Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

A bi-Lipschitz, volume preserving map from the unit ball onto a cube

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submitted: July 19, 2005

No. 1048

Berlin 2005

2000 *Mathematics Subject Classification.* Primary 53A55, Secondary 26B10, 57R50.

Key words and phrases. Lipschitz-homeomorphism, invariant sets, measure preserving maps.

Research partially supported by BMBF grant 03-GANGB-5.

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Abstract

We construct two bi-Lipschitz, volume preserving maps from Euclidean space onto itself which map the unit ball onto a cylinder and onto a cube, respectively. Moreover, we characterize invariant sets of these mappings.

1 Introduction

This paper is dedicated to a special case of the general question whether two manifolds with measure can be mapped onto each other by a measure preserving map which possesses, additionally, some continuity properties. This problem has a long history, for instance, in cartography, where surface preserving mappings from the two dimensional sphere onto the plane are required.

A variant of this problem has been treated by Moser [16], who proved that on a closed, smooth manifold any two smooth volume elements are diffeomorphic. Extensions of this result to noncompact manifolds are due to Greene and Shiokama [6], while Banyaga [1] proved it for smooth manifolds with boundary. Gromov [14] investigated these questions for real analytic manifolds with real analytic volume forms.

Zehnder [17] proved that certain Hölder and Lipschitz continuous volume elements can be mapped onto each other by means of C^1-mappings with Hölder and Lipschitz continuous first derivatives, respectively. For bounded domains with Hölder continuous volume forms and boundaries of class $C^{3,\alpha}$ Dacorogna and Moser [3] proved the existence of auto diffeomorphisms from the class $C^{1,\alpha}$ which provide the equivalence of the volume forms and are the identity map on the boundary of the manifold. Based upon this result Fonseca and Parry [4] proved that for any two elements from a class of star shaped domains in Euclidean space, there is a Lipschitz homeomorphism with constant Jacobian mapping these two domains onto each other. Fonseca and Parry's class contains in particular the ball, the cube, and the cylinder.

Our aim is to give a comprehensive proof of Fonseca and Parry's result for the special case of a ball and a cylinder by explicitly constructing the Lipschitz homeomorphism with constant Jacobian between the two domains. This special Lipschitz homeomorphism has additionally a variety of invariant sets and fixed points, which we characterize. Actually, our investigation has been spurred by just this additional demand on the mapping. Our construction comes to bear in applications of the concept of Gröger-regular sets in the theory of partial differential equations, see [12] and [7] for the concept itself, and [8], [10], [11], [13], and [15] for applications. In particular interpolation theory for function spaces on Gröger-regular sets relies on the mapping presented in this paper, see [9].

2 Results

We investigate two special cases of Fonseca and Parry's result [4, Ch. 5, Thm. 5.4], namely bi-Lipschitz mappings with constant Jacobian of a ball onto a cylinder and of a ball onto a cube. The special geometric situation allows to reduce the number of spatial variables by making use of rotational symmetry. Thus, we can formulate the constancy of the Jacobian by means of differential equations which can be explicitely solved. Finally, one obtains the sought-after mapping as a rational expression.

Theorem 1. *For any integer $d \geq 1$ there is a bi-Lipschitzian mapping Λ_{d+1} from $\mathbb{R}^{d+1}$ onto itself with the following properties:*

1. *Λ_{d+1} maps the unit ball B^{d+1} of $\mathbb{R}^{d+1}$ onto a cylinder with height 2 and radius 1:*

$$\Lambda_{d+1} : B^{d+1} \xrightarrow{\text{onto}} \big\{(x, y) \in \mathbb{R}^{d+1} : x \in B^d,\ |y| < 1\big\}.$$

2. *Λ_{d+1} maps the halfspace*

$$\big\{(x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} : x_j \geq 0\big\}$$

onto itself for each of the integers $l \in \{1, 2, \ldots, d+1\}$.

3. *Λ_{d+1} maps each hyperplane containing the rotation axis*

$$\big\{(x, y) \in \mathbb{R}^{d+1} : x = 0,\ y \in \mathbb{R}\big\}$$

onto itself.

4. *Λ_{d+1} is the identity map on the equatorial hyperplane*

$$\big\{(x, y) \in \mathbb{R}^{d+1} : x \in \mathbb{R}^d,\ y = 0\big\}.$$

5. *Both poles $(0, \ldots, 0, 1)$ and $(0, \ldots, 0, -1) \in \mathbb{R}^{d+1}$ are fixed points of Λ_{d+1}.*

6. *The map Λ_{d+1} is homogeneous of order 1:*

$$\Lambda_{d+1}(rx_1, rx_2, \ldots, rx_{d+1}) = r\Lambda_{d+1}(x_1, x_2, \ldots, x_{d+1})$$

for all $(x_1, x_2, \ldots, x_{d+1}) \in \mathbb{R}^{d+1}$, $r \geq 0$.

7. *The Jacobian of Λ_{d+1} is constant almost everywhere.*

We prove the theorem in Section 3.

Remark 2. In the two-dimensional case the mapping Λ_2 defined by

$$\Lambda_2(x,y) \stackrel{\text{def}}{=} \begin{cases} (0\,,\,0) & \text{if } x = y = 0, \\ \left(\sqrt{x^2+y^2}\,,\,\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{y}{x}\right) & \text{if } |y| \le x,\ x > 0, \\ \left(-\sqrt{x^2+y^2}\,,\,-\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{y}{x}\right) & \text{if } |y| \le -x,\ x < 0, \\ \left(\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{x}{y}\,,\,\sqrt{x^2+y^2}\right) & \text{if } |x| \le y,\ y > 0, \\ \left(-\frac{4}{\pi}\sqrt{x^2+y^2}\arctan\frac{x}{y}\,,\,-\sqrt{x^2+y^2}\right) & \text{if } |x| \le -y,\ y < 0, \end{cases}$$

with the inverse

$$\Lambda_2^{-1}(\xi,\eta) = \begin{cases} (0\,,\,0) & \text{if } \eta = \xi = 0, \\ \left(\xi\,\cos\frac{\pi}{4}\frac{\eta}{\xi}\,,\,\xi\,\sin\frac{\pi}{4}\frac{\eta}{\xi}\right) & \text{if } |\eta| \le |\xi|,\ \xi \ne 0, \\ \left(\eta\,\sin\frac{\pi}{4}\frac{\xi}{\eta}\,,\,\eta\,\cos\frac{\pi}{4}\frac{\xi}{\eta}\right) & \text{if } |\xi| \le |\eta|,\ \eta \ne 0, \end{cases}$$

meets the requirements of Theorem 1, see also Figure 1.

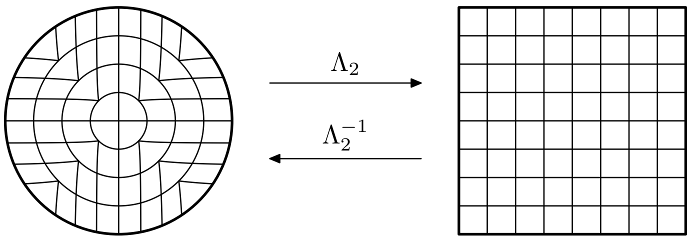

Figure 1: The map Λ_2 from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

Theorem 1 implies another special case of Fonseca and Parry's result.

Corollary 3. *There is a map from $\mathbb{R}^d$, $d \ge 1$, onto itself which is bi-Lipschitzian, has an almost everywhere constant Jacobian, and maps the unit ball onto the unit cube.*

Proof. In the one-dimensional case one can choose the identity map. In the two-dimensional case the mapping Λ_2 from Remark 2 is the right one. Now one deduces the statement by induction over the space dimension d, thereby making use of Theorem 1. $\square$

Corollary 4. *There is a bi-Lipschitzian, volume preserving map from $\mathbb{R}^d$, $d \ge 1$, onto itself which maps the unit ball onto a cube.*

Actually, Corollary 4 and Corollary 3 are equivalent. This follows from the fact that a homothecy has a constant Jacobian.

Remark 5. Due to Brouwer's invariance of domain theorem [2] the boundaries of the ball and of the cylinder as well as the boundaries of the ball and of the cube are mapped onto each other by the mappings from Theorem 1 and the corollaries, respectively.

3 Proof of the theorem

In the following we prove Theorem 1; we write the coordinates in $\mathbb{R}^{d+1}$ as $(x_1, ..., x_d, y)$, for short (x, y), $x \in \mathbb{R}^d$, $y \in \mathbb{R}$, and abbreviate the Euclidean norm $\|x\|_{\mathbb{R}^d}$ by $|x|_d$.

For $d = 1$ the mapping Λ_2 from Remark 2 satisfies the assertions of Theorem 1. Now, we regard the problem in $\mathbb{R}^{d+1}$ with $d \geq 2$ and make the following ansatz for Λ_{d+1}:

$$(x, y) \longmapsto \big(x_1 g(x, y), \ldots, x_d g(x, y), h(x, y)\big) \tag{1}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $y \in \mathbb{R}$. Moreover, we demand

$$\big(x_1 g(x, y), \ldots, x_d g(x, y), h(x, y)\big) = \big(x_1 g(x, -y), \ldots, x_d g(x, -y), -h(x, -y)\big). \tag{2}$$

As a consequence of this ansatz, hyperplanes which contain zero and whose normal vectors are orthogonal to the vector $(0, 0, \ldots, 0, 1)$ are mapped into themselves. If, additionally, $h(x, y) \geq 0$ for all $y \geq 0$, the halfspace

$$\mathbb{R}^{d+1}_+ \stackrel{\text{def}}{=} \big\{(x, y) \in \mathbb{R}^{d+1} : y \geq 0\big\}$$

is mapped into itself, and the hyperplane defined by $y = 0$ is mapped into itself. Thus, it suffices to define Λ_{d+1} on the upper halfspace $\mathbb{R}^{d+1}_+$. In order to do so, we partition $\mathbb{R}^{d+1}_+$:

$$C_\gamma^\triangledown \stackrel{\text{def}}{=} \big\{(x, y) \in \mathbb{R}^{d+1} : y \geq \gamma |x|_d\big\}, \tag{3}$$

$$C_\gamma^\bowtie \stackrel{\text{def}}{=} \big\{(x, y) \in \mathbb{R}^{d+1} : 0 \leq y \leq \gamma |x|_d\big\}, \tag{4}$$

and define Λ_{d+1}, thereby observing (1), (2), by the bi-Lipschitzian mappings

$$\Lambda_\gamma^\triangledown : C_\gamma^\triangledown \longrightarrow C_1^\triangledown, \qquad \Lambda_\gamma^\bowtie : C_\gamma^\bowtie \longrightarrow C_1^\bowtie, \tag{5}$$

such that the restrictions of these mappings coincide on the common boundary

$$\big\{(x, y) \in \mathbb{R}^{d+1} : y = \gamma |x|_d\big\}, \tag{6}$$

of $C_\gamma^\triangledown$ and $C_\gamma^\bowtie$. Here, $\gamma > 0$ is a constant, which we specify in Step 3 of the proof.

1. First, we construct a mapping on $C_\gamma^\triangledown$, see (3). We set $h^\triangledown(x, y) \stackrel{\text{def}}{=} \sqrt{|x|_d^2 + y^2}$. Now, we are looking for a function $g^\triangledown(x, y) = v\big(|x|_d / y\big)$ such that the Jacobian satisfies

$$\left| \frac{\partial\big(g^\triangledown(x, y)\, x, h^\triangledown(x, y)\big)}{\partial(x, y)} \right| = 1 \quad \text{for all } x \in \mathbb{R}^d,\ y > 0. \tag{7}$$

This determinant can be evaluated as follows

$$\begin{vmatrix} v + \frac{x_1^2}{y\,|x|_d} v' & \frac{x_1 x_2}{y|x|_d} v' & \dots & \frac{x_1 x_d}{y\,|x|_d} v' & -\frac{x_1 |x|_d}{y^2} v' \\ \frac{x_1 x_2}{y\,|x|_d} v' & v + \frac{x_2^2}{y\,|x|_d} v' & \dots & \frac{x_2 x_d}{y\,|x|_d} v' & -\frac{x_2\,|x|_d}{y^2} v' \\ \dots & \dots & \dots & \dots & \dots \\ \frac{x_1 x_d}{y\,|x|_d} v' & \frac{x_2 x_d}{y\,|x|_d} v' & \dots & v + \frac{x_d^2}{y\,|x|_d} v' & -\frac{x_d\,|x|_d}{y^2} v' \\ \frac{x_1}{\sqrt{|x|_d^2+y^2}} & \frac{x_2}{\sqrt{|x|_d^2+y^2}} & \dots & \frac{x_d}{\sqrt{|x|_d^2+y^2}} & \frac{y}{\sqrt{|x|_d^2+y^2}} \end{vmatrix}.$$

Adding suitable multiples of the first row to the others we get the determinant

$$\frac{1}{\sqrt{|x|_d^2 + y^2}} \begin{vmatrix} v + \frac{x_1^2}{y\,|x|_d} v' & \frac{x_1 x_2}{y|x|_d} v' & \dots & \frac{x_1 x_d}{y\,|x|_d} v' & -\frac{x_1 |x|_d}{y^2} v' \\ -\frac{x_2}{x_1} v & v & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{x_d}{x_1} v & 0 & \dots & v & 0 \\ x_1 & x_2 & \dots & x_d & y \end{vmatrix},$$

which can be simplified by adding multiples of the last column to the others

$$\frac{1}{\sqrt{|x|_d^2 + y^2}} \begin{vmatrix} v & 0 & \dots & 0 & -\frac{x_1 |x|_d}{y^2} v' \\ -\frac{x_2}{x_1} v & v & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{x_d}{x_1} v & 0 & \dots & v & 0 \\ x_1\left(1 + \frac{y^2}{|x|_d^2}\right) & x_2\left(1 + \frac{y^2}{|x|_d^2}\right) & \dots & x_d\left(1 + \frac{y^2}{|x|_d^2}\right) & y \end{vmatrix}.$$

Finally, adding suitable multiples of all columns to the first one we end up with

$$\frac{1}{\sqrt{|x|_d^2 + y^2}} \begin{vmatrix} v & 0 & \dots & 0 & -\frac{x_1 |x|_d}{y^2} v' \\ 0 & v & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & v & 0 \\ \frac{|x|_d^2}{x_1}\left(1 + \frac{y^2}{|x|_d^2}\right) & x_2\left(1 + \frac{y^2}{|x|_d^2}\right) & \dots & x_d\left(1 + \frac{y^2}{|x|_d^2}\right) & y \end{vmatrix}.$$

Developing the last determinant with respect to the first column, condition (7) leads to the following ordinary differential equation for v:

$$\frac{1}{\sqrt{\frac{|x|_d^2}{y^2}+1}}\, v^d\Big(\frac{|x|_d}{y}\Big) + \Big(\frac{|x|_d^2}{y^2}+1\Big)\frac{1}{\sqrt{\frac{y^2}{|x|_d^2}+1}}\, v^{d-1}\Big(\frac{|x|_d}{y}\Big)\, v'\Big(\frac{|x|_d}{y}\Big) = 1,$$

which transforms under the substitution $v^d = w$, $\frac{|x|_d}{y} = \zeta$, equivalently into

$$w' + \frac{d}{\zeta(\zeta^2+1)} w = \frac{d}{\zeta\sqrt{\zeta^2+1}}.$$

The general solution of this equation is

$$\zeta \longmapsto d\,\frac{(\zeta^2+1)^{d/2}}{\zeta^d}\left(\int_0^\zeta \frac{\alpha^{d-1}}{(\alpha^2+1)^{(d+1)/2}}\,\mathrm{d}\alpha + c\right) = d\,\frac{(\zeta^2+1)^{d/2}}{\zeta^d}\left(\int_0^{\arctan\zeta} \sin^{d-1}\alpha\,\mathrm{d}\alpha + c\right),$$

where c is an arbitrary real constant. As one has to avoid a singularity in $\zeta = 0$, one chooses $c = 0$. Thus, one obtains for g^∇:

$$g^\nabla(x,y) = \begin{cases} \sqrt{\frac{y^2}{|x|_d^2}+1}\left(d\int_0^{\arctan(|x|_d/y)} \sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{1/d} & \text{if } x \neq 0,\\ 1 & \text{if } x = 0.\end{cases} \tag{8}$$

Please note that

$$\lim_{x\to 0} \sqrt{\frac{y^2}{|x|_d^2}+1}\left(d\int_0^{\arctan(|x|_d/y)} \sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{1/d} = 1\,.$$

It should be noted that both h^∇ and g^∇ are rational transformations.

2. Next, we construct corresponding functions on $C_\gamma^{\bowtie}$, see (4). Because spheres have to pass into cylinder surfaces we define $g^{\bowtie}(x,y) \overset{\text{def}}{=} \sqrt{1+y^2/|x|_d^2}$. Now we are looking for a function $h^{\bowtie}(x,y) = u\big(|x|_d, y\big)$ such that the Jacobian satisfies

$$\left|\frac{\partial(g^{\bowtie}(x,y)\,x, h^{\bowtie}(x,y))}{\partial(x,y)}\right| = 1 \quad \text{for all } x\in\mathbb{R}^d,\ x\neq 0,\ y\geq 0. \tag{9}$$

It turns out that this condition on the Jacobian together with the requirement that u should vanish on the set $\big\{(x,y)\in\mathbb{R}^{d+1} : y = 0\big\}$ determines u uniquely. Using the substitution $|x|_d = \theta$ the Jacobian can be evaluated as follows

$$\begin{vmatrix} g^{\bowtie} - \frac{x_1^2 y^2}{g^{\bowtie}\,|x|_d^4} & -\frac{x_1x_2y^2}{g^{\bowtie}\,|x|_d^4} & \cdots & -\frac{x_1x_dy^2}{g^{\bowtie}\,|x|_d^4} & \frac{x_1y}{g^{\bowtie}\,|x|_d^2}\\ -\frac{x_1x_2y^2}{g^{\bowtie}\,|x|_d^4} & g^{\bowtie} - \frac{x_2^2y^2}{g^{\bowtie}\,|x|_d^4} & \cdots & -\frac{x_2x_dy^2}{g^{\bowtie}\,|x|_d^4} & \frac{x_2y}{g^{\bowtie}\,|x|_d^2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ -\frac{x_1x_dy^2}{g^{\bowtie}\,|x|_d^4} & -\frac{x_2x_dy^2}{g^{\bowtie}\,|x|_d^4} & \cdots & g^{\bowtie} - \frac{x_d^2y^2}{g^{\bowtie}\,|x|_d^4} & \frac{x_dy}{g^{\bowtie}\,|x|_d^2}\\ \frac{x_1}{|x|_d}\frac{\partial u}{\partial\theta} & \frac{x_2}{|x|_d}\frac{\partial u}{\partial\theta} & \cdots & \frac{x_d}{|x|_d}\frac{\partial u}{\partial\theta} & \frac{\partial u}{\partial y}\end{vmatrix}.$$

Adding suitable multiples of the first row to the others we get the determinant

$$\begin{vmatrix} g^{\bowtie} - \tfrac{x_1^2 y^2}{g^{\bowtie}\,|x|_d^4} & -\tfrac{x_1 x_2 y^2}{g^{\bowtie}\,|x|_d^4} & \dots & -\tfrac{x_1 x_d y^2}{g^{\bowtie}\,|x|_d^4} & \tfrac{x_1 y}{g^{\bowtie}\,|x|_d^2} \\ -\tfrac{x_2}{x_1} g^{\bowtie} & g^{\bowtie} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\tfrac{x_d}{x_1} g^{\bowtie} & 0 & \dots & g^{\bowtie} & 0 \\ \tfrac{x_1}{|x|_d} \tfrac{\partial u}{\partial \theta} & \tfrac{x_2}{|x|_d} \tfrac{\partial u}{\partial \theta} & \dots & \tfrac{x_d}{|x|_d} \tfrac{\partial u}{\partial \theta} & \tfrac{\partial u}{\partial y} \end{vmatrix},$$

which can be simplified by adding multiples of the last column to the others:

$$\begin{vmatrix} g^{\bowtie} & 0 & \dots & 0 & \tfrac{x_1 y}{g^{\bowtie}\,|x|_d^2} \\ -\tfrac{x_2}{x_1} g^{\bowtie} & g^{\bowtie} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -\tfrac{x_d}{x_1} g^{\bowtie} & 0 & \dots & g^{\bowtie} & 0 \\ \tfrac{x_1}{|x|_d}\left(\tfrac{\partial u}{\partial \theta} + \tfrac{y}{|x|_d}\tfrac{\partial u}{\partial y}\right) & \tfrac{x_2}{|x|_d}\left(\tfrac{\partial u}{\partial \theta} + \tfrac{y}{|x|_d}\tfrac{\partial u}{\partial y}\right) & \dots & \tfrac{x_d}{|x|_d}\left(\tfrac{\partial u}{\partial \theta} + \tfrac{y}{|x|_d}\tfrac{\partial u}{\partial y}\right) & \tfrac{\partial u}{\partial y} \end{vmatrix}.$$

Finally, adding suitable multiples of all columns to the first one we end up with

$$\begin{vmatrix} g^{\bowtie} & 0 & \dots & 0 & \tfrac{x_1 y}{g^{\bowtie}\,|x|_d^2} \\ 0 & g^{\bowtie} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & g^{\bowtie} & 0 \\ \tfrac{|x|_d}{x_1}\left(\tfrac{\partial u}{\partial \theta} + \tfrac{y}{|x|_d}\tfrac{\partial u}{\partial y}\right) & \tfrac{x_2}{|x|_d}\left(\tfrac{\partial u}{\partial \theta} + \tfrac{y}{|x|_d}\tfrac{\partial u}{\partial y}\right) & \dots & \tfrac{x_d}{|x|_d}\left(\tfrac{\partial u}{\partial \theta} + \tfrac{y}{|x|_d}\tfrac{\partial u}{\partial y}\right) & \tfrac{\partial u}{\partial y} \end{vmatrix}.$$

Developing the last determinant with respect to the first column, one easily obtains from (9) the condition

$$\left(1 + \frac{y^2}{|x|_d^2}\right)^{(d-2)/2} \left(\frac{\partial u}{\partial y} - \frac{y}{|x|_d}\frac{\partial u}{\partial \theta}\right) = 1.$$

This yields the partial differential equation

$$-y\,\frac{\partial u(\theta,y)}{\partial \theta} + \theta\,\frac{\partial u(\theta,y)}{\partial y} = \frac{\theta^{d-1}}{\left(\theta^2 + y^2\right)^{(d-2)/2}}$$

with the boundary condition

$$u(\theta,0) = 0 \quad \text{for } 0 \le \theta < +\infty.$$

By the method of characteristics, see for instance [5], one finds the solution

$$(\theta, y) \longmapsto \sqrt{\theta^2 + y^2} \int_0^{\arctan(y/\theta)} \cos^{d-1} \alpha \, \mathrm{d}\alpha,$$

and ends up with

$$h^{\bowtie}(x, y) = \sqrt{|x|_d^2 + y^2} \int_0^{\arctan(y/|x|_d)} \cos^{d-1} \alpha \, \mathrm{d}\alpha. \tag{10}$$

Again it should be noted that both $h^{\bowtie}$ and $g^{\bowtie}$ are rational transformations.

3. Up to now we have constructed two volume preserving mappings

$$(x_1 g^{\triangledown}, \ldots, x_d \, g^{\triangledown}, h^{\triangledown}) : \mathbb{R}_+^{d+1} \xrightarrow{\text{onto}} \mathbb{R}_+^{d+1},$$
$$(x_1 g^{\bowtie}, \ldots, x_d \, g^{\bowtie}, h^{\bowtie}) : \mathbb{R}_+^{d+1} \xrightarrow{\text{onto}} \mathbb{R}_+^{d+1},$$

which are homogeneous of order 1. These mappings do not depend on γ, see (5). We are now going to modify both mappings such that they coincide on the common boundary of the sets $C_\gamma^{\triangledown}$ and $C_\gamma^{\bowtie}$ for some $\gamma > 0$, see (6). To that end we introduce the functions

$$\tau : \lambda \longmapsto \left(d \int_0^{\arctan(1/\lambda)} \sin^{d-1} \alpha \, \mathrm{d}\alpha \right)^{1/d} \tag{11}$$

on $(0, +\infty)$ and

$$\varrho : \lambda \longmapsto \int_0^{\arctan \lambda} \cos^{d-1} \alpha \, \mathrm{d}\alpha \,, \tag{12}$$

on $[0, +\infty)$ and define mappings (5) by

$$\Lambda_\gamma^{\triangledown}(x, y) \stackrel{\text{def}}{=} \left(x_1 \frac{g^{\triangledown}(x, y)}{\tau(\gamma)}, \ldots, x_d \frac{g^{\triangledown}(x, y)}{\tau(\gamma)}, h^{\triangledown}(x, y) \right) \quad \text{for } (x, y) \in C_\gamma^{\triangledown}, \tag{13}$$

$$\Lambda_\gamma^{\bowtie}(x, y) \stackrel{\text{def}}{=} \left(x_1 \, g^{\bowtie}(x, y), \ldots, x_d \, g^{\bowtie}(x, y), \frac{h^{\bowtie}(x, y)}{\varrho(\gamma)} \right) \quad \text{for } (x, y) \in C_\gamma^{\bowtie}, \tag{14}$$

for all $\gamma > 0$. The Jacobians of these mappings are

$$\left| \frac{\partial \Lambda^{\triangledown}}{\partial(x, y)} \right| = \left(\frac{1}{\tau(\gamma)} \right)^d \quad \text{and} \quad \left| \frac{\partial \Lambda^{\bowtie}}{\partial(x, y)} \right| = \frac{1}{\varrho(\gamma)}, \tag{15}$$

for all $(x, y) \in \mathbb{R}^{d+1}$ with $x \neq 0$, $y > 0$. If

$$d \int_0^{\arctan(1/\gamma)} \sin^{d-1} \alpha \, \mathrm{d}\alpha \, = \int_0^{\arctan \gamma} \cos^{d-1} \alpha \, \mathrm{d}\alpha, \tag{16}$$

then the values of the Jacobians (15) are equal. There is exactly one $\gamma > 0$ which satisfies (16), and in the sequel γ shall be this solution of (16). From the monotonicity properties of τ and ϱ

one deduces that, in accordance with (5), $\Lambda_\gamma^\triangledown$ maps the set $C_\gamma^\triangledown$ onto $C_1^\triangledown$ and that $\Lambda_\gamma^\bowtie$ maps $C_\gamma^\bowtie$ onto $C_1^\bowtie$. Please note that

$$\begin{aligned}\tau'(\lambda) &= -\left(d\int_0^{\arctan(1/\lambda)} \sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{(1-d)/d} \sin^{d-1}\left(\arctan\frac{1}{\lambda}\right)\frac{1}{1+\lambda^2} \\ &= -\frac{1}{\tau(\lambda)^{d-1}(1+\lambda^2)^{(d+1)/2}}\end{aligned} \tag{17}$$

and

$$\varrho'(\lambda) = \cos^{d-1}(\arctan\lambda)\,\frac{1}{1+\lambda^2} = \frac{1}{(1+\lambda^2)^{(d+1)/2}}\,. \tag{18}$$

The inverse mappings to $\Lambda_\gamma^\triangledown$ and $\Lambda_\gamma^\bowtie$ are given by

$$(\Lambda_\gamma^\triangledown)^{-1}(\xi,\eta) = \frac{\left(\xi_1\frac{\eta}{|\xi|_d}\,,\ldots,\xi_d\frac{\eta}{|\xi|_d}\,,\eta\,\tau^{-1}\left(\frac{\tau(\gamma)|\xi|_d}{\eta}\right)\right)}{\sqrt{1+\left(\tau^{-1}\left(\frac{\tau(\gamma)|\xi|_d}{\eta}\right)\right)^2}} \tag{19}$$

in the interior of $C_1^\triangledown$, and

$$(\Lambda_\gamma^\bowtie)^{-1}(\xi,\eta) = \frac{\left(\xi_1,\ldots,\xi_d,|\xi|_d\,\varrho^{-1}\left(\frac{\varrho(\gamma)\eta}{|\xi|_d}\right)\right)}{\sqrt{1+\left(\varrho^{-1}\left(\frac{\varrho(\gamma)\eta}{|\xi|_d}\right)\right)^2}} \tag{20}$$

in the interior of $C_1^\bowtie$ plus continuous extension to $C_1^\triangledown$ and $C_1^\bowtie$, respectively. From the monotonicity properties of τ and ϱ follows that $(\Lambda_\gamma^\triangledown)^{-1}$ maps $C_1^\triangledown$ onto $C_\gamma^\triangledown$ and $(\Lambda_\gamma^\bowtie)^{-1}$ maps $C_1^\bowtie$ onto $C_\gamma^\bowtie$.

With respect to the solution γ of (16) we now define the sought-after mapping

$$\Lambda_{d+1}(x,y) \overset{\text{def}}{=} \begin{cases} \Lambda_\gamma^\triangledown(x,y) & \text{if } (x,y)\in C_\gamma^\triangledown, \\ S\Lambda_\gamma^\triangledown S(x,y) & \text{if } S(x,y)\in C_\gamma^\triangledown, \\ \Lambda_\gamma^\bowtie(x,y) & \text{if } (x,y)\in C_\gamma^\bowtie, \\ S\Lambda_\gamma^\bowtie S(x,y) & \text{if } S(x,y)\in C_\gamma^\bowtie, \end{cases}$$

where $S:\mathbb{R}^{d+1}\longmapsto\mathbb{R}^{d+1}$ is the reflection at the equatorial plane, given by $S(x,y)\overset{\text{def}}{=}(x,-y)$ for $(x,y)\in\mathbb{R}^{d+1}$.

4. Finally, we prove the Lipschitz properties of Λ_{d+1}. First we make sure, that

$$\Lambda_\gamma^\triangledown \in C^{0,1}(C_\gamma^\triangledown) \quad \text{and} \quad \Lambda_\gamma^\bowtie \in C^{0,1}(C_\gamma^\bowtie),$$

see Step 5. Then we can estimate

$$|\Lambda_{d+1}(x,y)-\Lambda_{d+1}(\tilde{x},\tilde{y})|_{d+1} \le \max\left\{\|\Lambda_\gamma^\triangledown\|_{C^{0,1}(C_\gamma^\triangledown)},\|\Lambda_\gamma^\bowtie\|_{C^{0,1}(C_\gamma^\bowtie)}\right\}\|(x,y)-(\tilde{x},\tilde{y})\|_{d+1}$$

for all (x,y), $(\tilde{x},\tilde{y})\in\mathbb{R}^{d+1}$. Please note that the segment connecting (x,y) and $(\tilde{x},\tilde{y})$ can be split up into finitely many parts in such a way, that each part belongs to one of the sets $C_\gamma^\triangledown$, $S[C_\gamma^\triangledown]$, $C_\gamma^\bowtie$, $S[C_\gamma^\bowtie]$.

Analogously one can prove the Lipschitz continuity of the inverse mapping Λ_{d+1}^{-1}, see also Step 6.

5. In the sequel we show that the transforming functions Λ_γ^∇ and $\Lambda_\gamma^\bowtie$ are Lipschitzian.

First, we regard Λ_γ^∇ on C_γ^∇, see (13): The function h^∇ is Lipschitzian due to the triangle inequality. Next we prove that the partial derivatives of

$$(x,y) \longmapsto x_k\, g^\nabla(x,y) = x_k \sqrt{\frac{y^2}{|x|_d^2}+1}\left(d\int_0^{\arctan(|x|_d/y)} \sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{1/d} \tag{21}$$

are bounded. We substitute $\lambda = \frac{y}{|x|_d}$. The cornerstone of the argument is the boundedness of the function

$$\lambda \longmapsto \tau(\lambda)\sqrt{1+\lambda^2} \tag{22}$$

on $(0,+\infty)$ from below and from above by strictly positive constants. Indeed, using the relation $\alpha/2 \le \sin\alpha \le \alpha$, we get

$$\frac{\sqrt{1+\lambda^2}}{2}\arctan\frac{1}{\lambda} \le \sqrt{1+\lambda^2}\left(d\int_0^{\arctan(1/\lambda)}\sin^{d-1}\alpha\,\mathrm{d}\alpha\right)^{1/d} \le \sqrt{1+\lambda^2}\arctan\frac{1}{\lambda}$$

for all $\lambda \in (0,+\infty)$. Hence, it remains to show that the following terms in the partial derivatives of (21) are bounded:

$$\begin{aligned}
|x|_d \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\tau(\lambda)\sqrt{1+\lambda^2}\right)\frac{\partial\lambda}{\partial x_j} &= -\frac{x_j y}{|x|_d^2}\left(\frac{\lambda\,\tau(\lambda)}{\sqrt{1+\lambda^2}} + \tau'(\lambda)\sqrt{1+\lambda^2}\right)\\
&= -\frac{x_j}{|x|_d}\left(\frac{\lambda^2\,\tau(\lambda)\sqrt{1+\lambda^2}}{1+\lambda^2} - \frac{\lambda}{\left(\tau(\lambda)\sqrt{1+\lambda^2}\right)^{d-1}\sqrt{1+\lambda^2}}\right)
\end{aligned}$$

and

$$\begin{aligned}
|x|_d \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\tau(\lambda)\sqrt{1+\lambda^2}\right)\frac{\partial\lambda}{\partial y} &= \frac{\lambda\,\tau(\lambda)}{\sqrt{1+\lambda^2}} + \tau'(\lambda)\sqrt{1+\lambda^2}\\
&= \frac{\lambda\,\tau(\lambda)}{\sqrt{1+\lambda^2}} - \frac{1}{\left(\tau(\lambda)\sqrt{1+\lambda^2}\right)^{d-1}\sqrt{1+\lambda^2}}\,.
\end{aligned}$$

In the calculations we have used (17). Owing to the boundedness of τ and the function (22), the expressions on the right-hand side are uniformly bounded for all $\lambda \in (0,+\infty)$.

Next, we regard $\Lambda_\gamma^\bowtie$ on $C_\gamma^\bowtie$, see (14): The function $h^\bowtie$, see (10), has bounded partial derivatives. Indeed, the second factor in (10) is bounded as well as the partial derivatives of the first factor. Hence, it remains to show that the following terms in the partial derivatives of $h^\bowtie$ are bounded:

$$\sqrt{|x|_d^2+y^2}\,\varrho'(\lambda)\frac{\partial\lambda}{\partial y} = \varrho'(\lambda)\sqrt{1+\lambda^2} = \frac{1}{(1+\lambda^2)^{d/2}}$$

and

$$\sqrt{|x|_d^2+y^2}\,\varrho'(\lambda)\frac{\partial\lambda}{\partial x_j} = -\frac{x_j y}{|x|_d^2}\varrho'(\lambda)\sqrt{1+\lambda^2} = -\frac{x_j y}{|x|_d^2}\frac{1}{(1+\lambda^2)^{d/2}}\,.$$

Here we have used (18). For $y \leq \gamma|x|_d$ these terms are bounded. Finally, we prove that the partial derivatives of the function

$$(x,y) \longmapsto x_k\, g^{\bowtie}(x,y) = \frac{x_k}{|x|_d} \sqrt{|x|_d^2 + y^2}\,, \tag{23}$$

are bounded. Because the first factor in (23) is bounded as well as the partial derivatives of the second factor, it suffices to note that

$$\sqrt{|x|_d^2 + y^2}\, \frac{\partial}{\partial x_j} \frac{x_k}{|x|_d} = \begin{cases} \left(\frac{1}{|x|_d} - \frac{x_j^2}{|x|_d^3}\right) \sqrt{|x|_d^2 + y^2} & \text{if } k = l, \\ -\frac{x_k x_j}{|x|^3} \sqrt{|x|_d^2 + y^2} & \text{if } k \neq l. \end{cases}$$

These terms are uniformly bounded on the set $\left\{(x,y) \in \mathbb{R}^{d+1} : y \leq \gamma|x|_d\right\}$.

6. In the sequel we show that the transforming functions $(\Lambda_\gamma^{\triangledown})^{-1}$and $(\Lambda_\gamma^{\bowtie})^{-1}$ are Lipschitzian.

First, we regard $(\Lambda_\gamma^{\triangledown})^{-1}$ on $C_1^{\triangledown}$, see (19), and define $s = \frac{\tau(\gamma)|\xi|_d}{\eta}$. In order to make sure that the partial derivatives of the function

$$(\xi,\eta) \longmapsto \frac{\eta\, \tau^{-1}\left(\frac{\tau(\gamma)|\xi|_d}{\eta}\right)}{\sqrt{1 + \left(\tau^{-1}\left(\frac{\tau(\gamma)|\xi|_d}{\eta}\right)\right)^2}}$$

are bounded, it suffices to consider the terms

$$\eta\, \frac{\mathrm{d}}{\mathrm{d}s} \frac{\tau^{-1}(s)}{\sqrt{1 + (\tau^{-1}(s))^2}} \frac{\partial s}{\partial \xi_j} = \frac{\tau(\gamma)\xi_j}{|\xi|_d} \frac{(\tau^{-1})'(s)}{\left(1 + (\tau^{-1}(s))^2\right)^{3/2}}$$

and

$$\eta\, \frac{\mathrm{d}}{\mathrm{d}s} \frac{\tau^{-1}(s)}{\sqrt{1 + (\tau^{-1}(s))^2}} \frac{\partial s}{\partial \eta} = -\frac{\tau(\gamma)|\xi|_d}{\eta} \frac{(\tau^{-1})'(s)}{\left(1 + (\tau^{-1}(s))^2\right)^{3/2}}\,.$$

These terms are bounded for $\eta \geq |\xi|_d$, since the function

$$s \longmapsto \frac{(\tau^{-1})'(s)}{1 + (\tau^{-1}(s))^2} = \frac{1}{1 + (\tau^{-1}(s))^2} \frac{1}{\tau'(\tau^{-1}(s))} \tag{24}$$

is bounded on $(0, \tau(\gamma)]$. Indeed, using (17), the right-hand side equals to

$$\frac{1}{1 + (\tau^{-1}(s))^2} \frac{1}{\tau'(\tau^{-1}(s))} = -\left(s\sqrt{1 + (\tau^{-1}(s))^2}\right)^{d-1}$$

which is bounded due to the boundedness of the function (22). Now, we investigate the partial derivatives of the function

$$(\xi,\eta) \longmapsto \frac{\frac{\eta}{|\xi|_d}\, \xi}{\sqrt{1 + \left(\tau^{-1}\left(\frac{\tau(\gamma)|\xi|_d}{\eta}\right)\right)^2}} = \frac{\tau(\gamma)}{s\sqrt{1 + (\tau^{-1}(s))^2}}\, \xi\,.$$

Please note that the fraction in front of ξ is bounded by the positive bounds of the function (22). Hence, it remains to treat the terms

$$|\xi|_d \,\frac{\mathrm{d}}{\mathrm{d}s}\,\frac{\tau(\gamma)}{s\sqrt{1+(\tau^{-1}(s))^2}}\,\frac{\partial s}{\partial \eta} = \frac{1}{\sqrt{1+\tau^{-1}(s)^2}} + \frac{s\,\tau^{-1}(s)(\tau^{-1})'(s)}{\big(1+(\tau^{-1}(s))^2\big)^{3/2}}$$

and

$$|\xi|_d \,\frac{\mathrm{d}}{\mathrm{d}s}\,\frac{\tau(\gamma)}{s\sqrt{1+(\tau^{-1}(s))^2}}\,\frac{\partial s}{\partial \xi_j} = -\frac{\tau(\gamma)\xi_j}{|\xi|_d}\left(\frac{1}{s\sqrt{1+(\tau^{-1}(s))^2}} + \frac{\tau^{-1}(s)(\tau^{-1})'(s)}{\big(1+(\tau^{-1}(s)^2)\big)^{3/2}}\right).$$

Both expressions are bounded for $\eta \geq |\xi|_d$, thanks to the boundedness of the functions (22) and (24).

Finally, we regard $(\Lambda_\gamma^{\bowtie})^{-1}$ on $C_1^{\bowtie}$, see (20), and define $t = \frac{\varrho(\gamma)\eta}{|\xi|_d}$. First, we investigate the partial derivatives of the function

$$(\xi,\eta) \longmapsto \frac{\xi}{\sqrt{1+\left(\varrho^{-1}\left(\frac{\varrho(\gamma)\eta}{|\xi|_d}\right)\right)^2}}\,.$$

The critical terms are

$$|\xi|_d \,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{\sqrt{1+(\varrho^{-1}(t))^2}}\,\frac{\partial t}{\partial \eta} = -\varrho(\gamma)\,\frac{\varrho^{-1}(t)(\varrho^{-1})'(t)}{\big(1+(\varrho^{-1}(t))^2\big)^{3/2}}$$

and

$$|\xi|_d \,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{1}{\sqrt{1+(\varrho^{-1}(t))^2}}\,\frac{\partial t}{\partial \xi_j} = \varrho(\gamma)\,\frac{\xi_j\eta}{|\xi|_d^2}\,\frac{\varrho^{-1}(t)(\varrho^{-1})'(t)}{\big(1+(\varrho^{-1}(t))^2\big)^{3/2}}\,.$$

For $\eta \leq |\xi|_d$ the boundedness of the right-hand side expressions is a consequence of the boundedness of the function

$$t \longmapsto \frac{(\varrho^{-1})'(t)}{1+(\varrho^{-1}(t))^2} = \frac{1}{1+(\varrho^{-1}(t))^2}\,\frac{1}{\varrho'(\varrho^{-1}(t))} \tag{25}$$

on the interval $[0,\varrho(\gamma)]$. Using (18), this follows from

$$\frac{1}{1+(\varrho^{-1}(t))^2}\,\frac{1}{\varrho'(\varrho^{-1}(t))} = \big(1+(\varrho^{-1}(t))^2\big)^{(d-1)/2}$$

and the fact that $\varrho^{-1}(t) \in [0,\gamma]$. Next, we investigate the partial derivatives of the function

$$(\xi,\eta) \longmapsto \frac{|\xi|_d\,\varrho^{-1}\left(\frac{\varrho(\gamma)\eta}{|\xi|_d}\right)}{\sqrt{1+\left(\varrho^{-1}\left(\frac{\varrho(\gamma)\eta}{|\xi|_d}\right)\right)^2}}\,.$$

The critical terms are

$$|\xi|_d \,\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\varrho^{-1}(t)}{\sqrt{1+(\varrho^{-1}(t))^2}}\,\frac{\partial t}{\partial \xi_j} = -\frac{\varrho(\gamma)\xi_j\eta}{|\xi|_d^2}\,\frac{(\varrho^{-1})'(t)}{\big((1+(\varrho^{-1}(t))^2\big)^{3/2}}$$

and
$$|\xi|_d \frac{\mathrm{d}}{\mathrm{d}t} \frac{\varrho^{-1}(t)}{\sqrt{1+(\varrho^{-1}(t))^2}} \frac{\partial t}{\partial \eta} = \frac{\varrho(\gamma)\,(\varrho^{-1})'(t)}{\left((1+(\varrho^{-1}(t))^2\right)^{3/2}}\,.$$
For $\eta \le |\xi|_d$ the boundedness of the right-hand side expressions follows from the boundedness of the function (25).

Remark 6. In the three-dimensional case the solution of (16) and the corresponding values of τ and ϱ, see (11), (12), are
$$\gamma = \frac{2}{\sqrt{5}}\,, \quad \tau(\gamma) = \frac{\sqrt{2}}{\sqrt{3}}\,, \quad \varrho(\gamma) = \frac{2}{3}\,.$$
The mappings g^∇ and $h^\bowtie$ are determined by
$$g^\nabla(x,y) = \sqrt{\frac{2\,|(x,y)|_3}{|(x,y)|_3 + |y|}} \quad \text{and} \quad h^\bowtie(x,y) = y\,.$$
Thus, we get
$$\Lambda_3(x,y) = \begin{cases} (0,0,0) & \text{if } x=0,\ y=0, \\ \left(x_1 \frac{|(x,y)|_3}{|x|_2}\,,\ x_2 \frac{|(x,y)|_3}{|x|_2}\,,\ \frac{3}{2}y\right) & \text{if } \frac{\sqrt{5}}{2}\,|y| \le |x|_2, \\ \left(x_1 \sqrt{\frac{3\,|(x,y)|_3}{|(x,y)|_3+|y|}}\,,\ x_2 \sqrt{\frac{3\,|(x,y)|_3}{|(x,y)|_3+|y|}}\,,\ |(x,y)|_3\right) & \text{if } \frac{\sqrt{5}}{2}\,y \ge |x|_2, \\ \left(x_1 \sqrt{\frac{3\,|(x,y)|_3}{|(x,y)|_3+|y|}}\,,\ x_2 \sqrt{\frac{3\,|(x,y)|_3}{|(x,y)|_3+|y|}}\,,\ -|(x,y)|_3\right) & \text{if } -\frac{\sqrt{5}}{2}\,y \ge |x|_2, \end{cases}$$
see also Figure 2.

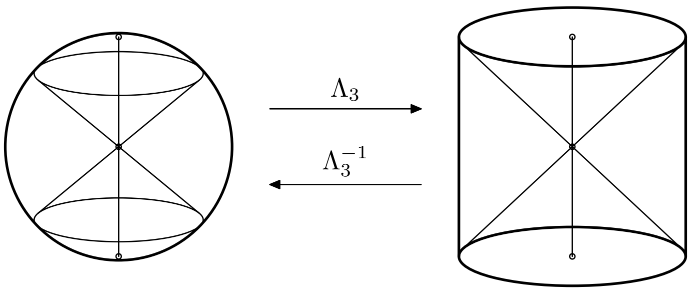

Figure 2: The map Λ_3 from $\mathbb{R}^3$ onto $\mathbb{R}^3$.

The inverse of Λ_3 is given by
$$\Lambda_3^{-1}(\xi,\eta) = \begin{cases} (0,0,0) & \text{if } \xi=0,\ \eta=0, \\ \left(\xi_1\sqrt{1-\frac{4}{9}\frac{\eta^2}{|\xi|_2^2}}\,,\ \xi_2\sqrt{1-\frac{4}{9}\frac{\eta^2}{|\xi|_2^2}}\,,\ \frac{2}{3}\eta\right) & \text{if } |\eta| \le |\xi|_2, \\ \left(\xi_1\sqrt{\frac{2}{3}-\frac{|\xi|_2^2}{9\eta^2}}\,,\ \xi_2\sqrt{\frac{2}{3}-\frac{|\xi|_2^2}{9\eta^2}}\,,\ \eta - \frac{|\xi|_2^2}{3\eta}\right) & \text{if } |\eta| \ge |\xi|_2. \end{cases}$$

Remark 7. In a way our investigation also is a contribution to the general knowledge about the unit cube, see Zong [18].

Problem 8. Are there other geometrical objects than the cylinder and the ball, such that a mapping of this object onto the unit cube exist and has the properties specified in our Theorem? Is there a complete geometrical characterization of these objects?

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