left(F(t/n) - I_{\mathfrak{h}} \right) f \right\| \le C_p \frac{t}{n} \|K\|_{\mathcal{B}(\mathfrak{h})} \|f\|, \quad f \in \mathfrak{h}.$$

Inserting this estimate into (3.9) we obtain

$$\left\| F(t/n)^n - e^{-tS(t/n)} \right\|_{\mathcal{B}(\mathfrak{h})} \le C_p \frac{t}{\sqrt{n}} \|K\|_{\mathcal{B}(\mathfrak{h})}$$

which yields

$$\lim_{n\to\infty} \left\| F(t/n)^n - e^{-tS(t/n)} \right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{3.10}$$

for any $t > 0$, uniformly in $t \in [0, t_0]$. Taking into account (3.7), (3.8) and (3.10) we arrive at the sought relation (3.5). □

Remark 3.3 Since $\phi(x) = e^{-ix}$, $x \in [0, \infty)$, is admissible we get from Theorem 3.1 that under the assumptions $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$ the original Zeno product formula (1.13) holds and that under the stronger assumptions $\sqrt{H}P$ is compact or $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H^{1+\alpha}})$, $\alpha > 0$, the original Zeno product formula (1.13) converges in the operator norm.

Remark 3.4 Obviously, the conclusion (3.5) is valid if $\mathfrak{h} \subseteq \mathrm{dom}(\sqrt{H})$ and $\mathfrak{h}$ is a finite dimensional subspace. Indeed, in this case the operator T is finite dimensional, and therefore compact. This gives an alternative proof of the result derived in Section 5 of [9] for the case $\phi(x) = e^{-ix}$.

Remark 3.5 In connection with the previous remark let us mention that in the finite-dimensional case there is one more way to prove the claim suggested by G.M. Graf and A. Guekos [13] for the special case $\phi(x) = e^{-ix}$. The argument is based on the observation that

$$\lim_{t\to 0} t^{-1} \left\| Pe^{-itH}P - Pe^{-itK}P \right\|_{\mathcal{B}(\mathfrak{h})} = 0 \tag{3.11}$$

implies $\left\| (Pe^{-itH/n}P)^n - e^{-itK} \right\|_{\mathcal{B}(\mathfrak{h})} = n\, o(t/n)$ as $n \to \infty$ by means of a natural telescopic estimate. To establish (3.11) one first proves that

$$t^{-1} \left[(f, Pe^{-itH}Pg) - (f,g) - it(\sqrt{H}Pf, \sqrt{H}Pg) \right] \longrightarrow 0$$

as $t \to 0$ for all f, g from $\mathrm{dom}(\sqrt{H}P)$ which coincides in this case with $\mathfrak{h}$ by assumption. The last expression is equal to

$$\left(\sqrt{H}Pf, \left[\frac{e^{-itH} - I}{tH} - i \right] \sqrt{H}Pg \right)$$

and the square bracket tends to zero strongly by the functional calculus, which yields the sought conclusion. We note that the operator in the square brackets is well-defined by the functional calculus even if H is not invertible. In the same way we find that

$$t^{-1} \left[(f, Pe^{-itK}Pg) - (f,g) - it(\sqrt{K}f, \sqrt{K}g) \right] \longrightarrow 0$$

holds as $t \to 0$ for any vectors $f, g \in \mathfrak{h}$. Next we note that $(\sqrt{K}f, \sqrt{K}g) = (\sqrt{H}Pf, \sqrt{H}Pg)$, and consequently, the expression contained in (3.11) tends to zero weakly as $t \to 0$, however, in a finite dimensional $\mathfrak{h}$ the weak and operator-norm topologies are equivalent.

Conjecture 3.6 Comparing the results of the present paper with those ones of [9] we conjecture that if we drop the assumption (1.9) in Theorem 1.2, then at least the convergence

$$\lim_{n\to\infty} \int_0^T \left\| (\phi(tH/n))^n f - e^{-itK} f \right\|^2 \, dt = 0 \tag{3.12}$$

holds for for each $f \in \mathfrak{h}$, $T > 0$ and arbitrary admissible functions ϕ. The proofs of [9] rely heavily on the analytic properties of the exponential function $\phi(x) = e^{-ix}$. For admissible functions analytic properties are not required which yields the necessity to look for a different proof idea.

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References

[1] A. Arias, S. Gudder, *Almost sharp quantum effects* J. Math. Phys. **45** (2004), 4196-4206.

[2] J. Beskow, J. Nilsson, *The concept of wave function and the irreducible representations of the Poincaré group, II. Unstable systems and the exponential decay law* Arkiv Fys. **34** (1967), 561-569.

[3] V. Cachia, *On a product formula for unitary groups* Bull. London Math. Soc. **37** (2005), no. 4, 621-626.

[4] P. R. Chernoff, *Note on product formulas for operator semigroups* J. Funct. Anal. **2** (1968), 238-242.

[5] P. R. Chernoff, *Product Formulas, Nonlinear Semigroups and Addition of Unbounded Operators* Mem. Amer. Math. Soc. **140**, Providence R.I, 1974.

[6] E. B. Davies, *Quantum Theory of Open Systems* Academic Press, London, 1976.

[7] E. B. Davies, *One-Parameter Semigroups* London Mathematical Society Monographs, Academic Press, London-New York, 1980.

[8] P. Exner, *Open Quantum Systems and Feynman Integrals* D. Reidel Publishing Co., Dordrecht, 1985.

[9] P. Exner, T. Ichinose, *A product formula related to quantum Zeno dynamics* Ann. H. Poincaré **6** (2005), 195-215.

[10] P. Facchi, G. Marmo, S. Pascazio, A. Scardicchio, E. C. G. Sudarshan, *Zeno's dynamics and constraints* J. Opt. B: Quantum Semiclass. Opt. **6** (2004), S492-S501.

[11] B. Sz.-Nagy, C. Foias, *Harmonic analysis of operators on Hilbert space* North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadmiai Kiad, Budapest, 1970.

[12] C. Friedman, *Semigroup product formulas, compressions, and continual observations in quantum mechanics* Indiana Math. J. **21** (1971/72), 1001-1011.

[13] G. M. Graf, A. Guekos, *private communication.*

[14] T. Ichinose, *A product formula and its application to the Schrödinger equation* Publ. RIMS, Kyoto Univ., **15** (1980), 585-600.

[15] T. Kato, *Perturbation Theory for Linear Operators* Springer, Berlin-Heidelberg-New York, 1966.

[16] T. Kato, *On the Trotter-Lie product formula* Proc. Japan Acad. **50** (1974), 694–698.

[17] T. Kato, *Trotter's product formula for an arbitrary pair of self-adjoint contraction semigroups* Topics in functional analysis, Adv. in Math. Suppl. Stud., vol. 3, Academic Press, New York-London, 1978.

[18] M. L. Lapidus, *Generalization of the Trotter-Lie formula* Int. Eq. Operator Theory **4** (1981), 366-415.

[19] M. L. Lapidus, *Product formula for imaginary resolvents with application to a modified Feynman integral* J. Funct. Anal. **63** (1985), 261-275.

[20] M. Matolcsi, R. Shvidkoy, *Trotter's product formula for projections* Arch. Math. **81** (2003), 309-317.

[21] B. Misra, E. C. G. Sudarshan, *The Zeno's paradox in quantum theory* J. Math. Phys. **18** (1977), 756-763.

[22] A. U. Schmidt, *Mathematics of the quantum Zeno effect* in Mathematical Physics Research on the Leading Edge, Ch. Benton ed., Nova Science Publ., Hauppauge NY, 2004; pp. 113-143.

[23] H. F. Trotter, *Approximation of semi-groups of operators* Pacific J. Math. **8** (1958), 887–919.

[24] H. F. Trotter, *On the product of semi-groups of operators* Proc. Amer. Math. Soc. **10** (1959), 545–551.

[25] V. A. Zagrebnov, *Topics in the Theory of Gibbs Semigroups* Leuven Notes in Mathematical and Theoretical Physics, vol. 10, Series A: Mathematical Physics; Leuven Univ. Press, 2003.