# Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.


## Self–averaging in a class of generalized Hopfield models


Anton Bovier



submitted: 30th June 1994



Institut für Angewandte
Analysis und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany



Preprint No. 105
Berlin 1994

Edited by
Institut für Angewandte Analysis und Stochastik (IAAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e-mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e-mail (Internet): preprint@iaas-berlin.d400.de

# SELF-AVERAGING IN A CLASS OF GENERALIZED HOPFIELD MODELS#

**Anton Bovier** [1]
*Institut für Angewandte Analysis und Stochastik*
*Mohrenstrasse 39, D-10117 Berlin, Germany*

**Abstract:** We prove the almost sure convergence to zero of the fluctuations of the free energy, resp. local free energies, in a class of disordered mean-field spin systems that generalize the Hopfield model in two ways: 1) Multi-spin interactions are permitted and 2) the random variables $\xi_i^\mu$ describing the 'patterns' can have arbitrary distributions with mean zero and finite $4+\epsilon$-th moments. The number of patterns, $M$, is allowed to be an arbitrary multiple of the systemsize. This generalizes a previous result of Bovier, Gayrard, and Picco [BGP3] for the standard Hopfield model, and improves a result of Feng and Tirozzi [FT] that required $M$ to be a finite constant. Note that the convergence of the mean of the free energy is *not* proven.

*Keywords:* Hopfield model, neural networks, self-averaging, law of large numbers.

# Work partially supported by the Commission of the European Communities under contract CHRX-CT93-0411

1 e-mail: bovier@iaas-berlin.d400.de

# I. Introduction

Over the last years some interesting properties of 'self-averaging' have been observed in two classes of 'spin-glass' type models of the mean field type, the Sherrington-Kirkpatrick model [SK] and the Hopfield model [FP, Ho]. The latter, largely used in the context of neural networks, is maybe particularly interesting, as it contains a parameter, the number $M$ of stored patterns as a function of the size of the system, $N$ which can be adjusted to alter the properties of the model. In a paper of Pastur and Shcherbina [PS], it was observed that the variance of the free energy of a finite system of size $N$ in the SK-model tends to zero like $1/N$, implying the convergence to zero in probability of the difference between the free energy and its mean. This result was later generalized to the Hopfield model by Shcherbina and Tirozzi [ST] under the assumption that the ratio $\alpha = M/N$ remains bounded as $N \uparrow \infty$. Further results of this type can be found in an interesting paper by Pastur, Shcherbina and Tirozzi [PST]. Self-averaging properties of the large deviation rate function as a function of the macroscopic parameters of the model (the so called 'overlap-parameters', see below) were used crucially in two papers by Bovier, Gayrard and Picco [BGP2, BGP3]. There, sharper than variance estimates were needed, and as a consequence [BGP3] contains in particular a proof of the almost sure convergence to zero of the difference between the free energy and its mean, both in the Hopfield model under the assumption the $M/N$ be bounded, and in the SK-model. Independently, Feng and Tirozzi [FT] have recently proven such a result in a class of generalized Hopfield models, however under the very restrictive assumption that $M$ itself be a bounded function of $N$. The purpose of the present note is to show that such a condition is in fact unnecessary.

Let us describe the class of models we will consider. We denote by $\mathcal{S}_N = \{-1,1\}^N$ the space of functions $\sigma : \Lambda \to \{-1,1\}$. We call $\sigma$ a *spin configuration* on $\Lambda$. $\mathcal{S} \equiv \{-1,1\}^{\mathbb{N}}$ denotes the space of half infinite sequences equipped with the product topology of the discrete topology on $\{-1,1\}$. We denote by $\mathcal{B}_\Lambda$ and $\mathcal{B}$ the corresponding Borel sigma algebras. We will define a random Hamiltonian function on the spaces $\mathcal{S}_\Lambda$ as follows. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space. Let $\xi \equiv \{\xi_i^\mu\}_{i,\mu\in\mathbb{N}}$ be a two-parameter family of independent, random variables on this space. We will specify we assumptions on their distribution later. In the context of neural networks, one assumes usually that $\mathbb{P}(\xi_i^\mu = 1) = \mathbb{P}(\xi_i^\mu = -1) = \frac{1}{2}$, but here we aim for more general distributions. We consider Hamiltonians of the form

$$H_N(\sigma) \equiv -\frac{1}{N^{r-1}} \sum_{\mu=1}^{M(N)} \sum_{i_1,\dots,i_r=1}^{N} \xi_{i_1}^\mu \dots \xi_{i_r}^\mu \sigma_{i_1} \dots \sigma_{i_r} \tag{1.1}$$

Here $r \geq 2$ is some chosen integer. The case $r = 2$ corresponds to the usual Hopfield model, and models with general $r$ were introduced by Lee et al. [Lee] and Peretto and Niez [PN]. Feng and Tirozzi [FT] also studied these models, but removed the terms in the sum where two or more indices coincide, which actually amounts to adding a term of the order of a constant to $H$ which does not alter the free energy. One may actually consider more general models in which the Hamiltonian is given as a linear combination of terms of the type (1.1) with different values of $r$. This only complicates but not really alters the proofs, and our results can easily be extended to this situation.

Let us introduce the so-called 'overlap-parameters'. This is the $M$-dimensional vector $m_N(\sigma)$ whose components are given by

$$m_N^\mu(\sigma) \equiv \frac{1}{N}\sum_{i=1}^N \xi_i^\mu \sigma_i \tag{1.2}$$

In terms of these quantities, the Hamiltonian can be written in the very pleasant form

$$H_N(\sigma) = -N\, \|m_N(\sigma)\|_r^r \tag{1.3}$$

Let us introduce a family $\chi_{\rho,\delta}$ of smooth functions satisfying

(1) $\chi_{\rho,\delta}(x) \geq 0$,

(2) $\left|\frac{\frac{d}{dx}\chi_{\rho,\delta}(x)}{\chi_{\rho,\delta}(x)}\right| \leq \delta^{-1}$,

(3) $\mathbb{1}_{\{|x|\leq\rho\}} \leq \chi_{\rho,\delta}(x) \leq \mathbb{1}_{\{|x|\leq\rho+\delta\}}$,

(4) $\ln \chi_{\rho,\delta}(x)$ is a concave function of $x$ (where we use the convention $\ln 0 \equiv -\infty$). (This condition actually only plays a rôle in the case $r = 2$). We define

$$Z_N(\tilde{m}) \equiv Z_{N,\beta,\rho,\delta}(\tilde{m}) \equiv \frac{1}{2^N}\sum_{\sigma\in S_N} e^{-\beta H_N(\sigma)}\chi_{\rho,1/\sqrt{N}}\left(\|m_N(\sigma)-\tilde{m}\|_2^2\right) \tag{1.4}$$

and

$$f_{N,\rho}(\tilde{m}) \equiv -\beta^{-1}\ln Z_{N,\rho}(\tilde{m}) \tag{1.5}$$

Observe that since we have chosen $\delta = 1/\sqrt{N}$, the function $\xi$ in (1.4) is essentially a characteristic function. Note that the free energy is given in terms of these quantities as

$$F_N(\beta) \equiv \frac{1}{N} f_{N,\infty}(0) \tag{1.6}$$

We will prove

**Theorem 1:** *Assume that* $\lim \frac{M(N)}{N} = \alpha < \infty$ *and that* $\mathbb{E}\xi_i^\mu = 0$, $\mathbb{E}\left(\xi_i^\mu\right)^2 = 1$ *and* $\mathbb{E}\left(\xi_i^\mu\right)^{4+\epsilon} < \infty$, *for some* $\epsilon > 0$. *Let* $\rho$ *and* $\|\tilde{m}\|_2$ *be bounded.*

*(i) If* $r = 2$, *for all* $n < \infty$ *there exists* $\tau_n < \infty$, *such that for all* $\tau \geq \tau_n$, *and for* $N$ *large enough,*

$$\mathbb{P}\left[\left|f_{N,\rho}(\tilde{m}) - \mathbb{E}f_{N,\rho}(\tilde{m})\right| \geq \tau(\ln N)^{3/2}N^{\frac{1}{2}}\right] \leq N^{-n} \tag{1.7}$$

*(ii) If $r \geq 3$, then there exist constants $C, c, c' > 0$ s.t.*

$$\mathbb{P}\left[|f_{N,\rho}(\tilde{m}) - \mathbb{E} f_{N,\rho}(\tilde{m})| \geq zN\right] \leq \begin{cases} e^{-cNz^2}, & \text{if } 0 \leq z < C \\ e^{-Nc'z}, & \text{if } z \geq C \end{cases} \tag{1.8}$$

We prove Theorem 1 in the next section. Before doing that, we will show that it implies

**Theorem 2:** *Under the assumptions of Theorem 1,*

$$\lim_{N\uparrow\infty} |F_N(\beta) - \mathbb{E} F_N(\beta)| = 0, \quad a.s. \tag{1.9}$$

**Remark:** Theorem 2 was proven under the additional assumption that $\xi_i^\mu = \pm 1$ for the case $r = 2$ in [BGP3]. In [FT] Theorem 2 was proven under the hypothesis $M(N) \leq M_0 < \infty$ and that $\mathbb{E}\left(\xi_i^\mu\right)^4 < \infty$.

**Remark:** Theorem 2 may in some way be regarded as a *strong law of large numbers*. We are, however, reluctant to employ this term, because the convergence of $\mathbb{E} F_N(\beta)$ to a limit is in general not proven. In the standard Hopfield model this was proven under the assumption $\lim_{N\uparrow\infty} \frac{M(N)}{N} = 0$ by Koch [K] (see also [BG]).

Theorem 2 would be immediate from Theorem 1, if we were allowed to set $\tilde{m} = 0$ and $\rho = \infty$. The latter is not really possible, since the constants $\tau$, resp. $c$ depend on $\rho$. However, essentially, $F_N(\beta)$ and $N f_{N,\rho}(0)$ differ only by an asymptotically negligible amount, if $\rho$ is chosen somewhat large. To see this, note first that

$$\|m_N(\sigma)\|_2^2 \leq \|A(N)\| \tag{1.10}$$

where $A(N)$ is the $N \times N$-matrix whose elements are

$$A_{ij}(N) \equiv \frac{1}{N} \sum_{\mu=1}^{M} \xi_i^\mu \xi_j^\mu \tag{1.11}$$

The crucial element linking Theorem 2 to Theorem 1 are bounds on the maximal eigenvalues of this matrix. The eigenvalue distribution of this matrix was first analyzed by Marchenko and Pastur [MP]. Girko [Gi] proved that under the hypothesis of Theorem 1, the maximal eigenvalue of $A(N)$ converges to $(1+\alpha)^2$ in probability. Adding the ideas used by Bai and Yin [BY] one can easily show that this convergence takes also place almost surely, and even in the case where only the 4-th moment of $\xi_i^\mu$ is finite. We will need additional estimates on the moments of $\|A(N)\|$ which we are only able to prove if we have a little more than four moments. The relevant estimate is formulated in the following lemma.

**Lemma 3:** *Assume that $\xi$ satisfies the hypothesis stated in Theorem 1. Then, for any $0 < \eta \leq 1/6$ and any $\delta > 0$, if $N$ is sufficiently large,*

$$\mathbb{P}\left[\|A(N)\| \geq (1+\sqrt{\alpha})^2(1+z)\right] \leq N(1+z)^{-N^{\gamma/\eta}\delta^{-1/\eta}} + \frac{c\alpha}{N^{\epsilon/4}\delta^{4+\epsilon}} \tag{1.12}$$

*where* $\alpha = \frac{M}{N}$, $\gamma = \frac{\epsilon}{4(4+\epsilon)}$.

**Remark:** The proof of Lemma 3 is in fact an adaptation of the the truncation idea in [BY] and fairly standard estimates on the traces of powers of $A$, as in [BY] (but see also [BGP1]). We will therefore not give the details of the proof of Lemma 3, but only mention that the second term is a bound on the probability that any of the $\xi_i^\mu$ exceeds the value $\sqrt{N}\delta$, while the first is a bound one would obtain *if* all $\xi_i^\mu$ satisfied this condition.

We conclude the introduction by giving the proof of Theorem 2, assuming Theorem 1.

***Proof:*** (of Theorem 2) Let us fix $\rho = 2(1+\sqrt{\alpha})^2$. Notice first that

$$\ln Z_N \mathbb{I}_{\{\|A(N)\|\leq\rho\}} = \ln Z_{N,\rho}(0)\mathbb{I}_{\{\|A(N)\|\leq\rho\}} \tag{1.13}$$

Therefore

$$\begin{aligned} \ln Z_N &= \ln Z_{N,\rho}(0)\mathbb{I}_{\{\|A(N)\|\leq\rho\}} + \ln Z_N \mathbb{I}_{\{\|A(N)\|>\rho\}} \\ &= \ln Z_{N,\rho}(0) + \ln Z_{N,\rho}(0)\mathbb{I}_{\{\|A(N)\|>\rho\}} + \ln Z_N \mathbb{I}_{\{\|A(N)\|>\rho\}} \end{aligned} \tag{1.14}$$

By Theorem 1 and the first Borel-Cantelli lemma it follows that

$$\lim_{N\uparrow\infty} \frac{1}{N} |\ln Z_{N,\rho}(0) - \mathbb{E} \ln Z_{N,\rho}(0)| = 0, \quad a.s. \tag{1.15}$$

Thus Theorem 2 will be proven if we can show that $(\ln Z_{N,\rho}(0) + \ln Z_N)\, \mathbb{I}_{\{\|A(N)\|>\rho\}} \downarrow 0$ both almost surely and in mean. The almost sure convergence follows easily, since

$$\mathbb{P}\left[(\ln Z_{N,\rho}(0) + \ln Z_N)\, \mathbb{I}_{\{\|A(N)\|>\rho\}} \neq 0 \text{ i.o.}\right] \leq \mathbb{P}\left[\|A(N)\| > \rho \text{ i.o.}\right] = 0 \tag{1.16}$$

where the last equality follows from applying Lemma 3.1 from Bai and Yin [BY]. Finally, to prove convergence of the mean, we use that first of all

$$\begin{aligned} |H_N(\sigma)| &\leq N\|m_N(\sigma)\|_2^2 \|m_N(\sigma)\|_\infty^{r-2} \\ &\leq N\|m_N(\sigma)\|_2^r \\ &\leq N\|A(N)\|^{r/2} \end{aligned} \tag{1.17}$$

and therefore,

$$\begin{aligned} |\ln Z_{N,\rho}(0) + \ln Z_N|\, \mathbb{I}_{\{\|A(N)\|>\rho\}} &\leq 2\ln Z_N \mathbb{I}_{\{\|A(N)\|>\rho\}} \\ &\leq 2N\beta\|A(N)\|^{r/2}\mathbb{I}_{\{\|A(N)\|>\rho\}} \end{aligned} \tag{1.18}$$

But

$$\begin{aligned} \mathbb{E}\|A(N)\|^x \mathbb{I}_{\{\|A(N)\|>\rho\}} &= \rho^x \mathbb{P}\left[\|A(N)\| > \rho\right] + \int_\rho^\infty x y^{x-1} \mathbb{P}\left[\|A(N)\| > y\right] dy \\ &\leq 2^x(1+\sqrt{\alpha})^{2x} \left(N 2^{-N^{\gamma/6}} + \frac{c\alpha}{N^{\epsilon/4}}\right) \\ &+ 2^x(1+\sqrt{\alpha})^{2x} \int_1^\infty x(1+y)^{x-1} \left(N e^{-yN^{\gamma/\eta(x)} y^{-x/(4\eta(x))}} + \frac{c\alpha}{N^{\epsilon/4} y^{x+\epsilon/4}}\right) dy \end{aligned} \tag{1.19}$$

To obtain the last expression we used Lemma 3 and made the choice $\delta = \delta(y,x) = y^{x/4}$ and $\eta(x) = \max(6, x/8)$. Obviously, the right hand side of (1.19) tends to zero as $N \uparrow \infty$, as desired. This concludes the proof of Theorem 2, assuming Theorem 1.◇

**Remark:** Note that the estimate in (1.19) implies in particular that

$$\mathbb{E}\|A(N)\| \leq C(1+\sqrt{\alpha})^2 \tag{1.20}$$

for some constant depending only on $\epsilon$. This is relevant for proving Theorem 1 in the case $r = 2$ (see [BGP3]).

## 2. Proof of Theorem 1

The basic idea of the proof is the same as in [BGP3] where the case $r = 2$ has been considered. The case $r \geq 3$ turns out to be considerably simpler, and we will only present this case here. Since the only properties of the random variables $\xi_i^\mu$ that are used in [BGP3] are estimates on the expectations of the norms of matrices of the type of $A(N)$, the generalization to random variables satisfying only the hypothesis of Theorem 1 is straightforward from the results presented in Section 1. We thus consider part (i) of Theorem 1 as proven.

The fact that in the case $r \geq 3$ sharper estimates can be obtained may justify the presentation of the details of the proof in that case. We fix $\rho$ and write for simplicity

$$f_N(m) \equiv f_{N,\rho}(m) \tag{2.1}$$

We now introduce the decreasing sequence of sigma-algebras $\mathcal{F}_k$ that are generated by the random variables $\{\xi_i^\mu\}_{i\geq k}^{\mu\in\mathbb{N}}$ and the corresponding martingale difference sequence (first introduced by Yurinskii [Yu])

$$\tilde{f}_N^{(k)}(\tilde{m}) \equiv \mathbb{E}\left[f_N(\tilde{m})\middle|\mathcal{F}_k\right] - \mathbb{E}\left[f_N(\tilde{m})\middle|\mathcal{F}_{k+1}\right] \tag{2.2}$$

Notice that we have the identity

$$f_N(\tilde{m}) - \mathbb{E} f_N(\tilde{m}) \equiv \sum_{k=1}^{N} \tilde{f}_N^{(k)}(\tilde{m}) \tag{2.3}$$

To get the sharpest possible estimates, we want to use an exponential inequality. To this end we observe that [BGP3]

$$\begin{aligned} \mathbb{P}\left[\left|\sum_{k=1}^{N} \tilde{f}_N^{(k)}(\tilde{m})\right| \geq Nz\right] &\leq 2 \inf_{t\in\mathbb{R}} e^{-|t|Nz} \mathbb{E} \exp\left\{t\sum_{k=1}^{N} \tilde{f}_N^{(k)}(\tilde{m})\right\} \\ &= 2 \inf_{t\in\mathbb{R}} e^{-|t|Nz} \mathbb{E}\left[\mathbb{E}\left[\ldots\mathbb{E}\left[e^{t\tilde{f}_N^{(1)}(\tilde{m})}\middle|\mathcal{F}_2\right] e^{t\tilde{f}_N^{(2)}(\tilde{m})}\middle|\mathcal{F}_3\right]\ldots e^{t\tilde{f}_N^{(N)}(\tilde{m})}\middle|\mathcal{F}_{N+1}\right] \end{aligned} \tag{2.4}$$

To make use of this inequality, we need bounds on the conditional Laplace transforms; namely, *if* we can show that, for some function $\mathcal{L}^{(k)}(t)$, $\ln I\!E\left[e^{t\tilde f_N^{(k)}(\tilde m)}|\mathcal{F}_{k+1}\right] \leq \mathcal{L}^{(k)}(t)$, uniformly in $\mathcal{F}_{k+1}$, then we obtain that

$$I\!P\left[\left|\sum_{k=1}^{N}\tilde f_N^{(k)}(\tilde m)\right| \geq Nz\right] \leq 2\inf_{t\in I\!R} e^{-|t|Nz+\sum_{k=1}^{N}\mathcal{L}^{(k)}(t)} \tag{2.5}$$

Note that this construction, so far, is completely model independent. In the estimation of the conditional Laplace transforms, a conventional trick [PS] is to introduce a continuous family of Hamiltonians, $\tilde H_N^{(k)}(\sigma,u)$, that are equal to the original one for $u=1$ and are independent of $\xi_k$ for $u=0$. We will do this in a slightly different form than in [ST, BGP2, BGP3, FT]. We first introduce the $M(N)$-dimensional vectors

$$m_N^{(k)}(\sigma,u) \equiv \frac{1}{N}\left(\sum_{\substack{i\\ i\neq k}}\xi_i\sigma_i + u\xi_k\sigma_k\right) \tag{2.6}$$

and then define

$$\tilde H_N^{(k)}(\sigma,u) = -N\left\|m_N^{(k)}(\sigma,u)\right\|_r^r \tag{2.7}$$

Note that this procedure can of course be used in all cases where the Hamiltonian is a function of the macroscopic order parameters. Naturally, we set

$$Z_N^{(k)}(\tilde m,u) \equiv \frac{1}{2^N}\sum_{\sigma\in\mathcal{S}_N} e^{-\beta\tilde H_N^{(k)}(\sigma,u)}\chi_{\delta,\rho}\left(\|m_N^{(k)}(\sigma,u)-\tilde m\|_2^2\right) \tag{2.8}$$

and finally

$$f_N^{(k)}(\tilde m,u) = -\beta^{-1}\left(\ln Z_N^{(k)}(\tilde m,u) - \ln Z_N^{(k)}(\tilde m,0)\right) \tag{2.9}$$

Since for the remainder of the proof, $\tilde m$ as well as $N$ will be fixed values, to simplify our notations we will write $f_k(u) \equiv f_N^{(k)}(\tilde m,u)$. Notice that

$$\tilde f_N^{(k)}(\tilde m) = I\!E\left[f_k(1)|\mathcal{F}_k\right] - I\!E\left[f_k(1)|\mathcal{F}_{k+1}\right] \tag{2.10}$$

and that, since

$$I\!E\left[f_k(0)|\mathcal{F}_k\right] - I\!E\left[f_k(0)|\mathcal{F}_{k+1}\right] = 0 \tag{2.11}$$

this implies

$$\tilde f_N^{(k)}(\tilde m) = \int_0^1 du\left(I\!E\left[f_k'(u)|\mathcal{F}_k\right] - I\!E\left[f_k'(u)|\mathcal{F}_{k+1}\right]\right) \tag{2.12}$$

To bound the Laplace transform, we use that, for all $x\in I\!R$,

$$e^x \leq 1 + x + \frac{1}{2}x^2 e^{|x|} \tag{2.13}$$

so that

$$\mathbb{E}\left[e^{t\tilde{f}_N^{(k)}(\tilde{m})}|\mathcal{F}_{k+1}\right] \leq 1 + \frac{1}{2}t^2\mathbb{E}\left[\left(\tilde{f}_N^{(k)}(\tilde{m})\right)^2 e^{|t\tilde{f}_N^{(k)}(\tilde{m})|}|\mathcal{F}_{k+1}\right] \tag{2.14}$$

Our strategy in [BGP3] was to use a rather *poor* uniform bound on $\tilde{f}_N^{(k)}(\tilde{m})$ in the exponent but to prove a better estimate on the remaining conditioned expectation of the square. Here it will turn out that the uniform bound is already sufficiently good to be used throughout in (2.14).

Namely

$$f_k'(u) = \mathcal{E}_{k,u}\left(\frac{\partial}{\partial u}H_N^{(k)}(\sigma,u) + \frac{2}{\beta N}\frac{\chi_{\rho,\delta}'(\|m_N^{(k)}(\sigma,u)-\tilde{m}\|_2^2)}{\chi_{\rho,\delta}(\|m_N^{(k)}(\sigma,u)-\tilde{m}\|_2^2)}\sum_\mu\left(m_N^{(k),\mu}(\sigma,u)-\tilde{m}^\mu\right)\xi_k^\mu\sigma_k\right) \tag{2.15}$$

where $\mathcal{E}_{k,u}$ denotes the expectation w.r.t. the probability measure

$$\frac{1}{Z_N^{(k)}(\tilde{m},u)}\chi_{\rho,\delta}(\|m_N^{(k)}(\sigma,u)-\tilde{m}\|_2^2)e^{-\beta \tilde{H}_N^{(k)}(\sigma,u)}d\sigma \tag{2.16}$$

Obviously,

$$|f_k'(u)| \leq \mathcal{E}_{k,u}\left|\frac{\partial}{\partial u}H_N^{(k)}(\sigma,u)\right| + \tfrac{2}{\beta\sqrt{N}}\mathcal{E}_{k,u}\left[\left|\sum_\mu\left(m_N^\mu(\sigma,u)-\tilde{m}^\mu\right)\xi_k^\mu\sigma_k\right|\right] \tag{2.17}$$

The second term in (2.17) is easy:

$$\begin{aligned}
&\tfrac{2}{\beta\sqrt{N}}\mathcal{E}_{k,u}\left[\left|\sum_\mu\left(m_N^\mu(\sigma,u)-\tilde{m}^\mu\right)\xi_k^\mu\sigma_k\right|\right]\\
&\leq \tfrac{2}{\beta\sqrt{N}}\mathcal{E}_{k,u}\left[\sum_\mu|m_N^\mu(\sigma,u)-\tilde{m}^\mu|\right]\\
&= \tfrac{2}{\beta\sqrt{N}}\mathcal{E}_{k,u}\|m_N(\sigma,u)-\tilde{m}\|_1 \leq \tfrac{2\sqrt{M}}{\beta\sqrt{N}}\mathcal{E}_{k,u}\|m_N(\sigma,u)-\tilde{m}\|_2\\
&\leq \tfrac{2\sqrt{M}}{\beta\sqrt{N}}\sqrt{\rho+N^{-1/2}}
\end{aligned} \tag{2.18}$$

so under our hypothesis it is bounded by a constant. What remains is to bound the expectation of the modulus of $\frac{\partial}{\partial u}H_N^{(k)}(\sigma,u)$. This is the only estimate that is model-dependent. Computing the derivative we get

$$\begin{aligned}
\left|\frac{\partial}{\partial u}H_N^{(k)}(\sigma,u)\right| &= \left|\sum_{\mu=1}^{M(N)} r\xi_k^\mu\sigma_k\left[m_N^{(k),\mu}(\sigma,u)\right]^{r-1}\right|\\
&\leq r\left|\sum_{\mu=1}^{M(N)}\xi_k^\mu\sigma_k\left[m_N^{(k),\mu}(\sigma,u)\right]^{r-1}\right|\\
&\leq r\left\|m_N^{(k)}(\sigma,u)\right\|_\infty^{r-3}\sum_{\mu=1}^{M(N)}\left[m_N^{(k),\mu}(\sigma,u)\right]^2\\
&\leq r\left\|m_N^{(k)}(\sigma,u)\right\|_\infty^{r-3}\|m_N^{(k)}(\sigma,u)\|_2^2
\end{aligned} \tag{2.19}$$

In the usual case, where $|\xi_i^\mu| \leq 1$, we can bound the sup-norms appearing in (2.19) by $\left\| m_N^{(k)}(\sigma, u) \right\|_\infty \leq 1$; in the case of unbounded $\xi$, we can still use that $\left\| m_N^{(k)}(\sigma, u) \right\|_\infty \leq \left\| m_N^{(k)}(\sigma, u) \right\|_2$. (Note that to get (2.19) we need that $r \geq 3$. In the case $r = 2$, we could only get a bound in terms of the $L^1$-norm of $m_N^{(k)}(\sigma, u)$, which is typically a factor of $\sqrt{M}$ larger than the $L^2$-norm). Hence we get

$$
\begin{aligned}
\mathcal{E}_{k,u} \left| \frac{\partial}{\partial u} H_N^{(k)}(\sigma, u) \right| &\leq r \mathcal{E}_{k,u} \| m_N^{(k)}(\sigma, u) \|_2^2 \\
&\leq r \left( \|\tilde{m}\|_2 + \sqrt{\rho + N^{-1/2}} \right)^2
\end{aligned}
\tag{2.20}
$$

if $|\xi_i^\mu| \leq 1$, and

$$
\begin{aligned}
\mathcal{E}_{k,u} \left| \frac{\partial}{\partial u} H_N^{(k)}(\sigma, u) \right| &\leq r \mathcal{E}_{k,u} \| m_N^{(k)}(\sigma, u) \|_2^{r-1} \\
&\leq r \left( \|\tilde{m}\|_2 + \sqrt{\rho + N^{-1/2}} \right)^{r-1}
\end{aligned}
\tag{2.21}
$$

in general. In both bounds we used the fact that the measure $\mathcal{E}_{k,u}$ has support on those $\sigma$ for which $\chi$ does not vanish. Putting (2.21) and (2.18) together, we see that indeed

$$
\left| \tilde{f}_N^{(k)}(\tilde{m}) \right| \leq C \tag{2.22}
$$

where $C \equiv C(\tilde{m}, \rho, \frac{M}{N})$ is same finite constant depending on $\tilde{m}$, $\rho$ and $\frac{M}{N}$. Using this bound in (2.14) we see that

$$
\mathcal{L}^{(k)}(t) \leq \frac{C^2}{2} t^2 e^{C|t|} \tag{2.23}
$$

To obtain (1.8), we insert this bound into (2.5) and bound the infimum over $t$ by its value for $t = \frac{z}{C^2}$, if $z < \ln 2C$, and by its value for $t = C^{-1}$, if $z \geq C \ln 2$. This concludes the proof of Theorem 1. ♢

## References


[BG] A. Bovier and V. Gayrard, "Rigorous results on the thermodynamics of the dilute Hopfield model", J. Stat. Phys. **69**: 597-627 (1993).

[BGP1] A. Bovier, V. Gayrard, and P. Picco, "Gibbs states of the Hopfield model in the regime of perfect memory", to appear in Prob. Theor. Rel. Fields **99** (1994).

[BGP2] A. Bovier, V. Gayrard, and P. Picco, "Large deviation principles for the Hopfield model and the Kac-Hopfield model", IAAS-preprint No. 95, submitted to Prob. Theor. Rel. Fields (1994).

[BGP3] A. Bovier, V. Gayrard, and P. Picco, "Gibbs states of the Hopfield model with extensively many patterns", IAAS-preprint No. 97, submitted to J. Stat. Phys. (1994).

[BY] Z.D. Bai and Y.Q. Yin, "Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix", Ann. Probab. **16**, 1729-1741 (1988).

[FP] L.A. Pastur and A.L. Figotin, "Exactly soluble model of a spin glass", Sov. J. Low Temp. Phys. **3**(6): 378-383 (1977); "On the theory of disordered spin systems", Theor. Math. Phys. **35**: 403-414 (1978).

[FT] J. Feng and B. Tirozzi, "The SLLN for the free energy in the Hopfield type model and Little model with finite number of patterns", preprint 1994.

[Gi] V.L. Girko, "Limit theorems for maximal and minimal eigenvalues of random matrices", Theor. Prob. Appl. **35**, 680-695 (1989).

[Ho] J.J. Hopfield,"Neural networks and physical systems with emergent collective computational abilities", Proc. Natl. Acad. Sci. USA **79**: 2554-2558 (1982).

[K] H. Koch, "A free energy bound for the Hopfield model", J. Phys. A : Math Gen. **26**: L353-L355 (1993).

[Lee] Y.C. Lee, G. Doolen, H.H. Chen, G.Z. Sun, T. Maxwell, H.Y. Lee, and C.L. Gilles, "Machine learning using higher order correlation networks", Physica D **22**, 276-306 (1986).

[MP] V.A. Marchenko and L.A. Pastur, "Eigenvalue distribution in some ensembles of random matrices", Math. USSR Sbornik **72**, 536-567 (1967).

[PN] P. Peretto and J.J. Niez, "Long term memory storage capacity of multiconnected neural networks", Biological Cybernetics **54**, 53-63 (1986).

[PS] L. Pastur and M. Shcherbina, "Absence of self-averaging of the order parameter in the Sherrington-Kirkpatrick model", J. Stat. Phys. **62** : 1-19 (1991).

[PST] L. Pastur, M. Shcherbina, and B. Tirozzi, "The replica symmetric solution without the replica trick for the Hopfield model", J. Stat. Phys. **74**: 1161-1183 (1994).

[SK] D. Sherrington and S. Kirkpatrick, "Solvable model of a spin glass", Phys. Rev. Lett. **35**: 1792-1796 (1972).

[ST] M. Shcherbina and B. Tirozzi, "The free energy for a class of Hopfield models", J. Stat. Phys. **72**: 113-125 (1992).

[Yu] V.V. Yurinskii, "Exponential inequalities for sums of random vectors", J. Multivariate Anal. **6**: 473-499 (1976).

# Recent publications of the Institut für Angewandte Analysis und Stochastik

## Preprints 1993

76. Wolfgang Dahmen, Bernd Kleemann, Siegfried Prößdorf, Reinhold Schneider: A multiscale method for the double layer potential equation on a polyhedron.

77. Hans–Günter Bothe: Attractors of non invertible maps.

78. Grigori Milstein, Michael Nussbaum: Autoregression approximation of a nonparametric diffusion model.

## Preprints 1994

79. Anton Bovier, Véronique Gayrard, Pierre Picco: Gibbs states of the Hopfield model in the regime of perfect memory.

80. Roland Duduchava, Siegfried Prößdorf: On the approximation of singular integral equations by equations with smooth kernels.

81. Klaus Fleischmann, Jean–François Le Gall: A new approach to the single point catalytic super–Brownian motion.

82. Anton Bovier, Jean–Michel Ghez: Remarks on the spectral properties of tight binding and Kronig–Penney models with substitution sequences.

83. Klaus Matthes, Rainer Siegmund–Schultze, Anton Wakolbinger: Recurrence of ancestral lines and offspring trees in time stationary branching populations.

84. Karmeshu, Henri Schurz: Moment evolution of the outflow–rate from nonlinear conceptual reservoirs.

85. Wolfdietrich Müller, Klaus R. Schneider: Feedback stabilization of nonlinear discrete–time systems.

86. Gennadii A. Leonov: A method of constructing of dynamical systems with bounded nonperiodic trajectories.

87. Gennadii A. Leonov: Pendulum with positive and negative dry friction. Continuum of homoclinic orbits.

88. Reiner Lauterbach, Jan A. Sanders: Bifurcation analysis for spherically symmetric systems using invariant theory.

89. Milan Kučera: Stability of bifurcating periodic solutions of differential inequalities in $\mathbb{R}^3$.

90. Peter Knabner, Cornelius J. van Duijn, Sabine Hengst: An analysis of crystal dissolution fronts in flows through porous media Part I: Homogeneous charge distribution.

91. Werner Horn, Philippe Laurençot, Jürgen Sprekels: Global solutions to a Penrose–Fife phase–field model under flux boundary conditions for the inverse temperature.

92. Oleg V. Lepskii, Vladimir G. Spokoiny: Local adaptivity to inhomogeneous smoothness. 1. Resolution level.

93. Wolfgang Wagner: A functional law of large numbers for Boltzmann type stochastic particle systems.

94. Hermann Haaf: Existence of periodic travelling waves to reaction–diffusion equations with excitable–oscillatory kinetics.

95. Anton Bovier, Véronique Gayrard, Pierre Picco: Large deviation principles for the Hopfield model and the Kac–Hopfield model.

96. Wolfgang Wagner: Approximation of the Boltzmann equation by discrete velocity models.

97. Anton Bovier, Véronique Gayrard, Pierre Picco: Gibbs states of the Hopfield model with extensively many patterns.

98. Lev D. Pustyl'nikov, Jörg Schmeling: On some estimations of Weyl sums.

99. Michael H. Neumann: Spectral density estimation via nonlinear wavelet methods for stationary non–Gaussian time series.

100. Karmeshu, Henri Schurz: Effects of distributed delays on the stability of structures under seismic excitation and multiplicative noise.

101. Jörg Schmeling: Estimates of Weyl sums over subsequences of natural numbers.

102. Grigori N. Milstein, Michael V. Tret'yakov: Mean–square approximation for stochastic differential equations with small noises.

103. Valentin Konakov: On convergence rates of suprema in the presence of non–negligible trends.

104. Pierluigi Colli, Jürgen Sprekels: On a Penrose–Fife model with zero interfacial energy leading to a phase–field system of relaxed Stefan type.