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Uniqueness in determining polygonal sound-hard obstacles with a single incoming wave

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ABSTRACT. We consider the two dimensional inverse scattering problem of determining a sound-hard obstacle by the far field pattern. We establish the uniqueness within the class of polygonal domains by a single incoming plane wave.

§1. Introduction and the main result.

Let $D \subset \mathbb{R}^2$ be a bounded domain such that $\mathbb{R}^2 \setminus \overline{D}$ is connected, and let $k > 0$ be the wave number. We consider scattering by the sound-hard obstacle D:

$$\Delta u + k^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \qquad \partial_\nu u = 0 \quad \text{on } \partial D, \tag{1.1}$$

$$u = u^i + u^s, \quad u^i(x) = \exp(ikx \cdot d), \qquad d \in S^1 \equiv \{x \in \mathbb{R}^2; |x| = 1\}, \tag{1.2}$$

and

$$\lim_{|x|\to\infty} \sqrt{|x|}(\partial_{|x|} u^s(x) - iku^s(x)) = 0. \tag{1.3}$$

Here we set $i = \sqrt{-1}$, and $d \in S^1$ is the direction of the incoming plane wave $\exp(ikx \cdot d)$. Throughout this paper, we exclusively assume that an obstacle D under consideration is a polygonal domain, that is, the boundary ∂D is composed of finitely many open segments and points (i.e., vertices).

Let $k > 0$ and $d \in S^1$ be arbitrarily fixed. There exists a unique solution $u(x) = u(D)(x) \in H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ to (1.1) - (1.3) (e.g., Chapter 9 in McLean [17]), and $u(D)$ is smooth on any compact set in $\mathbb{R}^2 \setminus \overline{D}$. Moreover, its far field pattern $u_\infty(D)$ is defined by

$$u^s(D)(x) = |x|^{-1/2} \exp(ik|x|)\{u_\infty(D)(x/|x|) + O(|x|^{-1})\} \quad \text{as } |x| \longrightarrow \infty \tag{1.4}$$

(e.g., Colton and Kress [6]). There is a vast literature on acoustic and electromagnetic scattering problems, and we refer the reader to Colton, Coyle and Monk [5], Colton and Kress [6], Kirsch [13], Lax and Phillips [15], Potthast [19], for example. In this paper, we will discuss the uniqueness in

Inverse scattering problem with sound-hard obstacles. Let D_1, D_2 be bounded polygonal domains such that $\mathbb{R}^2 \setminus \overline{D_1}$ and $\mathbb{R}^2 \setminus \overline{D_2}$ are connected. Does

$$u_\infty(D_1)(x) = u_\infty(D_2)(x), \qquad x \in S^1 \tag{1.5}$$

imply $D_1 = D_2$?

Now we state our uniqueness result.

Theorem. *Let $k > 0$ and $d \in S^1$ be arbitrarily fixed. Then (1.5) implies $D_1 = D_2$.*

Cheng and Yamamoto [3] proved the uniqueness by two incoming plane waves under an extra "non-trapping" condition, which could be removed in Elschner and Yamamoto [10]. A similar uniqueness result for the impedance boundary condition was obtained in Cheng and Yamamoto [4]. The above theorem asserts that we need not change incoming directions, so that a single choice of $d \in S^1$ already yields the uniqueness in the inverse Neumann problem. Earlier results in the sound-hard case concern the uniqueness for general C^2-domains and infinitely many incident waves (see Theorem 5.6 in Colton and Kress [6]) and the uniqueness for balls with a single incident direction (Yun [22]).

In the case of sound-soft obstacles where the boundary condition on ∂D is replaced by $u = 0$, Alessandrini and Rondi [1] recently proved that the far field pattern for a single incident direction determines polygonal (and even polyhedral) domains uniquely. Further uniqueness results for the inverse Dirichlet problem in general domains can be found in [6, Theorems 5.1 and 5.2], Colton and Sleeman [7], Kirsch and Kress [14], Liu [16], Sleeman [21]. Moreover, see Chapter 6 in Isakov [12], and Isakov [11], Rondi [20].

The proof of our uniqueness result is carried out in Section 3 and combines arguments in Cheng and Yamamoto [3] with an idea similar to the proof of Lemma 3.7 in Alessandrini and Rondi [1]. Section 2 is devoted to a sequence of preliminary results, which are needed in the proof of the theorem and are partly taken from [3].

§2. Preliminaries.

Henceforth, for two distinct points $P, Q \in \mathbb{R}^2$, let PQ denote the (non-empty) open segment with the boundary points P and Q. Moreover, for a polygonal domain D and a segment $PQ \in \mathbb{R}^2 \setminus \overline{D}$ with $Q \in \partial D$, by $\angle(PQ, \partial D)$ we denote the least angle among the two angles in $\mathbb{R}^2 \setminus \overline{D}$ formed by PQ and ∂D at Q. We note that the polygonal domains under consideration are always the complements of unbounded domains.

Lemma 1. *Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, and let OA be one of its sides such that Ω is located at one side of OA. Let Π be the symmetric transform in $\mathbb{R}^2$ with respect to the extended straight line of OA. Let $v \in H^1(\Omega)$ satisfy $\partial_\nu v = 0$ on OA and $\Delta v + k^2 v = 0$ in Ω. We set*

$$V(x_1, x_2) = \begin{cases} v(x_1, x_2), & (x_1, x_2) \in \Omega, \\ v(\Pi(x_1, x_2)), & (x_1, x_2) \in \Pi(\Omega). \end{cases}$$

Then $V \in H^1(\Omega \cup \Pi(\Omega) \cup OA)$ and $\Delta V + k^2 V = 0$ in $\Omega \cup \Pi(\Omega) \cup OA$. Moreover if $\partial_\nu v = 0$ on any other side BC of $\partial\Omega$, then $\partial_\nu v = 0$ on $\Pi(BC)$.

The proof is directly done by the definition of H^1-solutions and the even extension of v with respect to OA.

Lemma 2. *Let u satisfy (1.1) - (1.3). Then there do not exist two infinite straight half-lines $L_1, L_2 \in \mathbb{R}^2 \setminus \overline{D}$ such that L_1, L_2 are not parallel and $\partial_\nu u = 0$ on $L_1 \cup L_2$.*

Proof of Lemma 2. We set $u^s(x) = u(x) - \exp(ikx \cdot d)$. Then we can prove

$$\lim_{|x| \to \infty} |\nabla u^s(x)| = 0$$

(e.g., Lemma 9 in Cheng and Yamamoto [3]). Now assume contrarily that there exist such non-parallel infinite straight half-lines $L_1, L_2 \in \mathbb{R}^2 \setminus \overline{D}$. Without loss of generality, we can set $L_1 = \{(x_1, \alpha_1 x_1); x_1 > 0\}$ and $L_2 = \{(x_1, \alpha_2 x_1); x_1 > 0\}$ with $\alpha_1 \neq \alpha_2$. Therefore by $\partial_\nu u = 0$ on $L_1 \cup L_2$, we obtain

$$\lim_{|x| \to \infty, x \in L_j} |\partial_\nu \exp(ikx \cdot d)| = 0, \qquad j = 1, 2.$$

That is,

$$\lim_{|x|\to\infty, x\in L_j} \left| ik\left(d\cdot\begin{pmatrix}-\alpha_j\\1\end{pmatrix}\right)\exp(ikx\cdot d)\right| = 0, \quad j=1,2.$$

Hence, since $k \neq 0$, we have

$$d\cdot\begin{pmatrix}-\alpha_j\\1\end{pmatrix} = 0, \quad j=1,2.$$

Since $\alpha_1 \neq \alpha_2$ and $|d|=1$, this is impossible. Thus the proof of Lemma 2 is complete.

Lemma 3. *Let $E \subset \mathbb{R}^2$ be a domain and let $v \in H^1_{loc}(E)$ satisfy $\Delta v + k^2 v = 0$ in E. Let $L_0 \subset L \subset E$ be two segments. Then $\partial_\nu v = 0$ on L_0 implies $\partial_\nu v = 0$ on L.*

This follows easily from the fact that the solution v to the homogeneous Helmholtz equation is real analytic in E (e.g., [6]).

We will further state two lemmas, which are proved similarly to Lemmas 6 and 7 in Cheng and Yamamoto [3]. We omit the proofs.

Lemma 4. *Let $A=(\varepsilon,0)$, $O=(0,0)$, $B=(\varepsilon\cos\theta,\varepsilon\sin\theta)$, $E=\{x\in\mathbb{R}^2; 0<\arg x<\theta,\ |x|<\varepsilon\}$ for $\varepsilon>0$ and $0<\theta<2\pi$. We take $P\in E$ and set $\phi=\angle AOP\in(0,\theta)$. We assume that*

$$\frac{\phi}{\theta} \notin \mathbb{Q}. \tag{2.1}$$

Moreover, let $\widehat{E}\subset\mathbb{R}^2$ be an unbounded domain such that $E\subset\widehat{E}$. If $v\in H^1_{loc}(\widehat{E})$ satisfies

$$\Delta v + k^2 v = 0 \qquad \text{in } \widehat{E} \tag{2.2}$$

$$\partial_\nu v = 0 \quad \text{on } OA\cup OB \tag{2.3}$$

$$\partial_\nu v = 0 \quad \text{on } OP, \tag{2.4}$$

then $v(x)-\exp(ikx\cdot d)$ does not satisfy the Sommerfeld radiation condition (1.3).

Lemma 5. *Let the sector E and the points A, B, O be defined as in Lemma 4, and let $P\in E$ and $\phi=\angle AOP\in(0,\theta)$. Let $v\in H^1(E)$ satisfy (2.2) - (2.4) and let us assume that*

$$\frac{\phi}{\theta}=\frac{n}{m}\in\mathbb{Q},$$

where $m,n\in\mathbb{N}$, $1\le n\le m-1$, and the greatest common divisor of m and n is one. Then:
(i) There exist $m-1$ points $P^j\in E$, $1\le j\le m-1$, such that $\angle AOP^j=\frac{j}{m}\theta$ and $\partial_\nu v=0$ on OP^j.
(ii) There exists a point $Q\in E$ such that $\angle AOP=\angle BOQ$ and $\partial_\nu v=0$ on OQ.

By $\lambda_2(\Omega)$ we denote the second smallest eigenvalue of $-\Delta$ in a bounded domain Ω with the homogeneous Neumann boundary condition. We note that the smallest eigenvalue is always 0. Now we derive a lower bound for $\lambda_2(\Omega)$ for a triangular domain Ω. Henceforth $\triangle PQR$ denotes the interior of the triangle with the vertices P,Q,R (which are assumed to be not collinear).

Lemma 6. *Let $diam(\triangle PQR) = \max\{|PQ|, |PR|, |QR|\}$. Then there exists an absolute constant $c_0 > 0$ such that*

$$\lambda_2(\triangle PQR) \geq \frac{c_0}{|diam(\triangle PQR)|^2}$$

for an arbitrary triangle $\triangle PQR$.

The lower estimate is related with the constant in the Poincaré inequality, and there are many papers on this topic. Two relevant papers are Payne and Weinberger [18] and Bebendorf [2], where an explicit expression for the constant c_0 is given for a general convex domain, and a gap in the proof in [18] is fixed in [2]. For completeness, we will give an easy proof for triangles which does not specify the contant $c_0 > 0$, but is sufficient for our purpose.

Proof of Lemma 6. Without loss of generality, let PQ be the longest side, and we choose P as the origin $O = (0,0)$ and take the x_1x_2-coordinates such that $Q = (q, 0)$ with $q > 0$ and $R = (r, h)$ with $h > 0$. Since PQ is the longest side, we have $\mathrm{diam}\,(\triangle PQR) = q$ and $0 \leq r \leq q$. In fact, if $r > q$, then $|PR| = \sqrt{r^2 + h^2} > q$, which is impossible because $\mathrm{diam}\,(\triangle PQR) = q$.

By the maximum-minimum principle (e.g., Courant and Hilbert [8]), we have

$$\lambda_2(\triangle PQR) = \inf\left\{ \frac{\int_{\triangle PQR} \left(\left| \frac{\partial u}{\partial x_1} \right|^2 + \left| \frac{\partial u}{\partial x_2} \right|^2 \right) dx_1 dx_2}{\int_{\triangle PQR} u^2 dx_1 dx_2}; \right.$$
$$\left. u \neq 0, \in H^1(\triangle PQR), \quad \int_{\triangle PQR} u dx_1 dx_2 = 0 \right\}.$$

Introducing the new independent variables $y_1 = x_1/q$ and $y_2 = x_2/h$, we set $v(y_1, y_2) = u(x_1, x_2)$, $Q_1 = (1, 0)$, $R_1 = (\rho, 1)$, $\rho = r/q \in [0, 1]$. Then, by $\frac{q^2}{h^2} \geq 1$ and the maximum-minimum principle, we obtain

$$\lambda_2(\triangle PQR) = \frac{1}{q^2} \inf\left\{ \frac{\int_{\triangle OQ_1R_1} \left(\left| \frac{\partial v}{\partial y_1} \right|^2 + \frac{q^2}{h^2} \left| \frac{\partial v}{\partial y_2} \right|^2 \right) dy_1 dy_2}{\int_{\triangle OQ_1R_1} v^2 dy_1 dy_2}; \right.$$
$$\left. v \neq 0, \in H^1(\triangle OQ_1R_1), \quad \int_{\triangle OQ_1R_1} v dy_1 dy_2 = 0 \right\}$$
$$\geq \frac{1}{q^2} \inf\left\{ \frac{\int_{\triangle OQ_1R_1} \left(\left| \frac{\partial v}{\partial y_1} \right|^2 + \left| \frac{\partial v}{\partial y_2} \right|^2 \right) dy_1 dy_2}{\int_{\triangle OQ_1R_1} v^2 dy_1 dy_2}; \right.$$
$$\left. v \neq 0, \in H^1(\triangle OQ_1R_1), \quad \int_{\triangle OQ_1R_1} v dy_1 dy_2 = 0 \right\}$$
$$= \frac{1}{q^2} \lambda_2(\triangle OQ_1R_1).$$

Since $\triangle OQ_1R_1$ is parametrized by $\rho \in [0,1]$, we denote $\lambda_2(\triangle OQ_1R_1)$ by $\lambda_2(\rho)$. By Courant and Hilbert [8, Chapter VI.2.6], we see that $\lambda_2(\rho)$ is a continuous function in ρ and $\lambda_2(\rho) > 0$ for $\rho \in [0,1]$. Therefore $c_0 \equiv \min_{0\le\rho\le1} \lambda_2(\rho) > 0$, which completes the proof of Lemma 6.

We conclude this section with the following fundamental property of a connected set; see Theorem 3.19.9 in Dieudonné [9, p.70] for the proof.

Lemma 7. *Let E be a metric space, $A \subset E$ a subset, $B \subset E$ a connected set such that $A \cap B \neq \emptyset$ and $(E \setminus A) \cap B \neq \emptyset$. Then $\partial A \cap B \neq \emptyset$.*

§3. Proof of Theorem.

First Step. Assume contrarily that $D_1 \neq D_2$. For simplicity, we set

$$u_j = u(D_j), \qquad j = 1, 2.$$

By the Rellich theorem (e.g., Lemma 2.11 in [6]), we see from $u_\infty(D_1) \equiv u_\infty(D_2)$ that (e.g., Theorem 2.13 in [6])

$$\text{(3.1)} \qquad u_1 = u_2 \qquad \text{in the unbounded connected component of } \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)},$$

which is denoted by Ω. Moreover, we note that if $\partial\Omega \subset \overline{D_1} \cup \overline{D_2}$, then $\overline{D_1} = \overline{D_2} = \mathbb{R}^2 \setminus \Omega$. This follows from the fact that both $\mathbb{R}^2 \setminus \overline{D_1}$ and $\mathbb{R}^2 \setminus \overline{D_2}$ are connected. Indeed, we obviously have $\Omega \subset \mathbb{R}^2 \setminus (\overline{D_1} \cup \overline{D_2}) \subset \mathbb{R}^2 \setminus \overline{D_j}$, $j = 1, 2$, and if there exists $x_j \in \mathbb{R}^2 \setminus \overline{D_j}$ such that $x_j \notin \Omega$, we obtain $\partial\Omega \cap (\mathbb{R}^2 \setminus \overline{D_j}) \neq \emptyset$ by Lemma 7.

Hence, by $D_1 \neq D_2$, there exists an open segment PQ which is on $\partial\Omega \cap (\mathbb{R}^2 \setminus \overline{D_1})$ or on $\partial\Omega \cap (\mathbb{R}^2 \setminus \overline{D_2})$. Without loss of generality, we may assume the former case and so

$$\text{(3.2)} \qquad \text{there is an open segment } PQ \subset \partial\Omega \cap (\mathbb{R}^2 \setminus \overline{D_1}) \text{ with } \partial_\nu u_1 = 0 \text{ on } PQ,$$

in view of (3.1) and $\partial_\nu u_2 = 0$ on ∂D_2. Then, by Lemma 3, we have $\partial_\nu u_1 = 0$ on the maximum extension of PQ, provided that the extension is in $\mathbb{R}^2 \setminus \overline{D_1}$.

Henceforth we set

$$\text{(3.3)} \qquad \begin{cases} \mathcal{G}_1 = \{S;\ S \text{ is a finite open segment extended to maximum length} \\ \text{in } \mathbb{R}^2 \setminus \overline{D_1} \text{ such that } \partial_\nu u_1 = 0 \text{ on } S\}, \\ \mathcal{G}_2 = \{S;\ S \text{ is an infinite open segment in } \mathbb{R}^2 \setminus \overline{D_1} \text{ such that} \\ \partial_\nu u_1 = 0 \text{ on } S\}. \end{cases}$$

We now prove the following crucial

Lemma 8. *The set $\mathcal{G}_1$ is non-empty and consists of finitely many segments.*

Proof of Lemma 8. If the segment PQ from (3.2) cannot be extended to an infinite half-line in $\mathbb{R}^2 \setminus \overline{D_1}$, then Lemma 3 implies that the extension of PQ is in $\mathcal{G}_1$, hence $\mathcal{G}_1 \neq \emptyset$.

If PQ can be extended to an infinite open segment in $\mathbb{R}^2 \setminus \overline{D_1}$, then by $PQ \subset \partial\Omega \cap (\mathbb{R}^2 \setminus \overline{D_1})$, it follows that there exists a vertex R of $\partial\Omega$ such that $R \in \mathbb{R}^2 \setminus \overline{D_1}$. In fact, any side of $\partial\Omega$ is a finite segment, and so the side containing PQ has to be separated from the infinite extended line of PQ at some point R. Then R is a vertex of $\partial\Omega$.

Hence there exists another point R_1 such that the segment $RR_1 \subset \partial\Omega \cap (\mathbb{R}^2 \setminus \overline{D_1})$ is not parallel to PQ, and by (3.1) and $\partial_\nu u_2 = 0$ on ∂D_2, we have $\partial_\nu u_1 = 0$ on RR_1. If RR_1 can be extended to an infinite open segment in $\mathbb{R}^2 \setminus \overline{D_1}$, then Lemma 3 yields two non-parallel infinite half-lines in $\mathbb{R}^2 \setminus \overline{D_1}$ where $\partial_\nu u_1 = 0$. This contradicts Lemma 2. Consequently, RR_1 cannot be extended to an infinite open segment in $\mathbb{R}^2 \setminus \overline{D_1}$, so that $\mathcal{G}_1 \neq \emptyset$.

Next we will prove the finiteness of $\mathcal{G}_1$. The proof is similar to [3]. Assume on the contrary that $\mathcal{G}_1$ contains infinitely many segments. Then we can choose sequences of points $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$ such that

$$P_j \neq P_{j'} \quad \text{if } j \neq j', \quad P_j, Q_j \in \partial D_1,\ P_jQ_j \in \mathbb{R}^2 \setminus \overline{D_1} \tag{3.4}$$

and

$$\partial_\nu u_1 = 0 \quad \text{on } P_jQ_j, \quad j \in \mathbb{N}. \tag{3.5}$$

Here we note that $\{Q_j\}_{j\in\mathbb{N}}$ may not be mutually distinct.

Since the length of the curve ∂D_1 is finite and $P_j \neq P_{j'}$ if $j \neq j'$, we can choose subsequences $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$, which are denoted by the same letters, such that

$$\lim_{j\to\infty} P_j = P_\infty, \quad \lim_{j\to\infty} Q_j = Q_\infty. \tag{3.6}$$

Without loss of generality, by further taking subsequences of $\{P_j\}_{j\in\mathbb{N}}$ and $\{Q_j\}_{j\in\mathbb{N}}$, we may assume that

$$\begin{aligned} &P_j,\ Q_j,\ j \in \mathbb{N}, \text{ are located at one side of } P_\infty,\ Q_\infty \text{ respectively} \\ &\text{and } P_j \text{ are not vertices of } D_1. \end{aligned} \tag{3.7}$$

Then we note that

$$P_jP_{j+1}, \quad Q_jQ_{j+1} \subset \partial D_1 \quad \text{for sufficiently large } j \in \mathbb{N}. \tag{3.8}$$

Moreover, we can verify that

$$\frac{\angle(Q_jP_j, \partial D_1)}{\pi} \neq \frac{1}{2}, \in \mathbb{Q}, \quad j \in \mathbb{N}, \tag{3.9}$$

provided that we extract subsequences if necessary.

In fact, let $\frac{\angle(Q_jP_j,\partial D_1)}{\pi} \notin \mathbb{Q}$ for some $j \in \mathbb{N}$. Then, by Lemma 4, the scattered field $u_1(x) - \exp(ikx \cdot d)$ cannot satisfy (1.3), which is a contradiction. Next let us assume without loss of generality that $\frac{\angle(Q_mP_m,\partial D_1)}{\pi} = \frac{\pi}{2}$ for $m \in \mathbb{N}$. Then, since $\partial_\nu u_1 = 0$ on P_mQ_m for $m \in \mathbb{N}$, and $\lim_{m\to\infty} |P_{m+1}P_m| = 0$, we repeat applications of Lemma 1 with respect to the symmetry axes P_mQ_m, $m \in \mathbb{N}$, so that we can prove the following: There is a family $\{\ell_j\}_{j\in\mathbb{N}}$ of segments with $\partial_\nu u_1 = 0$ on ℓ_j, $\ell_j \parallel P_mQ_m$ for all $j, m \in \mathbb{N}$, and such that $\cup_{j\in\mathbb{N}}\ell_j$ is dense in the set $U \equiv \{P; |PP_\infty| < \delta\} \cap (\mathbb{R}^2 \setminus \overline{D_1})$ with sufficiently small $\delta > 0$. Since the Laplace operator is invariant with respect to a rotation, we may take ℓ_j, $j \in \mathbb{N}$, parallel to the x_2-axis, and may assume that, near P_∞, the boundary

∂D_1 is on the x_1-axis. Then $|\partial_\nu u_1| = \left|\frac{\partial u_1}{\partial x_1}\right| = 0$ on ℓ_j for all $j \in \mathbb{N}$. Hence, since $\frac{\partial u_1}{\partial x_1}$ is continuous in $\mathbb{R}^2 \setminus \overline{D_1}$, we have that $\frac{\partial u_1}{\partial x_1} = 0$ in the open set $U \subset \mathbb{R}^2 \setminus \overline{D_1}$ defined above. Since $\Delta\left(\frac{\partial u_1}{\partial x_1}\right) + k^2\left(\frac{\partial u_1}{\partial x_1}\right) = 0$ in U, by the classical unique continuation, we then see that $u_1(x_1, x_2) = v(x_2)$ for $(x_1, x_2) \in \mathbb{R}^2 \setminus \overline{D_1}$. Moreover, from (1.2) we obtain $\frac{\partial v}{\partial x_2}(0) = 0$. Therefore, by (1.1), $v(x_2) = \alpha \cos kx_2$ for some $\alpha \in \mathbb{C}$. On the other hand, condition (1.4) yields that $\lim_{|x|\to\infty} |u_1(x_1, x_2) - \exp(ikx \cdot d)| = 0$, that is, $\lim_{|x|\to\infty} |\alpha \cos kx_2 - \exp(ikx \cdot d)| = 0$. In particular, we can set $x = \left(x_1, \frac{\pi}{2k}\right)$ and let $x_1 \to \infty$. Then $\lim_{x_1\to\infty} \left|\exp\left(ik\left(x_1 d_1 + \frac{\pi}{2k} d_2\right)\right)\right| = 0$, which is impossible. Thus the proof of (3.9) is complete.

By [3], under condition (3.9), we can construct triangles $\triangle P_j P_{j+1} R_j \subset \mathbb{R}^2 \setminus \overline{D_1}$, $j \in \mathbb{N}$, which satisfy

$$\Delta u_1 + k^2 u_1 = 0 \quad \text{in} \quad \triangle P_j P_{j+1} R_j, \tag{3.10}$$

$$\partial_\nu u_1 = 0 \quad \text{on } \partial(\triangle P_j P_{j+1} R_j) \tag{3.11}$$

and

$$\lim_{j\to\infty} \operatorname{diam}\left(\triangle P_j P_{j+1} R_j\right) = 0. \tag{3.12}$$

For completeness, we will give the construction of the triangles at the end of the proof of Lemma 8.

Then we can yield a contradiction as follows, which completes the proof of Lemma 8. If u_1 identically vanishes in $\triangle P_j P_{j+1} R_j$ for some $j \in \mathbb{N}$, then the classical unique continuation yields that $u_1 = 0$ in $\mathbb{R}^2 \setminus \overline{D_1}$. On the other hand, (1.4) means that $\lim_{|x|\to\infty} |u_1(x_1, x_2) - \exp(ikx \cdot d)| = 0$, which is not compatible with $u_1 \equiv 0$. Therefore u_1 does not vanish identically in $\triangle P_j P_{j+1} R_j$ for any $j \in \mathbb{N}$. Hence $k^2 > 0$ is an eigenvalue of $-\Delta$ in $\triangle P_j P_{j+1} R_j$ with the homogeneous Neumann boundary condition.

By Lemma 6, we have

$$\lambda_2(\triangle P_j P_{j+1} R_j) \geq c_0 |\operatorname{diam}\left(\triangle P_j P_{j+1} R_j\right)|^{-2},$$

where $c_0 > 0$ does not depend on j. In terms of (3.12), we then obtain

$$\lim_{j\to\infty} \lambda_2(\triangle P_j P_{j+1} R_j) = \infty. \tag{3.13}$$

Since $k \neq 0$ and $\lambda_2(\triangle P_j P_{j+1} R_j)$ is the smallest positive eigenvalue of $-\Delta$ with the boundary condition $\partial_\nu u = 0$, we see that $k^2 \geq \lambda_2(\triangle P_j P_{j+1} R_j)$, $j \in \mathbb{N}$, in terms of (3.10) and (3.11). This is impossible by (3.13). To complete the proof of Lemma 8, we now give

Construction of $\triangle P_j P_{j+1} R_j$ satisfying (3.10) - (3.12).
We consider the following two cases separately.
Case a. $P_\infty = Q_\infty$.
Case b. $P_\infty \neq Q_\infty$.

Case a. By extracting a subsequence if necessary, we can assume that $Q_j \neq Q_{j'}$ if $j \neq j'$. Otherwise $Q_j = Q_\infty$ for $j \in \mathbb{N}$, which is impossible because $P_jP_\infty = P_jQ_j \subset \mathbb{R}^2 \setminus \overline{D_1}$. By $Q_j \neq Q_{j'}$ if $j \neq j'$, we may assume that Q_j are not vertices of ∂D_1, by extracting a subsequence if necessary. Hence, by (3.7) and (3.8), we have P_jP_∞, $Q_jQ_\infty \subset \partial D_1$. Hence, since $P_jQ_j \subset \mathbb{R}^2 \setminus \overline{D_1}$ by (3.4), we see that the three points P_j, Q_j, P_∞ are not collinear, that is, they form a triangle. Moreover $\triangle P_jQ_jP_\infty \subset \mathbb{R}^2 \setminus \overline{D_1}$. Therefore, setting $R_j = P_\infty$ for $j \in \mathbb{N}$, we see that $\triangle P_jQ_jP_\infty$ satisfies (3.10), (3.11) and (3.12). In fact, (3.10) and (3.11) are straightforward from (3.4) - (3.6). Finally, since $\lim_{j\to\infty} |P_jP_\infty| = \lim_{j\to\infty} |Q_jP_\infty| = 0$ by (3.6), the lengths of all the sides of $\triangle P_jQ_jP_\infty$ tend to 0 as $j \to \infty$, so that (3.12) follows.

Case b. Let L be the side of D_1 including $P_\infty P_j$, $j \in \mathbb{N}$. With (3.6) and (3.7), by further taking subsequences, we can assume that

$$|P_jP_\infty| \text{ and } |Q_jQ_\infty| \text{ are monotonically decreasing in } j \in \mathbb{N}. \tag{3.14}$$

In terms of (3.6), if we choose the minor angle or the major angle suitably, then

$$\lim_{j\to\infty} \angle(Q_jP_j, L) = \angle(Q_\infty P_\infty, L). \tag{3.15}$$

By (3.9), there exist $m_j, n_j \in \mathbb{N}$ such that the greatest common divisor of m_j and n_j is one, $n_j/m_j \neq 1/2$, $1 \leq n_j \leq m_j - 1$ and

$$\angle(Q_jP_j, L) = \frac{n_j}{m_j}\pi, \quad j \in \mathbb{N}. \tag{3.16}$$

In view of (3.15), the sequence n_j/m_j, $j \in \mathbb{N}$, converges. We have the two cases:
Case b-(i). $\sup_{j\in\mathbb{N}} m_j = \infty$.
Case b-(ii). $\sup_{j\in\mathbb{N}} m_j < \infty$.

Case b-(i). We choose a subsequence if necessary, so that $m_j > 2$ and $m_j \to \infty$ as $j \to \infty$. Since D_1 is a polygon, we can choose a point A such that $\triangle P_\infty AP_1 \subset \mathbb{R}^2 \setminus \overline{D_1}$.

Henceforth $j \in \mathbb{N}$ are arbitrary but sufficiently large. We can apply Lemma 5 twice, choosing $(O, A, B, P) = (P_j, P_1, P_\infty, Q_j)$, $(P_{j+1}, P_1, P_\infty, Q_{j+1})$. Then there exist points $R_j \in \mathbb{R}^2 \setminus \overline{D_1}$ such that $\angle R_jP_{j+1}P_j = \frac{1}{m_{j+1}}\pi$, $\angle R_jP_jP_{j+1} = \frac{1}{m_j}\pi$ and $\partial_\nu u_1 = 0$ on $R_jP_{j+1} \cup R_jP_j$. Since $P_jP_{j+1} \subset P_\infty P_1$ and $\angle R_jP_{j+1}P_j \to 0$, $\angle R_jP_jP_{j+1} \to 0$ as $j \to \infty$, we see that $\triangle P_jP_{j+1}R_j \subset \triangle P_\infty AP_1 \subset \mathbb{R}^2 \setminus \overline{D_1}$ for large $j \in \mathbb{N}$. Therefore (3.10) and (3.11) follow. Since $\angle R_jP_jP_{j+1} \to 0$ and $\angle R_jP_{j+1}P_j \to 0$ as $j \to \infty$, we see that P_jP_{j+1} is the longest side for large j. Therefore (3.12) also follows.

Case b - (ii). If necessary, we can again choose subsequences, so that we can assume that for some $m, n \in \mathbb{N}$,

$$\angle(Q_jP_j, L) = \frac{n}{m}\pi, \quad j \in \mathbb{N}, \qquad \frac{n}{m} \neq \frac{1}{2} \tag{3.17}$$

in terms of (3.9) and (3.15).

In this case, $P_jQ_jQ_{j+1}P_{j+1}$ forms a quadrilateral, because $P_jQ_j \parallel P_{j+1}Q_{j+1}$. Henceforth $P_jQ_jQ_{j+1}P_{j+1}$ means the interior of the quadrilateral. Then we can prove that, for all j sufficiently large,

$$P_jQ_jQ_{j+1}P_{j+1} \subset \mathbb{R}^2 \setminus \overline{D_1}. \tag{3.18}$$

In fact, we may assume that P_j and Q_j are on one side of the polygonal boundary ∂D_1 respectively. Then the trapezoidal domain $T_j = P_j Q_j Q_\infty P_\infty$ lies entirely in $\mathbb{R}^2 \setminus \overline{D_1}$ if j is large enough. This follows from the fact that T_j cannot contain an open segment of ∂D_1 with one end point on the closed segment $\overline{P_\infty Q_\infty}$. Otherwise $P_\infty Q_\infty$ cannot be approached by the segments $P_m Q_m \subset \mathbb{R}^2 \setminus \overline{D_1}$ as $m \to \infty$. Thus (3.18) follows.

Let L_j be the infinite half-line starting at P_j such that L_j is not parallel to $P_j Q_j$ and the angle between L_j and L is $\frac{n}{m}\pi$. Since $\angle(Q_j P_j, \partial D_1) = \frac{n}{m}\pi, \neq \frac{\pi}{2}$ by (3.9), such a straight line L_j exists. Then L_{j+1}, $P_j P_{j+1}$ and the half-line passing Q_j and starting at P_j, or L_j, $P_j P_{j+1}$ and the half-line passing Q_{j+1} and starting at P_{j+1} form a triangle $\triangle P_j P_{j+1} R_j$. By (3.6) and $P_\infty \neq Q_\infty$, we have

$$\inf_{j \in \mathbb{N}} |P_j Q_j| > 0. \tag{3.19}$$

Moreover, we see that $\angle R_j P_{j+1} P_j = \angle R_j P_j P_{j+1} = \frac{n}{m}\pi$, so that $|P_j R_j| = |P_{j+1} R_j|$ and

$$\lim_{j \to \infty} |P_j R_j| = \lim_{j \to \infty} \frac{|P_j P_{j+1}|}{2} \left(\cos \frac{n}{m}\pi \right)^{-1} = 0 \tag{3.20}$$

by $\lim_{j \to \infty} |P_j P_{j+1}| = 0$.

It follows from (3.19) and (3.20) that R_j is on the segment $P_j Q_j$ or $P_{j+1} Q_{j+1}$. Therefore (3.18) implies that $\triangle P_j P_{j+1} R_j \subset \mathbb{R}^2 \setminus \overline{D_1}$, $j \in \mathbb{N}$. Then Lemma 5 yields $\partial_\nu u_1 = 0$ on $P_{j+1} R_j$, and so (3.10) and (3.11) follow. Finally, by (3.6) and (3.20), condition (3.12) is seen. Thus the construction of $\triangle P_j P_{j+1} R_j$ satisfying (3.10) - (3.12) is complete.

Second Step. In this step, we will prove that the set $\mathcal{G}_2$ defined in (3.3) is not empty. More precisely, we will find an infinite straight half-line Σ such that $\Sigma \subset \mathbb{R}^2 \setminus \overline{D_1}$ and $\partial_\nu u_1 = 0$ on Σ. We will use an idea similar to the proof of Lemma 3.7 in Alessandrini and Rondi [1]. By Lemma 8, we can set $\mathcal{G}_1 = \{S_1, ..., S_N\}$, where S_j, $1 \leq j \leq N$, are finite segments. We note that, recalling (3.3),

$$\begin{aligned} &S_j \subset \mathbb{R}^2 \setminus \overline{D_1}, \text{ the both end points are on } \partial D_1 \text{ and} \\ &\partial_\nu u_1 = 0 \quad \text{on } S_j,\ 1 \leq j \leq N. \end{aligned} \tag{3.21}$$

Let Ω_∞ be the unbounded connected component of $(\mathbb{R}^2 \setminus \overline{D_1}) \setminus \cup_{j=1}^N S_j$. Note that the latter set has only one unbounded component since its boundary is a bounded set. In fact, outside a sufficiently large disk, there cannot be a continuous curve connecting points from two different components, which would intersect the boundary of $(\mathbb{R}^2 \setminus \overline{D_1}) \setminus \cup_{j=1}^N S_j$ in view of Lemma 7.

We obviously have

$$\Omega_\infty \cap \bigcup_{j=1}^N S_j = \emptyset. \tag{3.22}$$

Choose a point $P \in \partial \Omega_\infty$ lying on a segment S of $\mathcal{G}_1$. We note that $P \in \mathbb{R}^2 \setminus \overline{D_1}$. Let G^+ be the unbounded connected component of $(\mathbb{R}^2 \setminus \overline{D_1}) \setminus S$, and let G^- be its bounded connected component. Here the bounded component G^- is also uniquely determined.

In fact, the segment S cannot divide the connected open set $\mathbb{R}^2 \setminus \overline{D_1}$ into more than two connected components; compare the first steps in the proof of Jordan's curve theorem in [9, Chap. 9, Appendix 4].

Let Π be the symmetric transform with respect to the extended straight line $\widetilde{S}$ of S, and let us define E^+ as the connneced component of $G^+ \cap \Pi(G^-)$ and E^- as the connected component of $G^- \cap \Pi(G^+)$ whose closures contain P. We set $E = E^+ \cup E^- \cup S$. Then ∂E consists of segments of ∂D_1, $\Pi(\partial D_1)$ and their end points, and since u_1 is symmetric with respect to $\widetilde{S}$, by Lemma 1 we have $\partial_\nu u_1 = 0$ on ∂E. Since G^- is bounded and $E^+ = \Pi(E^-)$, we see that E^+ is also bounded. Therefore, since Ω_∞ is the complement of some closed bounded connected set, it follows that $\mathbb{R}^2 \setminus E^+$ and Ω_∞ contain $\{x; |x| > \rho\}$ for sufficiently large $\rho > 0$, that is, $(\mathbb{R}^2 \setminus E^+) \cap \Omega_\infty \neq \emptyset$.

Moreover, we have $E^+ \cap \Omega_\infty \neq \emptyset$. In fact, for sufficiently small $\varepsilon > 0$, we see that $B_\varepsilon(P) \equiv \{x \in \mathbb{R}^2; |x - P| < \varepsilon\} \cap E^+ \neq \emptyset$ by the definition of E^+, because $P \in S \subset \partial G^-$ and Π is the symmetric transform with respect to $\widetilde{S}$. Furthermore, by $P \in \partial\Omega_\infty$, we have $B_\varepsilon(P) \cap \Omega_\infty \neq \emptyset$.

Consequently, by Lemma 7, we obtain

$$\partial E^+ \cap \Omega_\infty \neq \emptyset. \tag{3.23}$$

Moreover, since ∂E^+ is composed of finitely many segments and points, there exists an open segment $\ell \subset \Omega_\infty \cap \partial E^+$ such that $\partial_\nu u_1 = 0$ on ℓ. Henceforth by a ray we mean an infinite open straight half-line. Using Lemma 3 and (3.22), it is now easy to see that the segment ℓ can be extended to a ray $\Sigma \subset \mathbb{R}^2 \setminus \overline{D_1}$ belonging to the set $\mathcal{G}_2$. In fact, assume contrarily that the extension of ℓ to maximum length in $\mathbb{R}^2 \setminus \overline{D_1}$ belongs to $\mathcal{G}_1$, so that $\ell \subset \cup_{j=1}^N S_j$. Then $\ell \subset \Omega_\infty \cap (\cup_{j=1}^N S_j)$, which contradicts (3.22).

Third Step. In this step, we will find a ray $\Sigma_1 \in \mathcal{G}_2$ which is not parallel to Σ.

Case 1. Let the ray $\Sigma \supset \ell$ lie entirely in Ω_∞. Then, since ∂E^+ is bounded and forms the boundary of a polygonal domain, there exist a point $P_0 \in \Sigma$ and a segment $\ell_0 \subset \Omega_\infty \cap \partial E^+$ starting at P_0, which is not on Σ. Again, by Lemma 3 and (3.22), the extension Σ_1 of ℓ_0 belongs to $\mathcal{G}_2$. Note that Σ_1 is not parallel to Σ.

Case 2. Let $\Sigma \not\subset \Omega_\infty$. Then there exists an intersection point of the ray Σ with $\cup_{j=1}^N S_j$. Since $\mathcal{G}_1$ consists of finitely many segments, the set of the intersection points of Σ and $\cup_{j=1}^N S_j$ is also finite. Hence there is a "last" intersection point P_0, so that the subray $\Sigma_0 \subset \Sigma$ starting at P_0 lies entirely in Ω_∞. In fact, $\Sigma_0 \cap \cup_{j=1}^N S_j = \emptyset$, and so $\Sigma_0 \subset (\mathbb{R}^2 \setminus \overline{D_1}) \setminus \cup_{j=1}^N S_j$. Since Ω_∞ is the unbounded connected component of $(\mathbb{R}^2 \setminus \overline{D_1}) \setminus \cup_{j=1}^N S_j$, we have that $\Sigma_0 \subset \Omega_\infty$. Let $S_0 \in \mathcal{G}_1$ be a segment with $P_0 \in S_0$.

We now repeat the reflection argument in the second step with S_0 in place of S, and obtain the corresponding bounded polygonal domains: E_0^-, $E_0^+ = \Pi_0(E_0^-)$ and $E_0 = E_0^- \cup E_0^+ \cup S_0$, where Π_0 is the symmetric transform with respect to the extended straight line of S_0. Arguing as in the proof of (3.23), with replacing P by P_0 and Ω_∞ by Σ_0, we have that $E_0^+ \cap \Sigma_0 \neq \emptyset$ and $(\mathbb{R}^2 \setminus E_0^+) \cap \Sigma_0 \neq \emptyset$. Since Σ_0 is connected, Lemma 7 yields that $\partial E_0^+ \cap \Sigma_0 \neq \emptyset$.

Since ∂E_0^+ is the boundary of a bounded polygonal domain, there exist a point $Q_0 \in \partial E_0^+ \cap \Sigma_0$ and a segment $\ell_0 \subset \Omega_\infty \cap \partial E_0^+$ which starts at Q_0 and is not on Σ_0. Again by Lemma 3 and (3.22), similarly to the second step, we can conclude that the segment ℓ_0 can be extended to a ray $\Sigma_1 \in \mathcal{G}_2$, which is not parallel to Σ.

Thus, in terms of Lemma 2, the assumption $D_1 \neq D_2$ yields a contradiction. Hence, by the reduction to absurdity, the proof of the theorem is complete.

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