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Exponential bounds for the probability deviations of sums of random fields

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Abstract

Non-asymptotic exponential upper bounds for the deviation probability for a sum of independent random fields are obtained under Bernstein's condition and assumptions formulated in terms of Kolmogorov's metric entropy. These estimations are constructive in the sense that all the constants involved are given explicitly. In the case of moderately large deviations, the upper bounds have optimal log-asymptotices. The exponential estimations are extended to the local and global continuity modulus for sums of independent samples of a random field.

1. Introduction

It is well known that the large deviation probability estimations for sums of random fields play an important role in the theory of empirical functions and measures, see e.g., [1], in the dependence sampling Monte Carlo technique [15],[14], [13], in the statistical error estimations when solving PDEs with random coefficients [2].

In this paper we suggest an exponential upper bound for the deviation probability (1.1) in the metric of the space $C(T)$, T being an arbitrary parametric set. To derive such estimations, it is natural to assume exponential decrease of the tales of the one-point distributions of the random field. The exponential upper bound will be obtained under additional assumptions formulated in terms of Kolmogorov's metric entropy. These estimations are valid for arbitrary values of $x \geq 0$ and n (i.e., they are not asymptotic estimations), and they are constructive in the sense that all the constants involved are given explicitly. We note that the rough asymptotics of the estimations is optimal in the interval of moderate large deviations. The approach used is based on the theory of subgaussian random variables and fields developed in [6], [9].

Let (Ω, A, P) be a probability space, $F(t) = F(t,\omega)$, $t \in T$ a random function with an index set T, such that $f(t) \equiv EF(t) = \int_\Omega F(t,\omega)\, P(d\omega) < \infty$ for each $t \in T$. The aim of this paper is the estimation of the probability of deviations:

$$P\left\{\sup_{t\in T}\left|f(t) - \frac{1}{n}\sum_{i=1}^{n} F_i(t)\right| \geq x\right\} = P\left\{\sup_{t\in T}\left|\frac{1}{n}\sum_{i=1}^{n}\left(F_i(t) - EF(t)\right)\right| \geq x\right\}. \qquad (1.1)$$

Considering the centered random field $\xi(t) = F(t) - EF(t)$, $t \in T$, we can reformulate our problem as the estimation of $P\{\sup_{t\in T}|\frac{1}{n}\sum_{i=1}^{n}\xi_i(t)| \geq x\}$ for independent samples $\xi_1(t), \xi_2(t), \dots \xi_n(t)$ of $\xi(t)$.

Thus let $\xi_1(t), \xi_2(t), ..., \xi_n(t),\ t \in T$ be independent samples of a centered random field $\xi(t),\ t \in T$, with an arbitrary parametric set T. Denote

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(t), \quad p_n(x) \equiv P\Big\{ \sup_{t \in T} |S_n(t)| \geq x \Big\}. \tag{1.2}$$

In this paper we will obtain the following type of exponential inequalities for $p_n(x)$:

$$p_n(x) \leq \exp(-\phi_n(x)), \quad x \geq 0, \tag{1.3}$$

where $\phi_n(x)$ is a convex function of x, parametrically depending on n, such that $\phi_n(x) \to \infty$ as $x \to \infty$. From the convexity of $\phi_n(x)$ it follows that $\phi_n(x) \geq c_0 + c_1 x \quad (c_1 > 0)$, for x large enough. Therefore, for validity of an estimation of type (1.3) the following condition on tails of one point distributions of $\xi(t)$ should be assumed:

$$\sup_{t \in T} P\{|\xi(t)| \geq x\} \leq c_2 \exp(-c_3 x)$$

for some positive constants c_2, c_3 which is, in turn, equivalent to the following *Generalized Kramer's condition*: there exists some positive constant γ such that

$$\sup_{t \in T} E \exp\{\gamma |\xi(t)|\} < \infty. \tag{1.4}$$

The problem of derivation of the exponential estimations of type (1.3) is well studied in the theory of empirical functions (e.g., see [1] and the review therein). In these studies, it is assumed that the samples $\xi(t)$ are uniformly bounded in the following sense: $P\{sup_{t \in T} |\xi(t)| \leq L\} = 1$ is valid for some constant L. In a more general case when the generalized Kramer's condition (1.4) is satisfied, the estimation of type (1.3) can be obtained using the theory of large deviations for sums of independent random elements in Banach spaces [16] and theory of subgaussian types of random fields [9]. Here the Banach space can be chosen as $C(T)$, the space of continuous functions on a metric space (T, ρ) where ρ is an appropriate metric in T which guarantees, with probability 1, the continuity of samples of the random field. A drawback of this approach is that the assumptions about the random field $\xi(t)$ is formulated in terms of of statistical moments of $\|\xi\| = sup_{t \in T} |\xi(t)|$ rather than using the finite-point distributions of the field. As a consequence, the rough asymptotics of the estimations of type (1.3) obtained by this approach are not optimal.

In [3] a different approach was suggested which lead to more exact estimation

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp\left\{ -n\, \phi^* \left(\frac{x(1-p)}{\sigma \sqrt{n}} \right) + \frac{C_2 + \kappa\, |\ln(\sigma p)|}{1-p} \right\}, \quad x \geq 0, \tag{1.5}$$

where $\phi^*(x)$ is the Legendre transformation of the function

$$\phi(\lambda) \equiv \sup_{t \in T} \max_{z = \pm 1} \ln E \exp\{z \lambda \xi(t)\},$$

and it is assumed that Kolmogorov's ϵ entropy $H(\epsilon)$ of the metric space (T, d) with pseudometric

$$d(t,s) \equiv \sup_{\lambda>0} \frac{1}{\lambda} \phi^{(-1)} \left(\sup_{t \in T} \max_{z=\pm 1} \ \ln E \exp\{z\lambda(\xi(t)-\xi(s))\} \right)$$

satisfies the condition $H(\epsilon) \leq C_1 + \kappa|\ln \epsilon|, \ \epsilon > 0$. Here $\phi^{(-1)}$ is the inverse function to ϕ.

A remarkable property of the estimation (1.5) is that in the interval of moderate large deviations (1.7) it has an optimal (i.e., not improvable) rough asymptotics:

$$\lim_{n\to\infty} \frac{\phi_n(x_n)}{|\ln \ p_n(x_n)|} = 1 \tag{1.6}$$

where $\{x_n\}_{n=1}^{\infty}$ is an arbitrary sequence of positive numbers in the interval of moderate large deviations:

$$x_n \to \infty, \text{ and } \frac{x_n}{\sqrt{n}} \to 0 \ \text{ as } n \to \infty, \tag{1.7}$$

and $\phi_n(x)$ is an absolute value of the logarithm of the right-hand side of (1.5). Note that an application of this estimation is difficult since it is hard to evaluate the functions $\phi(\lambda)$ and $H(\epsilon)$.

Summarizing, the present research deals with the construction of proper explicitly defined majorants for the function $\phi(\lambda)$ and metric $d(t,s)$ so that the obtained estimation has an optimal rough asymptotics in the interval of moderate large deviations (1.7). In constructing such majorants, it turns out that the use of Bernstein condition appears to be more convenient than the equivalent generalized Kramer's condition.

2. Bernstein's inequality for sums of random fields

A real-valued random variable ξ is said to satisfy *Bernstein's condition* if there exist positive constants σ and b such that

$$E|\xi|^k \leq \sigma^2 b^{k-2} \frac{k!}{2}, \quad k = 2, 3, \ldots. \tag{2.1}$$

In the following assertion the Bernstein's inequality for probability of deviations of sums of random variables is given (see, e.g., [16], p. 90). Another form of this type of inequality can be found, for example, in [11] (see p. 52).

Bernstein's inequality. Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent samples of a centered random variable ξ (i.e., $E\xi = 0$) satisfying Bernstein's condition (2.1), then

$$P\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \geq x \right\} \leq \exp\left\{ -\frac{x^2}{2\sigma^2} \left(1 + \frac{bx}{\sigma^2\sqrt{n}} \right)^{-1} \right\}, \quad \forall x \geq 0. \tag{2.2}$$

An immediate consequence of (2.2) is the following inequality:

$$P\left\{ \left| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \right| \geq x \right\} \leq 2\exp\left\{ -\frac{x^2}{2\sigma^2} \left(1 + \frac{bx}{\sigma^2\sqrt{n}} \right)^{-1} \right\}, \quad \forall x \geq 0 \tag{2.3}$$

The aim of this section is the generalization of inequality (2.3) for a sum of independent samples of a random field. In order to formulate an estimation for $p_n(x)$ we need some definitions. Let $\xi(t)$, $t \in T$ be a centered random field with a parametric set T. Define a pseudometric $\rho_1(t,s)$ (i.e., $\rho_1(t,s) = 0$ does not necessarily imply $t = s$) on T by

$$\rho_1(t,s) \equiv \|\xi(t) - \xi(s)\|_{(1)},$$

where for a random variable ξ the norm $\|\xi\|_{(1)}$ is defined by

$$\|\xi\|_{(1)} \equiv \sup_{k \geq 2} \left(\frac{2\, E|\xi|^k}{k!} \right)^{1/k}.$$

Let us denote by $H_1(\epsilon)$ Kolmogorov's metric ϵ entropy of (T, ρ_1), i.e. the natural logarithm of N_ϵ, the minimal integer such that T can be covered by N_ϵ balls of radius ϵ. In what follows we will assume that the random field $\xi(t)$ is separable in the metric space (T, ρ_1).

We will assume $\int_0^1 H_1(\epsilon)\, d\epsilon < \infty$, which ensures [9] the sample continuity of the random field $\xi(t)$, $t \in T$, and of the sum $S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \xi_i(t)$. Therefore, the function $p_n(x) = P\{sup_{t \in T} |S_n(t)| \geq x\}$ is well defined.

Note that if the random field $\xi(t)$ is generally non-separable there is an example with $p_n(x_n) = 1$ for $x_n = \sqrt{n}$, see [4], page 14.

Thus we will further assume that the random field $\xi(t)$ is separable in the metric space (T, ρ_1).

The definition of separable random fields can be found in [7], p. 203.
For a fixed positive constant β, let $\psi_\beta(\mu)$ be a function defined on the interval $0 \leq \mu < 1/\beta$ by

$$\psi_\beta(\mu) = \frac{\mu^2}{2(1 - \beta\mu)}.$$

Denote by $\psi^*_\beta(x)$ the Legendre transformation of $\psi_\beta(\mu)$:

$$\psi^*_\beta(x) = \sup_{0 \leq \mu \leq 1/\beta} (x\mu - \psi_\beta(\mu)) = \frac{1}{2\beta^2} \left(\sqrt{1 + 2\beta x} - 1 \right)^2, \quad x \geq 0.$$

We will need the following estimation:

$$\psi^*_\beta(x) > \frac{x^2}{2(1 + \beta\, x)}, \quad x \geq 0. \tag{2.4}$$

which is true since the right-hand side of it equals to $x\mu' - \psi_\beta(\mu')$ for $\mu' = x/(x + \beta x) \in [0, 1/\beta)$.

In what follows we will use the notation

$$[x]_1 = max\{1, x\},\ x \geq 0.$$

The following assertion is the main result of this paper. It is a direct generalization of Bernstein's inequality (2.3) for a sum of independent samples of a random field.

Theorem 1. Let $\xi(t)$ be a centered random field on a parametric set T such that:
(i) $\xi(t)$ is separable on the metric space (T, ρ_1);
(ii) there exist positive constants σ and b such that for each $t \in T$ the random variable $\xi = \xi(t)$ satisfies Bernstein's condition (2.1);
(iii) the metric space (T, ρ_1) is precompact and $\int_0^1 H_1(\epsilon)\, d\epsilon < \infty$;
Then

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp\left\{ -n\, \psi_\beta^* \left(\frac{x(1-p)}{\sigma\sqrt{n}} \right) + \frac{1}{\sigma p} \int\limits_0^{\sigma p} H_1(\epsilon)\, d\epsilon \right\} \tag{2.5}$$

for each $\beta \geq [b/\sigma]_1$ and $x \geq 0$.

The following assertion is an immediate consequence of Theorem 1 and inequality (2.4).

Corollary 1. Let $\xi(t)$ be a random field satisfying all the conditions of Theorem 1. Then for each $\beta \geq [b/\sigma]_1$ and each $x \geq 0$

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp\left\{ -\frac{x^2}{2\sigma^2}(1-p)^2 \left(1 + \beta \frac{x(1-p)}{\sigma\sqrt{n}} \right)^{-1} + \frac{1}{\sigma p} \int\limits_0^{\sigma p} H_1(\epsilon)\, d\epsilon \right\} . \tag{2.6}$$

Remark 1. It is easy to verify (provided $b \geq \sigma$, $\beta = b/\sigma$), that the inequality (2.3) can be derived by (2.6) if we take into account that a random variable can be considered as a random field given in a specific one element parametric set. Therefore $H_1(\epsilon) = 0$ for each $\epsilon > 0$ and letting in (2.6) $p \to 0$ one obtains the inequality (2.3).

Remark 2. Note that in the interval of moderate large deviations (1.7) the estimations (2.5), (2.6) are equivalent since in this interval, the asymptotics in their right-hand sides coincide. For large x (when $\beta x >> \sqrt{n}$) the estimation (2.5) is more exact than the estimation (2.6) since asymptotics of the logarithm of the right-hand sides differ by factor 2, as $x \to \infty$.

The following assertion is an immediate consequence of Theorem 1.

Corollary 2. Let $\xi(t)$ be a random field satisfying all the conditions of Theorem 1. Assume that there exist positive constants C_1 and κ such that

$$H_1(\epsilon) \leq C_1 + \kappa\, |\ln \epsilon| \tag{2.7}$$

for each $\epsilon > 0$. Then for each $\beta \geq [b/\sigma]_1$ and each $x \geq 0$

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp\left\{ -n\, \psi_\beta^* \left(\frac{x(1-p)}{\sigma\sqrt{n}} \right) + C_1 + \kappa\, [|\ln(\sigma p)| + 1] \right\} . \tag{2.8}$$

Proof of Theorem 1

To prove the theorem we need a result due to Ostrovsky (see [10]). Let $\psi : [0,\Lambda) \to R_+ = [0,\infty)$ be a convex and continuous function ($\Lambda \le \infty$), such that

$$0 < \lim_{\mu\to 0} \frac{\psi(\mu)}{\mu^2} < \infty, \quad \lim_{\mu\to\Lambda-0} \frac{\psi(\mu)}{\mu} = \infty.$$

Let $\psi^*(x) \equiv \sup\limits_{\mu\ge 0}(\mu x - \psi(\mu))$ be the Legendre transformation of ψ. Let (T,ρ) be a pre-compact pseudometric space, $H(\epsilon)$ is Kolmogorov's metric ϵ entropy of (T,ρ). Assume that $\int_0^1 H(\epsilon)\,d\epsilon < \infty$.

Ostrovskiy's Theorem (see [10]). Let $\eta(t)$, $t \in T$ be a centered and separable on (T,ρ) random field such that

$$\ln E\exp\{\lambda\eta(t)\} \le \psi(\sigma|\lambda|), \quad \lambda \in R^1,\ t \in T;$$

$$\ln E\exp\{\lambda(\eta(t)-\eta(s))\} \le \psi(\sigma|\lambda|\rho(t,s)), \quad \lambda \in R^1,\ t,s \in T$$

for some $\sigma > 0$. Then

$$P\{\sup_{t\in T}|\eta(t)| \ge \sigma x\} \le 2\exp\left\{-\psi^*(x(1-p)) + \sum_{k=1}^{\infty}(1-p)p^{k-1}H(p^k)\right\} \tag{2.9}$$

for each $x \ge 0$ and $p \in (0,1)$.

Now let us continue the proof of Theorem 1. If $0 \le \lambda \le 1/b$, it follows from (2.1) that

$$E\{\exp(\lambda\xi(t))\} = 1+\sum_{k=2}^{\infty}\frac{\lambda^k E\xi^k(t)}{k!} \le 1+\frac{1}{2}\sum_{i=2}^{\infty}\lambda^k\sigma^2 b^{k-2} = 1+\frac{\sigma^2\lambda^2}{2(1-b\lambda)} \le \exp\left(\frac{\sigma^2\lambda^2}{2(1-b\lambda)}\right).$$

From this and independency of $\xi_1(t),\xi_2(t),\ldots,\xi_n(t)$ it follows that:

$$E\exp\{\lambda S_n(t)\} \le \exp\left\{\frac{\sigma^2\lambda^2}{2(1-b|\lambda|/\sqrt{n})}\right\}, \quad |\lambda| \le \frac{\sqrt{n}}{b}.$$

Taking into account $\beta \ge b/\sigma$ we have

$$E\exp\{\lambda S_n(t)\} \le \exp\left\{\frac{\sigma^2\lambda^2}{2(1-\beta\frac{|\lambda|\sigma}{\sqrt{n}})}\right\} = \exp\{\psi_n(\sigma|\lambda|)\}, \quad \lambda \in R^1, \tag{2.10}$$

where $\psi_n(\mu) \equiv n\psi_\beta(\mu/\sqrt{n})$. Here and below we assume that $\psi_\beta(\mu) = \infty$ if $\mu \ge 1/\beta$. By definition of the norm $\|\cdot\|_{(1)}$ and pseudometric $\rho_1(t,s)$ we have

$$E|\xi(t)-\xi(s)|^k \le \frac{k!\rho_1^k(t,s)}{2}, \quad k = 2,3,\ldots.$$

Therefore

$$E\exp\{\lambda(S_n(t)-S_n(s))\} = \left(E\exp\left\{\frac{\lambda}{\sqrt{n}}(\xi(t)-\xi(s))\right\}\right)^n \leq \left(1+\sum_{k=2}^{\infty}\frac{|\lambda|^k\rho_1^k(t,s)}{2n^{k/2}}\right)^n$$
$$= (1+\psi_1(\mu))^n \leq \exp\{n\psi_1(\mu)\}, \quad \text{where } \mu = \frac{|\lambda|\rho_1(t,s)}{\sqrt{n}},$$

and taking into account that $\beta \geq 1$ (which implies $\psi_1(\mu) \leq \psi_\beta(\mu)$ for each $\mu \geq 0$) we have for $\lambda \in R^1$:

$$E\exp\{\lambda(S_n(t)-S_n(s))\} \leq \exp\left\{n\psi_\beta(\frac{|\lambda|\rho_1(t,s)}{\sqrt{n}})\right\} = \exp\{\psi_n(|\lambda|\rho_1(t,s))\}\ . \qquad (2.11)$$

Thus, if we put $\eta(t) \equiv S_n(t)$, $\psi(\lambda) \equiv \psi_n(\lambda)$ and $\rho(t,s) \equiv \rho_1(t,s)/\sigma$, then it follows from (2.10)- (2.11) that all the assumptions of Theorem 2 are fulfilled. Therefore it follows from (2.9) that

$$p_n(x) \leq 2\cdot \inf_{p\in(0,1)} \exp\left\{-n\,\psi_\beta^*\left(\frac{x(1-p)}{\sigma\sqrt{n}}\right) + \sum_{k=1}^{\infty}(1-p)p^{k-1}H(p^k)\right\}\ .$$

From the fact that the Kolmogorov metric ϵ entropy is a monotonically decreasing function of ϵ it follows that

$$\sum_{k=1}^{\infty}(1-p)p^{k-1}H(p^k) \leq \frac{1}{p}\int_0^p H(\epsilon)\,d\epsilon \leq \frac{1}{\sigma p}\int_0^{\sigma p} H_1(\epsilon)\,d\epsilon$$

which completes the proof.

3. Random fields with parametric set $T \subset R^k$

Let us consider the case when $T \subset R^k$ is a bounded (therefore T is precompact) subset of the k dimensional Euclidean space R^k. Denote by $\|\cdot\|$ the norm $\|t\| = \max_{i=1,k}|t_i|$. Let $F(t), t \in T$ be a random field, which is assumed to be separable on the metric space (T,ρ), where $\rho(t,s) = \|t-s\|$. Then the following assertion holds

Theorem 2. Let $F_1(t)$, $F_2(t), \ldots, F_n(t)$ be independent samples of the random field $F(t)$. Assume that
(i) there exists a positive constant γ, such that

$$A \equiv \sup_{t\in T} E\exp\{\gamma|F(t)|\} < \infty$$

(ii) there exist constants $\gamma_0 > 0$, $\alpha \in (0,1]$ and a positive random variable η_0 satisfying the condition $A_0 \equiv E\exp\{\gamma_0\eta_0\} < \infty$ such that

$$P\left\{|F(t)-F(s)| \leq \eta_0\|t-s\|^\alpha\right\} = 1 \qquad \text{for each } t,s \in T;$$

(iii) there exists positive σ such that

$$\sup_{t\in T} E(F(t)-EF(t))^2 \le \sigma^2;$$

then for each $x \ge 0$

$$P\left\{\sup_{t\in T} |\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(F_i(t)-EF(t))| \ge x\right\} \le \tag{3.1}$$

$$2\inf_{p\in(0,1)} \exp\left\{-\frac{n}{2\beta_0^2}\left(\sqrt{1+\frac{2\beta_0 x(1-p)}{\sigma\sqrt{n}}}-1\right)^2 + k\ln(1+D\,C_3^{1/\alpha}) + \frac{k}{\alpha}(1+|\ln(\sigma p)|)\right\},$$

where

$$\beta_0 = [b/\sigma]_1,\ b = \frac{1}{\gamma}\left[\frac{2(A^2-1)}{\gamma^2\sigma^2}\right]_1,\quad D = \sup_{t,s\in T}\|t-s\|,\ C_3 = \frac{\ln A_0}{\gamma_0} + \frac{1}{\gamma_0}[2(A_0-1)]_1^{1/2}.$$

Proof. First we need in the following assertion:

Lemma 1. Let ξ be a random variable such that $E\exp\{\gamma|\xi|\} < \infty$ and $E|\xi|^2 \le \sigma^2$ for some positive constants γ and σ, then
(i) $\|\xi\|_{(1)} \le \frac{1}{\gamma}[2(E\exp\{\gamma|\xi|\}-1)]_1^{1/2}$; and
(ii) Bernstein's condition (2.1) is valid with

$$b = \frac{1}{\gamma}\left[\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^2\sigma^2}\right]_1.$$

Proof. Since

$$\frac{\gamma^k E|\xi|^k}{k!} \le E\exp\{\gamma|\xi|\}-1, \quad k=2,3,\ldots$$

we have

$$\|\xi\|_{(1)} = \sup_{k\ge 2}\left(\frac{2E|\xi|^k}{k!}\right)^{1/k} \le \sup_{k\ge 2}\left(\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^k}\right)^{1/k} = \frac{1}{\gamma}[2(E\exp\{\gamma|\xi|\}-1)]_1^{1/2},$$

which completes the proof of (i).

To prove (ii) we note that

$$b = \frac{1}{\gamma}\sup_{i\ge 3}\left\{\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^2\sigma^2}\right\}^{\frac{1}{i-2}}$$

and therefore

$$\sigma^2 b^{k-2} \ge \sigma^2\frac{1}{\gamma^{k-2}}\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^2\sigma^2} \ge \frac{2E|\xi|^k}{k!}, \quad k=2,3,\ldots$$

which completes the proof of Lemma 1.

Now let us continue the proof of Theorem 2. Define the centered random field $\xi(t) = F(t) - EF(t)$, $t \in T$. Taking into account $|\xi(t)| \le |F(t)| + |EF(t)|$ we have

$$E \exp\{\gamma|\xi(t)|\} \le E \exp\{\gamma(|F(t)| + |EF(t)|)\} \le A^2.$$

From this inequality and by Lemma 1 it follows that the random field $\xi(t)$ satisfies Bernstein's condition (2.1) with $b = \frac{1}{\gamma}\left[\frac{2(A^2-1)}{\gamma^2\sigma^2}\right]_1$.

Now let us estimate the ϵ entropy $H_1(\epsilon)$ of the metric space (T, ρ_1) where $\rho_1(t,s) = \|\xi(t) - \xi(s)\|_{(1)}$. From

$$|\xi(t) - \xi(s)| \le |F(t) - F(s)| + |EF(t) - EF(s)|$$

and by the assumptions of the Theorem it follows that

$$\|\xi(t) - \xi(s)\|_{(1)} \le (\|\eta_0\|_{(1)} + E\eta_0)\|t-s\|^\alpha.$$

Using Lemma 1 and Jensen's inequality we have

$$\|\eta_0\|_{(1)} \le \frac{1}{\gamma_0}[2(A_0-1)]_1^{1/2} \text{ and } \exp\{\gamma_0 E\eta_0\} \le E\exp\{\gamma_0\eta_0\} = A_0,$$

respectively. These inequalities show that

$$\rho_1(t,s) = \|\xi(t) - \xi(s)\|_{(1)} \le C_3\|t-s\|^\alpha.$$

Hence for each $\epsilon > 0$

$$H_1(\epsilon) \le H(\delta), \quad \text{where } \delta = (\epsilon/C_3)^{1/\alpha}.$$

Therefore taking into account $H(\delta) \le k\ln(1 + D/\delta)$ we have

$$H_1(\epsilon) \le k\ln\left(1 + D\frac{C_3^{1/\alpha}}{\epsilon^{1/\alpha}}\right) \le k\left(\ln(1 + D\,C_3^{1/\alpha}) + \frac{1}{\alpha}|\ln\epsilon|\right).$$

Here we applied the following simple inequality $\ln(1 + a/x) \le \ln(1+a) + |\ln x|$ for each positive a and x.

Hence the inequality (3.1) follows from that of Corollary 1 if we put $C_1 = k\ln(1 + D\,C_3^{1/\alpha})$ and $\kappa = k/\alpha$. This completes the proof.

4. Asymptotic behaviour for moderately large deviations

In this section we study asymptotic behaviour of the right- and the left-hand sides of the inequality (2.8) in the interval of moderately large deviations (1.7).

Remark 4. Let us explain the importance of moderately large deviations. We consider the equality

$$P\left\{\sup_{t\in T}|\frac{1}{n}\sum_{i=1}^n(F_i(t) - EF(t))| \ge \epsilon_n\right\} = p_n(x_n), \quad x_n = \epsilon_n\sqrt{n}.$$

From one side, it makes a sense to consider such ϵ_n that satisfies the condition $\epsilon_n \to 0$ as $n \to \infty$ since ϵ_n is the measure of the error in the approximation

$$E\,F(t) \simeq \frac{1}{n}\sum_{i=1}^{n} F_i(t)\ . \tag{4.1}$$

Therefore $\epsilon_n = x_n/\sqrt{n} \to 0$ as $n \to \infty$. From the other side it is meaningful to consider such ϵ_n which ensures the convergence of the probability $p_n(x_n)$ to zero since this probability characterizes the confidence of the approximation (4.1). Therefore it should be assumed that $x_n \to \infty$ as $n \to \infty$.

In the domain of moderately large deviations (1.7), the asymptotic behaviour of probabilities of deviations is quite similar to that of Gaussian distributions. In this section we will use the following known result of the theory of large deviations (e.g., see [12]).

Theorem 3. Let $\xi_1,\ \xi_2, \ldots$ be a sequence of independent identically distributed centered random variables satisfying Bernstein's condition (2.1) and $x_n, n = 1, 2, \ldots$ a sequence satisfying the condition (1.7). Assume that $E\xi_i^2 = \sigma^2$, then

$$P\left\{ \mid \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \xi_i \mid \geq x_n \right\} = \exp\left\{-\frac{x_n^2}{2\sigma^2}(1+\delta_n)\right\},$$

where $\delta_n, n = 1, 2, \ldots$ is a sequence satisfying the condition $\delta_n \to 0$ as $n \to \infty$.

Now let us consider the estimation (2.8). For the brevity of notations let us rewrite the inequality (2.8) in the form (1.3).

Let $\xi(t), t \in T$ be a random field satisfying all the conditions of Corollary 1, and the following condition

$$\exists t_0 \in T \text{ such that } \sigma^2 = E\xi^2(t_0). \tag{4.2}$$

Then it follows from Theorem 3 that

$$p_n(x_n) \geq P\left\{ \mid \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \xi_i(t_0) \mid \geq x_n \right\} = \exp\left\{-\frac{x_n^2}{2\sigma^2}(1+\delta_n)\right\}. \tag{4.3}$$

From the definition of $\psi_\beta^*(x)$ it follows that

$$n\,\psi_\beta^*\left(\frac{x_n(1-p)}{\sigma\sqrt{n}}\right) = \frac{x_n^2}{2\sigma^2}(1+\delta_n')(1-p)^2, \tag{4.4}$$

where $\delta_n', n = 1, 2, \ldots$ is a sequence satisfying the condition $\delta_n' \to 0$ as $n \to \infty$.

Due to $x_n \to \infty$ as $n \to \infty$ we can choose a sequence $\{p_n\} \subset (0, 1)$ such that $p_n \to 0$ and $|\ln p_n|/x_n^2 \to 0$ as $n \to \infty$. Therefore, taking into account (4.4) we have

$$\phi_n(x_n) \geq \frac{x_n^2}{2\sigma^2}(1+\delta_n''),$$

where $\delta_n'', n = 1, 2, \ldots$ is a sequence satisfying the condition $\delta_n'' \to 0$ as $n \to \infty$. From this inequality and (4.3) we come to the following conclusion

$$\exp\left\{-\frac{x_n^2}{2\sigma^2}(1+\delta_n)\right\} \le p_n(x_n) \le \exp\{-\phi_n(x_n)\} \le \exp\left\{-\frac{x_n^2}{2\sigma^2}(1+\delta_n'')\right\}.$$

Thus we establish that in the range of moderately large deviations under the conditions of Corollary 2 and the condition (4.2) the asymptotic behaviour of the right-hand side of (2.8) is optimal in the sense that (1.6) is true.

5. Deviation probability for the continuity modulus

In some applications, there is interest in evaluation of the distribution of sample continuity modulus for a sum of independent random fields. We mention here for example the study of the convergence rate in the functional central limit theorem [5], the error estimations in discrete stochastic procedures of functional Monte Carlo methods [14], [13].

Let us recall that we deal with a sequence of independent samples $\xi_1(t), \xi_2(t), \ldots \xi_n(t)$ of a centered random field $\xi(t)$, $t \in T$, and $S_n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i(t)$; (T, ρ_1) is a metric space with the metric $\rho_1(t,s) = ||\xi(t) - \xi(s)||_{(1)}$; $H_1(\epsilon)$ is Kolmogorov's metric ϵ entropy of (T, ρ_1).

Let us introduce the notations:

$$\bar{H}_1(\epsilon) = \ln(1 + \exp\{H_1(\epsilon)\}), \quad \Omega_n(\delta) = \int_0^\delta \left[\frac{1}{\sqrt{n}}\bar{H}_1(\epsilon) + (2\bar{H}_1(\epsilon))^{1/2}\right] d\epsilon$$

In the following statement we deal with the local and global sample continuity modulus of the random field $S_n(t)$ in the metric $\rho_1(t,s)$ defined as

$$\omega_{S_n}(\delta; t_0) \equiv \sup_{t \in T : \rho_1(t, t_0) < \delta} |S_n(t) - S_n(t_0)|,$$

$$\omega_{S_n}(\delta) \equiv \sup_{t,s \in T : \rho_1(t,s) < \delta} |S_n(t) - S_n(s)|.$$

Theorem 4. Let $\xi(t), t \in T$ be a centered random field which is separable on (T, ρ_1), and $\int_0^1 H_1(\epsilon)d\epsilon < \infty$. Then for each $n = 1, 2, \ldots$, and $x > 0$ the following estimations are true

$$P\{\omega_{S_n}(\delta, t_0) \ge 18e\Omega_n(\delta)x\} \le 2\exp\left\{-n\psi_1^*(\frac{x}{\sqrt{n}})\right\}, \tag{5.1}$$

$$P\{\omega_{S_n}(\delta) \ge 54e\Omega_n(\delta)x\} \le 2\exp\left\{-n\psi_1^*(\frac{x}{\sqrt{n}})\right\}, \tag{5.2}$$

where $\psi_1^*(x) = \frac{1}{2}(\sqrt{1+2x} - 1)^2$.

Proof. We need some auxiliary results from the theory of random variables of subgaussian type and random variables in Orlicz spaces [9] .

Let $\psi : [0, \Lambda) \to R_+ = [0, \infty)$ $(\Lambda \le \infty)$ be a continuous convex function such that

$$0 < \lim_{\mu \to 0} \frac{\psi(\mu)}{\mu^2} < \infty, \quad \lim_{\mu \to \Lambda - 0} \frac{\psi(\mu)}{\mu} = \infty.$$

If $\Lambda < \infty$ we prolong the function ψ on the half-axis $[\Lambda, \infty)$ by $\psi(\mu) = \infty$ for $\mu \ge \lambda$. We denote by $\psi^*(x)\,(x \ge 0)$, the function $\psi^*(x) \equiv \sup_{\mu \ge 0}(\mu x - \psi(\mu))$ which is the Legendre transformation of the function ψ. Let us prolong the function $\psi^*(x)$ on the left half-axis by $\psi^*(x) = \psi^*(-x)$ if $x \le 0$.

Let us define a norm for a centered random variable ξ by

$$||\xi||_\psi \equiv \inf\{C > 0 : \ln Ee^{\lambda\xi} \le \psi(C|\lambda|),\ \lambda \in R^1\}.$$

In [9] it is shown that the class of random variables

$$B_\psi(\Omega) \equiv \{\xi : \Omega \to R^1;\ E\xi = 0, ||\xi||_\psi < \infty\}$$

is a linear space, and the pair $(B_\psi(\Omega), ||\cdot||_\psi)$ is a real Banach space.

Let $U(x) \equiv \exp\{\psi^*(x)\} - 1$, $x \in R^1$. By the definition it follows that the function $U(x)$ is a Young function i.e., $U(x)$ is a continuous convex function satisfying the conditions $U(0) = 0$, $\lim_{|x| \to \infty} U(x) = \infty$.

Therefore, for the random variable ξ the following norm can be defined by

$$L_U(\xi) \equiv \inf\{r > 0,\ EU(\xi/r) \le 1\},$$

which is called in the theory of Orlicz spaces as a Luxemburg norm [8].

In [9] it is proven that for $\xi \in B_\psi(\Omega)$ it is necessary and sufficient that $E\xi = 0$ and $L_U(\xi) < \infty$. In [9] it is shown that the Luxemburg norm and the norm $\|\cdot\|_\psi$ are equivalent in the space $B_\psi(\Omega)$.

Further we will use the following statement which is obtained in [9]: For any centered random variable ξ the following estimation is true:

$$L_U(\xi) \le 3||\xi||_\psi, \quad (\text{ where } U(x) \equiv \exp\{\psi^*(x)\} - 1). \tag{5.3}$$

To prove Theorem 4, we need the following statement

Lemma 2. For any centered random variable ξ the following estimation is true:

$$||\xi||_{\psi_1} \le ||\xi||_{(1)}, \tag{5.4}$$

where

$$\psi_1(\lambda) = \frac{\lambda^2}{2(1-\lambda)}.$$

Proof. By $E|\xi|^k \le r^k k!/2$, $k = 2, 3, \ldots$, where $r = ||\xi||_{(1)}$ we get

$$Ee^{\lambda\xi} \le 1 + \frac{1}{2}\sum_{k=2}^{\infty} r^k |\lambda|^k = 1 + \frac{r^2\lambda^2}{2(1 - r|\lambda|)} \le e^{\psi_1(r\lambda)}.$$

From this we get (5.4), by the definition of the norm $||\cdot||_{\psi_1}$.

Let us define a sequence of functions $U_n : R^1 \to R_+$, $n = 1, 2, \ldots$ by

$$U_n(x) \equiv \exp\{\psi_n^*(x)\} - 1, \quad \psi_n^*(x) \equiv n\psi_1^*\left(\frac{x}{\sqrt{n}}\right), \quad x \in R^1,\, n = 1, 2, \ldots .$$

where $\psi_1^*(x) = \frac{1}{2}\left(\sqrt{1+2x} - 1\right)^2$ is the Legendre transformation of the function $\psi_1(\mu)$.

Lemma 3. The following estimation is true

$$L_{U_n}(S_n(t) - S_n(s)) \leq 3\rho_1(t,s), \quad t, s \in T;\, n = 1, 2, \ldots . \tag{5.5}$$

Proof. We fix n, and define the function $\psi_n(\lambda) \equiv n\psi_1(\lambda/\sqrt{n})$. Therefore, the Legendre transformation $\psi_n^*(x)$ of the function $\psi_n(\lambda)$ coincides with the function $n\psi_1^*(x/\sqrt{n})$. Therefore, by the estimation (5.3) we get

$$L_{U_n}(S_n(t) - S_n(s)) \leq 3||S_n(t) - S_n(s)||_{\psi_n}. \tag{5.6}$$

From the independence of the fields $\xi_1(t), \xi_2(t), \ldots, \xi_n(t)$ it follows $||S_n(t) - S_n(s)||_{\psi_n} = ||\xi(t) - \xi(s)||_{\psi_1}$.

Hence, in view of Lemma 2, we conclude

$$||S_n(t) - S_n(s)||_{\psi_n} = ||\xi(t) - \xi(s)||_{\psi_1} \leq \rho_1(t,s).$$

From this and by (5.6) we obtain (5.5). This completes the proof of Lemma.

Proof of Theorem 4. The convexity of $\psi_n^*(x)$ implies that the function $U_n(x) = \exp\{\psi_n^*(x)\} - 1$ has the following property

$$U_n^2(x) \leq U_n(2x), \quad x \geq 0 . \tag{5.7}$$

This property in turn implies that (see [8]):

$$U_n(x)U_n(y) \leq U_n(xy), \quad x \geq 2,\, y \geq 2. \tag{5.8}$$

Let

$$\bar{\omega}_{S_n}(\delta; t_0) \equiv \sup_{t\in T:\rho^{(n)}(t,t_0)<\delta} |S_n(t) - S_n(t_0)|,$$

$$\bar{\omega}_{S_n}(\delta) \equiv \sup_{t,s\in T:\rho^{(n)}(t,s)<\delta} |S_n(t) - S_n(s)|.$$

be the local and global sample continuity modula of $S_n(t)$ in the metric $\rho^{(n)}(t,s) \equiv L_{U_n}(S_n(t) - S_n(s))$, respectively.

From (5.7)-(5.8) and Theorems 2 and 3 of [8] it follows

$$L_{U_n}(\bar{\omega}_{S_n}(\delta; t_0)) \leq 6R_n \int_0^\delta \hat{U}_n(N^{(n)}(\epsilon))d\epsilon, \tag{5.9}$$

$$L_{U_n}(\bar{\omega}_{S_n}(\delta)) \leq 18R_n \int_0^\delta \hat{U}_n(N^{(n)}(\epsilon))d\epsilon, \tag{5.10}$$

where $\hat{U}_n(y)$ is the inverse function to $U_n(x)$, and $N^{(n)}(\epsilon)$ is a minimal number of balls of radius ϵ in the metric $\rho^{(n)}$ covering the set T, and

$$R_n = \max\left(\exp\{\psi_n^*(2)\}, \frac{\psi_n^*(2)-1}{\hat{U}_n(1)}\right).$$

It is not difficult to derive the equality

$$\hat{U}_n(y) = \sqrt{2\ln(1+y)} + \frac{1}{\sqrt{n}}\ln(1+y), \quad y \geq 0. \tag{5.11}$$

Therefore, $\hat{U}_n(1) \geq \sqrt{2\ln 2}$ is true for each n. Thus since $\psi_n^*(2) < 1$ for each n, we get

$$R_n \leq \max\{e, (e-1)/\sqrt{2\ln 2}\} = e, \quad n = 1, 2, \ldots \tag{5.12}$$

In view of Lemma 3, $N^{(n)}(\epsilon) \leq N_1(\epsilon/3), \ n = 1, 2, \ldots$.
From this, we conclude by (5.9)-(5.12)

$$L_{U_n}(\bar{\omega}_{S_n}(\delta; t_0)) \leq 18e\Omega_n(\delta/3), \quad L_{U_n}(\bar{\omega}_{S_n}(\delta)) \leq 54e\Omega_n(\delta/3), \ \delta > 0, \ n = 1, 2, \ldots \tag{5.13}$$

By Lemma 3,

$$\{(t,s) : \rho_1(t,s) < \delta\} \subset \{(t,s) : \rho^{(n)}(t,s) < 3\delta\}, \quad n = 1, 2, \ldots,$$

hence,

$$P\{\omega_{S_n}(\delta; t_0) \leq \bar{\omega}_{S_n}(3\delta; t_0)\} = 1, \quad P\{\omega_{S_n}(\delta) \leq \bar{\omega}_{S_n}(3\delta)\} = 1$$

From this we get by (5.13)

$$L_{U_n}(\omega_{S_n}(\delta; t_0)) \leq 18e\Omega_n(\delta), \quad L_{U_n}(\omega_{S_n}(\delta)) \leq 54e\Omega_n(\delta), \ \delta > 0, \ n = 1, 2, \ldots$$

From these estimations we conclude (5.1), (5.2) using the Chebyshev inequality. The proof of Theorem 4 is complete.

Remark 5. From the inequality $\bar{H}_1(\epsilon) = \ln(1+\exp\{H_1(\epsilon)\}) \leq \ln 2 + H_1(\epsilon)$ and by the definition of $\Omega_n(\delta)$ it follows

$$\Omega_n(\delta) \leq \left(\frac{\ln 2}{\sqrt{n}} + \sqrt{2\ln 2}\right)\delta + \int_0^\delta \left(\frac{1}{\sqrt{n}}H_1(\epsilon) + (2H_1(\epsilon))^{1/2}\right) d\epsilon \equiv \Omega_n'(\delta).$$

Now we note that the estimations (5.1), (5.2) remain true if we replace $\Omega_n(\delta)$ with $\Omega_n'(\delta)$.

6. Conclusions

Exponential upper bounds for the probability of deviations of a sum of independent random fields are obtained under Bernstein's condition and assumptions formulated in terms of Kolmogorov's metric entropy. These estimations are not asymptotic estimations, and they are constructive in the sense that all the constants involved are given explicitly. The proposed estimation for moderately large deviations has optimal log-asymptotics. The exponential estimations are extended to the local and global sample continuity modulus.

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