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Convergence of Coercive Approximations for Strictly Monotone Quasistatic Models in the Inelastic Deformation Theory

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Abstract

This article studies coercive approximation procedures in the infinitesimal inelastic deformation theory. For quasistatic, strictly monotone, viscoplastic models using the Young measures approach a convergence theorem in general Orlicz spaces is proved.

1 Introduction and formulation of the problem

In this article we study well-posedness of systems, which model viscoplastic deformation behaviour of solids at small strain in quasistatic setting of the problem. Let us start with the formulation of the initial-boundary value problem, which we are going to investigate. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We have to find the displacement field $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$, the Cauchy stress tensor $T : \Omega \times \mathbb{R}_+ \to \mathcal{S}^3 = \mathbb{R}^{n\times n}_{\rm sym}$ and the inelastic deformation tensor $\varepsilon^p : \Omega \times \mathbb{R}_+ \to \mathbb{R}^N$ satisfying the following system of equations

$$\begin{aligned} \operatorname{div}_x T(x,t) &= -F(x,t)\,, \\ T(x,t) &= \mathcal{D}(\varepsilon(u(x,t)) - \varepsilon^p(x,t))\,, \\ \varepsilon(u(x,t)) &= \frac{1}{2}(\nabla_x u(x,t) + \nabla_x^T u(x,t))\,, \\ \varepsilon^p_t(x,t) &= \mathcal{G}\Big(PT(x,t)\Big)\,, \end{aligned} \tag{MS}$$

where the function $F : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ describes the external forces acting on the material, $\mathcal{D} : \mathcal{S}^3 \to \mathcal{S}^3$ is the elasticity tensor which is assumed to be constant in time and space, symmetric and positive definite. Moreover, $\mathcal{G} : \mathcal{S}^3 \to P\mathcal{S}^3$ is the inelastic constitutive function and the map P is defined by $PT = T - \frac{1}{3}\mathrm{tr}\, T \cdot I$. We investigate here only models of monotone type (for the definition see [1]). Flow rule (MS4) is of monotone type if the function $\mathcal{G}$ is monotone and $\mathcal{G}(0) = 0$. Additionally, we assume that $\mathcal{G}$ is strictly monotone

$$\forall\, \sigma_1, \sigma_2 \in \mathcal{S}^3 \qquad \sigma_1 \neq \sigma_2 \;\Rightarrow\; (\mathcal{G}(\sigma_1) - \mathcal{G}(\sigma_2), \sigma_1 - \sigma_2) > 0$$

and continuous. We call monotone models with a strictly monotone inelastic constitutive function also **strictly monotone**.
We consider system (MS) with the following boundary condition of mixed type: the Dirichlet boundary condition on $\Gamma_1 \subset \partial\Omega$

$$u(x,t) = g_D(x,t) \;\text{ for } x \in \Gamma_1 \;\text{ and } t \geq 0 \tag{1.1}$$

and the Neumann boundary condition on $\Gamma_2 \subset \partial\Omega$

$$T(x,t) \cdot n(x) = g_N(x,t) \;\text{ for } x \in \Gamma_2 \;\text{ and } t \geq 0 \tag{1.2}$$

where $n(x)$ is the exterior unit normal vector to the boundary $\partial\Omega$ at the point x, Γ_1 and Γ_2 are open in $\partial\Omega$, disjoint, "smooth enough" sets satisfying $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $\mathcal{H}_2(\Gamma_1) > 0$, where $\mathcal{H}_2$ denotes the 2-dimensional Hausdorff measure. Moreover, the functions g_D, g_N are given boundary data. Finally, the initial condition for the inelastic strain tensor is in the form

$$\varepsilon^p(x,0) = \varepsilon^{p,0}(x) \tag{1.3}$$

with a given initial data $\varepsilon^{p,0} : \Omega \to P\mathcal{S}^3$.
System (MS) with the assumptions on the inelastic constitutive function written above belongs to the class of monotone models defined in the monograph [1]. The vector of internal variables z contains the inelastic strain tensor only and the free energy function associated with the system is in the form

$$\rho\psi(\varepsilon,\varepsilon^p) = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon^p)\cdot(\varepsilon - \varepsilon^p)\,,$$

where ρ is the mass density which we assume to be constant. We see that the quadratic form ψ is semi-positive definite only and the considered model is not coercive. It was shown in [1] and in [2] that in the inelastic deformation theory coercive models (models with positive defined free energy) are $\mathbb{L}^2$-well-posed. In the noncoercive case in the article [5] an approximation procedure was proposed. The idea of the approximation was very simple. A noncoercive model was approximated by a sequence of coercive models. Therefore this process is called in the literature a coercive approximation. Convergence of this procedure in the dynamical setting of the problem, assuming homogeneous boundary conditions and a polynomial growth condition for the inelastic constitutive function, was studied for system (MS) in [6]. The main mathematical tool used in [6] was the Minty-Browder method. By this monotonicity trick the weak limit of the nonlinear term appearing in the system was characterized. In the quasistatic case, using variational inequalities techniques, system (MS) with homogeneous boundary conditions was investigated in [17].
Assuming that $\mathcal{G}$ is strictly monotone, continuous and satisfies a nondegeneration condition (see Section 4 for the definition of this condition) we prove in this article a convergence result of the coercive approximation process for system (MS) without any growth conditions for the function $\mathcal{G}$ in Orlicz spaces associated with the inelastic constitutive function. To characterize the weak limit of the nonlinear term in (MS) we use the Young measures approach. The main mathematical tool used in our method is a generalisation of the compactness result published in [11].

2 Coercive approximation

In this section we formulate the coercive approximation process for system (MS) and present an existence and uniqueness result for the approximated problem. Let us write system (MS) in the following form

$$\begin{aligned} \mathrm{div}_x \rho\frac{\partial\psi}{\partial\varepsilon}(\varepsilon(x,t),\varepsilon^p(x,t)) &= -F(x,t)\,, \\ \varepsilon(u(x,t)) &= \frac{1}{2}(\nabla_x u(x,t) + \nabla_x^T u(x,t))\,, \end{aligned} \tag{2.4}$$

$$\varepsilon^p_t(x,t) \;=\; \mathcal{G}\Big(- P\rho \frac{\partial \psi}{\partial \varepsilon^p}(\varepsilon(x,t), \varepsilon^p(x,t))\Big)\,,$$

where

$$\rho \frac{\partial \psi}{\partial \varepsilon}(\varepsilon, \varepsilon^p) = \mathcal{D}(\varepsilon - \varepsilon^p) = -\rho \frac{\partial \psi}{\partial \varepsilon^p}(\varepsilon, \varepsilon^p)\,.$$

Hence, we see that our system of equations possesses a symmetry property given by the last two equalities. The coercive approximation procedure destroys this property slightly. Let k be a positive natural number. We define the following approximate free energy function

$$\rho \psi^k(\varepsilon, \varepsilon^p) = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon^p)\cdot(\varepsilon - \varepsilon^p) + \frac{1}{2k}\mathcal{D}\varepsilon\cdot\varepsilon\,.$$

This positive definite quadratic form is associated with the following approximate system

$$\begin{aligned}
\operatorname{div}_x T^k(x,t) &= -F(x,t)\,,\\
T^k(x,t) &= \mathcal{D}(\varepsilon(u^k(x,t)) - \varepsilon^{p,k}(x,t) + \frac{1}{k}\varepsilon(u^k(x,t))\,,\\
\varepsilon(u^k(x,t)) &= \frac{1}{2}(\nabla_x u^k(x,t) + \nabla_x^T u^k(x,t))\,,\\
\varepsilon^{p,k}_t(x,t) &= \mathcal{G}\Big(P\hat{T}^k(x,t)\Big)\,,
\end{aligned} \tag{CA}$$

where $\hat{T}^k = \mathcal{D}(\varepsilon^k - \varepsilon^{k,p}) = T^k - \frac{1}{k}\mathcal{D}\varepsilon(u^k)$. System (CA) will be studied with the boundary conditions

$$u^k(x,t) = g_D(x,t) \quad \text{for} \quad x\in\Gamma_1 \ \text{and}\ t\geq 0 \tag{2.1}$$

$$T^k(x,t)\cdot n(x) = g_N(x,t) \quad \text{for} \quad x\in\Gamma_2 \ \text{and}\ t\geq 0 \tag{2.2}$$

and with the initial condition

$$\varepsilon^{p,k}(x,0) = \varepsilon^{p,0}(x)\,, \tag{2.3}$$

where the given data $g_D, g_N, F, \varepsilon^{p,0}$ are the same as used in system (MS). Let us assume that $\varepsilon^{p,0} \in \mathbb{L}^2(\Omega; P\mathcal{S}^3)$, $F(x,0)\in \mathbb{L}^2(\Omega;\mathbb{R}^3)$, $g_D(x,0)\in \mathbb{H}^{\frac{1}{2}}(\Gamma_1;\mathbb{R}^3)$ and $g_N(x,0)\in\mathbb{H}^{-\frac{1}{2}}(\Gamma_2;\mathbb{R}^3)$, where $\mathbb{H}^s$ denotes the standard Sobolev space constructed over $\mathbb{L}^2$. Moreover, we use the notation $\mathbb{W}^{k,p}$ for the Sobolev spaces over $\mathbb{L}^p$. The initial function $\varepsilon^{p,0}$ generates initial values for the stress and the displacement. Let us denote by $T^{k,0}$ and by $u^{k,0}$ the unique solution of the linear problem

$$\begin{aligned}
\operatorname{div}_x T^{k,0}(x) &= -F(x,0)\\
T^{k,0}(x) &= \mathcal{D}\Big(\varepsilon(u^{k,0}(x)) - \varepsilon^{p,0}(x) + \frac{1}{k}\varepsilon(u^{k,0}(x))\Big)\\
u^{k,0}(x)_{|\Gamma_1} &= g_D(x,0)\,, \quad T^{k,0}(x)\cdot n(x)_{|\Gamma_2} = g_N(x,0)\,.
\end{aligned} \tag{2.4}$$

We see that the initial values $T^{k,0}, u^{k,0}$ are not constant in the approximation procedure. Nevertheless, using the standard elliptic estimates for the differences $u^{k,0} - u^{l,0}$ we conclude that

$$\|u^{k,0} - u^{l,0}\|_{\mathbb{H}^1(\Omega)} \leq C\Big(\frac{1}{k} + \frac{1}{l}\Big)\,. \tag{2.5}$$

Hence, the sequence $\{u^{k,0}\}$ is a Cauchy sequence in the space $\mathbb{H}^1(\Omega;\mathbb{R}^3)$ and converges to some function u^0. Moreover, from the definition of $T^{k,0}$ we conclude that $T^{k,0} \to T^0$ in the space $\mathbb{L}^2(\Omega;\mathcal{S}^3)$. Additionally, (u^0, T^0) is the unique solution of the problem

$$\begin{aligned} \operatorname{div}_x T^0(x) &= -F(x,0) \\ T^0(x) &= \mathcal{D}\Big(\varepsilon(u^0(x)) - \varepsilon^{p,0}(x)\Big) \\ u^0(x)_{|\Gamma_1} &= g_D(x,0)\,, \quad T^0(x)\cdot n(x)_{|\Gamma_2} = g_N(x,0)\,. \end{aligned} \tag{2.6}$$

Next, we present an existence and uniqueness result for system (CA). A proof of this result can be found in [2] or in [8].

Theorem 2.1 (existence for each approximation step) *Let us assume that the given data have the following regularity*

$$F \in \mathbb{W}^{2,\infty}((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3))$$
$$g_D \in \mathbb{W}^{3,\infty}((0,T);\mathbb{H}^{\frac{1}{2}}(\Gamma_1;\mathbb{R}^3))\,, \quad g_N \in \mathbb{W}^{2,\infty}((0,T);\mathbb{H}^{-\frac{1}{2}}(\Gamma_2;\mathbb{R}^3))\,.$$

Moreover, assume that $\varepsilon^{p,0} \in \mathbb{L}^2(\Omega;P\mathcal{S}^3)$ implies that for all k $\mathcal{G}(PT^{k,0}) \in \mathbb{L}^2(\Omega;P\mathcal{S}^3)$ where the initial stress $T^{k,0}$ is defined as the solution of the system (2.4). If the constitutive function $\mathcal{G}$ is monotone, continuous and satisfies $\mathcal{G}(0)=0$ then for each positive number k the problem (CA) with the boundary conditions (2.1), (2.2) and the initial condition (2.3) possesses a global in time, unique solution

$$(u^k, T^k, \varepsilon^{p,k}) \in \mathbb{W}^{1,\infty}((0,T);\mathbb{H}^1(\Omega;\mathbb{R}^3)\times\mathbb{L}^2(\Omega;\mathcal{S}^3\times P\mathcal{S}^3)) \ \textit{for all } T>0\,.$$

In fact from [2] and [8] follows that problem (CA) is $\mathbb{L}^2$-well-posed which means that the solution depends continuously on given data. Next, we have to obtain some estimates for the approximate sequence to conclude a convergence result of this sequence. We will see that the free energy function ψ^k can be controlled in the space $\mathbb{L}^\infty(\mathbb{L}^1)$ by a constant which does not depend on k. Unfortunately, if $k\to\infty$ then the limit free energy ψ is not coercive and we loose a control of strains in $\mathbb{L}^\infty(\mathbb{L}^2)$. This is the main problem appearing in the theory of inelastic deformations.

3 Energy estimates

Next, we are going to obtain a convergence result for the approximation procedure defined in the last section. In the dynamical setting for all monotone and viscoplastic models (noncoercive models for which the inelastic constitutive function do not blow up on finite domains) in the article [7] weak convergence of strains in $\mathbb{L}^1(\Omega\times(0,T))$ was obtained, provided that the given data satisfy the so called save load condition. We are going to follow this idea for system (MS). In this section we prove the main estimates for the approximate sequence.

Definition 3.1 *We say that the given data F, g_D, g_N satisfy the weak save load condition if the unique solution (u^*, T^*) of the linear system*

$$\begin{aligned}\operatorname{div}_x T^*(x,t) &= -F(x,t)\\ T^*(x,t) &= \mathcal{D}\varepsilon(u^*(x,t))\\ u^*(x)_{|\Gamma_1} &= g_D(x,t)\,, \quad T^*(x)\cdot n(x)_{|\Gamma_2} = g_N(x,t)\,.\end{aligned} \tag{3.1}$$

have the regularity:
for all $T>0$ $u^* \in \mathbb{W}^{1,\infty}((0,T);\mathbb{H}^1(\Omega;\mathbb{R}^3))$, $T^* \in \mathbb{W}^{1,\infty}((0,T);\mathbb{L}^2(\Omega;\mathcal{S}^3))$ and

$$\mathcal{G}(PT^*) \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;P\mathcal{S}^3))\,.$$

The first estimate which we are going to prove is the energy estimate for the approximate sequence. Let us define the energy function associated with system (MS) by

$$\mathcal{E}(\varepsilon,\varepsilon^p)(t) = \int_\Omega \rho\psi(\varepsilon(x,t),\varepsilon^p(x,t))\,dx \tag{3.2}$$

and the energy associated with system (CA) by

$$\mathcal{E}^k(\varepsilon,\varepsilon^p)(t) = \int_\Omega \rho\psi^k(\varepsilon(x,t),\varepsilon^p(x,t))\,dx\,. \tag{3.3}$$

Theorem 3.1 (energy estimate) *Assume that the given data $F, g_D, g_N, \varepsilon^{p,0}$ satisfy the requirements from Theorem 2.1 and additionally F, g_D, g_N have the weak save load property. Then there exists a positive constant $C(T)$ not depending of k such that*

$$\begin{aligned}&\mathcal{E}^k(\varepsilon^k-\varepsilon^*,\varepsilon^{p,k})(t) \qquad (3.4)\\ &+\int_0^t\int_\Omega \Big(\mathcal{G}(P\hat{T}^k(x,\tau)) - \mathcal{G}(PT^*(x,\tau), P\hat{T}^k(x,\tau) - PT^*(x,\tau)\Big)\,dx\,d\tau \le C(T)\,,\end{aligned}$$

where $\varepsilon^k = \varepsilon(u^k)$, $\varepsilon^* = \varepsilon(u^*)$ and (u^*, T^*) is the solution of system (3.1).

Proof
Calculating the time derivative of the energy $\mathcal{E}^k(\varepsilon^k-\varepsilon^*,\varepsilon^{p,k})(t)$ we have

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}^k(\varepsilon^k-\varepsilon^*,\varepsilon^{p,k})(t) =\\ &\int_\Omega \mathcal{D}(\varepsilon^k-\varepsilon^*-\varepsilon^{p,k})\cdot(\varepsilon^k_t-\varepsilon^*_t-\varepsilon^{p,k}_t)\,dx + \frac{1}{k}\int_\Omega \mathcal{D}(\varepsilon^k-\varepsilon^*)\cdot(\varepsilon^k_t-\varepsilon^*_t)\,dx =\\ &\int_\Omega \mathcal{D}(\varepsilon^k-\varepsilon^*-\varepsilon^{p,k}+\frac{1}{k}(\varepsilon^k-\varepsilon^*))\cdot(\nabla u^k_t - \nabla u^*_t)\,dx - \int_\Omega (\hat{T}^k - T^*)\cdot\varepsilon^{p,k}_t\,dx =\\ &-\int_\Omega \operatorname{div}(T^k-(1+\frac{1}{k})T^*)(u^k_t-u^*_t)\,dx - \int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k-PT^*)\,dx\\ &+\int_{\partial\Omega}(T^k-(1+\frac{1}{k})T^*)n(u^k_t-u^*_t)\,dS(x)\,.\end{aligned} \tag{3.5}$$

Using the definition of the pair (u^*, T^*) we obtain that

$$\frac{d}{dt}\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t) = -\int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k - PT^*)\,dx$$
$$+\frac{1}{k}\int_\Omega F(u_t^k - u_t^*)\,dx - \frac{1}{k}\int_{\Gamma_2} g_N(u_t^k - u_t^*)\,dS(x) =$$
$$-\int_\Omega(\mathcal{G}(P\hat{T}^k) - \mathcal{G}(PT^*))(P\hat{T}^k - PT^*)\,dx - \int_\Omega \mathcal{G}(PT^*)(P\hat{T}^k - PT^*)\,dx$$
$$+\frac{1}{k}\int_\Omega F(u_t^k - u_t^*)\,dx - \frac{1}{k}\int_{\Gamma_2} g_N(u_t^k - u_t^*)\,dS(x)\,. \tag{3.6}$$

Next we integrate in time over $(0, t)$ and estimate the last three integrals from the right hand side of (3.6).

$$\int_0^t \int_\Omega \mathcal{G}(PT^*)(P\hat{T}^k - PT^*)\,dx\,d\tau$$
$$\leq \frac{1}{2}\int_0^t \|\mathcal{G}(PT^*)\|^2_{\mathbb{L}^2(\Omega)}d\tau + \frac{1}{2}\int_0^t \mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})d\tau \tag{3.7}$$

and the first term on the right hand side is bounded by the weak save load condition.

$$\frac{1}{k}\int_0^t \int_\Omega F(u_t^k - u_t^*)\,dx\,d\tau =$$
$$-\frac{1}{k}\int_0^t \int_\Omega F_t(u^k - u^*)\,dx\,d\tau$$
$$+\frac{1}{k}\int_\Omega F(u^k - u^*)\,dx - \frac{1}{k}\int_\Omega F(0)(u^{k,0} - u^*(0))\,dx\,. \tag{3.8}$$

Using regularity of $F, u^{k,0}$ and u^* we see that the last integral in (3.8) is bounded. Moreover, on Γ_1 we have $u^k - u^* = 0$ hence, by the Korn inequality $\|u^k - u^*\|_{\mathbb{H}^1(\Omega)} \leq C(\Omega)\|\varepsilon^k - \varepsilon^*\|_{\mathbb{L}^2(\Omega)}$. This allows us to write that

$$\frac{1}{k}\int_0^t \int_\Omega F(u_t^k - u_t^*)\,dx\,d\tau \leq \frac{1}{2}\int_0^t \|F_t\|^2_{\mathbb{L}^2}\,d\tau$$
$$+C\int_0^t \mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})d\tau + C(\alpha)\|F\|^2_{\mathbb{L}^2} + \alpha\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k}) + C\,, \tag{3.9}$$

where $\alpha > 0$ is any positive number, $C(\alpha)$ do not depend on k and C is a global positive constant. Similarly we estimate the last integral

$$-\frac{1}{k}\int_0^t \int_{\Gamma_2} g_N(u_t^k - u_t^*)\,dS(x)\,d\tau = \frac{1}{k}\int_0^t \int_{\Gamma_2} g_{N,t}(u^k - u^*)\,dS(x)\,d\tau$$
$$-\frac{1}{k}\int_{\Gamma_2} g_N(u^k - u^*)\,dS(x) + \frac{1}{k}\int_{\Gamma_2} g_N(0)(u^{k,0} - u^*(0))\,dS(x)\,. \tag{3.10}$$

Using the continuity of the trace operator and again the Korn inequality we arrive at the inequality

$$-\frac{1}{k}\int_0^t \int_{\Gamma_2} g_N(u_t^k - u_t^*)\,dS(x)\,d\tau \leq \frac{1}{2}\int_0^t \|g_{N,t}\|_{\mathbb{L}^2(\Gamma_2)}\,d\tau$$
$$+C\int_0^t \mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})d\tau + C(\beta)\|g_N\|_{\mathbb{L}^2(\Gamma_2)} + \beta\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k}) + C\,, \tag{3.11}$$

where $\beta > 0$ is any positive number, $C(\beta)$ do not depend on k and C is a global positive constant. Finally, we choose α and β so small that $\alpha + \beta < 1$, insert (3.7), (3.9) and (3.11) into the time integral of (3.6) and use the Gronwall Lemma. ■

Next step is an estimate for the time derivatives of the approximate sequence. This is the main estimate in the existence theory. In the dynamical setting of the problem this was done for general monotone models in the article [7]. In the quasistatic case for coercive and self-controlling models (for the definition of the class containing self-controlling models we refer to [5]). System (MS) does not have the self-controlling structure and therefore we are going to follow the idea from [7].

Theorem 3.2 (energy estimate for time derivatives) *Assume that the given data $F, g_D, g_N, \varepsilon^{p,0}$ satisfy all assumptions from Theorem 2.1. Additionally suppose that the boundary data g_N and the external force F posses the regularity*

$$\forall\ T > 0 \qquad g_{N,tt}, g_{N,t} \in \mathbb{L}^{\infty}(\Omega \times (0,T)) , \quad F_t, F_{tt} \in \mathbb{L}^{\infty}((0,T); \mathbb{L}^3(\Omega; \mathcal{S}^3)) ,$$

and the sequence $\mathcal{G}(PT^{k,0})$ is bounded in $\mathbb{L}^2(\Omega; P\mathcal{S}^3)$. Then the energy function $\mathcal{E}^k$ for the time derivatives can be estimated as follows: for all $t \in (0,T)$

$$\mathcal{E}^k(\varepsilon^k_t, \varepsilon^{p,k}_t)(t) \leq D(T)(1 + \sup_{t\in(0,T)} \|\varepsilon^{k,p}_t\|_{\mathbb{L}^1(\Omega)}) , \tag{3.12}$$

where the positive constant $D(T)$ does not depend on k.

Proof
Let us denote by $(\varepsilon^k_h, \varepsilon^{p,k}_h)$ the shifted functions $(\varepsilon^k(x, t+h), \varepsilon^{p,k}(x, t+h))$ for $h \in (0,T)$. Calculating the time derivative of the function $\mathcal{E}^k(\varepsilon^k_h - \varepsilon^k, \varepsilon^{p,k}_h - \varepsilon^{p,k})(t)$ in the same manner as in the proof of Theorem 3.1 we arrive at the equality

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}^k(\varepsilon^k_h - \varepsilon^k, \varepsilon^{p,k}_h - \varepsilon^{p,k})(t) = \\ &\quad -\int_\Omega \operatorname{div}(T^k_h - T^k)(v^k_h - v^k)\, dx + \int_{\partial\Omega} (T^k_h - T^k)n(v^k_h - v^k)\, dS(x) + \\ &\quad -\int_\Omega (\mathcal{G}(P\hat{T}^k_h) - \mathcal{G}(\hat{T}^k)(P\hat{T}^k_h - P\hat{T}^k)\, dx \leq \\ &\quad \int_\Omega (F^k_h - F^k)(v^k_h - v^k)\, dx + \int_{\partial\Omega} (T^k_h - T^k)n(v^k_h - v^k)\, dS(x) \end{aligned} \tag{3.13}$$

(the last inequality follows by monotonicity of the function $\mathcal{G}$). Here $v^k = u^k_t$ and T^k_h, v^k_h, F_h are shifted functions T^k, v^k, F respectively. Using the boundary conditions we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}^k(\varepsilon^k_h - \varepsilon^k, \varepsilon^{p,k}_h - \varepsilon^{p,k})(t) &\leq \int_\Omega (F^k_h - F^k)(v^k_h - v^k)\, dx \\ + \int_{\Gamma_1} (T^k_h - T^k)n(g^t_{D,h} - g^t_D)\, dS(x) &+ \int_{\Gamma_2} (g_{N,h} - g_N)(v^k_h - v^k)\, dS(x) , \end{aligned} \tag{3.14}$$

where $g^t_D = \partial_t g_D$ and $g^t_{D,h}, g_{N,h}$ are shifted functions g^t_D, g_N. Next we integrate (3.14) with respect to t, shift all difference operators onto given data, divide by h^2

and pass to the limit $h \to 0^+$. Then we obtain the inequality

$$\begin{aligned}
\mathcal{E}^k(\varepsilon_t^k, \varepsilon_t^{p,k})(t) &\le \mathcal{E}^k(\varepsilon_t^k, \varepsilon_t^{p,k})(0) + \int_0^t \|F_{tt}\|_{\mathbb{L}^3(\Omega)} \|v^k\|_{\mathbb{L}^{\frac{3}{2}}(\Omega)} d\tau \\
&+ C(T)\Big(\sup_{t\in(0,T)} \|g^t_{D,tt}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma_1)} + \sup_{t\in(0,T)} \|g^t_{D,t}\|_{\mathbb{H}^{\frac{1}{2}}(\Gamma_1)}\Big) \sup_{t\in(0,T)} \|T^k\cdot n\|_{\mathbb{H}^{-\frac{1}{2}}(\partial\Omega)} \\
&+ C(T)\Big(\sup_{t\in(0,T)} \|g_{N,tt}\|_{\mathbb{L}^\infty(\Gamma_2)} + \sup_{t\in(0,T)} \|g_{N,t}\|_{\mathbb{L}^\infty(\Gamma_2)}\Big) \sup_{t\in(0,T)} \|v^k\|_{\mathbb{L}^1(\partial\Omega)} \\
&+ \sup_{t\in(0,T)} \|F_t\|_{\mathbb{L}^3(\Omega)} \|v^k\|_{\mathbb{L}^{\frac{3}{2}}(\Omega)} \qquad (3.15)
\end{aligned}$$

where the positive constant $C(T)$ do not depend on k. Using the continuous embedding $\mathbb{LD}(\Omega) \subset \mathbb{L}^{\frac{3}{2}}(\Omega)$, where $\mathbb{LD}(\Omega)$ consists of integrable functions u for which the weak derivative $\varepsilon(u)$ is also integrable, we have

$$\|v^k\|_{\mathbb{L}^{\frac{3}{2}}(\Omega)} \le C(\Omega)\Big(\|\varepsilon_t^k\|_{\mathbb{L}^1(\Omega)} + \int_{\Gamma_1} |g_{D,t}|\, dS(x)\Big)\,. \qquad (3.16)$$

By the trace theorem in the space $\mathbb{LD}(\Omega)$ we can estimate the boundary norm of v^k

$$\|v^k\|_{\mathbb{L}^1(\partial\Omega)} \le C(\Omega)\Big(\|\varepsilon_t^k\|_{\mathbb{L}^1(\Omega)} + \int_{\Gamma_1} |g_{D,t}|\, dS(x)\Big)\,. \qquad (3.17)$$

Finally, by the trace theorem in the space $\mathbb{L}^2(\mathrm{div}\,)$ we obtain that

$$\|T^k\cdot n\|_{\mathbb{H}^{-\frac{1}{2}}(\partial\Omega)} \le C(\Omega)\Big(\|T^k\|_{\mathbb{L}^2(\Omega)} + \|F\|_{\mathbb{L}^2(\Omega)}\Big)\,. \qquad (3.18)$$

Inserting (3.16), (3.17) and (3.18) into (3.15), observing that the sequence $\mathcal{E}^k(\varepsilon_t^k, \varepsilon_t^{p,k})(0)$ is bounded and using the following inequality $\|\varepsilon_t^k\|_{\mathbb{L}^1(\Omega)} \le C(\|\varepsilon_t^{p,k}\|_{\mathbb{L}^1(\Omega)} + \|T_t^k\|_{\mathbb{L}^2(\Omega)})$ we complete the proof. ∎

To close the energy estimate for the time derivatives we have to prove the boundedness of the strains in the space $\mathbb{L}^\infty((0,T);\mathbb{L}^1(\Omega;\mathcal{S}^3))$. To do this we use an idea from the article [9]. First we define a stronger save load condition.

Definition 3.2 *We say that the given data F, g_D, g_N satisfy the save load condition if the unique solution (u^*, T^*) of the linear system (3.1) have the regularity required in Definition 3.1 and additionally there exists $\delta > 0$ such that for $\sigma \in P\mathcal{S}^3$*

$$\sup_{|\sigma|\le\delta} |\mathcal{G}(PT^* + \sigma)| \in \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathbb{R}_+))\,.$$

Theorem 3.3 *Let us assume that all requirements from Theorem 3.2 hold and the given data satisfy the save load condition. Then the sequences $\{\varepsilon_t^{p,k}\}, \{\varepsilon_t^k\}$ are bounded in the space $\mathbb{L}^\infty((0,T);\mathbb{L}^1(\Omega;\mathcal{S}^3))$.*

Proof
Let us fix $\delta > 0$ from the save load condition and fix $\sigma \in P\mathcal{S}^3$ with $|\sigma| \le \delta$. By the monotonicity of the inelastic constitutive function we have

$$(\mathcal{G}(P\hat{T}^k) - \mathcal{G}(PT^* + \sigma), P\hat{T}^k - PT^* - \sigma) \ge 0\,. \qquad (3.19)$$

We rewrite (3.20) in the form

$$\varepsilon_t^{p,k} \cdot \sigma \le (\mathcal{G}(P\hat{T}^k), P\hat{T}^k - PT^*) - (\mathcal{G}(PT^* + \sigma), P\hat{T}^k - PT^* - \sigma)\,. \tag{3.20}$$

Next we take the supremum with respect to $|\sigma| \le \delta$ and integrate over Ω. Hence, we conclude that

$$\begin{aligned}\int_\Omega |\varepsilon_t^{p,k}|\, dx \le \frac{1}{\delta}\int_\Omega (\mathcal{G}(P\hat{T}^k), P\hat{T}^k - PT^*)\, dx \\ + \frac{1}{\delta}\int_\Omega \sup_{|\sigma|\le\delta} |\mathcal{G}(PT^* + \sigma)|(|P\hat{T}^k| + |PT^*| + \delta)\, dx\end{aligned} \tag{3.21}$$

According to the save load condition and to the energy estimate from Theorem 3.1 we see that the last integral in the right hand side of (3.22) is bounded in time. Hence, to end the proof we have to estimate the previous integral. By equality (3.6) we have

$$\begin{aligned}\int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k - PT^*)\, dx = -\frac{d}{dt}\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t) \\ + \frac{1}{k}\int_\Omega F(u_t^k - u_t^*)\, dx - \frac{1}{k}\int_{\Gamma_2} g_N(u_t^k - u_t^*)\, dS(x)\,.\end{aligned} \tag{3.22}$$

Next, we observe that

$$\Big|\frac{d}{dt}\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t)\Big| \le \alpha\mathcal{E}^k(\varepsilon_t^k, \varepsilon_t^{p,k})(t) + C(\alpha)\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t) + C(T)\,, \tag{3.23}$$

where $\alpha > 0$ is arbitrary and the positive constants $C(\alpha), C(T)$ do not depend on k. Moreover, on the set Γ_1 we have $u_t^k - u_t^* = 0$ which allows us to use the Korn inequality in the form $\|u_t^k - u_t^*\|_{\mathbb{H}^1(\Omega)} \le C(\Omega)\|\varepsilon_t^k - \varepsilon_t^*\|_{\mathbb{L}^2(\Omega)}$. These observations imply that

$$\begin{aligned}\int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k - PT^*)\, dx \\ \le \beta\mathcal{E}^k(\varepsilon_t^k, \varepsilon_t^{p,k})(t) + C(\beta)\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t) + C(T)\,,\end{aligned} \tag{3.24}$$

where $\beta > 0$ is arbitrary and the positive constants $C(\beta), C(T)$ do not depend on k. Choosing β so small that $2\beta D(T) < 1$, where $D(T)$ is the constant from Theorem 3.2 we obtain

$$\int_\Omega |\varepsilon_t^{p,k}|\, dx \le \frac{1}{2}\sup_{t\in(0,T)}\int_\Omega |\varepsilon_t^{p,k}|\, dx + C(T)$$

where $C(T)$ is independent of k. This inequality completes the proof immediately. ■

Remark The boundedness of the energy $\mathcal{E}^k(\varepsilon_t^k, \varepsilon_t^{p,k})(t)$ implies that the functions

$$\int_\Omega (\mathcal{G}(P\hat{T}^k) - \mathcal{G}(PT^*))(P\hat{T}^k - PT^*)\, dx \quad \text{and} \quad \int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k - PT^*)\, dx$$

are also bounded on finite time intervals.

In [7] in dynamical setting of the problem was proved that if the function PT^* is bounded then the sequences of strains are relatively weakly precompact in the space $\mathbb{L}^1(\Omega \times (0,T); \mathcal{S}^3)$. Note that the boundedness of PT^* automatically implies the condition from Definition 3.2. In this article we obtain a similar result in the next section.

4 Convergence in Orlicz spaces

We start this section with some definitions and results concerning vector-valued Orlicz spaces generated by $\mathcal{N}$-functions which are not necessary spherical symmetric. For more information and proofs we refer to [17].

Definition 4.1 *Let $M : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex and differentiable function.*
(a) If M satisfies the condition $\lim_{\lambda\to\infty} \lambda^{-1} M(\lambda p) = \infty$ for all $p \in \mathbb{R}^n \setminus \{0\}$ and additionally for all $p \in \mathbb{R}^n$ $M(p) = M(-p)$ then we say that M is an $\mathcal{N}$-function.
(b) The Legendre transformation of M is called the dual conjugate to M and is denoted by M^*.
(c) If M is an $\mathcal{N}$-function we denote by $\mathcal{L}_M(\Omega;\mathbb{R}^n)$ the set of all functions $p : \Omega \to \mathbb{R}^n$ from $\mathbb{L}^1(\Omega;\mathbb{R}^n)$ such that

$$\int_\Omega M(p(x))\, dx < \infty\,.$$

(d) The Orlicz space $L_M(\Omega;\mathbb{R}^n)$ consists of all functions $p \in \mathbb{L}^1(\Omega;\mathbb{R}^n)$ such that

$$\|p\|_{L_M} = \sup\left\{\Big|\int_\Omega (p(x), q(x))\, dx\Big| : q \in \mathcal{L}_{M^*}(\Omega;\mathbb{R}^n) \text{ and } \int_\Omega M^*(q(x))\, dx \le 1\right\}$$

is finite.
(e) We say that M satisfies the Δ_2-condition if there exist positive constant c, λ such that $M(2p) \le cM(p)$ for all $|p| > \lambda$.

Theorem 4.1 *Let M be an $\mathcal{N}$-function.*
(a) The space $L_M(\Omega;\mathbb{R}^n)$ with the norm $\|p\|_{L_M}$ is a Banach space.
(b) If $p \in \mathcal{L}_M(\Omega;\mathbb{R}^n)$ then $p \in L_M(\Omega;\mathbb{R}^n)$ and

$$\|p\|_{L_M} \le 1 + \int_\Omega M(p(x))\, dx\,.$$

(c) If $p \in L_M(\Omega;\mathbb{R}^n)$ and $\|p\|_{L_M} \le 1$ then $p \in \mathcal{L}_M(\Omega;\mathbb{R}^n)$ and

$$\int_\Omega M(p(x))\, dx \le \|p\|_{L_M}\,.$$

(d) If $p \in L_M(\Omega;\mathbb{R}^n)$ and $q \in L_{M^*}(\Omega;\mathbb{R}^n)$ then the function (p, q) is integrable and the following version of the Hölder inequality holds

$$\int_\Omega |(p(x), q(x))|\, dx \le \|p\|_{L_M} \|q\|_{L_{M^*}}\,.$$

(e) If M satisfies the Δ_2-condition then $L_M(\Omega;\mathbb{R}^n) = \mathcal{L}_M(\Omega;\mathbb{R}^n)$ and the space $\mathbb{L}^\infty(\Omega;\mathbb{R}^n)$ is dense in $L_M(\Omega;\mathbb{R}^n)$.
(f) If M satisfies the Δ_2-condition then the dual space to the Orlicz space $L_M(\Omega;\mathbb{R}^n)$ is the Orlicz space $L_{M^*}(\Omega;\mathbb{R}^n)$.

In this section we want to prove that the weak limit of the coercive approximation obtained in the last section satisfies system (MS). To do this we assume that the inelastic constitutive function $\mathcal{G}$ satisfies the following nondegeneration condition.

Definition 4.2 *We say that the function $\mathcal{G}$ satisfies the nondegeneration condition if there exists an $\mathcal{N}$-function M and positive constant c such that*

$$\forall p \in \mathcal{S}^3 \qquad M(p) + M^*(\mathcal{G}(p)) \le c\,(\mathcal{G}(p), p)$$

and the dual conjugate M^* satisfies the Δ_2-condition.

Note that if $\mathcal{G}$ is equal to the derivative of some $\mathcal{N}$-function then $\mathcal{G}$ satisfies immediately the nondegeneration condition, provided that the dual conjugate satisfies the Δ_2-condition. This is a consequence of the equality $M(p) + M^*(DM(p)) = (DM(p), p)$ which is satisfied by all $\mathcal{N}$-functions. This condition implies that $\mathcal{G}$ cannot behave extremely weird. Compare this condition with similar conditions from [6] and from [17].

Theorem 4.2 *Suppose that the inelastic constitutive function $\mathcal{G}$ satisfies the nondegeneration condition and all assumptions from Theorem 3.3 hold. Additionally, assume that the function PT^* defined by the save load condition possesses the regularity*

$$\forall\ T > 0 \qquad PT^* \in \mathbb{L}^\infty(\Omega \times (0,T); P\mathcal{S}^3)\,.$$

Then the sequences of strains and of time derivatives of strains converges weakly in the space $\mathbb{L}^1(\Omega \times (0,T); \mathcal{S}^3))$ and the weak limit of $\varepsilon_t^{p,k}$ belongs to the space $\mathbb{L}^\infty((0,T); L_{M^*}(\Omega; \mathcal{S}^3))$, where M is the $\mathcal{N}$-function from the nondegeneration condition.

Proof
We want to obtain an estimate for the sequence of the time derivatives of inelastic strains in a space in which bounded sets are weakly precompact in $\mathbb{L}^1(\Omega\times(0,T); \mathcal{S}^3))$. Let us start with the following observation:

$$\int_\Omega \mathcal{G}(P\hat{T}^k)P\hat{T}^k\,dx \le \Big|\int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k - PT^*)\,dx\Big| + \int_\Omega \mathcal{G}(P\hat{T}^k)PT^*\,dx\,. \tag{4.1}$$

By the nondegeneration condition, the remark at the end of Section 3 and the additional regularity of PT^* we obtain

$$\begin{aligned}\int_\Omega M(P\hat{T}^k)\,dx &+ \int_\Omega M^*(\mathcal{G}(P\hat{T}^k))\,dx \\ &\le c\int_\Omega \mathcal{G}(P\hat{T}^k)P\hat{T}^k\,dx \le C(T) + c\|\mathcal{G}(P\hat{T}^k)\|_{\mathbb{L}^1(\Omega)}\|PT^*\|_{\mathbb{L}^\infty(\Omega)}\,.\end{aligned} \tag{4.2}$$

Consequently, the sequence $\{\varepsilon_t^{p,k}\}$ is bounded in $\mathbb{L}^\infty((0,T); L_{M^*}(\Omega; P\mathcal{S}^3))$ and the sequence $\{PT^k\}$ is bounded in $\mathbb{L}^\infty((0,T); L_M(\Omega; P\mathcal{S}^3))$. This yields that the sequence $\{\varepsilon_t^{p,k}\}$ is bounded in $L_{M^*}(\Omega \times (0,T); P\mathcal{S}^3))$ and in this space bounded sets are relatively weakly precompact in $\mathbb{L}^1(\Omega \times (0,T); \mathcal{S}^3))$. Hence, there a subsequence (which will be denoted by $\{\varepsilon_t^{p,k}\}$ again) such that $\varepsilon_t^{p,k} \rightharpoonup \varepsilon_t^p$ in $\mathbb{L}^1(\Omega \times (0,T); \mathcal{S}^3))$ where ε^p is the weak limit of the sequence $\{\varepsilon^{p,k}\}$. Moreover, without loss of generality we can assume that $PT^k \overset{*}{\rightharpoonup} PT$ in $\mathbb{L}^\infty((0,T); L_M(\Omega; P\mathcal{S}^3))$ where T is the weak limit of the sequence $\{T^k\}$ in the space $\mathbb{L}^\infty((0,T); \mathbb{L}^2(\Omega; \mathcal{S}^3))$. Using the convexity of the function M^* we conclude that $\varepsilon^p \in \mathbb{L}^\infty((0,T); L_{M^*}(\Omega; \mathcal{S}^3))$. Finally, by the equality $(1+\frac{1}{k})\varepsilon_t^k = \mathcal{D}^{-1}T_t^k + \varepsilon_t^{p,k}$ we obtain a convergence result for the sequence $\{\varepsilon_t^k\}$. ■

From the last theorem we deduce that the approximate sequence $(u^k, T^k, \varepsilon^{p,k})$ converges weakly to a limit (u, T, ε^p). These functions satisfy the system of equations

$$\begin{aligned}
\operatorname{div}_x T(x,t) &= -F(x,t) \quad \text{in} \quad \mathbb{L}^\infty((0,T);\mathbb{L}^2(\Omega;\mathbb{R}^3))\,,\\
\varepsilon(u(x,t)) &= \frac{1}{2}(\nabla u(x,t) + \nabla^T u(x,t)) \quad \text{in} \quad \mathbb{L}^\infty((0,T);\mathbb{L}^1(\Omega;\mathcal{S}^3))\,,\\
\varepsilon_t^p(x,t) &= w - \lim_{k\to\infty} \varepsilon_t^{p,k}(x,t) = \chi(x,t) \quad \text{in} \quad \mathbb{L}^\infty((0,T);L_{M^*}(\Omega;\mathcal{S}^3))\,,
\end{aligned}$$

the boundary condition (1.1) and (1.2) and the initial condition (1.3). Hence, it remains to prove that

$$\chi(x,t) = \mathcal{G}(PT(x,t)) \quad \text{for a.e. } (x,t) \in \Omega \times (0,T)\,. \tag{4.3}$$

In [7] equality (4.3) was proved using the gradient structure of the inelastic constitutive function. In [5], provided that $\mathcal{G}$ possesses a polynomial growth only, the function χ was characterized by the Minty-Browder method. Here we are going to use the following general theorem using the Young measures approach.

Theorem 4.3 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set of finite measure and let a function $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy the following conditions:*

(i) *$A(\xi)$ is continuous.*

(ii) *For all $\xi_1, \xi_2 \in \mathbb{R}^n$, $\xi_1 \neq \xi_2$*

$$[A(\xi_1) - A(\xi_2)] \cdot [\xi_1 - \xi_2] > 0.$$

(iii) *There exist positive constants c_1, c_2 and an $\mathcal{N}$-function such that for all ξ it holds*

$$A(\xi)\cdot\xi \geq c_1\{M(\xi) + M^*(A(\xi))\}$$

and

$$|A(\xi)| \leq c_2 M^*(A(\xi)),$$

where M^* is the dual conjugate function to M.

Let $z^n : \Omega \to \mathbb{R}^n$ be a sequence of measurable functions such that

(iv) *$\{A(z^n)\cdot z^n\}$ is uniformly bounded in $\mathbb{L}^1(\Omega)$,*

(v) *$z^n \overset{*}{\rightharpoonup} z$ in $L_M(\Omega)$ and $A(z^n) \rightharpoonup \bar{A}$ in $\mathbb{L}^1(\Omega)$,*

(vi)

$$\limsup_{n\to\infty} \int_\Omega A(z^n)\cdot z^n\,dx \leq \int_\Omega \bar{A}\cdot z\,dx.$$

Then

$$z^n \to z \quad \textit{in measure.}$$

We postpone a proof of this theorem (it will be done in the last section) and prove that by this general tool equality (4.3) follows. We set $\mathcal{G} = A$ and $z^n = P\hat{T}^n$ and see that we have only to show that (vi) holds to satisfy the all requirements of this theorem. Moreover, we immediately have that if the sequence $\{P\hat{T}^k\}$ converges in measure then there exists a subsequence (again denoted with the same symbol) that $P\hat{T}^k(x,t) \to PT(x,t)$ for a.e. $(x,t) \in \Omega \times (0,T)$. Hence, the continuity of $\mathcal{G}$ implies (4.3). In the next theorem we prove condition (vi).

Theorem 4.4

$$\limsup_{k\to\infty} \int_0^t \int_\Omega \mathcal{G}(P\hat{T}^k)P\hat{T}^k \, dx\, d\tau \leq \int_0^t \int_\Omega \chi \cdot PT \, dx\, d\tau \,. \tag{4.4}$$

Proof

From Theorem 3.1 we have

$$\begin{aligned}\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t) = \mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(0) - \int_0^t \int_\Omega \mathcal{G}(P\hat{T}^k)(P\hat{T}^k - PT^*) \, dx \\ + \frac{1}{k}\int_0^t \int_\Omega F(u_t^k - u_t^*) \, dx - \int_0^t \frac{1}{k}\int_{\Gamma_2} g_N(u_t^k - u_t^*) \, dS(x) \,. \end{aligned} \tag{4.5}$$

From Theorem 3.2 we conclude that two last integrals on the right hand side of (4.5) converge to zero if k tends to infinity. Moreover, in the same manner as in the proof of Theorem 3.1 we obtain

$$\mathcal{E}(\varepsilon - \varepsilon^*, \varepsilon^p)(t) = \mathcal{E}(\varepsilon - \varepsilon^*, \varepsilon^p)(0) - \int_0^t \int_\Omega \chi \cdot PT \, dx\, d\tau \,. \tag{4.6}$$

A comparison of the initial energies yields

$$\mathcal{E}^k(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(0) = \mathcal{E}(\varepsilon - \varepsilon^*, \varepsilon^p)(0) + \frac{1}{2k}\int_\Omega \mathcal{D}\varepsilon^0 \cdot \varepsilon^0 \, dx \,.$$

Consequently we arrive at the inequality

$$\begin{aligned}\mathcal{E}(\varepsilon^k - \varepsilon^*, \varepsilon^{p,k})(t) + \int_0^t \int_\Omega \mathcal{G}(P\hat{T}^k)P\hat{T}^k \, dx\, d\tau + R_k(t) \\ \leq \mathcal{E}(\varepsilon - \varepsilon^*, \varepsilon^p)(t) + \int_0^t \int_\Omega \chi \cdot PT \, dx\, d\tau \,,\end{aligned}$$

where $R_k(t)$ converges to zero uniformly on bounded time intervals. Finally, the convexity of the energy function completes the proof. ■

5 Young measures tools

For the convenience of the reader we collect below all the necessary tools concerning Young measures used in the proof of Theorem 4.3. For more details and the proofs, we refer to [16, Corollaries 3.2-3.4], and [3, Theorem 2.9], see also [13, 15].

Lemma 5.1 *Suppose that the sequence of maps $z^j : \Omega \to \mathbb{R}^d$ generates the Young measure ν. Let $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function (i.e. measurable in the first argument and continuous in the second). Let also assume that the negative part $F^-(x, z^j(x))$ is weakly relatively compact in $\mathbb{L}^1(\Omega)$. Then*

$$\liminf_{j\to\infty} \int_\Omega F(x, z^j(x))dx \geq \int_\Omega \int_{\mathbb{R}^d} F(x, \lambda)d\nu_x(\lambda)dx.$$

If, in addition, the sequence of functions $x \mapsto |F|(x, z^j(x))$ is weakly relatively compact in $\mathbb{L}^1(\Omega)$ then

$$F(\cdot, z^j(\cdot)) \rightharpoonup \int_{\mathbb{R}^d} F(x, \lambda)d\nu_x(\lambda) \quad \text{in} \quad \mathbb{L}^1(\Omega)$$

Remark The second part of the above theorem can be easily extended to vector valued functions F.

Lemma 5.2 *Suppose that a sequence z^j of measurable functions from Ω to $\mathbb{R}^d$ generates the Young measure $\nu : \Omega \to \mathcal{M}(\mathbb{R}^d)$. Then*

$$z^j \to z \quad \textit{in measure if and only if} \quad \nu_x = \delta_{z(x)} \quad \textit{a.e.}.$$

6 Proof of Theorem 4.3

We apply Lemma 5.1 to the function $A(z^n) \cdot z^n$. The coercivity condition from assumption (iii) of the theorem assures that the negative part of this function is equal to zero; thus it is certainly weakly relatively compact in $\mathbb{L}^1(\Omega)$. This allows to conclude that

$$\liminf_{n\to\infty} \int_\Omega A(z^n) \cdot z^n \, dx \geq \int_\Omega \int_{\mathbb{R}^n} A(\xi) \cdot \xi \, d\nu_x(s, \xi)dx \,, \tag{6.7}$$

where μ_x is the Young measure generated by the sequence $\{z^n\}$. Since the sequence $\{A(z^n)\}$ is uniformly bounded in $L_{M^*}(\Omega)$, it is weakly relatively compact in $\mathbb{L}^1(\Omega)$, which implies $\bar{A} = \int_{\mathbb{R}^n} A(\xi)d\nu_x(\xi)$. Thus, from assumption (vi), the following inequality holds

$$\int_\Omega \int_{\mathbb{R}^n} A(\xi) \, d\nu_x(\xi) \cdot \int_{\mathbb{R}^n} \xi d\nu_x(\xi) \, dx \geq \int_\Omega \int_{\mathbb{R}^n} A(\xi) \cdot \xi \, d\nu_x(\xi) \, dx. \tag{6.8}$$

From the monotonicity of A we have that

$$\int_{\Omega}\int_{\mathbb{R}^n} h(x,\xi)d\nu_x(\xi)dx \geq 0, \tag{6.9}$$

where h is defined by

$$h(x,\xi) := \left[A(\xi) - A(\textstyle\int_{\mathbb{R}^n} \xi d\nu_x(\xi))\right] \cdot \left[\xi - \textstyle\int_{\mathbb{R}^n} \xi d\nu_x(\xi)\right].$$

Simple calculations imply that

$$\int_{\Omega}\int_{\mathbb{R}^n} h(x,\xi)d\nu_x(\xi)dx = \int_{\Omega}\int_{\mathbb{R}^n} A(\xi)\cdot\xi d\nu_x(\xi)dx - \int_{\Omega}\int_{\mathbb{R}^n} A(\xi)d\nu_x(\xi)\cdot\int_{\mathbb{R}^n}\xi d\nu_x(\xi)dx,$$

which, together with (6.8), assures that

$$\int_{\Omega}\int_{\mathbb{R}^n} h(x,\xi)d\nu_x(\xi)dx \leq 0. \tag{6.10}$$

Then, (6.9) and (6.10) imply that $\int_{\mathbb{R}^n} h(x,\xi)d\nu_x(\xi) = 0$ for a.e. $x \in \Omega$. Moreover, since $\nu_x \geq 0$, we have

$$\text{supp}\{\nu_x\} \overset{\text{a.e}}{=} \left\{\int_{\mathbb{R}^n} \xi d\nu_x(\xi)\right\}.$$

Note that the single point in the right-hand side set is located a.e. in the point $z(x)$, where z is the weak limit of the sequence $\{z^n\}$. Finally we can conclude that $\nu_x = \delta_{z(x)}$ a.e.. A direct application of Lemma 5.2 yields that $z^n \to z$ in measure. ■

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