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Lower deviation probabilities for supercritical Galton-Watson processes

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ABSTRACT. There is a well-known sequence of constants c_n describing the growth of supercritical Galton-Watson processes Z_n. With "lower derivation probabilities" we refer to $\mathbf{P}(Z_n = k_n)$ with $k_n = o(c_n)$ as n increases. We give a detailed picture of the asymptotic behavior of such lower deviation probabilities. This complements and corrects results known from the literature concerning special cases. Knowledge on lower deviation probabilities is needed to describe large deviations of the ratio Z_{n+1}/Z_n. The latter are important in statistical inference to estimate the offspring mean. For our proofs, we adapt the well-known Cramér method for proving large deviations of sums of independent variables to our needs.

CONTENTS

1. Introduction and statement of results

1.1. **On the growth of supercritical processes.** Let $Z = (Z_n)_{n \geq 0}$ denote a Galton-Watson process with offspring generating function

$$f(s) \;=\; \sum_{j \geq 0} p_j s^j, \qquad 0 \leq s \leq 1, \tag{1}$$

which is required to be non-degenerate, that is, $p_j < 1$, $j \geq 0$. Suppose that Z is supercritical, i.e. $f'(1) =: m \in (1, \infty)$. For simplicity, the initial state $Z_0 \geq 1$ is always assumed to be deterministic, and, if not noted otherwise (as by an application of the Markov property), we set $Z_0 = 1$.

It is well-known (see, e.g., Asmussen and Hering (1983) [1, § 3.5]) that

$$\text{there are } c_n > 0 \text{ such that a.s. } c_n^{-1} Z_n \underset{n \uparrow \infty}{\longrightarrow} \text{some non-degenerate } W. \tag{2}$$

In this sense, the sequence of constants c_n describes the order of growth of Z. But, $\mathbf{P}(W = 0) = q$, with $q \in [0, 1)$ the smallest root of $f(s) = s$, that is, the extinction probability of Z. On the other hand, W restricted to $(0, \infty)$ has a (strictly) positive continuous density function denoted by w. Therefore the following *global limit theorem* holds:

$$\lim_{n \uparrow \infty} \mathbf{P}(Z_n \geq x c_n) \;=\; \int_x^\infty w(t)\,\mathrm{d}t, \qquad x > 0. \tag{3}$$

The normalizing sequence $(c_n)_{n \geq 0}$ can be chosen to have the following additional properties:

$$c_0 = 1 \text{ and } c_n < c_{n+1} \leq m\,c_n\,, \;\; n \geq 0, \tag{4a}$$

$$c_n \;=\; m^n L(m^n) \text{ with } L \text{ slowly varying at infinity,} \tag{4b}$$

$$\lim_{x \uparrow \infty} L(x) \text{ exists; it is positive if and only if } \mathbf{E} Z_1 \log Z_1 < \infty. \tag{4c}$$

Because of (4b,c), we may (and subsequently shall) take

$$c_n \;:=\; m^n \quad \text{if} \quad \mathbf{E} Z_1 \log Z_1 < \infty. \tag{5}$$

1.2. **Asymptotic local behavior of Z, purpose.** A local limit theorem related to (3) is due to Dubuc and Seneta (1976) [10], see also [1, §3.7]. To state it we need the following definition.

Definition 1 (Type (d, μ)). We say the offspring generating function f *is of type* (d, μ), if $d \geq 1$ is the greatest common divisor of the set $\{j - \ell : j \neq \ell,\ p_j p_\ell > 0\}$, and $\mu \geq 0$ is the minimal j for which $p_j > 0$. ◇

Here is the announced *local limit theorem.* Suppose f is of type (d, μ). Take $x > 0$, and consider integers $k_n \geq 1$ such that $k_n / c_n \to x$ as $n \uparrow \infty$. Then, for each $j \geq 1$,

$$\lim_{n \uparrow \infty} \Big(c_n\, \mathbf{P}\big\{Z_n = k_n \,\big|\, Z_0 = j\big\} - d\, 1_{\{k_n \equiv j\,\mu^n (\mathrm{mod}\, d)\}}\, w_j(x) \Big) \;=\; 0, \tag{6}$$

where $w_j := \sum_{\ell=1}^{j} \binom{j}{\ell} q^{j-\ell} w^{*\ell}$.

In particular, in our standard case $Z_0 = 1$ and if additionally $k_n \equiv \mu^n \pmod d$, then

$$\mathbf{P}(Z_n = k_n) \;\sim\; d\, c_n^{-1}\, w(k_n / c_n) \quad \text{as } n \uparrow \infty \tag{7}$$

(with the usual meaning of the symbol $\sim$ as the ratio converges to 1).

Statement (6) [and especially (7)] can be considered as describing the local behavior of supercritical Galton-Watson processes in the region of *normal* deviations (from the growth of the c_n; 'deviations' are meant here in a multiplicative sense, related to the multiplicative nature of branching). But what about $\mathbf{P}(Z_n = k_n)$ when $k_n/c_n \to 0$ or ∞? In these cases we speak of *lower* and *upper* (local) deviation probabilities, respectively.

Lower deviations of Z_n are closely related to large deviations of Z_{n+1}/Z_n (see Ney and Vidyashankar (2004) [15, Section 2.3]). The latter are important in statistical inference for supercritical Galton-Watson processes, since Z_{n+1}/Z_n is the well-known Lotka-Nagaev estimator of the offspring mean.

The *main purpose* of the present paper is to study lower deviation probabilities in their own and to provide a detailed picture (see Theorems 4 and 5 below). As a starting point we discuss a relevant claim in [15] concerning an important special case (see Sections 1.4 and 1.5 below). Applications of our results for large deviations of Z_{n+1}/Z_n and also to subcritical Galton-Watson processes are postponed to a future paper.

Here is the program for the remaining introduction. After introducing a basic dichotomy, we review in Sections 1.4 and 1.5 what is known on lower deviations from the literature, before we state our results in Sections 1.6 and 1.7.

1.3. **A dichotomy for supercritical processes.** Recalling that f denotes the offspring generating function, q the extinction probability, and m the mean,

$$\text{set} \quad \gamma := f'(q), \quad \text{and define } \alpha \text{ by} \quad \gamma = m^{-\alpha}. \tag{8}$$

Note that $\gamma \in [0,1)$ and $\alpha \in (0,\infty]$. We introduce the following notion, reflecting a crucial dichotomy for supercritical Galton-Watson processes.

Definition 2 (Schröder and Böttcher case). For our supercritical offspring law we distinguish between the *Schröder* and the *Böttcher* case, in dependence on whether $p_0 + p_1 > 0$ or $= 0$. ◇

Obviously, f is of Schröder type if and only if $\gamma > 0$, if and only if $\alpha < \infty$.

Next we want to collect a few basic facts from the literature concerning that dichotomy. Clearly, f can be considered as a function on D, where D denotes the closed unit disc in the complex plane. As usual, denote by f_n the n^{th} iterate of f. We start with the *Schröder case*. Here it is well-known (see, e.g., [1, Lemma 3.7.2 and Corollary 3.7.3]) that

$$\mathsf{S}_n(z) := \frac{f_n(z) - q}{\gamma^n} \underset{n\uparrow\infty}{\longrightarrow} \text{some } \mathsf{S}(z) =: \sum_{j=0}^{\infty} \nu_j z^j, \qquad z \in D. \tag{9}$$

Moreover, the convergence is uniform on each compact subsets of the interior D° of D. Furthermore, the function S restricted to the reals is the unique solution of the so-called *Schröder functional equation* (see, e.g., Kuczma (1968) [13, Theorem 6.1, p.137]),

$$\mathsf{S}\big(f(s)\big) = \gamma\, \mathsf{S}(s), \qquad 0 \le s \le 1, \tag{10}$$

satisfying

$$\mathsf{S}(q) = 0 \quad \text{and} \quad \lim_{s\to q} \mathsf{S}'(s) = 1. \tag{11}$$

As a consequence of (9),

$$\lim_{n\uparrow\infty} \gamma^{-n}\, \mathbf{P}(Z_n = k) \;=\; \nu_k\,, \qquad k \ge 1. \tag{12}$$

Consequently, in the Schröder case, these extreme (k is fixed) lower deviation probabilities $\mathbf{P}(Z_n = k)$ are positive and decay to 0 with order γ^n. On the other hand, the characteristics $\alpha \in (0,\infty)$ describes the behavior of the limiting quantities $w(x)$ and $\mathbf{P}(W \le x)$ as $x \downarrow 0$. In fact, according to Biggins and Bingham (1993) [5], there is a continuous, positive multiplicatively periodic function V such that

$$x^{1-\alpha}\, w(x) \;=\; V(x) + o(1) \quad \text{as } x \downarrow 0. \tag{13}$$

Dubuc (1971) [7] has shown that the function V can be replaced by a constant $V_0 > 0$ if and only if

$$\mathsf{S}\big(\varphi(h)\big) \;=\; K_0\, h^{-\alpha}, \qquad h \ge 0, \tag{14}$$

for some constant $K_0 > 0$, where $\varphi = \varphi_W$ denotes the Laplace function of W,

$$\varphi_W(h) \;:=\; \mathbf{E} e^{-hW}, \qquad h \ge 0. \tag{15}$$

We mention that condition (14) is certainly fulfilled if Z is embeddable (see [1, p.96]) into a continuous-time Galton-Watson process (as in the case of a geometric offspring law, see Example 3 below).

Now we turn to the *Böttcher case.* Here $\mu \ge 2$ (recall Definition 1). Clearly, opposed to (12), extreme lower deviation probabilities disappear, even $\mathbf{P}(Z_n < \mu^n) = 0$ for all $n \ge 1$. Evidently,

$$\mathbf{P}(Z_n = \mu^n) \;=\; \mathbf{P}(Z_{n-1} = \mu^{n-1})\, p_\mu^{(\mu^{n-1})}. \tag{16}$$

Hence,

$$\mathbf{P}(Z_n = \mu^n) \;=\; \prod_{j=0}^{n-1} p_\mu^{(\mu^j)} \;=\; \exp\Big[\frac{\mu^n - 1}{\mu - 1} \log p_\mu\Big]. \tag{17}$$

Next, $\mathbf{P}(Z_n = \mu^n + 1) = \mathbf{P}(Z_{n-1} = \mu^{n-1})\, \mu^{n-1}\, p_{\mu+1}\, p_\mu^{\mu^{n-1}-1}$. Thus, from (16),

$$\mathbf{P}(Z_n = \mu^n + 1) \;=\; p_\mu^{-1}\, p_{\mu+1}\, \mu^{n-1}\, \mathbf{P}(Z_n = \mu^n). \tag{18}$$

For simplification, consider for the moment the special case $p_{\mu+j} > 0,\ j \ge 0$. Then, as in the previous representation, for fixed $k \ge 0$ and some positive constants C_k,

$$\mathbf{P}(Z_n = \mu^n + k) \;\sim\; C_k\, \mu^{nk}\, \mathbf{P}(Z_n = \mu^n) \quad \text{as} \quad n \uparrow \infty. \tag{19}$$

Consequently, in contrast to (12) in the Schröder case, here the lower positive deviation probabilities $\mathbf{P}(Z_n = \mu^n + k)$ do *not* have a uniform order of decay. But by (19),

$$\mu^{-n} \log \mathbf{P}(Z_n = \mu^n + k) \;\underset{n\uparrow\infty}{\longrightarrow}\; \log p_\mu, \qquad k \ge 0. \tag{20}$$

That is, on a *logarithmic* scale, we gain again a uniform order, namely $-\mu^n$.

Turning back to the general Böttcher case,

$$\lim_{n\uparrow\infty} \big(f_n(s)\big)^{(\mu^{-n})} \;=:\; \mathsf{B}(s), \qquad 0 \le s \le 1, \tag{21}$$

exists, is continuous, positive, and satisfies the *Böttcher functional equation*

$$\mathsf{B}\big(f(s)\big) = \mathsf{B}^\mu(s) \qquad 0 \le s \le 1, \tag{22}$$

with boundary conditions

$$\mathsf{B}(0) = 0 \quad \text{and} \quad \mathsf{B}(1) = 1 \tag{23}$$

(see, e.g., Kuczma (1968) [13, Theorem 6.9, p.145]).

Recalling that $\mu \geq 2$, define $\beta \in (0,1)$ by

$$\mu = m^{\beta}. \tag{24}$$

According to [5, Theorem 3], there exists a positive and multiplicatively periodic function V^* such that

$$-\log \mathbf{P}(W \leq x) = x^{-\beta/(1-\beta)}\, V^*(x) + o(x^{-\beta/(1-\beta)}) \quad \text{as } x \downarrow 0. \tag{25}$$

If additionally $\log \varphi_W(h) \sim -\kappa h^{\beta}$ as $h \uparrow \infty$ for some constant $\kappa > 0$, then by Bingham (1988) [6, formula (4)],

$$-\log \mathbf{P}(W \leq x) \sim \beta^{-1}(1-\beta)(\kappa\beta)^{1/(1-\beta)}\, x^{-\beta/(1-\beta)} \quad \text{as } x \downarrow 0. \tag{26}$$

1.4. **Lower deviation probabilities in the literature.** What else is known in the literature on lower deviation probabilities of Z? In the *Schröder case* ($0 < \alpha < \infty$), Athreya and Ney (1970) [2] proved that in case of mash $d = 1$ and $\mathbf{E}Z_1^2 < \infty$, for every $\varepsilon \in (0, \eta)$, where

$$\eta := m^{\alpha/(3+\alpha)} > 1, \tag{27}$$

there exists a positive constant C_ε such that for all $k \geq 1$,

$$\Big| m^n \mathbf{P}(Z_n = k) - w(k/m^n) \Big| \leq C_\varepsilon \frac{\eta^{-n}}{km^{-n}} + (\eta - \varepsilon)^{-n}. \tag{28}$$

The estimate (28) allows to get some information on lower deviation probabilities. Indeed, in the general Schröder case, from (13),

$$w(x) \asymp x^{\alpha-1} \quad \text{as } x \downarrow 0 \tag{29}$$

(meaning that there are positive constants C_1 and C_2 such that $C_1\, x^{\alpha-1} \leq w(x) \leq C_2\, x^{\alpha-1}$, $0 < x \leq 1$). Together with (28) this implies

$$\mathbf{P}(Z_n = k_n) = m^{-n}\, w(k_n/m^n) \Big[1 + O\Big(\frac{m^{\alpha n}}{k_n^{\alpha}\eta^n} + \frac{m^{(\alpha-1)n}}{k_n^{\alpha-1}(\eta-\varepsilon)^n} \Big) \Big] \quad \text{as } n \uparrow \infty. \tag{30}$$

We want to show that in important special cases the O–expression is actually an $o(1)$. Recalling the definition (27) of η, one easily verifies that $m^{\alpha n}/k_n^{\alpha}\eta^n \to 0$ (as $n \uparrow \infty$) if and only if $k_n/m^{n(2+\alpha)/(3+\alpha)} \to \infty$. Concerning the second O-term, if additionally $\alpha \leq 1$, then $m^{(\alpha-1)n}/k_n^{\alpha-1} \leq 1$ provided that $k_n \leq m^n$. Hence, here $m^{(\alpha-1)n}/\big(k_n^{\alpha-1}(\eta-\varepsilon)^n\big)$ converges to zero if $\eta - \varepsilon > 1$. On the other hand, if $\alpha > 1$ and $k_n/m^{n(2+\alpha)/(3+\alpha)} \to \infty$ (which we needed for the first term), then $m^{(\alpha-1)n}/\big(k_n^{\alpha-1}(\eta-\varepsilon)^n\big) \to 0$ provided that additionally $\varepsilon \leq m^{\alpha/(3+\alpha)} - m^{(\alpha-1)/(3+\alpha)}$. Altogether, under the assumptions in [2],

$$\mathbf{P}(Z_n = k_n) = m^{-n}\, w(k_n/m^n)\, \big(1 + o(1)\big) \quad \text{as } n \uparrow \infty \tag{31}$$

provided that both $k_n \leq m^n$ and $k_n/m^{n(2+\alpha)/(3+\alpha)} \to \infty$.

In [2] it is also mentioned that according to an unpublished manuscript of S. Karlin, in the Schröder case, for each embeddable processes Z of finite second moment,

$$\lim_{n\uparrow\infty} \frac{m^{\alpha n}}{k_n^{\alpha-1}}\, \mathbf{P}(Z_n = k_n) \text{ exists in } (0,\infty), \text{ provided that } k_n = o(m^n). \tag{32}$$

In the present situation, as we remarked after (13), $w(x) \sim V_0\, x^{\alpha-1}$ as $x \downarrow 0$ with $V_0 > 0$. Hence, from (32), for some constant $C > 0$,

$$\mathbf{P}(Z_n = k_n) \sim C\, m^{-n}\, w(k_n/m^n) \quad \text{as } n \uparrow \infty, \tag{33}$$

which is compatible with (31).

Intuitively, the asymptotic behavior of lower deviation probabilities should be more related to characteristics as α and β than to the tail of the offspring distribution. Thus one can expect that it is possible to describe lower deviation probabilities successfully without the second moment assumption used in [2]. Actually, in [15, Theorem 1] one finds the following *claim.*

Suppose $p_0 = 0$ and $\mathbf{E} Z_1 \log Z_1 < \infty$. Then there exist positive constants $C_1 < C_2$ such that for $k_n \to \infty$ with $k_n = O(m^n)$ as $n \uparrow \infty$,

$$C_1 \;\leq\; \liminf_{n\uparrow\infty} \frac{\mathbf{P}(Z_n = k_n)}{A_n} \;\leq\; \limsup_{n\uparrow\infty} \frac{\mathbf{P}(Z_n = k_n)}{A_n} \;\leq\; C_2, \tag{34}$$

where

$$A_n \;:=\; \begin{cases} p_1^n\, k_n^{\alpha-1} & \text{if } \alpha < 1, \\ \theta_n\, p_1^n & \text{if } \alpha = 1, \\ m^{-n} & \text{if } 1 < \alpha \leq \infty, \end{cases} \tag{35}$$

and $\theta_n := \big[n + 1 - \log k_n / \log m\big]$. Furthermore, if $k_n = m^{n-\ell_n}$ for natural numbers $\ell_n = O(n)$ as $n \uparrow \infty$, then

$$\lim_{n\uparrow\infty} A_n^{-1}\, \mathbf{P}(Z_n = k_n) \;=:\; C_{\mathrm{lim}} \text{ exists in } (0, \infty). \tag{36}$$

1.5. **Contradictions.** Let us test that claim by an example which allows explicit calculations.

Example 3 (Geometric offspring law). Consider the offspring generating function

$$f(s) \;=\; \frac{s}{m - (m-1)s} \;=\; \sum_{j=1}^{\infty} m^{-1}\,(1 - m^{-1})^{j-1}\, s^j, \qquad 0 \leq s \leq 1, \tag{37}$$

(with mean $m > 1$). Obviously, here $q = 0$, $\gamma = m^{-1}$, hence $\alpha = 1$. For the n^{th} iterate one easily gets

$$f_n(s) \;=\; \frac{s}{m^n - (m^n - 1)s} \;=\; \sum_{j=1}^{\infty} m^{-n}\,(1 - m^{-n})^{j-1}\, s^j. \tag{38}$$

Thus,

$$\mathbf{P}(Z_n = k) \;=\; m^{-n}\,(1 - m^{-n})^{k-1} \;\leq\; m^{-n}, \tag{39}$$

for all $n, k \geq 1$. On the other hand, since $p_1 = m^{-1}$, by claim (34) there is a constant $C > 0$ such that for the considered k_n,

$$\mathbf{P}(Z_n = k_n) \;\geq\; C\,\theta_n\, m^{-n} \tag{40}$$

for n large enough. If, for example, $k_n = m^{n/2}$ then $\theta_n \to \infty$, and (40) contradicts (39). Consequently, the left-hand part of claim (34) cannot be true in the case $\alpha = 1$. ◇

Next we compare the claim with our discussion in the previous section on lower deviation probabilities based on [2]. In fact, under the assumptions in [2], if additionally $k_n = o(m^n)$ but $k_n / m^{n(2+\alpha)/(3+\alpha)} \to \infty$ as $n \uparrow \infty$, then by (31) and (29),

$$\mathbf{P}(Z_n = k_n) \;\asymp\; m^{-n} \left(\frac{k_n}{m^n}\right)^{\alpha-1}. \tag{41}$$

Thus, in the case $1<\alpha<\infty$ we get $\mathbf{P}(Z_n=k_n)=o(m^{-n})$ which contradicts the positivity of C_{lim} in claim (36), hence of C_1 in claim (34).

Here is one more consideration. According to claim (34), under $1<\alpha\leq\infty$,

$$\mathbf{P}(Z_n=k)\ \geq\ C\,m^{-n} \tag{42}$$

for all $k\in[m^{\varepsilon n},m^{(1-\varepsilon)n}]$, $\varepsilon\in(0,1/2)$, and all n large enough. Here and later, C refers to a generic positive constant which might change its value from place to place. Hence,

$$\begin{aligned}\mathbf{E}Z_n^{-1}\ &\geq\ \sum_{k=m^{\varepsilon n}}^{m^{(1-\varepsilon)n}} k^{-1}\,\mathbf{P}(Z_n=k) \\ &\geq\ C\,m^{-n}\sum_{k=m^{\varepsilon n}}^{m^{(1-\varepsilon)n}} k^{-1}\ =\ C\,(1-2\varepsilon)\,n\,m^{-n}\,\bigl(1+o(1)\bigr)\quad\text{as}\quad n\uparrow\infty.\end{aligned} \tag{43}$$

But by Ney and Vidyashankar (2003) [14, Theorem 1], $\mathbf{E}Z_n^{-1}$ is asymptotically equivalent to m^{-n} (in the case $1<\alpha\leq\infty$), getting one more contradiction.

Looking into details of the proof of [15, Theorem 1], the following formulas are claimed to be true:

$$2\pi\,C_{\mathrm{lim}}\ = \tag{44}$$

$$\begin{cases}\displaystyle\sum_{j\geq 1}\nu_j\,w^{*j}(1), & \alpha<1,\\ \displaystyle\int_{\pi/m}^{\pi}\Bigl[\mathsf{S}\bigl(\psi(u)\bigr)-\mathsf{S}\bigl(\psi(-u)\bigr)\Bigr]\,\mathrm{d}u, & \alpha=1,\\ \displaystyle\sum_{\ell\geq 0}m^{\ell}\int_{\pi/m}^{\pi}\Bigl[f_\ell\bigl(\psi(u)\bigr)+f_\ell\bigl(\psi(-u)\bigr)\Bigr]\,\mathrm{d}u+\int_{-\pi/m}^{\pi/m}\psi(u)\,\mathrm{d}u, & 1<\alpha<\infty,\\ \displaystyle\int_{-\pi/m}^{\pi/m}\psi(u)\,\mathrm{d}u, & \alpha=\infty,\end{cases}$$

with S from (9) and where $\psi=\psi_W$ denotes the characteristic function of W,

$$\psi_W(u)\ :=\ \mathbf{E}e^{iuW},\qquad u\in\mathbb{R}. \tag{45}$$

Recall that $C_{\mathrm{lim}}>0$ according to the claim. Now, if $\alpha<1$, the positiveness of C_{lim} is obvious from this formula, since the density function w is positive. But the point is that the claim $C_{\mathrm{lim}}>0$ is *not* true in all other cases.

In fact, consider first the case $1<\alpha<\infty$. It is well-known that ψ solves the equation

$$\psi(mu)\ =\ f\bigl(\psi(u)\bigr),\qquad u\in\mathbb{R}, \tag{46}$$

(e.g. [1, formula (6.1)]). Iterating, we obtain

$$\psi(m^\ell u)\ =\ f_\ell\bigl(\psi(u)\bigr)\,,\qquad u\in\mathbb{R},\quad \ell\geq 1. \tag{47}$$

Thus,

$$\int_{\pi/m}^{\pi}\Bigl[f_\ell\bigl(\psi(u)\bigr)+f_\ell\bigl(\psi(-u)\bigr)\Bigr]\,\mathrm{d}u\ =\ m^{-\ell}\int_{\pi m^{\ell-1}}^{\pi m^{\ell}}\bigl[\psi(u)+\psi(-u)\bigr]\,\mathrm{d}u. \tag{48}$$

Therefore,

$$\left|\sum_{\ell\geq 0} m^\ell \int_{\pi/m}^{\pi} \Big[f_\ell\big(\psi(u)\big)+f_\ell\big(\psi(-u)\big)\Big]\,\mathrm{d}u\right| \leq \int_{\pi/m}^{\infty}\Big[\,\big|\psi(u)\big|+\big|\psi(-u)\big|\,\Big]\,\mathrm{d}u, \tag{49}$$

which is finite, since in the Schröder case (see, for example, [3], p.83, Lemma 1),

$$\big|\psi(u)\big| \;\leq\; c\,|u|^{-\alpha}, \qquad u\in\mathbb{R}. \tag{50}$$

Hence,

$$\sum_{\ell\geq 0} m^\ell \int_{\pi/m}^{\pi}\Big[f_\ell\big(\psi(u)\big)+f_\ell\big(\psi(-u)\big)\Big]\,\mathrm{d}u \;=\; \bigg(\int_{-\infty}^{-\pi/m}+\int_{\pi/m}^{\infty}\bigg)\psi(u)\,\mathrm{d}u, \tag{51}$$

and, consequently, by (44),

$$C_{\mathrm{lim}} \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(u)\,\mathrm{d}u \tag{52}$$

in the present $\alpha\in(1,\infty)$ case. Inverting (45) gives

$$\int_{-\infty}^{\infty} e^{-iux}\,\psi(u)\,\mathrm{d}u \;=\; 2\pi w(x), \qquad x>0. \tag{53}$$

But by (13) there is a (positive) constant C such that $w(x)\leq C\,x^{\alpha-1}$, $0<x\leq 1$. Hence, $w(0)=0$, and (52) implies $C_{\mathrm{lim}}=0$.

In the case $\alpha=\infty$, the proof of Lemma 5 in [15] is incorrect. In fact, the statement (82) there is wrong. But we can start from (79) there (setting $\eta(r,s)\equiv 1$) to define

$$I^{(2)}_{r-j}(r,s) \;:=\; \int_{\pi/m}^{\pi} e^{-ium^{-r+j}} f_j\big(\psi_{s+r-j}(u)\big)\,\mathrm{d}u, \qquad r,s\geq 1, \quad 0\leq j\leq r, \tag{54}$$

where in this section by an abuse of notation,

$$\psi_\ell(u) \;:=\; f_\ell\big(e^{iu/m^\ell}\big) \;=\; \mathbf{E}e^{iuZ_\ell/m^\ell}, \qquad \ell\geq 0, \quad u\in\mathbb{R}. \tag{55}$$

By the global limit theorem (3), for $u\in\mathbb{R}$ and $j\geq 0$ we get $\lim_{r,s\to\infty}\psi_{s+r-j}(u) = \psi(u)$ with $\psi=\psi_W$ from (45), yielding $\lim_{r,s\to\infty} f_j\big(\psi_{s+r-j}(u)\big) = f_j\big(\psi(u)\big)$. Thus, by dominated convergence, for $j\geq 0$,

$$\lim_{r,s\to\infty} I^{(2)}_{r-j}(r,s) \;=\; \int_{\pi/m}^{\pi} f_j\big(\psi(u)\big)\,\mathrm{d}u. \tag{56}$$

Using this and the bound (81) there, one can easily verify that

$$\lim_{r,s\to\infty}\sum_{j=0}^{r} I^{(2)}_{r-j}(r,s) \;=\; \sum_{j=0}^{\infty} m^j\int_{\pi/m}^{\pi} f_j\big(\psi(u)\big)\,\mathrm{d}u. \tag{57}$$

This gives for C_{lim} in the case $\alpha=\infty$ the same formula as written in (44) for the case $1<\alpha<\infty$. Now, instead of (50), in the Böttcher case we have

$$\big|\psi(u)\big| \;\leq\; e^{-Cu^{\beta}}, \qquad u\in\mathbb{R}, \tag{58}$$

for some constant C, see [8, Theorem 23]. Therefore we get again (49) and (52) also in the Böttcher case. Finally, by our Remark 6 below, $w(0)=0$ and again we arrive at $C_{\mathrm{lim}}=0$.

It remains to discuss the case $\alpha = 1$. Here in the last formula at p.1156 of [15] there is a sign error: It must be read as $\int_{\pi/m}^{\pi}[\mathsf{S}(\psi(u)) + \mathsf{S}(\psi(-u))]\,\mathrm{d}u$, which equals indeed the true value of $C_{\lim}$. Now, at least if Z is embeddable into a continuous-time Galton-Watson process then analogously to (14) we get the identity $\mathsf{S}(\psi(u)) = K_0\,(iu)^{-1}$ for some constant $K_0 > 0$, implying $\mathsf{S}(\psi(u)) + \mathsf{S}(\psi(-u)) \equiv 0$. Then $C_{\lim} = 0$ for this class of processes.

Altogether, all these contradictions to the quoted claim from [15, 'Theorem 1'] (and its generalization [15, 'Theorem 2']) had been rather unexpected for us. Of course, they gave us some more motivation to ask for the right and general picture on lower deviation probabilities. Actually, it is wrong to distinguish between velocity cases as in (35). The only needed velocity case differentiation is the mentioned dichotomy of Definition 2. This we will explain in the next two sections. In the end of Section 1.7 we then discuss the influence of [15, 'Theorem 1'] to other results in [15].

1.6. **Lower deviations in the Schröder case.** We start by stating our results on lower deviation probabilities in the Schröder case. Recall that here $\mu = 0$ or 1.

Theorem 4 (**Schröder case**). *Let the offspring law be of the Schröder type and of type* (d, μ). *Then for all* $k_n \equiv \mu \pmod d$ *with* $k_n \to \infty$ *but* $k_n = o(c_n)$,

$$\mathbf{P}(Z_n = k_n) \;=\; \frac{d}{m^{n-a_n}\,c_{a_n}}\; w\Big(\frac{k_n}{m^{n-a_n}\,c_{a_n}}\Big)\big(1 + o(1)\big) \tag{59}$$

and

$$\mathbf{P}(0 < Z_n \le k_n) \;=\; \mathbf{P}\Big(0 < W < \frac{k_n}{m^{n-a_n}\,c_{a_n}}\Big)\big(1 + o(1)\big) \tag{60}$$

as $n \uparrow \infty$, *where for* $n \ge 1$ *fixed we put* $a_n := \min\{\ell \ge 1 : c_\ell \ge k_n\}$.

The appearing of the a_n in the theorem, depending on the c_n and k_n looks a bit disturbing, so we have to discuss it. First assume additionally that $\mathbf{E}Z_1 \log Z_1 < \infty$. Since here we set $c_n = m^n$, from (59) we obtain the a_n-free formula

$$\mathbf{P}(Z_n = k_n) \;=\; d\,m^{-n}\,w(k_n/m^n)\,\big(1 + o(1)\big). \tag{61}$$

Also, comparing this with (7), we see that under this $Z_1 \log Z_1$–moment condition in the Schröder case, $m^{-n}\,w(k_n/m^n)$ describes not only normal deviation probabilities but also lower ones.

On the other hand, without this additional moment condition, recalling property (4b), $c_n = m^n\,L(m^n)$ with L slowly varying at infinity. Hence, we have

$$\frac{1}{m^{n-a_n}\,c_{a_n}} \;=\; \frac{1}{c_n}\,\frac{L(m^n)}{L(m^{a_n})}, \quad \text{thus} \quad \frac{k_n}{c_{a_n}\,m^{n-a_n}} \;=\; \frac{k_n}{c_n}\,\frac{L(m^n)}{L(m^{a_n})}. \tag{62}$$

Therefore, from (59),

$$\frac{c_n\,\mathbf{P}(Z_n = k_n)}{d\,w(k_n/c_n)} \;=\; \frac{L(m^n)}{L(m^{a_n})}\,\frac{w\big(k_n L(m^n)/c_n L(m^{a_n})\big)}{w(k_n/c_n)}\,\big(1 + o(1)\big). \tag{63}$$

Using now (13), we find

$$\frac{c_n\,\mathbf{P}(Z_n = k_n)}{d\,w(k_n/c_n)} \;=\; \Big(\frac{L(m^n)}{L(m^{a_n})}\Big)^{\alpha}\,\frac{V\big(k_n L(m^n)/c_n L(m^{a_n})\big)}{V(k_n/c_n)}\,\big(1 + o(1)\big). \tag{64}$$

Next we want to expel the disturbing a_n from this formula.

It is well-known (Seneta (1976) [17, p.23]) that the regularly varying function $x \mapsto xL(x)$ asymptotically equals a (strictly) increasing, continuous, regularly varying function $x \mapsto R(x) := xL_1(x)$ with slowly varying L_1. Hence, $L(x) \sim L_1(x)$ as $x \uparrow \infty$. Using now [17, Lemma 1.3], we conclude that the inverse function R^* of R equals $x \mapsto xL^*(x)$, where L^* is again a slowly varying function.

Put $x_n := R^*(k_n)$. Then $k_n = x_n L_1(x_n)$ by the definition of R^*. Recalling that $x_n = k_n L^*(k_n)$, we get the identity

$$L^*(k_n)\, L_1(x_n) \;=\; 1, \qquad n \geq 1. \tag{65}$$

For n fixed, define $b_n := \min\bigl\{\ell \geq 1 : m^\ell L_1(m^\ell) \geq k_n\bigr\}$. Combined with $x_n L_1(x_n) = k_n$ we get

$$m^{b_n}\, L_1(m^{b_n}) \;\geq\; x_n L_1(x_n) \;>\; m^{b_n-1}\, L_1(m^{b_n-1}). \tag{66}$$

But $x \mapsto xL_1(x)$ is increasing, and the previous chain of inequalities immediately gives

$$m^{b_n} \;\geq\; x_n \;>\; m^{b_n-1}. \tag{67}$$

By (4b),

$$c_{b_n+1} \;=\; m^{b_n+1}\, L(m^{b_n+1}) \;=\; m\, \frac{L(m^{b_n+1})}{L_1(m^{b_n})}\, m^{b_n}\, L_1(m^{b_n}) \;\geq\; k_n \tag{68}$$

for all n sufficiently large. Here, in the last step we used $m > 1$, that the slowly varying functions L and L_1 are asymptotically equivalent, and the definition of b_n. Now $c_{b_n+1} \geq k_n$ implies

$$b_n + 1 \geq a_n\,, \tag{69}$$

by the definition of a_n. On the other hand,

$$m^{a_n+1}\, L_1(m^{a_n+1}) \;=\; m\, \frac{L_1(m^{a_n+1})}{L(m^{a_n})}\, c_{a_n} \;\geq\; k_n \tag{70}$$

for all n sufficiently large. Here, in the last step we used the definition of a_n. This gives

$$a_n + 1 \geq b_n\,, \tag{71}$$

by the definition of b_n. Entering with (71) and (69) into (67), we get

$$m^{a_n+1} \;\geq\; x_n \;>\; m^{a_n-2} \quad \text{for all } n \text{ sufficiently large.} \tag{72}$$

Therefore, recalling (65),

$$L(m^{a_n}) \;\sim\; L(x_n) \;\sim\; L_1(x_n) \;\sim\; \frac{1}{L^*(k_n)} \quad \text{as } n \uparrow \infty. \tag{73}$$

Entering this into (64) gives

$$\frac{c_n\, \mathbf{P}(Z_n = k_n)}{d\, w(k_n/c_n)} \;=\; \bigl[L(m^n)\, L^*(k_n)\bigr]^\alpha\; \frac{V\bigl(k_n\, L(m^n)\, L^*(k_n)/c_n\bigr)}{V(k_n/c_n)}\, \bigl(1 + o(1)\bigr), \tag{74}$$

which contains L^* instead of the a_n.

Note also that such reformulation of (59) reminds the classical Cramér theorem (see, for example, Petrov (1975) [16, §VIII.2]) on large deviations for sums of independent random variables. There the ratio of a tail probability of a sum of independent variables and the corresponding normal law expression is considered. The crucial role in Cramér's theorem is played by the so-called Cramér series $\lambda(s) := \sum_{k=0}^\infty \lambda_k s^k$, where the coefficients λ_k depend on the cumulants of the summands. For the lower deviation probabilities of supercritical Galton-Watson processes we have a more complex situation: It is not at all clear, how to find the input data L, L^*, V [entering into (74)] based only on the knowledge of the offspring generating function f.

It was already noted after (13) that if Z is embeddable into a continuous-time Galton-Watson process then $V(x) \equiv V_0$. Consequently, for embeddable processes, (74) takes the slightly simpler form

$$\frac{c_n\,\mathbf{P}(Z_n = k_n)}{d\,w(k_n/c_n)} \;=\; \bigl[L(m^n)\,L^*(k_n)\bigr]^{\alpha}\,\bigl(1+o(1)\bigr). \tag{75}$$

On the other hand, if V is not constant, the influence of this function on the asymptotic behavior of the ratio $c_n\,\mathbf{P}(Z_n = k_n)/w(k_n/c_n)$ is relatively small. Indeed, from continuity and multiplicatively periodicity of $V(x)$ we see that $0 < V_1 \leq V(x) \leq V_2 < \infty,\ x > 0,$ for some constants V_1, V_2. Therefore, from (74),

$$\begin{aligned} \frac{V_1}{V_2}\bigl[L(m^n)\,L^*(k_n)\bigr]^{\alpha}\bigl(1+o(1)\bigr) \;&\leq\; \frac{c_n\,\mathbf{P}(Z_n = k_n)}{d\,w(k_n/c_n)} \qquad (76)\\ &\leq\; \frac{V_2}{V_1}\bigl[L(m^n)\,L^*(k_n)\bigr]^{\alpha}\bigl(1+o(1)\bigr). \end{aligned}$$

Note also that for many offspring distributions the bounds V_1 and V_2 may be chosen close to each other. This "near-constancy" phenomenon was studied by Dubuc (1982) [9] and by Biggins and Bingham (1991, 1993) [4, 5].

1.7. **Lower deviations in the Böttcher case.** Recall that $\mu \geq 2$ in the Böttcher case.

Theorem 5 (**Böttcher case**)**.** *Let the offspring law be of the Böttcher type and of type* (d,μ)*. Then there exist positive constants* B_1 *and* B_2 *such that for all* $k_n \equiv \mu^n \,(\mathrm{mod}\, d)$ *with* $k_n \geq \mu^n$ *but* $k_n = o(c_n)$,

$$\begin{aligned} -B_1 \;&\leq\; \liminf_{n\uparrow\infty} \mu^{b_n - n}\log\bigl[c_n\,\mathbf{P}(Z_n = k_n)\bigr] \qquad (77a)\\ &\leq\; \limsup_{n\uparrow\infty} \mu^{b_n - n}\log\bigl[c_n\,\mathbf{P}(Z_n = k_n)\bigr] \;\leq\; -B_2, \qquad (77b) \end{aligned}$$

where $b_n := \min\{\ell : c_\ell\,\mu^{n-\ell} \geq 2k_n\}$*. The inequalities remain true if one replaces* $c_n\,\mathbf{P}(Z_n = k_n)$ *by* $\mathbf{P}(Z_n \leq k_n)$*.*

Let us add at this place the following remark.

Remark 6 (**Behavior of** w **at** 0)**.** In analogy with (29), in the Böttcher case one has

$$\log w(x) \;\asymp\; -x^{-\beta/(1-\beta)} \quad \text{as } x \downarrow 0 \tag{78}$$

with β from (24). This can be shown using techniques from the proof of Theorem 5; see Remark 16 below. ◇

Our results in the Böttcher case are much weaker than the results in the Schröder case: We got only logarithmic bounds. But this is not unexpected, recall our discussion around (20).

Repeating arguments as we used to obtain (74), from Theorem 5 we get

$$\frac{\log\bigl[c_n\,\mathbf{P}(Z_n = k_n)\bigr]}{(k_n/c_n)^{-\beta/(1-\beta)}} \;\asymp\; -\Bigl[L^*(k_n/m^{\beta n})\,L^{1/(1-\beta)}(m^n)\Bigr]^{\beta} \quad \text{as } n\uparrow\infty, \tag{79}$$

where L^* is such that $R_1(x) := x^{(1-\beta)}L(x)$ and $R_2(x) := x^{1/(1-\beta)}L^*(x)$ are asymptotic inverses, i.e. $R_1(R_2(x)) \sim x$ and $R_2(R_1(x)) \sim x$ as $x \uparrow \infty$.

Taking into account (78), we conclude that

$$\frac{\log\bigl[c_n\,\mathbf{P}(Z_n = k_n)\bigr]}{\log w(k_n/c_n)} \asymp \Bigl[L^*(k_n/m^{\beta n})\,L^{1/(1-\beta)}(m^n)\Bigr]^\beta \quad \text{as } n \uparrow \infty. \tag{80}$$

Let us continue our discussion of the paper [15]. The main reason to study there lower deviation probabilities is the application to large deviation probabilities for the ratio Z_{n+1}/Z_n, stated as Theorems 3 and 4 there. Using our Theorem 4 (instead of 'Theorem 1' there) in the proof of [15, Theorem 3] concerning large deviation probabilities in the Schröder case, one can easily verify that one needs only to change the quantity B in [15, Theorem 3] to be $-\log p_1$ for *all* $\alpha \in (0,\infty)$, in order to get the right picture. On the other hand, [15, Theorem 4] concerning large deviation probabilities in the Böttcher case is true as it is stated, since 'Theorem 1' was used only to show that

$$\lim_{n\uparrow\infty}\frac{1}{k_n}\log\bigl[m^n\,\mathbf{P}(Z_n = k_n)\bigr] = 0 \quad \text{if} \quad \frac{\mu^n}{k_n} \underset{n\uparrow\infty}{\longrightarrow} 0, \tag{81}$$

see [15, p.1163]. Recalling that $c_n = m^n$ and $L(x) \equiv L^*(x) \equiv 1$ under $\mathbf{E}Z_1 \log Z_1 < \infty$, using our (79), one obtains

$$\frac{1}{k_n}\log\bigl[m^n\,\mathbf{P}(Z_n = k_n)\bigr] \asymp -\Bigl(\frac{m^{\beta n}}{k_n}\Bigr)^{1/(1-\beta)} \quad \text{as } n \uparrow \infty. \tag{82}$$

But $m^\beta = \mu$ by definition (24) of β, and (81) follows indeed.

2. Cramér transforms applied to Galton-Watson processes

Our way to prove Theorems 4 and 5 is based on the well-known Cramér method (see, e.g., [16, Chapter 8]), which was developed to study large deviations for sums of independent random variables. A key in this method is the so-called *Cramér transform* defined as follows. A random variable $X(h)$ is called a Cramér transform (with parameter $h \in \mathbb{R}$) of the random real variable X if

$$\mathbf{E}e^{itX(h)} = \frac{\mathbf{E}e^{(h+it)X}}{\mathbf{E}e^{hX}}, \qquad t \in \mathbb{R}. \tag{83}$$

Of course, this transformation is well-defined if $\mathbf{E}e^{hX} < \infty$.

In what follows, we will *always assume* that our offspring law additionally satisfies $p_0 = 0$. This condition is not crucial but allows a bit simplified exposition of auxiliary results formulated in Lemma 11 below and of the proof of Theorem 4 in Section 3.1 (see also Remark 15 below).

2.1. **Basic estimates.**

Fix an offspring law of type (d,μ). Let $n \geq 1$. Since $Z_n > 0$, the Cramér transforms $Z_n(-h/c_n)$ exist for all $h \geq 0$. Clearly, $\mathbf{E}e^{itZ_n(-h/c_n)} = f_n(e^{-h/c_n+it})/f_n(e^{-h/c_n})$. We want to derive upper bounds of $f_n(e^{-h/c_n+it})$ on $\bigl\{t \in \mathbb{R}:\ c_n^{-1}\pi d^{-1} \leq |t| \leq \pi d^{-1}\bigr\}$. For this purpose, it is convenient to decompose the latter set into $\bigcup_{j=1}^n J_j$ where

$$J_j := \bigl\{t:\ c_j^{-1}\pi d^{-1} \leq |t| \leq c_{j-1}^{-1}\pi d^{-1}\bigr\}, \qquad j \geq 1. \tag{84}$$

To prepare for this, we start with the following generalization of [10, Lemma 2].

Lemma 7 (Preparation). *Fix* $\varepsilon \in (0,1)$. *There exists* $\theta = \theta(\varepsilon) \in (0,1)$ *such that*

$$\bigl|f_\ell(e^{-h/c_\ell + it/c_\ell})\bigr| \le \theta, \quad \ell \ge 0, \;\; h \ge 0, \;\; t \in J_\varepsilon := \bigl\{t:\; \varepsilon\pi d^{-1} \le |t| \le \pi d^{-1}\bigr\}.$$

Proof. Put $g_{h,t}(x) := e^{-hx+itx}$, $h, x \ge 0$, $t \in \mathbb{R}$. Evidently,

$$\begin{aligned} \bigl|g_{h,t}(x) - g_{h,t}(y)\bigr| &= \bigl|e^{-hx}(e^{itx} - e^{ity}) + e^{ity}(e^{-hx} - e^{-hy})\bigr| \\ &\le |e^{itx} - e^{ity}| + |e^{-hx} - e^{-hy}| \le \bigl(h + |t|\bigr)\,|x - y|. \end{aligned} \tag{85}$$

It means that for $H \ge 1$ and $T \ge \pi d^{-1}$ fixed, $\mathcal{G} := \{g_{h,t};\; 0 \le h \le H,\; |t| \le T\}$ is a family of uniformly bounded and equi-continuous functions on $\mathbb{R}_+$. Therefore, by (2),

$$f_\ell(e^{-h/c_\ell + it/c_\ell}) = \mathbf{E} g_{h,t}(Z_\ell/c_\ell) \to \mathbf{E} g_{h,t}(W) \quad \text{as } \ell \uparrow \infty, \tag{86}$$

uniformly on $\mathcal{G}$ (see, e.g., Feller (1971) [11, Corollary in Chapter VIII, §1, p.252]). Since $W > 0$ has an absolutely continuous distribution, and $t \in J_\varepsilon$ implies $|t| \le T$,

$$\sup_{0 \le h \le H,\; t \in J_\varepsilon} \bigl|\mathbf{E} e^{-hW + itW}\bigr| < 1. \tag{87}$$

From (86) and (87) it follows that there exist $\delta_1 \in (0,1)$ and ℓ_0 such that

$$\sup_{0 \le h \le H,\; t \in J_\varepsilon} \bigl|f_\ell(e^{-h/c_\ell + it/c_\ell})\bigr| \le \delta_1, \qquad \ell > \ell_0. \tag{88}$$

On the other hand, $\bigcup_{\ell=0}^{\ell_0} \bigl\{e^{-h/c_\ell + it/c_\ell};\; h \ge 0,\; t \in J_\varepsilon\bigr\}$ is a subset of a compact subset K of the unit disc D, where K does not contain the d^{th} roots of unity. Thus for some $\delta_2 \in (0,1)$,

$$\sup_{0 \le h \le H,\; t \in J_\varepsilon} \bigl|f_\ell(e^{-h/c_\ell + it/c_\ell})\bigr| \le \delta_2, \qquad \ell \le \ell_0. \tag{89}$$

In fact, from Definition 1,

$$f_\ell(z) = \sum_{j=0}^{\infty} \mathbf{P}(Z_\ell = \mu^\ell + jd)\, z^{\mu^\ell + jd}, \qquad \ell \ge 0, \quad z \in D, \tag{90}$$

implying

$$\bigl|f_\ell(z)\bigr| \le \Bigl|\sum_{j=0}^{\infty} \mathbf{P}(Z_\ell = \mu^\ell + jd)\, z^{jd}\Bigr|. \tag{91}$$

But the latter sum equals 1 if and only if z is a d^{th} root of unity, that is, if it is of the form $e^{2\pi i/d}$.

Combining (88) and (89) gives the claim in the lemma under the addition that $h \le H$. Consider now any $h > H$. In this case

$$\bigl|f_\ell(e^{-h/c_\ell + it/c_\ell})\bigr| \le f_\ell(e^{-1/c_\ell}). \tag{92}$$

By (2) we have

$$f_\ell(e^{-h/c_\ell}) = \mathbf{E} e^{-hZ_\ell/c_\ell} \to \mathbf{E} e^{-hW} \in (0,1] \text{ as } \ell \uparrow \infty, \tag{93}$$

uniformly in h from compact subsets of $\mathbb{R}_+$. In particular,

$$\sup_{\ell \ge 1} f_\ell(e^{-1/c_\ell}) < 1. \tag{94}$$

This completes the proof. □

The following lemma generalizes [10, Lemma 3].

Lemma 8 (Estimates on $J_1, \ldots J_n$). *There are constants $A > 0$ and $\theta \in (0,1)$ such that for $h \geq 0$, $t \in J_j$, and $1 \leq j \leq n$,*

$$\left|f_n(e^{-h/c_n+it})\right| \;\leq\; \begin{cases} A\, p_1^{n-j+1} & \text{in the Schröder case,} \\ \theta^{(\mu^{n-j+1})} & \text{in all cases.} \end{cases} \tag{95}$$

Proof. By (4a), we have $\varepsilon := \inf_{\ell\geq 1} c_{\ell-1}/c_\ell \in (0,1)$. If $t \in J_j$, $j \geq 1$, then evidently,

$$\pi d^{-1} \;\geq\; c_{j-1}\,|t| \;\geq\; c_{j-1}\,c_j^{-1}\pi d^{-1} \;\geq\; \varepsilon\pi d^{-1}, \tag{96}$$

hence $c_{j-1}t \in J_\varepsilon$. Thus, by Lemma 7,

$$U \;:=\; \bigcup_{j=1}^{\infty}\Big\{f_{j-1}(e^{-h+it});\;\; h\geq 0,\;\; t\in J_j\Big\} \;\subseteq\; \theta D \quad \text{with } 0<\theta<1. \tag{97}$$

From the representation (90), $f_\ell(z) \leq |z|^{(\mu^\ell)}$ for all $\ell \geq 0$ and $|z| \leq 1$. Hence, for all $z \in U \subseteq \theta D$ we have the bound $\left|f_\ell(z)\right| \leq \theta^{(\mu^\ell)}$. Thus, for $h \geq 0$, $t \in J_j$, and $1 \leq j \leq n$,

$$\left|f_n(e^{-h/c_n+it})\right| \;\leq\; f_{n-j+1}\Big(\left|f_{j-1}(e^{-h/c_n+it})\right|\Big) \;\leq\; \theta^{(\mu^{n-j+1})}, \tag{98}$$

which is the second claim in (95).

If additionally $p_1 > 0$, then by (9) (and our assumption $p_0 = 0$) we have that $p_1^{-\ell} f_\ell(z)$ converges as $\ell \uparrow \infty$, uniformly on each compact $K \subset D^\circ$. Therefore, there exists a constant $C = C(K)$ such that

$$\left|f_\ell(z)\right| \leq C\,p_1^\ell, \qquad \ell\geq 0, \quad z\in K. \tag{99}$$

Consequently, iterating as in (98),

$$\left|f_n(e^{-h/c_n+it})\right| \leq\; C\,p_1^{n-j+1}, \qquad h\geq 0, \quad t\in J_j, \quad 1\leq j\leq n, \tag{100}$$

finishing the proof. □

2.2. **On concentration functions.** Fix for the moment $h \geq 0$ and $n \geq 1$. Denote by $\left\{X_j(h,n)\right\}_{j\geq 1}$ a sequence of independent random variables which equal in law the Cramér transform $Z_n(-h/c_n)$, that is

$$\mathbf{P}\big(X_1(h,n)=k\big) \;=\; \frac{e^{-kh/c_n}}{f_n(e^{-h/c_n})}\,\mathbf{P}(Z_n=k), \qquad k\geq 1. \tag{101}$$

Put

$$S_\ell(h,n) \;:=\; \sum_{j=1}^{\ell} X_j(h,n), \qquad \ell\geq 1. \tag{102}$$

Note that

$$\mathbf{E}e^{itS_\ell(h,n)} \;=\; \big(f_n(e^{-h/c_n+it})/f_n(e^{-h/c_n})\big)^\ell. \tag{103}$$

Recall notation $\alpha \in (0,\infty]$ from (8).

Lemma 9 (A concentration function estimate). *For every $h \geq 0$, there is a constant $A(h)$ such that*

$$\sup_{n,k\geq 1} c_n\,\mathbf{P}\big(S_\ell(h,n)=k\big) \;\leq\; \frac{A(h)}{\ell^{1/2}}, \qquad \ell\geq \ell_0 := 1+[1/\alpha]. \tag{104}$$

Proof. It is known (see, for example, [16, Lemma III.3, p.38]) that for arbitrary (real-valued) random variables X and every $\lambda, T > 0$,

$$Q(X;\lambda) \;:=\; \sup_y \mathbf{P}(y \le X \le y+\lambda) \;\le\; \Big(\frac{96}{95}\Big)^2 \max(\lambda, T^{-1}) \int_{-T}^{T} \big|\psi_X(t)\big|\,\mathrm{d}t \quad (105)$$

(with ψ_X the characteristic function of X). Applying this inequality to $X = S_{\ell_0}(h,n)$ and with $T = \pi d^{-1}$ and $\lambda = 1/2$, using (103) we have

$$\sup_{k\ge 1} \mathbf{P}\big(S_{\ell_0}(h,n) = k\big) \;\le\; C \int_{-\pi d^{-1}}^{\pi d^{-1}} \frac{\big|f_n(e^{-h/c_n + it})\big|^{\ell_0}}{f_n^{\ell_0}(e^{-h/c_n})}\,\mathrm{d}t \quad (106)$$

for some constant C independent of h, n. By (93), for h fixed, $f_n(e^{-h/c_n})$ is bounded away from zero, and consequently, there is a positive constant $C(h)$ such that

$$\sup_{k\ge 1} \mathbf{P}\big(S_{\ell_0}(h,n) = k\big) \;\le\; C(h) \int_{-\pi d^{-1}}^{\pi d^{-1}} \big|f_n(e^{-h/c_n + it})\big|^{\ell_0} \mathrm{d}t. \quad (107)$$

Fist assume that $\alpha < \infty$ (Schröder case). Using the first inequality in (95), we get for $1 \le j \le n$,

$$\int_{J_j} \big|f_n(e^{-h/c_n + it})\big|^{\ell_0}\,\mathrm{d}t \;\le\; A^{\ell_0}\, p_1^{(n-j+1)\ell_0} |J_j| \;\le\; 2\pi d^{-1} A^{\ell_0} p_1^{(n-j+1)\ell_0} c_{j-1}^{-1}. \quad (108)$$

On the other hand,

$$\int_{-\pi d^{-1}/c_n}^{\pi d^{-1}/c_n} \big|f_n(e^{-h/c_n + it})\big|^{\ell_0}\,\mathrm{d}t \;\le\; 2\pi d^{-1}/c_n\,. \quad (109)$$

From (108) and (109), for some constant C,

$$c_n \int_{-\pi d^{-1}}^{\pi d^{-1}} \big|f_n(e^{-h/c_n + it})\big|^{\ell_0}\,\mathrm{d}t \;\le\; C\Big(1 + \sum_{j=1}^{n} p_1^{(n-j+1)\ell_0} c_n\, c_{j-1}^{-1}\Big). \quad (110)$$

But by (4a),

$$c_n \;\le\; m^{n-j+1} c_{j-1}, \qquad 1 \le j \le n. \quad (111)$$

Also, by the definition of ℓ_0 in (104) and α in (8), $p_1^{\ell_0} m = p_1^{1+[1/\alpha]-1/\alpha} < 1$. Hence the right hand side of (110) is bounded in n. Thus, from (107) it follows that

$$\sup_{n,k\ge 1} c_n\, \mathbf{P}\big(S_{\ell_0}(h,n) = k\big) \;\le\; C(h). \quad (112)$$

This estimate actually holds also in the Böttcher case, where $\ell_0 = 1$. Indeed, proceeding in the same way but using the second inequality in (95) instead, the sum expression in (110) has to be replaced by

$$\sum_{j=1}^{n} \theta^{(\mu^{n-j+1})} c_n\, c_{j-1}^{-1} \;\le\; \sum_{j=1}^{n} \theta^{(\mu^{n-j+1})}\, m^{n-j+1} \;=\; \sum_{j=1}^{n} \theta^{(\mu^{j})}\, m^{j}, \quad (113)$$

which again is bounded in n.

Note that (112) is (104) restricted to $\ell = \ell_0$. Hence, from now on we may restrict our attention to $\ell > \ell_0$. Let $Y_1, \ldots, Y_j$ be independent identically distributed random variables. Then by Kesten's inequality (see, e.g., [16, p.57], there is a constant C such that for $0 < \lambda' < 2\lambda$ the concentration function inequality

$$Q(Y_1 + \ldots + Y_j; \lambda) \;\le\; \frac{C\lambda}{\lambda' j^{1/2}}\; Q(Y_1;\lambda)\big[1 - Q(Y_1;\lambda')\big]^{-1/2} \quad (114)$$

holds. We specialize to $Y_1 = S_{\ell_0}(h,n)$ and $\lambda' = \lambda = 1/2$. Note that $Q(Y_1;1/2) = \sup_{k\geq 1} \mathbf{P}\big(S_{\ell_0}(h,n) = k\big) < 1$ in this case, since the random variable $X_1(h,n)$ is non-degenerate. But also as $n \uparrow \infty$ this quantity is bounded away from 1, which follows from (112). Consequently, $\inf_{n\geq 1}[1 - Q(Y_1;1/2)] > 0$. Thus, using again (112), we infer

$$\sup_{n,k\geq 1} \mathbf{P}\big(S_{j\ell_0}(h,n) = k\big) \;\leq\; \frac{C_1(h)}{j^{1/2}} \;=\; \frac{C_2(h)}{(j\ell_0)^{1/2}}\,, \qquad j \geq 1, \tag{115}$$

for some positive constants $C_1(h)$ and $C_2(h)$. If X and Y are independent random variables, then, $Q(X+Y\,;\lambda) \leq Q(X;\lambda)$ (s. [16, Lemma III.1]). Thus for every $\ell > \ell_0$ we have the inequality

$$\sup_{n,k\geq 1} c_n\, \mathbf{P}\big(S_\ell(h,n) = k\big) \;\leq\; \sup_{n,k\geq 1} c_n\, \mathbf{P}\big(S_{[\ell/\ell_0]\ell_0}(h,n) = k\big). \tag{116}$$

Combining this bound once more with (115), the proof is finished. □

Remark 10 (**Special case** $h = 0$)**.** Note that $S_\ell(0,n)$ equals in law to Z_n conditioned to $Z_0 = \ell$. Therefore, by Lemma 9,

$$\sup_{k\geq 1} \mathbf{P}(Z_n = k \,|\, Z_0 = \ell) \;\leq\; \frac{A(0)}{\ell^{1/2}\, c_n}\,, \qquad n \geq 1, \quad \ell \geq \ell_0\,. \tag{117}$$

In particular, if $\alpha > 1$, implying $\ell_0 = 1$, in (117) all initial states Z_0 are possible. Especially, if $Z_0 = 1$, then inequality (117) generalizes the upper estimate in [15, (10)] to processes without $Z_1 \log Z_1$-moment condition. ◇

Lemma 9 can also be used to get very useful bounds for $\mathbf{P}(Z_n = k \,|\, Z_0 = \ell)$ which are not uniform in k. This will be achieved in the next lemma by specializing Lemma 9 to $h = 1$.

Lemma 11 (**Non-uniform bounds**)**.** *There exist two positive constants A and δ such that*

$$c_n\, \mathbf{P}(Z_n = k \,|\, Z_0 = \ell) \;\leq\; A\, e^{k/c_n} \ell^{-1/2}\, e^{-\delta\ell}, \qquad n,k \geq 1, \quad \ell \geq \ell_0\,, \tag{118}$$

[*with ℓ_0 defined in* (104)].

Proof. By the branching property and the definition (102) of $S_\ell(h,n)$,

$$\mathbf{P}(Z_n = k \,|\, Z_0 = \ell) \;=\; e^{kh/c_n} \big[f_n(e^{-h/c_n})\big]^\ell\, \mathbf{P}\big(S_\ell(h,n) = k\big). \tag{119}$$

Putting here $h = 1$ and multiplying both sides by c_n, we have

$$c_n\, \mathbf{P}(Z_n = k \,|\, Z_0 = \ell) \;\leq\; e^{k/c_n} \big[f_n(e^{-1/c_n})\big]^\ell \max_{n,k\geq 1} c_n\, \mathbf{P}\big(S_\ell(1,n) = k\big). \tag{120}$$

Using Lemma 9 gives

$$c_n\, \mathbf{P}(Z_n = k \,|\, Z_0 = \ell) \;\leq\; A(1)\, \ell^{-1/2}\, e^{k/c_n} \big[f_n(e^{-1/c_n})\big]^\ell. \tag{121}$$

From (94) the existence of a $\delta > 0$ follows such that $f_n(e^{-1/c_n}) \leq e^{-\delta}$ for all $n \geq 1$. Entering this into (121) finishes the proof. □

2.3. **On the limiting density function** w. Recall from Section 1.1 that w denotes the density function of W, and $\psi = \psi_W$ its characteristic function.

Lemma 12 (**Bounds for the limiting density**). *There is a constant* $A > 0$ *such that*

$$w^{*\ell}(x) \ \le\ A\Big(\int_0^x w(t)\,\mathrm{d}t\Big)^{\ell-\ell_0}, \qquad x > 0, \quad \ell \ge \ell_0\,. \tag{122}$$

Proof. Suppose $\alpha < \infty$, the case $\alpha = \infty$ can be treated similarly. By the inversion formula,

$$w^{*\ell_0}(x) \ =\ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\psi^{\ell_0}(t)\,\mathrm{d}t, \qquad x > 0. \tag{123}$$

Hence,

$$A \ :=\ \sup_{x>0} w^{*\ell_0}(x) \ \le\ \frac{1}{2\pi}\int_{-\infty}^{\infty} \big|\psi(t)\big|^{\ell_0}\,\mathrm{d}t. \tag{124}$$

We want to convince ourselves that $A < \infty$. For $j \ge 0$,

$$\int_{m^j}^{m^{j+1}} \big|\psi(t)\big|^{\ell_0}\,\mathrm{d}t \ =\ m^j\int_1^m \big|\psi(tm^j)\big|^{\ell_0}\,\mathrm{d}t \ =\ m^j\int_1^m \big|f_j\big(\psi(t)\big)\big|^{\ell_0}\,\mathrm{d}t, \tag{125}$$

where we used (47). Since $W > 0$ has an absolute continuous law, $\big|\psi(t)\big| \le C < 1$ for $t \in [1, m]$. Moreover, by (99), $\big|f_j(z)\big| \le C\,p_1^j$ for z in a compact subset of D°. Therefore,

$$\int_{m^j}^{m^{j+1}} \big|\psi(t)\big|^{\ell_0}\,\mathrm{d}t \ \le\ C\,m^j p_1^{j\ell_0} \ =\ C\,m^{j(1-\alpha\ell_0)} \tag{126}$$

by definition (8) of α. Consequently,

$$\int_1^{\infty} \big|\psi(t)\big|^{\ell_0}\,\mathrm{d}t \ \le\ C\sum_{j=0}^{\infty} m^{j(1-\alpha\ell_0)} \ <\ \infty, \tag{127}$$

since $1 - \alpha\ell_0 < 0$. Analogously,

$$\int_{-\infty}^{-1} \big|\psi(t)\big|^{\ell_0}\,\mathrm{d}t \ <\ \infty. \tag{128}$$

Hence, A in (124) is finite. But $w^{*(\ell+1)}(x) = \int_0^x w^{*\ell}(x-y)\,w(y)\,\mathrm{d}y$, $x > 0$, and the claim follows by induction. □

2.4. **A local central limit theorem.** Recall notation (102) of $S_\ell(h,n)$, $h \ge 0$, $\ell, n \ge 1$. By an abuse of notation, denote by $\psi_\ell = \psi_\ell^{h,n}$ the characteristic function of the random variable

$$\ell^{-1/2}\,\sigma^{-1}(h,n)\,\big(S_\ell(h,n) - \mathbf{E}S_\ell(h,n)\big), \tag{129}$$

where $\sigma(h,n) := \sqrt{\mathbf{E}\,\big(X_1(h,n) - \mathbf{E}X_1(h,n)\big)^2}$. Note that by (103),

$$\psi_\ell^{h,n}(t) \ =\ \bigg(e^{-it\ell^{-1/2}\sigma^{-1}(h,n)\mathbf{E}X_1(h,n)}\,\frac{f_n\big(e^{-h/c_n + it\ell^{-1/2}\sigma^{-1}(h,n)}\big)}{f_n\big(e^{-h/c_n}\big)}\bigg)^{\ell}. \tag{130}$$

Lemma 13 (**An Esseen type Inequality**). *If* $0 < h_1 \le h_2 < \infty$, *then there exist positive constants* $C = C(h_1,h_2)$ *and* $\varepsilon = \varepsilon(h_1,h_2) < 1$ *such that*

$$\sup_{h\in[h_1,h_2],\ n\ge 1} \big|\psi_\ell^{h,n}(t) - e^{-t^2/2}\big| \ \le\ C\,\ell^{-1/2}\,|t|^3\,e^{-t^2/3}, \quad |t| < \varepsilon\,\ell^{1/2}, \ \ \ell \ge 1. \tag{131}$$

Proof. Put $\bar{X}_j(h,n) := X_j(h,n) - \mathbf{E}X_j(h,n)$. Using the global limit theorem from (3) one easily verifies that for some positive constants $C_1,\dots,C_4$,

$$C_1 \;\le\; \frac{\sigma(h,n)}{c_n} \;\le\; C_2 \quad \text{uniformly in } h \in [h_1,h_2] \text{ and } n \ge 1 \tag{132}$$

and

$$C_3 \;\le\; \frac{\mathbf{E}\big|\bar{X}_1(h,n)\big|^3}{c_n^3} \;\le\; C_4 \quad \text{uniformly in } h \in [h_1,h_2] \text{ and } n \ge 1. \tag{133}$$

Consequently, the Lyapunov ratio $\mathbf{E}\big|\bar{X}_1(h,n)\big|^3/\sigma^3(h,n)$ is bounded away from zero and infinity. Applying now Lemma V.1 from [16] to the random variables $\bar{X}_1(h,n),\dots,\bar{X}_\ell(h,n)$ we get the desired result. □

The next lemma is a key step in our development concerning the Böttcher case. Recall notations $S_\ell := S_\ell(h,n)$ and $\sigma := \sigma(h,n)$ defined in (102) and after (129), respectively.

Lemma 14 (**Local central limit theorem**). *Suppose the offspring law is of type (d,μ). If $0 < h_1 \le h_2 < \infty$, then*

$$\sup_{\substack{h\in[h_1,h_2]\\ n\ge 1}} \; \sup_{k:\,k\equiv \ell\mu^n (\mathrm{mod}\, d)} \left| \ell^{1/2}\,\sigma(h,n)\,\mathbf{P}\big(S_\ell(h,n)=k\big) - \frac{d}{\sqrt{2\pi}}\, e^{-x^2_{k,\ell}(h,n)/2} \right| \underset{\ell\uparrow\infty}{\longrightarrow} 0,$$

where $x_{k,\ell} := x_{k,\ell}(h,n) := \ell^{-1/2}\,\sigma^{-1}(h,n)\,\big(k - \ell\,\mathbf{E}X_1(h,n)\big)$.

Note that a local limit theorem, which would correspond to our case $h=0$ but concerning an offspring law with finite variance and with initial state tending to ∞, was derived by Höpfner [12, Theorem 1]. The following proof of our lemma is a bit simpler, since for $h>0$ the random variables $X_1(h,n)$ have finite moments of all orders (also if the underlying Z does not have finite variance).

Proof of Lemma 14. By (103) and the inversion formula,

$$\mathbf{P}\big(S_\ell = k\big) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itk}\Big[\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\Big]^\ell \mathrm{d}t. \tag{134}$$

Decomposing the unit circle,

$$\big\{e^{it} : -\pi < t \le \pi\big\} \;=\; \bigcup_{j=0}^{d-1}\big\{\varrho^j\, e^{it} : -\pi d^{-1} < t \le \pi d^{-1}\big\}, \tag{135}$$

where $\varrho := e^{2\pi i/d}$, the latter integral equals

$$\sum_{j=0}^{d-1}\int_{-\pi d^{-1}}^{\pi d^{-1}} \varrho^{-jk} e^{-itk}\Big[\frac{f_n(\varrho^j e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\Big]^\ell \mathrm{d}t. \tag{136}$$

It is known (see, for instance, [1, p.105]) that for an offspring law of type (d,μ) we have

$$f_n(\varrho^j z) \;=\; \varrho^{j\mu^n} f_n(z), \qquad n,j\ge 1, \quad z\in D. \tag{137}$$

Therefore the latter sum equals

$$\int_{-\pi d^{-1}}^{\pi d^{-1}} e^{-itk}\Big[\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\Big]^\ell \mathrm{d}t \,\sum_{j=0}^{d-1}\varrho^{-j(k-\ell\mu^n)}. \tag{138}$$

But $\varrho^{-j(k-\ell\mu^n)} \equiv 1$ for $k \equiv \ell\mu^n \pmod d$. Altogether, for (134) we get

$$\mathbf{P}\big(S_\ell = k\big) \;=\; \frac{d}{2\pi}\int_{-\pi d^{-1}}^{\pi d^{-1}} e^{-itk}\Big[\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\Big]^\ell \mathrm{d}t, \qquad k \equiv \ell\mu^n \pmod d. \quad (139)$$

Using the substitution $t \to t/\ell^{1/2}\sigma$ and (130), we arrive at

$$\mathbf{P}\big(S_\ell = k\big) \;=\; \frac{d}{2\pi\ell^{1/2}\sigma}\int_{-\pi d^{-1}\ell^{1/2}\sigma}^{\pi d^{-1}\ell^{1/2}\sigma} e^{-itx_{k,\ell}}\psi_\ell(t)\,\mathrm{d}t, \qquad k \equiv \ell\mu^n \pmod d. \quad (140)$$

Fix $0 < h_1 \le h_2 < \infty$. Recall from (132) that

$$C_1 \;\le\; \inf_{h\in[h_1,h_2],\,n\ge 1}\frac{\sigma(h,n)}{c_n} \;\le\; \sup_{h\in[h_1,h_2],\,n\ge 1}\frac{\sigma(h,n)}{c_n} \;\le\; C_2 \quad (141)$$

for some $0 < C_1 < C_2$ (depending on h_1, h_2). Choose a positive

$$\varepsilon \;=\; \varepsilon(h_1,h_2) \;<\; C_1\pi d^{-1} \quad (142)$$

as in Lemma 13. Take any $A = A(h_1,h_2) > \varepsilon$ (to be specified later). Then the identity $\int_{-\infty}^{\infty} e^{-itx-t^2/2}\,\mathrm{d}t = \sqrt{2\pi}\;e^{-x^2/2}$ and representation (140) imply that

$$\sup_{k:\,k\equiv\ell\mu^n \,(\mathrm{mod}\, d)}\Big|\ell^{1/2}\,\sigma\,\mathbf{P}\big(S_\ell = k\big) - \frac{d}{\sqrt{2\pi}}\;e^{-x_{k,\ell}^2/2}\Big| \;\le\; d\,(I_1+I_2+I_3+I_4), \quad (143)$$

where

$$I_1 \;:=\; \int_{-\varepsilon\ell^{1/2}}^{\varepsilon\ell^{1/2}}\big|\psi_\ell(t) - e^{-t^2/2}\big|\,\mathrm{d}t, \quad I_2 \;:=\; \int_{|t|>\varepsilon\ell^{1/2}} e^{-t^2/2}\,\mathrm{d}t, \quad (144)$$

$$I_3 \;:=\; \int_{\varepsilon\ell^{1/2}<|t|<A\ell^{1/2}}\big|\psi_\ell(t)\big|\,\mathrm{d}t, \quad I_4 \;:=\; \int_{A\ell^{1/2}<|t|<\pi d^{-1}\ell^{1/2}\sigma}\big|\psi_\ell(t)\big|\,\mathrm{d}t.$$

[Of course, I_4 disappears if $A(h_1,h_2) > \pi d^{-1}\sigma(h,n)$.]

Trivially, $I_2 \to 0$ as $\ell \uparrow \infty$. Further, due to Lemma 13, there is a $C = C(h_1,h_2)$ such that

$$I_1 \;\le\; C\,\ell^{-1/2}\int_0^{\varepsilon\ell^{1/2}} t^3\,e^{-t^2/3}\,\mathrm{d}t \;\le\; C\,\ell^{-1/2} \;\underset{\ell\uparrow\infty}{\longrightarrow}\; 0. \quad (145)$$

Thus, it remains to show that the integrals I_3 and I_4 converge to zero as $\ell \uparrow \infty$, uniformly in the considered h and n.

First of all, using again (130) and substituting $t \to t\ell^{1/2}\sigma/c_n$, by (141) we obtain the following estimates

$$I_3 \;\le\; C_2\,\ell^{1/2}\int_{\varepsilon/C_2<|t|<A/C_1}\Big|\frac{f_n(e^{-h/c_n+it/c_n})}{f_n(e^{-h/c_n})}\Big|^\ell dt, \quad (146a)$$

$$I_4 \;\le\; C_2\,\ell^{1/2}\int_{A/C_2<|t|<\pi d^{-1}c_n}\Big|\frac{f_n(e^{-h/c_n+it/c_n})}{f_n(e^{-h/c_n})}\Big|^\ell dt. \quad (146b)$$

First we fix our attention to I_3. By (86),

$$f_n(e^{-h/c_n+it/c_n}) \;\to\; \mathbf{E}e^{-hW+itW} \quad \text{as } n\uparrow\infty, \quad (147)$$

uniformly in $h \in [0,h_2]$ and $t \in [0, A/C_1]$ [recall (142)]. Also, by (93),

$$f_n(e^{-h/c_n}) \;\to\; \mathbf{E}e^{-hW} \quad \text{as } n\uparrow\infty, \text{ uniformly in } h\in[0,h_2]. \quad (148)$$

It follows that

$$\frac{f_n(e^{-h/c_n+it/c_n})}{f_n(e^{-h/c_n})} \underset{n\uparrow\infty}{\longrightarrow} \frac{\mathbf{E}e^{-hW+itW}}{\mathbf{E}e^{-hW}} = \mathbf{E}e^{itW(-h)}, \tag{149}$$

uniformly in $h \in [0, h_2]$ and $t \in [0, A/C_1]$ (with $W(-h)$ the Cramér transform of W). Since the $W(-h)$ have absolutely continuous laws, we have $|\mathbf{E}e^{itW(-h)}| < 1$ for all $h \geq 0$ and $|t| > 0$. This inequality and continuity of $(h,t) \mapsto \mathbf{E}e^{itW(-h)}$ imply that

$$\sup_{0\leq h\leq h_2\,,\ \varepsilon/C_2\leq|t|\leq A/C_1} \frac{|\mathbf{E}e^{-hW+itW}|}{\mathbf{E}e^{-hW}} < 1. \tag{150}$$

Using (149) and (150) we infer the existence of a positive constant $\eta = \eta(h_1, h_2) < 1$ and an $n_1 = n_1(h_1, h_2) \geq 1$ such that for $n \geq n_1$,

$$\sup_{0\leq h\leq h_2,\ \varepsilon/C_2\leq|t|\leq A/C_1} \left|\frac{f_n(e^{-h/c_n+it/c_n})}{f_n(e^{-h/c_n})}\right| \leq \eta. \tag{151}$$

Applying (151) to the bound of I_3 in (146a), we conclude that

$$I_3 \leq CA\,\ell^{1/2}\eta^\ell \to 0 \quad \text{as } \ell\uparrow\infty, \tag{152}$$

uniformly in $h \in [h_1, h_2]$ and $n \geq n_1$. (The remaining n will be considered below.)

Next, we prepare for the estimation of I_4. Since $f_n(e^{-h/c_n}) \geq f_n(e^{-h_2/c_n})$ for $0 \leq h \leq h_2$, and $f_n(e^{-h_2/c_n}) \to \mathbf{E}e^{-h_2W} > 0$ as $n\uparrow\infty$ [recall (148)], there is a positive constant $C = C(h_2)$ such that

$$\left|\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\right| \leq C\left|f_n(e^{-h/c_n+it})\right| \tag{153}$$

for all $t \in \mathbb{R}$, $0 \leq h \leq h_2$, and $n \geq 1$.

At this point we have to distinguish between Schröder and Böttcher cases. Actually, we proceed with the Böttcher case $\alpha = \infty$, which is the only case we need later, and leave the other case for the reader. Applying the second case of (95) to (153), we obtain the estimate

$$\left|\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\right| \leq C\exp\bigl[-\mu^{n-j+1}\log\theta^{-1}\bigr], \tag{154}$$

$0 \leq h \leq h_2$, $t \in J_j$, and $1 \leq j \leq n$. Since $\mu \geq 2$, there exists an $n_2 = n_2(h_2)$ such that

$$\left|\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\right| \leq \exp\bigl[-\mu^{n-j}\log\theta^{-1}\bigr], \tag{155}$$

if $0 \leq h \leq h_2$, $t \in J_j$, and $1 \leq j \leq n - n_2$. But $|J_j| \leq 2c_{j-1}^{-1}\pi d^{-1}$, hence

$$\int_{J_j}\left|\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\right|^\ell \mathrm{d}t \leq 2c_{j-1}^{-1}\pi d^{-1}\exp\bigl[-\ell\,\mu^{n-j}\log\theta^{-1}\bigr]. \tag{156}$$

Summing over the considered j gives

$$\int_{c_{n-n_2}^{-1}\pi d^{-1}\leq|t|\leq\pi d^{-1}}\left|\frac{f_n(e^{-h/c_n+it})}{f_n(e^{-h/c_n})}\right|^\ell \mathrm{d}t \leq 2\pi d^{-1}\sum_{j=1}^{n-n_2} c_{j-1}^{-1}\exp\bigl[-\ell\,\mu^{n-j}\log\theta^{-1}\bigr],$$

$0 \le h \le h_2$ and $n \ge n_2$. Substituting $t \to t/c_n$ and using (111), we arrive at

$$\int_{\pi d^{-1} m^{n_2} \le |t| \le \pi d^{-1} c_n} \left| \frac{f_n(e^{-h/c_n + it/c_n})}{f_n(e^{-h/c_n})} \right|^{\ell} dt \tag{157}$$

$$\le 2\pi d^{-1} \sum_{j=1}^{n-n_2} m^{n-j+1} \exp\left[-\ell\, \mu^{n-j} \log \theta^{-1}\right]$$

$$\le 2\pi d^{-1} \sum_{j=1}^{\infty} m^{j+1} \exp\left[-\ell\, \mu^{j} \log \theta^{-1}\right] \le C\, e^{-C'\ell}$$

with constants C, C', uniformly in $h \in [h_1, h_2]$ and $n \ge n_2$. Choosing now A so large that $\pi d^{-1} m^{n_2} \le A/C_2$, we conclude from (146b) that

$$I_4 \le C\, \ell^{1/2}\, e^{-C'\ell} \to 0 \quad \text{as } \ell \uparrow \infty, \tag{158}$$

uniformly in $h \in [h_1, h_2]$ and $n \ge n_2$.

Finally, we consider all $n \le n^* := n_1 \vee n_2$. By definition, as in (90),

$$\frac{f_n(e^{-h/c_n + it/c_n})}{f_n(e^{-h/c_n})} = \sum_{j=0}^{\infty} \mathbf{P}\big(X_1(h,n) = \mu^n + jd\big)\, e^{(it/c_n)(\mu^n + jd)}. \tag{159}$$

Hence, since the set $\big\{e^{-it/c_n} : t \in [\varepsilon/C_2, \pi d^{-1} c_n]\big\}$ does not contain the d^{th} roots of unity,

$$\sup_{t \in [\varepsilon/C_2,\, \pi d^{-1} c_n]} \left| \frac{f_n(e^{-h/c_n + it/c_n})}{f_n(e^{-h/c_n})} \right| =: \theta_n(h) < 1. \tag{160}$$

From the continuity $(h,t) \to f_n(e^{-h/c_n + it/c_n})$ it follows that the function θ_n is continuous, too. Therefore,

$$\sup_{h \in [h_1, h_2]} \theta_n(h) =: \bar\theta_n < 1. \tag{161}$$

Combining (160) and (161),

$$\max_{n \le n^*} \sup_{\substack{h \in [h_1, h_2] \\ t \in [\varepsilon/C_2,\, \pi d^{-1} c_n]}} \left| \frac{f_n(e^{-h/c_n + it/c_n})}{f_n(e^{-h/c_n})} \right| \le \bar\theta \tag{162}$$

for some $\bar\theta < 1$. Substituting this into (146) gives

$$I_3 + I_4 \le C\, \ell^{1/2}\, \bar\theta^{\ell} \to 0 \quad \text{as } \ell \uparrow \infty, \tag{163}$$

and the proof is finished. □

3. Proof of the main results

3.1. **Schröder case (proof of Theorem 4).** Let f, k_n, and a_n be as in Theorem 4. Fix n_0 such that $c_n > k_n \ge 1$ and $n > a_n \ge 1$ for all $n \ge n_0$, and consider only such n. Recall that $p_0 = 0$ by our convention. By the Markov property,

$$\mathbf{P}(Z_n = k_n) = \sum_{\ell=1}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, \mathbf{P}(Z_{a_n} = k_n \mid Z_0 = \ell). \tag{164}$$

and

$$\mathbf{P}(Z_n \le k_n) = \sum_{\ell=1}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, \mathbf{P}(Z_{a_n} \le k_n \mid Z_0 = \ell). \tag{165}$$

Step 1° (*Proof of* (59)). Using Lemma 11 we get for $N \geq \ell_0$ the estimate

$$c_{a_n} \sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, \mathbf{P}(Z_{a_n} = k_n \,|\, Z_0 = \ell) \;\leq\; C\, \frac{e^{k_n/c_{a_n}}}{N^{1/2}}\, f_{n-a_n}(e^{-\delta}) \tag{166}$$

for some constant $\delta > 0$. By (4a), and since $c_{a_n-1} < k_n \leq c_{a_n}$ by the definition of a_n,

$$m^{-1} \;\leq\; \frac{c_{a_n-1}}{c_{a_n}} \;\leq\; \frac{k_n}{c_{a_n}} \;\leq\; 1. \tag{167}$$

On the other hand, by (99),

$$f_{n-a_n}(e^{-\delta}) \;\leq\; C\, p_1^{n-a_n}. \tag{168}$$

Thus, from (166),

$$p_1^{a_n-n} c_{a_n} \sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, \mathbf{P}(Z_{a_n} = k_n \,|\, Z_0 = \ell) \;\leq\; \frac{C}{N^{1/2}}. \tag{169}$$

By [10, Lemma 9],

$$\lim_{n\uparrow\infty} \frac{1}{2\pi} \int_{-\pi d^{-1} c_n}^{\pi d^{-1} c_n} f_n^{\ell}(e^{it/c_n})\, e^{-itx}\, \mathrm{d}t \;=\; w^{*\ell}(x) \tag{170}$$

uniformly in $x \in [m^{-1}, 1]$. This together with

$$\begin{aligned} & c_{a_n} \mathbf{P}(Z_{a_n} = k_n \,|\, Z_0 = \ell) \\ &= \frac{d}{2\pi} \int_{-\pi d^{-1} c_n}^{\pi d^{-1} c_n} f_{a_n}^{\ell}(e^{it/c_n})\, e^{-itk_n/c_{a_n}}\, \mathrm{d}t, \qquad \ell \equiv k_n \pmod d, \end{aligned} \tag{171}$$

(see [1, p.105]) and (167) gives

$$\lim_{n\uparrow\infty} \Big(c_{a_n} \mathbf{P}(Z_{a_n} = k_n \,|\, Z_0 = \ell) - d\, w^{*\ell}(k_n/c_{a_n}) \Big) \;=\; 0, \qquad \ell \equiv k_n \pmod d. \tag{172}$$

Since $k_n \equiv 1 \pmod d$, the previous statement holds for all $\ell \equiv 1 \pmod d$. For other ℓ, the probabilities $\mathbf{P}(Z_{n-a_n} = \ell)$ disappear. Thus, by (172),

$$\begin{aligned} & \sum_{\ell=1}^{N-1} \mathbf{P}(Z_{n-a_n} = \ell)\, \mathbf{P}(Z_{a_n} = k_n \,|\, Z_0 = \ell) \\ &= d\, c_{a_n}^{-1} \Big[\sum_{\ell=1}^{N-1} \mathbf{P}(Z_{n-a_n} = \ell)\, w^{*\ell}(k_n/c_{a_n}) \Big] \big(1 + o_N(1)\big) \end{aligned} \tag{173}$$

with $o_N(1) \to 0$ as $n \uparrow \infty$, for each fixed N. Further, using Lemma 12, one can easily verify that there exist two constants C and $\eta \in (0,1)$ such that $w^{*\ell}(k_n/c_{a_n}) \leq C\,\eta^{\ell}$ for all $\ell \geq 1$ and n. Thus,

$$\sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, w^{*\ell}(k_n/c_{a_n}) \;\leq\; C \sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, \eta^{\ell}. \tag{174}$$

But for every $\eta_1 \in (\eta, 1)$,

$$\sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, \eta^{\ell} \;\leq\; \Big(\frac{\eta}{\eta_1}\Big)^{N} f_{n-a_n}(\eta_1) \;\leq\; C \Big(\frac{\eta}{\eta_1}\Big)^{N} p_1^{n-a_n}, \tag{175}$$

where in the last step we used (99). Inequalities (174) and (175) imply

$$\sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n}=\ell)\, w^{*\ell}(k_n/c_{a_n}) \;\le\; C\, p_1^{n-a_n}\, e^{-\delta N} \tag{176}$$

for all n, N and some constant $\delta > 0$. Combining (164), (173), (169) and (176), we have

$$\begin{aligned}\mathbf{P}(Z_n=k_n) \;=\; & d\, c_{a_n}^{-1}\Big[\sum_{\ell=1}^{\infty} \mathbf{P}(Z_{n-a_n}=\ell)\, w^{*\ell}(k_n/c_{a_n})\Big]\big(1+o_N(1)\big) \\ & + O\Big(c_{a_n}^{-1}\, p_1^{n-a_n} N^{-1/2}\Big),\end{aligned} \tag{177}$$

where the O-term applies to both $n, N \uparrow \infty$. By (47),

$$m^{-j}\, w(x/m^j) \;=\; \sum_{\ell=1}^{\infty} \mathbf{P}(Z_j=\ell)\, w^{*\ell}(x), \qquad j \ge 1, \quad x > 0. \tag{178}$$

Putting here $j = n - a_n$, $x = k_n/c_{a_n}$, and substituting into (177), we arrive at

$$\mathbf{P}(Z_n=k_n) \;=\; d\, c_{a_n}^{-1}\, m^{a_n-n}\, w(k_n m^{a_n-n}/c_{a_n})\,\big(1+o_N(1)\big) + O\Big(c_{a_n}^{-1}\, p_1^{n-a_n} N^{-1/2}\Big).$$

By (29), (167), and the definition (8) of α,

$$d\, c_{a_n}^{-1}\, m^{a_n-n}\, w(k_n m^{a_n-n}/c_{a_n}) \;\ge\; C\, c_{a_n}^{-1}\, m^{\alpha(a_n-n)} \;=\; C\, c_{a_n}^{-1}\, p_1^{n-a_n}, \quad \text{for all } n. \tag{179}$$

Therefore,

$$\mathbf{P}(Z_n=k_n) \;=\; d\, c_{a_n}^{-1}\, m^{a_n-n}\, w(k_n m^{a_n-n}/c_{a_n})\,\Big(1+o_N(1)+O(N^{-1/2})\Big), \tag{180}$$

where the O-term now applies to $N \uparrow \infty$, uniformly in n. Letting first $n \uparrow \infty$ and then $N \uparrow \infty$, we see that (59) is true.

Step 2° (*Proof of* (60)). Trivially, for independent and identically distributed non-negative random variables $X_1, \ldots, X_n$ we have

$$\mathbf{P}(X_1+\ldots+X_n < x) \;\le\; \mathbf{P}(\max_j X_j < x) \;=\; \mathbf{P}^n(X_1 < x), \qquad x \ge 0. \tag{181}$$

Hence,

$$\mathbf{P}(Z_{a_n} \le k_n \,|\, Z_0=\ell) \;\le\; \mathbf{P}^{\ell}(Z_{a_n} \le k_n). \tag{182}$$

Further, from (167) and (3),

$$\mathbf{P}(Z_{a_n} \le k_n) \;\le\; \mathbf{P}(c_{a_n}^{-1} Z_{a_n} \le 1) \;\underset{n\uparrow\infty}{\longrightarrow}\; \int_0^1 w(x)\, dx. \tag{183}$$

Therefore, since $w > 0$ on all of $(0, \infty)$, there exists an $\eta \in (0,1)$ such that $\mathbf{P}(Z_{a_n} \le k_n) \le \eta$ for all n large enough. Thus,

$$\sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n}=\ell)\, \mathbf{P}(Z_{a_n} \le k_n \,|\, Z_0=\ell) \;\le\; \sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n}=\ell)\, \eta^{\ell} \tag{184}$$

for all N sufficiently large. Taking into account (175), we conclude that

$$\sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n}=\ell)\, \mathbf{P}(Z_{a_n} \le k_n \,|\, Z_0=\ell) \;\le\; C\, p_1^{n-a_n}\, e^{-\delta N} \tag{185}$$

for N sufficiently large and some $\delta > 0$. By the same arguments,

$$\sum_{\ell=N}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, F^{*\ell}(k_n/c_{a_n}) \;\le\; C\, p_1^{n-a_n}\, e^{-\delta N}, \tag{186}$$

where $F(x) := \mathbf{P}(W < x),\ \ x \ge 0$.

On the other hand, the continuity of F and (3) yield that $\mathbf{P}(Z_{a_n} \le c_{a_n} x \,|\, Z_0 = \ell) \to F^{*\ell}(x)$ uniformly in $x \ge 0$. Therefore,

$$\lim_{n\uparrow\infty} \sup_{k\ge 1} \Big| \mathbf{P}(Z_{a_n} \le k \,|\, Z_0 = \ell) - F^{*\ell}(k/c_{a_n}) \Big| \;=\; 0. \tag{187}$$

Combining (165), (185), (186), and (187), we arrive at

$$\begin{aligned} &\mathbf{P}(Z_n \le k_n) \\ &\quad= \Big[\sum_{\ell=1}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, F^{*\ell}(k_n/c_{a_n})\Big] \big(1 + o_N(1)\big) \;+\; O(p_1^{n-a_n}\, e^{-\delta N}) \end{aligned} \tag{188}$$

with the same meaning of o_N and the O-term as in the previous step of proof. Since $\mathbf{P}(Z_{n-a_n} = 1) = p_1^{n-a_n}$ and $F(k_n/c_{a_n}) \ge F(m^{-1}) > 0$ by (167), we obtain

$$p_1^{n-a_n}\, e^{-\delta N} \;\le\; C\, e^{-\delta N} \sum_{\ell=1}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, F^{*\ell}(k_n/c_{a_n}). \tag{189}$$

Combining this inequality with (188) gives

$$\mathbf{P}(Z_n \le k_n) \;=\; \Big[\sum_{\ell=1}^{\infty} \mathbf{P}(Z_{n-a_n} = \ell)\, F^{*\ell}(k_n/c_{a_n})\Big] \big(1 + o_N(1) + O(e^{-\delta N})\big). \tag{190}$$

Integrating both parts of (178), one has

$$F(y/m^k) \;=\; \sum_{\ell=1}^{\infty} \mathbf{P}(Z_k = \ell)\, F^{*\ell}(y), \qquad k \ge 1, \quad y > 0. \tag{191}$$

Thus,

$$\mathbf{P}(Z_n \le k_n) \;=\; F\Big(\frac{k_n}{c_{a_n}\, m^{n-a_n}}\Big) \big(1 + o_N(1) + O(e^{-\delta N})\big). \tag{192}$$

Letting again first $n \uparrow \infty$ and then $N \uparrow \infty$ finishes the proof. □

Remark 15 (Proof in the case $p_0 > 0$). We indicate now how to proceed with the proof of Theorem 4 in the remaining case $p_0 > 0$. Here in the representation (164) one has additionally to take into account that

$$\begin{aligned} &\mathbf{P}(Z_{a_n} = k_n \,|\, Z_0 = \ell) \\ &\quad= \sum_{j=1}^{\ell} \binom{\ell}{j} f_{a_n}^{\ell-j}(0)\, \big(1 - f_{a_n}(0)\big)^j\, \mathbf{P}\Big\{ \sum_{i=1}^{j} Z_{a_n}^{(i)} = k_n \;\Big|\; Z_{a_n}^{(i)} > 0,\ 1 \le i \le j \Big\}, \end{aligned} \tag{193}$$

where the $Z^{(1)}, Z^{(2)}, \ldots$ are independent copies of Z. Then instead of Lemma 11 we need

$$c_n\, \mathbf{P}\Big\{ \sum_{i=1}^{j} Z_{a_n}^{(i)} = k_n \;\Big|\; Z_{a_n}^{(i)} > 0,\ 1 \le i \le j \Big\} \;\le\; A\, e^{k/c_n} j^{-1/2}\, e^{-\delta \ell}, \quad n, k \ge 1,\ j \ge \ell_0\,.$$

But this is valid by

$$\mathbf{P}\big\{z^{Z_n^{(1)}} \,\big|\, Z_n^{(1)} > 0\big\} \;=\; \frac{f_n(z) - f_n(0)}{1 - f_n(0)} \underset{n\uparrow\infty}{\longrightarrow} \frac{\mathsf{S}(z) - \mathsf{S}(0)}{1 - q}\,, \qquad (194)$$

uniformly in z from compact subsets of D°. This indeed follows from (9). $\diamond$

3.2. **Böttcher case (proof of Theorem 5).** From the Markov property,

$$\mathbf{P}(Z_n = k_n) \;=\; \sum_{\ell=\mu^{n-b_n}}^{\infty} \mathbf{P}(Z_{n-b_n} = \ell)\, \mathbf{P}(Z_{b_n} = k \,|\, Z_0 = \ell). \qquad (195)$$

Using (119) and Lemma 9, we obtain the following estimate

$$c_{b_n}\mathbf{P}(Z_{b_n} = k_n \,|\, Z_0 = \ell) \;\le\; A(h)\, \ell^{-1/2}\big[e^{hk_n/\ell c_{b_n}} f_{b_n}(e^{-h/c_{b_n}})\big]^{\ell}. \qquad (196)$$

From the definition of b_n it immediately follows that

$$2k_n \;\le\; c_{b_n}\mu^{n-b_n} \;=\; c_{b_n-1}\mu^{n-b_n+1}\Big(\frac{c_{b_n}}{\mu c_{b_n-1}}\Big) \;\le\; 2k_n\frac{m}{\mu}\,. \qquad (197)$$

Hence,

$$\frac{hk_n}{\ell c_{b_n}} \;\le\; \frac{h}{2} \qquad (198)$$

for $\ell \ge \mu^{n-b_n}$. Therefore,

$$c_{b_n}\mathbf{P}(Z_{b_n} = k_n \,|\, Z_0 = \ell) \;\le\; A(h)\, \ell^{-1/2}\big[e^{h/2} f_{b_n}(e^{-h/c_{b_n}})\big]^{\ell}. \qquad (199)$$

It is known (see, for example, [1], Corollary III.5.7), that $\mathbf{E}W = 1$ if $\mathbf{E}Z_1 \log Z_1 < \infty$ and $\mathbf{E}W = \infty$ otherwise. It means, that for the Laplace function $\varphi = \varphi_W$ of W we have $e^{h/2}\varphi(h) < 1$ for all small enough h. Thus, due to the global limit theorem (3), there exist $\delta < 1$ and $h_0 > 0$ such that $e^{h_0/2} f_n(e^{-h_0/c_n}) \le e^{-\delta}$ for all large enough n. Hence,

$$c_{b_n}\mathbf{P}(Z_{b_n} = k_n \,|\, Z_0 = \ell) \;\le\; A\, \ell^{-1/2}\, e^{-\delta\ell}. \qquad (200)$$

Inserting (200) into (195), we obtain

$$c_{b_n}\mathbf{P}(Z_n = k_n) \;\le\; A\, \mu^{-(n-b_n)/2} f_{n-b_n}(e^{-\delta}), \qquad (201)$$

consequently,

$$\mu^{b_n-n}\log\big[c_n\, \mathbf{P}(Z_n = k_n)\big] \;\le\; \mu^{b_n-n}C + \mu^{b_n-n}\log\Big(\frac{c_n}{c_{b_n}}\Big) + \frac{\log f_n(e^{-\delta})}{\mu^{n-b_n}}\,. \qquad (202)$$

Since $c_n/c_{b_n} \le m^{n-b_n}$ and $\mu^{n-b_n} = m^{\beta(n-b_n)}$, $\mu^{b_n-n}\log(c_n/c_{b_n}) \to 0$ as $n \uparrow \infty$. Thus,

$$\limsup_{n\uparrow\infty} \mu^{b_n-n}\log\big[c_n\, \mathbf{P}(Z_n = k_n)\big] \;\le\; \limsup_{n\uparrow\infty} \frac{\log f_{n-b_n}(e^{-\delta})}{\mu^{n-b_n}}\,. \qquad (203)$$

Using (21), we arrive at the desired upper bound.

We show now that (77b) holds for $\log \mathbf{P}(Z_n \le k_n)$. First of all we note that for arbitrary non-negative random variable X and all $x, h \ge 0$

$$\mathbf{P}(X \le x) \;\le\; e^{hx}\, \mathbf{E}e^{-hX}. \qquad (204)$$

Applying this bound to the process Z starting from ℓ individuals and taking into account (198), we have

$$\mathbf{P}(Z_{b_n} \le k_n \,|\, Z_0 = \ell) \;\le\; \big[e^{hk_n/\ell c_{b_n}} f_{b_n}(e^{-h/c_{b_n}})\big]^{\ell} \;\le\; \big[e^{h/2} f_{b_n}(e^{-h/c_{b_n}})\big]^{\ell}. \qquad (205)$$

As we argued in the derivation of (200), this gives

$$\mathbf{P}(Z_{b_n} \le k_n \,|\, Z_0 = \ell) \ \le\ e^{-\delta \ell}. \tag{206}$$

Consequently, by the Markov property,

$$\mathbf{P}(Z_n \le k_n) \ \le\ f_{n-b_n}(e^{-\delta}). \tag{207}$$

Taking logarithm and using (21), we obtain (77b).

Let us verify the lower bounds in Theorem 5. By (195),

$$\mathbf{P}(Z_n = k_n) \ \ge\ \mathbf{P}(Z_{n-b_n} = \mu^{n-b_n})\, \mathbf{P}(Z_{b_n} = k_n \,|\, Z_0 = \mu^{n-b_n}). \tag{208}$$

From (119),

$$\mathbf{P}(Z_{b_n} = k_n \,|\, Z_0 = \mu^{n-b_n}) \ >\ \big[f_{b_n}(e^{-h/c_{b_n}})\big]^{\ell_n}\, \mathbf{P}\big(S_{\ell_n}(h,b_n) = k_n\big), \tag{209}$$

where $\ell_n = \mu^{n-b_n}$.

Consider the equation

$$c_{b_n}^{-1}\mathbf{E}X_1(h,b_n) \ =\ \frac{f'_{b_n}(e^{-h/c_{b_n}})\, e^{-h/c_{b_n}}}{c_{b_n} f_{b_n}(e^{-h/c_{b_n}})} \ =\ x. \tag{210}$$

Evidently,

$$\frac{f'_{b_n}(e^{-h/c_{b_n}})\, e^{-h/c_{b_n}}}{f_{b_n}(e^{-h/c_{b_n}})}\Big|_{h=0} \ =\ m^{b_n} \tag{211}$$

and

$$\frac{f'_{b_n}(e^{-h/c_{b_n}})\, e^{-h/c_{b_n}}}{f_{b_n}(e^{-h/c_{b_n}})}\Big|_{h=\infty} \ =\ \mu^{b_n}. \tag{212}$$

From these identities and monotonicity of $f'_{b_n}(e^{-h/c_{b_n}})\, e^{-h/c_{b_n}} / f_{b_n}(e^{-h/c_{b_n}})$ it follows that (210) has a unique solution $h_n(x)$ for $\mu^{b_n} c_{b_n}^{-1} < x < m^{b_n} c_{b_n}^{-1}$. Analogously one shows that the equation $\varphi'(h)/\varphi(h) = -x$ has also a unique solution $h(x)$. By the integral limit theorem (3), the right-hand side in (210) converges to $-\varphi'(h)/\varphi(h)$ and consequently, $h_n(x) \to h(x)$ as $n \uparrow \infty$. Further, by (197),

$$\frac{\mu}{2m} \ \le\ x_n \ :=\ \frac{k_n}{c_{b_n}\ell_n} \ \le\ \frac{1}{2}\,. \tag{213}$$

Thus,

$$h(\mu/2m) \ \le\ \liminf_{n\uparrow\infty} h_n \ \le\ \liminf_{n\uparrow\infty} h_n \ \le\ h(1/2), \tag{214}$$

where $h_n := h_n(x_n)$. It means that there exist h_* and h^* such that $h_* \le h_n \le h^*$ for all $n \ge 1$. From the definition of h_n and (210) immediately follows that $\mathbf{E}S_{\ell_n}(h_n,b_n) = k_n$. Thus, applying Lemma 14, we get

$$\lim_{n\uparrow\infty} \left| \ell_n^{1/2}\, \sigma(h_n,b_n)\, \mathbf{P}\big(S_{\ell_n}(h_n,b_n) = k_n\big) \ -\ \frac{d}{\sqrt{2\pi}} \right| \ =\ 0. \tag{215}$$

Recall that by (132) we have $\sigma(h_n,b_n) \ge C\, c_{b_n}$. Hence,

$$\liminf_{n\uparrow\infty} \ell_n^{1/2} c_{b_n}\, \mathbf{P}\big(S_{\ell_n}(h_n,b_n) = k_n\big) \ \ge C \ >\ 0. \tag{216}$$

Moreover, since $f_{b_n}(e^{-h_n/c_{b_n}}) \ge f_{b_n}(e^{-h^*/c_{b_n}})$ and $f_j(e^{-h^*/c_j}) \to \mathbf{E}e^{-h^*W} > 0$, there exists a $\theta > 0$ such that

$$f_{b_n}(e^{-h/c_{b_n}}) \ \ge\ \theta \tag{217}$$

for all n. Applying these bounds to the right-hand side in (209), we find that

$$\liminf_{n\uparrow\infty} \mu^{b_n-n} \log\left[c_n\ \mathbf{P}(Z_{b_n} = k_n \,|\, Z_0 = \mu^{n-b_n})\right] \ \geq\ -C. \tag{218}$$

Using this inequality and (21) to bound the right-hand side in (208), we conclude that

$$\liminf_{n\uparrow\infty} \mu^{b_n-n} \log\bigl[c_n\, \mathbf{P}(Z_n = k_n)\bigr] \ \geq\ -C, \tag{219}$$

i.e. (77a) is proved.

Next we want to extend this result to $\mathbf{P}(Z_n \leq k_n)$. Obviously,

$$\mathbf{P}(Z_n \leq k_n) \ \geq\ \mathbf{P}(Z_{n-b_n} = \ell_n)\, \mathbf{P}(Z_{b_n} \leq k_n \,|\, Z_0 = \ell_n). \tag{220}$$

Then, using (119) with $h = h_n$, we have

$$\mathbf{P}(Z_n \leq k_n) \ \geq\ \mathbf{P}(Z_{n-b_n} = \ell_n)\left[f_n(e^{-h_n/c_{b_n}})\right]^{\ell_n} \mathbf{P}\bigl(S_{\ell_n}(h, b_n) \leq k_n\bigr). \tag{221}$$

By the central limit theorem,

$$\lim_{n\uparrow\infty} \mathbf{P}\bigl(S_{\ell_n}(h, b_n) \leq k_n\bigr) \ =\ \frac{1}{2}\,. \tag{222}$$

From this statement and (217) it follows that

$$\liminf_{n\uparrow\infty} \mu^{b_n-n} \log \mathbf{P}(Z_n \leq k_n) \ \geq\ \liminf_{n\uparrow\infty} \mu^{b_n-n} \log \mathbf{P}(Z_{n-b_n} = \mu^{n-b_n}) + \log\theta. \tag{223}$$

Recalling (17), the proof of Theorem 5 is complete.

Remark 16 (**To the proof of Remark** 6)**.** To prove (78) one can use the methods from the proof of Theorem 5. But some changes are needed, since in Remark 6 we deal with absolutely continuous distributions.

Instead of (195) we shall use (178). Putting there $x = ym^k$ and $k = k_y = \max\{j \geq 1 : m^j \leq \mu^j/2y\}$ we obtain

$$w(y) \ =\ m^{k_y} \sum_{\ell=\mu^{k_y}}^{\infty} \mathbf{P}(Z_{k_y} = \ell)\, w^{*\ell}(ym^{k_y}). \tag{224}$$

For every $h \geq 0$ we may define the density function

$$w_h(x) \ :=\ \frac{e^{-hx}}{\varphi(h)}\, w(x), \tag{225}$$

corresponding to the Cramér transform of W. By Lemma 12, $C_w := \sup_{x\geq 0} w(x) < \infty$ in the present Böttcher case. Hence, $\sup_{x\geq 0} w_h(x) \leq C_w/\varphi(h)$. By induction (analogously to Lemma 9),

$$\sup_{x\geq 0} w_h^{*\ell}(x) \ \leq\ \frac{C_w}{\varphi(h)}\,, \qquad \ell \geq 1. \tag{226}$$

It is easy to see that

$$w_h^{*l}(x) \ =\ \frac{e^{-hx}}{\varphi^{\ell}(h)}\, w^{*\ell}(x), \qquad \ell \geq 1. \tag{227}$$

From this identity and (226) it follows that

$$w^{*\ell}(x) \ \leq\ C_w\, \varphi^{\ell-1}(h)\, e^{hx}. \tag{228}$$

Therefore, for all $\ell \geq \mu^{k_y}$,

$$w^{*\ell}(ym^{k_y}) \;\leq\; \frac{C_w}{\varphi(h)}\,\Big[e^{hym^{k_y}/\mu^{k_y}}\,\varphi(h)\Big]^{\ell}. \tag{229}$$

Further, by the definition of k_y,

$$\frac{\mu}{2my} \leq \frac{m^{k_y}}{\mu^{k_y}} \leq \frac{1}{2y}, \tag{230}$$

and consequently,

$$w^{*\ell}(ym^{k_y}) \;\leq\; \frac{C_w}{\varphi(h)}\,\big[e^{h/2}\varphi(h)\big]^{\ell}. \tag{231}$$

Before (200) we showed that $e^{h_0/2}\varphi(h_0) \leq e^{-\delta}$. As a result we have the bound

$$w^{*\ell}(ym^{k_y}) \;\leq\; \frac{C_w}{\varphi(h_0)}\;e^{-\delta\ell}. \tag{232}$$

Entering this into (224) gives

$$w(y) \;\leq\; C\,m^{k_y}\,f_{k_y}(e^{-\delta}). \tag{233}$$

Taking logarithm and using (21), we see that

$$\limsup_{y\to 0}\,\mu^{-k_y}\log w(y) \;\leq\; \log B(e^{-\delta}). \tag{234}$$

Now we deal with a corresponding lower bound of $\log w(y)$. By (224) and (227),

$$\begin{aligned} w(y) &> m^{k_y}\,\mathbf{P}(Z_{k_y} = \mu^{k_y})\,w^{*\mu^{k_y}}(ym^{k_y}) \\ &> \mathbf{P}(Z_{k_y} = \mu^{k_y})\,\varphi^{\mu^{k_y}}(h)\,w_h^{*\mu^{k_y}}(ym^{k_y}), \qquad h > 0. \end{aligned} \tag{235}$$

Recalling that $h(x)$ is the unique solution of the equation $\varphi'(h)/\varphi(h) = -x$ and using (230), one gets the inequality $h(ym^{k_y}/\mu^{k_y}) \leq h(\mu/2m)$. Thus, by monotonicity of φ,

$$\varphi^{\mu^{k_y}}\big(h(ym^{k_y}/\mu^{k_y})\big) > \varphi^{\mu^{k_y}}\big(h(\mu/2m)\big) = \exp[-C\mu^{k_y}]. \tag{236}$$

If in (225) we set $h = h(ym^{k_y}/\mu^{k_y})$, then $w_h^{*\mu^{k_y}}(ym^{k_y})$ is the value of the density function of the sum $\sum_{j=1}^{\mu^{k_y}} W_j(-h)$ at the point $\mathbf{E}\sum_{j=1}^{\mu^{k_y}} W_j(-h)$. Thus, by the central limit theorem for densities ([16, Theorem VII.7]),

$$\lim_{y\to 0} w_h^{*\mu^{k_y}}(ym^{k_y}) \;=\; \frac{1}{\sqrt{2\pi}}. \tag{237}$$

Putting $h = h(ym^{k_y}/\mu^{k_y})$ in (235) and using (17), (236), and (237), we obtain

$$\liminf_{y\to 0}\,\mu^{-k_y}\log w(y) \;\geq\; -C. \tag{238}$$

Combining (234) and (238) we get

$$\log w(y) \;\asymp\; -\mu^{k_y}. \tag{239}$$

Then the relation $\mu^{k_y} \asymp y^{-\beta/(1-\beta)}$ finishes the proof. $\diamond$

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