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Rate-independent damage processes in nonlinear elasticity

Alexander Mielke [1 2], Tomáš Roubíček [3 4]

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[1] Institut für Analysis,
Dynamik und Modellierung
Universität Stuttgart
Pfaffenwaldring 57
D – 70569 Stuttgart
Germany
E-Mail: mielke@wias-berlin.de

[2] Weierstraß-Institut
für Angewandte Analysis
und Stochastik
Mohrenstraße 39
D – 10117 Berlin
Germany

[3] Mathematical Institute
Charles University
Sokolovská 83
CZ – 186 75 Praha 8
Czech Republic
E-Mail: tomas.roubicek@mff.cuni.cz

[4] Institute of Information,
Theory and Automation
Academy of Sciences
Pod vodárenskou věží 4
CZ – 182 08 Praha 8
Czech Republic

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax:	+ 49 30 2044975
E-Mail:	preprint@wias-berlin.de
World Wide Web:	http://www.wias-berlin.de/

Abstract

Damage of an elastic body undergoing large deformations by a "hard-device" loading possibly combined with an impact (modelled by a unilateral frictionless contact) of another, ideally rigid body is formulated as an activated, rate-independent process. The damage is assumed to absorb a specific and prescribed amount of energy. A solution is defined by energetic principles of stability and balance of stored and dissipated energies with the work of external loading, realized here through displacement on a part of the boundary. Rigorous analysis by time discretization is performed.

1 Introduction

Damage as a special sort of inelastic response of solid materials due to their microstructural changes under mechanical load receives nowadays great attention in engineering mainly because of wide and important applications and amenability for computational simulations although mostly without being supported by rigorous mathematical and numerical analysis. There are many models, often combining damage with plasticity, viscosity, fatigue, and other effects, and accordingly there are hundreds or rather thousands papers addressing damage in engineering or materials-science literature.

Mathematical investigation is, however, much more modest and engineering literature typically replaces rigorous convergence proofs at best by computer simulations showing certain mesh insensitivity on specific examples. This lack of mathematical support is the main motivation of this paper. Henceforth, as the mathematical analysis is technically not trivial, we confine ourselves to a relatively simple model neglecting other inelastic phenomena. Thus, on the one hand, we are able to avoid unnecessary technicalities and, on the other hand, we still keep the essential phenomena in play and allow for reasonable applications.

In particular, we consider damage in the context of nonlinear elasticity as e.g. in [12, 21, 25, 26, 31] which is certainly a relevant concept especially because damaged materials may allow indeed very large deformations. On the other hand, it is important for our mathematical method that we consider only materials with quasi-convex stored energy of a polynomial growth $p > 3$ (as Ogden's type materials). In an axisymmetrical case, $p > 2$ would be allowed but we consider a general physically relevant 3-dimensional situation.

Moreover, we consider damage as a *rate-independent* process. This is an assumption which can be discussed and certainly not all applications are well fitted into this framework. However, it is often an appropriate concept and has applications in a variety of industrially important materials, especially to concrete [14, 17, 32], filled polymers [12], or filled rubbers

[20, 25, 26]. Being rate-independent, it is necessarily an *activated* process, i.e. to trigger a damage the mechanical stress must achieve a certain activation threshold. Of course, not every activated process must be rate independent but we will base our mathematical technique just on the rate independency assumption of damage and absence of any other rate dependent processes like viscosity and inertia.

Simple models use one scalar damage parameter (which is what we will use here) which corresponds to an *isotropic damage*. Anyhow models with two damage parameters (cf. [14, Sect.12.5] or [18, 25]) or tensor damage parameters (cf. [7]) are popular in engineering literature to reflect anisotropy or distinguish between tension and compression in small-strain models. A generalization of the model presented here and its analysis in this direction seems well possible.

In agreement with experiments, one other aspect is often built into damage models, namely the *gradient of damage*, cf. [10, 14, 17, 18, 21, 24]. This expresses certain nonlocality in the sense that damage of a particular spot is to some extent influenced by its surrounding, leading to possible hardening or softening-like effects, and introducing a certain internal length scale eventually preventing damage microstructure development. From the mathematical viewpoint, the damage gradient has a compactifying character and opens possibilities for successful analysis of the model. It should be, however, remarked that various others non-local mechanisms based on gradient terms have been proposed in engineering literature, cf. [19], without mathematical justification.

An important issue is a way how the external load can be applied. To keep applications wide, we admit loading by "hard-device", i.e. by prescribing displacement on some part of the boundary as a function of time, or loading through an impact of a rigid body with prescribed motion, i.e. unilateral contact boundary conditions on some part of the boundary, or combination of both regimes. This seems to cover indeed a large variety of applications.

It should be emphasized that rate-independent models based on small-strains, which are quite popular in engineering [5, 6, 14, 18], are automatically covered as a special case, too.

Mathematical results, especially those involving the rate-independent case, seem to be very rare. Only local-in-time existence for a simplified scalar model or for a rate dependent 0- or 1-dimensional model has been recently performed in [6, 11, 15, 16]. The following analysis seems to provide a first mathematical existence result in space dimension 3.

In Section 2, we formulate the announced rate-independent model, introduce a certain transformation that makes the treatment of the impacting rigid body easier, and specify basic assumptions. Then, in Section 3, we derive a weak (here we say "energetic") formulation of the problem and prove existence of such a solution by approximating it by a time-discretization method in a regular case that even completely damaged material still does not completely disintegrate. Finally, Section 4 outlines some results without this hypothesis.

2 The model

Let us first specify our notation as far as geometry concerns, cf. Figure 1. The elastic body will occupy a reference domain $\Omega \subset \mathbb{R}^3$ assumed open, bounded with the Lipschitz boundary $\Gamma = \partial\Omega$. Some part of the boundary $\Gamma_0 \subset \Gamma$ is assumed to be loaded by time-dependent hard-device loading, i.e. Dirichlet boundary conditions. Besides, the body can be loaded also by time-dependent unilateral boundary conditions, which is used to describe a frictionless unilateral contact during an impact of another body B whose movement is prescribed, as a rigid body motion; B is assumed a bounded open set with a C^1-boundary.

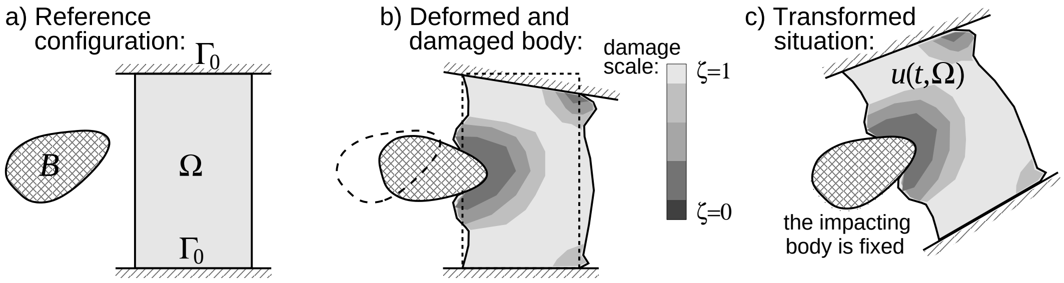

Figure 1. A schematic situation of a moving rigid obstacle B impacting an elastic body undergoing possibly a damage due to this impact and the hard-device load on the part Γ_0 of the boundary $\partial\Omega$.

For simplicity, we will confine ourselves to materials with a single damage quantity (like in [14, Sect.12.4]) but we admit large deformations and also strains, which are important in some specific applications. Moreover, the stored-energy density may be nonconvex; we only assume quasi-convexity and hence may also impose frame-indifference. Small displacements, which are often considered in damage models, are thus covered, too. Generalization to more than one damage quantity (like in [14, Sect.12.5]) is, in principle, straightforward. Besides, we consider the damage process temperature-independent or slow enough so that the produced heat is transferred out to keep temperature variations unimportant, which allows us to consider *isothermal* situation and speak about stored energy instead of the free energy.

2.1 Stored energy.

At a fixed time, the state of the system will be considered as $q = (u, \zeta)$ where $u : \Omega \to \mathbb{R}^3$ is the *deformation* related with the *displacement* considered on the reference body configuration $\Omega \subset \mathbb{R}^n$, let us denote it by w. This means $u(x) = x + w(x)$. Hence, the deformation gradient is $\nabla u(x) = \mathrm{I} + \nabla w$, where $\mathrm{I} \in \mathbb{R}^{n\times n}$ denotes the identity matrix; to simplify most of the formulae we will work in terms of the deformation u only. As to $\zeta : \Omega \to [0, 1]$, it is a *damage parameter* indicating how much of the material is already destroyed at a reference point $x \in \Omega$: 1 means 100% quality of the material, 0 means that the material is completely damaged at the current point $x \in \Omega$, and $0 < \zeta(x) < 1$ means that some portion of material is already damaged due to, e.g., microcracks or microvoids.

The *stored energy* density $\varphi(x, F, z)$ is then a function of deformation gradient $F = \nabla u$ and the damage variable z while dependence on $x \in \Omega$ expresses a possible inhomogeneity of the material. We may assume that $\varphi(x,\cdot,z)$ is frame-indifferent so that it essentially depends only on the right Cauchy-Green tensor $F^\top F$ rather than F itself. Moreover, we assume that $\varphi(x,\cdot,z)$ is composed from two parts $\varphi_0,\varphi_1:\Omega\times\mathbb{R}^{3\times3}\to\mathbb{R}$ and has the form

$$\varphi(x,F,z):=\varphi_0(x,F)+\psi(x,z)\varphi_1(x,F) \tag{2.1}$$

where $\psi:\Omega\times[0,1]\to[0,+\infty)$. Examples used in engineering literature are $\psi(z)=z$ or $\psi(z)=z^2$ (see e.g. [5]) or $\mathrm{e}^{(1-z)/z_0}$ which admits that damage can never be complete (see e.g. [25]). The former case corresponds to a so-called "$1-d$" model with d having the meaning of density of microcracks or microvoids, which is very popular in engineering; in this context, we put $z:=1-d$, which is occasionally used in mathematical literature, cf. [6, 15, 16]. By considering φ_0 nonconstant, we can describe the phenomenon that even a completely damaged material can still resist certain load, typically related with volume changes at least. Hence, in specific case, φ_0 is a *volumetric* and φ_1 an *isochoric* contribution to the stored energy, cf. [25, Formula (2.18)]. Resistance to pure pressure can be described, for example, by putting $\varphi_0(x,F):=\max(0,1-\det(F))^{p/3}$.

For notational simplicity, we confine ourselves to homogeneous media and omit dependence of φ on x. Later, in (2.16), we will assume $F\mapsto\varphi(F,z)$ quasi-convex. However, for mathematical reasons, we do not cover the case that $\varphi(F,z)$ blows up to $+\infty$ if $\det(F)\to0$. As announced in Section 1, nonlocality of the damage will be described by involving gradients of ζ. The simplest possibility is to augment the elastic stored energy by a term like $|\nabla\zeta(x)|^r$ so that the overall stored energy is then

$$V(u,\zeta):=\int_\Omega\varphi\big(\nabla u(x),\zeta(x)\big)+\frac{\kappa}{r}|\nabla\zeta(x)|^r\,\mathrm{d}x \tag{2.2}$$

where $\kappa>0$ is a so-called factor of influence of damage. In engineering literature $r=2$ is used [10, 14, 18] and for example, in the case of concrete, a definite value of $\kappa=0.2\,\mathrm{J/m}$ has been used in [18], see also [14, Sect.12.6]. Here, for some analytical reasons, namely Lemmas 3.8(ii) and 3.9 which rely on the embedding $W^{1,r}(\Omega)\subset L^\infty(\Omega)$, we confine ourselves to $r>3$.

Remark 2.1 (*Alternative regularization.*) Modifying (2.2) by considering

$$V(u,\zeta):=\int_\Omega\varphi\big(\nabla u(x),\zeta(x)\big)+\frac{\kappa}{2}|\nabla\zeta(x)|^2+\frac{\varepsilon}{r}|\nabla\zeta(x)|^r\,\mathrm{d}x \tag{2.3}$$

with $r>3$ and ε only a small regularizing parameter would satisfy our analytical needs as well as engineering expectations. For notational simplicity, we confined ourselves to (2.2).

Remark 2.2 (*Ogden-type material.*) An example for such φ from (2.1), in fact polyconvex in F, is the Ogden-type material

$$\begin{aligned}\varphi(F,z)\;=\;&c_{01}\mathrm{tr}(C)^{p/2}+c_{02}\mathrm{tr}\big(\mathrm{cof}(C){-}\mathrm{I}\big)^{p_1/2}+\phi_0\big(\det(F)\big)\\&+\psi(z)\Big[c_{11}\mathrm{tr}(C)^{p/2}+c_{12}\mathrm{tr}\big(\mathrm{cof}(C){-}\mathrm{I}\big)^{p_1/2}+\phi_1\big(\det(F)\big)\Big]\end{aligned} \tag{2.4}$$

where $C=F^\top F$ is the Cauchy-Green stretch tensor, $c_{11},c_{12}>0$, $c_{01},c_{02}\geq0$, $p_1\leq p/2$, and $\phi_0,\phi_1:[0,+\infty]\to[0,+\infty]$ continuous convex functions. Depending whether c_{01} is positive or not we distinguish the non-degenerate case (when even for $z = 0$ the damage of the material is incomplete, see Section 3) or the degenerate case (when for $z = 0$ the damage causes the material to completely disintegrate). Yet, we still admit that even completely disintegrated material still keeps certain rigidity under compression, described by φ_0 increasing when $\det(F)$ decays to 0.

Remark 2.3 (*Nonlocal terms.*) Instead of $\int_\Omega \frac{\kappa}{r}|\nabla\zeta(x)|^r\,\mathrm{d}x$, an integral term $\int_{\Omega\times\Omega} K(x,\xi)|\zeta(x) - \zeta(\xi)|^2\mathrm{d}x\mathrm{d}\xi$ with a singular kernel K has sometimes been used to get a regularizing effect like the gradient term in (2.2) but to admit spatial jumps in ζ and, moreover, to allow for easier implementation on computers after numerical discretization, cf. [2]. In context of damage, such integral nonlocal term has been proposed in [5] with an interpretation as nonlocal hardening of the damage activation threshold, but it has the same effect as in our model, cf. the complementarity problems [5, Formula (51e)] and (2.15) below. However, such modification would bring mathematical troubles because the technique we will use in Lemmas 3.8(ii) and 3.9 relies on an embedding into $C(\bar{\Omega})$ which essentially just excludes jumps in ζ.

2.2 Dissipation.

Dissipative mechanisms are routinely described by a (pseudo)*potential of dissipative forces*, here denoted by R, as a function of the rate of $q = q(t)$. The only dissipation of energy we will consider is due to the damage and, on the microscopical level, it is related with irreversible structural changes of the material starting with microcracks and ending by its complete disintegration. In particular, the spatial gradient of damage does not cause any dissipation, which is a concept suggested in [17]. We allow for the simplification that this process can be described with good accuracy, beside the function ψ from (2.1), by a single phenomenological parameter $d = d(x)$ having the meaning of a specific energy (per volume, i.e. in physical units Jm^{-3} =Pa) needed for complete damage of the unit volume of the material at a point $x \in \Omega$, i.e. the energy needed to switch $\zeta(x)$ from 1 to 0. This energy is irreversibly dissipated to the mentioned structural change of the material. In other words, the damage process in our model is *rate independent*, and in particular it is an *activated process*. This is related with the maximum-dissipation principle and, as already mentioned in Sect. 1, is accepted as an adequate model for many materials.

Again, for notational simplicity we assume homogeneity of the medium and omit x-dependence of d. The specific dissipation then includes only the rate of damage parameter ζ but not of the displacement u, and has the form

$$\varrho(\dot{z}) := \begin{cases} -d\dot{z} & \text{if } \dot{z} \le 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.5}$$

The mentioned irreversibility of the delamination process is related with the phenomenon that, if once damaged, it cannot be healed back and it is reflected by the non-symmetry $\varrho(\dot{z}) \ne \varrho(-\dot{z})$, cf. (2.5). The consequence of the assumed rate-independency is that ϱ is homogeneous of degree 1. In particular, ϱ is nonsmooth at 0, which is related to the activation phenomena.

The overall (non-symmetric) dissipation potential is then defined as

$$R(\dot{q}) \;:=\; \int_\Omega \varrho\big(\dot{\zeta}(x)\big)\,\mathrm{d}x. \tag{2.6}$$

here $\dot{q} = (\dot{y};\dot{\zeta})$ stands for the rate of q. The above formula is to be understood in the sense that $R(\dot{q}) = +\infty$ if $\varrho(\dot{\zeta}(\cdot))$ is not in $L^1(\Omega)$. The important property of R is that it satisfies the triangle inequality, i.e.

$$\forall q_1, q_2, q_3 \in Q: \quad R(q_1 - q_3) \le R(q_1 - q_2) + R(q_2 - q_3) \tag{2.7}$$

where Q denotes the set of admissible states specified later.

Let us remark that sometimes, see e.g. [14, 18], the energy d needed to complete damage is not counted as dissipated but as stored, and then $d = 0$ in (2.5) but a term dz occurs additionally in (2.1). Due to irreversibility of the damage process, this alternative understanding is well possible and leads to equivalent equations. However, our energetic formulation clearly shows the advantage of our distinction of stored and dissipated energies.

2.3 Unilateral contact problem: a coordinate transformation.

Now, for a given parameter t (=time), let us assume that the obstacle $B \subset \mathbb{R}^3$ underwent a prescribed movement and also the hard-device loading through prescribed displacement on some part of the boundary Γ_0 evolved in time, cf. Figure 1b. As we neglect all inertial effect, we can take the liberty to choose the coordinate system arbitrarily up to translation and rotation (dependent on time). Moreover, as we consider the body B ideally rigid, we can fix the coordinate system just with B, cf. Figure 1c. This is an important trick which, beside simplifying the notation, makes the analysis of the problem easier because the work that external loading possibly makes is only through the (thus transformed) hard-device loading but not through the (thus fixed) body B.

Henceforth, we consider the Dirichlet boundary conditions for displacement on Γ_0:

$$u(t)|_{\Gamma_0} = w_{\mathrm{D}}(t) \tag{2.8}$$

where $w_{\mathrm{D}} = w_{\mathrm{D}}(t,x)$ is a given prescribed function $[0,T] \times \Gamma_0 \to \mathbb{R}^3$ and $(\cdot)|_{\Gamma_0}$ is to be understood in the sense of traces. Later, for mathematical reasons we will also work with a suitable extension u_{D} of w_{D}, i.e. $u_{\mathrm{D}}(t)|_{\Gamma_0} = w_{\mathrm{D}}(t)$. The boundary conditions due to the impacting (now, after the transformation, fixed) body can be prescribed, at a current time t, by a rather abstract way as a *nonpenetration* of the deformed Ω with the rigid B, namely:

$$\forall x \in \bar{\Omega}: \qquad u(t,x) \notin B \tag{2.9}$$

where $\bar{\Omega}$ denotes the closure of Ω and B is assumed open. One should realize that, as we do not allow φ to blow up as $\det(F)$ approaches zero and but allow the damage parameter to approach zero, we cannot exclude situations when a point from the interior of Ω is deformed up to the boundary of B. This is why we formulated the nonpenetration (2.9) for the whole $\bar{\Omega}$. This approach was also used by Schuricht [33]. On the other hand, we formulated the Dirichlet boundary conditions (2.8) in a conventional way, having in mind applications where interior points of Ω are not displaced across the boundary $u(t,\Gamma_0)$, again as in [33].

Considering (2.8) and (2.9), the overall (Gibbs-type) stored energy is then

$$G(t,q):=\begin{cases} V(u,\zeta) & \text{if } u(\bar\Omega)\cap B=\emptyset,\ u|_{\Gamma_0}{=}w_{\rm D}(t),\ \text{and } \zeta\ge 0 \text{ a.e. on } \Omega,\\ +\infty & \text{otherwise.}\end{cases} \tag{2.10}$$

Example 2.4 (Γ_0 *fixed,* B *moving.*) An example for the fixation of Ω on Γ_0 and moving B into a position $B(t):=\mathcal{R}(t)B+r(t)$ by a rotation $\mathcal{R}(t)$ and a translation $r(t)$ results, after the above proposed transformation, to the Dirichlet data:

$$w_{\rm D}(t,x)=\mathcal{R}(t)^{-1}\big(x-r(t)\big),\qquad \text{with } r(t)\in\mathbb{R}^3,\ R(t)\in\mathrm{SO}(3)\subset\mathbb{R}^{3\times3}. \tag{2.11}$$

We assume that $\mathcal{R}(\cdot)$ and $r(\cdot)$ are continuous (or even smooth, cf. (2.16f) below) and $r(0)=0$ and $\mathcal{R}(0)=\mathrm{I}$ says that the movement of B really starts from its reference configuration as on Figure 1a such that $\bar\Omega\cap B\ne\emptyset$ implies that $u(0)=\mathrm{id}$ and $\zeta(0)=1$ are suitable initial data.

2.4 Governing equations in the classical formulation

Now, we will let t ranging $[0,T]$ with $T>0$ a fixed time horizon. Hence now we will write $q=q(t)$, and also $u=u(t,x)$, and $\zeta=\zeta(t,x)$. The deformed reference domain is then $u(t,\Omega)$, cf. Figure 1c.

Taking into account our Gibbs energy and the dissipation potential, the classical considerations in rational thermodynamics leads to the generalized force $f\in-\partial_qG(t,q(t))$ to belong to $\partial R(\frac{\mathrm{d}}{\mathrm{d}t}q)$, where the notation ∂ stands for subdifferential of the involved convex functionals. This, at least formally, leads to the classical formulation (cf. [13]) consisting in the balance of Piola-Kichhoff stress and the evolution of the damage parameter:

$$-\,\mathrm{div}\big(\varphi_0'(\nabla u)+\psi(\zeta)\varphi_1'(\nabla u)\big)=\mathfrak{r}(x), \tag{2.12a}$$

$$\partial\varrho\Big(\frac{\partial\zeta}{\partial t}\Big)-\kappa\Delta_r\zeta+\psi'(\zeta)\varphi_1(\nabla u)+\partial\chi_{[0,+\infty)}(\zeta)\ni 0 \tag{2.12b}$$

on Ω. The notation $\chi_{[0,+\infty)}$ stands for the indicator function of the interval $[0,+\infty)$ where the damage parameter ranges. As usual, in (2.12b) we abbreviated

$$\Delta_r\zeta:=\mathrm{div}\big(|\nabla\zeta|^{r-2}\nabla\zeta\big) \tag{2.13}$$

in (2.12b). The boundary condition (2.8) is to be completed by suitable normal-stress condition on $\Gamma\setminus\Gamma_0$ on the normal stress $\varphi_F'(\nabla u,\zeta)\nu=\varphi_0'(\nabla u)\nu+\psi(\zeta)\varphi_1'(\nabla u)\nu$ and zero normal damage flux on Γ, hence altogether

$$\left.\begin{array}{ll}\varphi_F'(\nabla u,\zeta)\nu=0 & \text{for } x\in\Gamma\setminus\Gamma_0,\ \ u(x)\notin\bar B,\\ \varphi_F'(\nabla u,\zeta)\nu=\mathfrak{r}(x) & \text{for } x\in\Gamma\setminus\Gamma_0,\ \ u(x)\in\bar B,\\ u(x)=w_{\rm D}(x) & \text{for } x\in\Gamma_0,\end{array}\right\} \tag{2.14a}$$

$$\nabla\zeta\cdot\nu=0 \qquad \text{for } x\in\Gamma \tag{2.14b}$$

where here ν denotes the unit normal to Γ. The unilateral (boundary) conditions (2.9), and the right-hand side $\mathfrak{r}$ of (2.12a) and of (2.14a) represents a certain "residuum" which in most reasonable cases will presumably vanish on Ω (being concentrated on $\Gamma \setminus \Gamma_0$ and entering the boundary conditions (2.14a)) only, otherwise it represents the reaction force that may arises if a point of Ω touches the obstacle B; cf. (3.5) below together with Proposition 3.1. Also, $\partial\varrho$ in (2.12b) denotes the subdifferential of ϱ. In view of (2.5), the inclusion (2.12b) thus means the following complementarity problem

$$\left.\begin{array}{rcl} \dfrac{\partial\zeta}{\partial t} & \leq & 0, \\ \psi'(\zeta)\varphi_1(\nabla u) - r_\zeta & \leq & d + \kappa\Delta_r\zeta, \\ \dfrac{\partial\zeta}{\partial t}\Big(d - \psi'(\zeta)\varphi_1(\nabla u) + \kappa\Delta_r\zeta + r_\zeta\Big) & = & 0 \end{array}\right\} \tag{2.15}$$

on Ω, where $r_\zeta \in \partial\chi_{[0,+\infty)}(\zeta)$ is an additional force balancing the constraint $\zeta \geq 0$; let us remark that the constraint $\zeta \leq 1$ is satisfied automatically due to $\zeta \leq \zeta_0 = 1$ as ensured by $R(\zeta - \zeta_0) < +\infty$ and by (2.17b) below. The second inequality in (2.15) can bear the interpretation that the *driving force* for the damage process can be identified as the specific energy $\psi'(\zeta)\varphi_1(\nabla u)$ and the damage evolves if it reaches the *activation threshold* d modified by the term $\kappa\Delta_r\zeta(x)$ which reflect in some way *hardening-like* effects (if the spot x is surrounded by a less damaged material) or softening (in an opposite case); for the hardening interpretation we refer to [5]. For another interpretation of (2.15) see Remark 2.5 below.

There are, however, substantial troubles with giving a rigorous sense to the classical formulation (2.12) not only because the usual loss of smoothness of weak solutions, but here additionally because of a possible loss of meaning of the deformation $u(x)$ at those x where damage is complete and also because $\mathfrak{r}$ can be a measure, cf. (3.5) below. Therefore, the solution has to be defined carefully to hit rather the energetics of the process and not involving quantities that may not be well defined, and we will do it later in Section 4.

Let us now summarize the basic assumptions we pose throughout the paper:

$$\Omega \subset \mathbb{R}^3 \text{ a bounded Lipschitz domain,} \tag{2.16a}$$
$$B \subset \mathbb{R}^3 \text{ an open set with a } C^1\text{-boundary,} \tag{2.16b}$$
$$\varphi_0, \varphi_1 : \mathbb{R}^{3\times 3} \to \mathbb{R} \ \text{ continuously differentiable and quasi-convex,} \tag{2.16c}$$
$$\varphi_0 \geq 0, \quad \varphi_1 \geq 0, \quad \varphi_0(\mathrm{I}) = \varphi_1(\mathrm{I}) = 0, \tag{2.16d}$$
$$\psi : [0,1] \to \mathbb{R} \ \text{ continuous and increasing,} \quad \psi(0) = 0, \tag{2.16e}$$
$$w_{\rm D} \in W^{1,1}\big(0,T; W^{1,\infty}(\Gamma_0;\mathbb{R}^3)\big), \tag{2.16f}$$
$$\mathrm{meas}_2(\Gamma_0) > 0, \quad \bar{B} \cap w_{\rm D}(t,\Gamma_0) = \emptyset \quad \forall t \in [0,T], \tag{2.16g}$$
$$r > 3 \quad \text{and} \quad \kappa > 0 \quad \text{in} \ \ (2.2). \tag{2.16h}$$

Note that we do not exclude the case $B = \emptyset$ when the loading is only via the boundary Γ_0.

Moreover, as we want to address the initial-value problem, we have to prescribe an initial state $q_0 = (u_0, \zeta_0) \in Q$. Without narrowing possible applications as outlined on Figure 1a, we assume that a non-damaged and non-deformed body Ω is initially placed away B, i.e.

$$\bar{B} \cap \bar{\Omega} = \emptyset, \tag{2.17a}$$
$$\forall (\text{a.a.})\ x \in \Omega: \qquad w_{\rm D}(0,x) = x, \quad u_0(x) = x, \quad \zeta_0(x) = 1; \tag{2.17b}$$

in view of (2.16d), it means also that the body is initially non-stressed. They guarantees $G(0,\mathrm{I},1) = 0$, which further guarantees, if $\varphi \geq 0$ and $\varrho \geq 0$ are taken into account, the so-called stability of $q_0 = (\mathrm{I}, 1)$, i.e.

$$\forall \tilde{q} \in Q: \qquad 0 = G(0, q_0) \leq G(0, \tilde{q}) + R(\tilde{q} - q_0) \tag{2.18}$$

where we put

$$Q := W^{1,p}(\Omega; \mathbb{R}^3) \times L^1(\Omega) \tag{2.19}$$

with p referring to (3.1b).

Remark 2.5 (*Maximum-dissipation principle.*) The dissipation mechanism through the convex, homogeneous potential ϱ (and thus R) is intimately related with Hill's maximum-dissipation principle. In fact, (2.12b) can be written as the system of two inclusions: $\kappa\Delta_r\zeta - \psi'(\zeta)\varphi_1(\nabla u) - \omega \in \partial\chi_{[0,+\infty)}(\zeta)$ and $\omega \in \partial\varrho(\frac{\partial}{\partial t}\zeta)$. The latter relation means equivalently that $\langle \omega(x) - z, \frac{\partial}{\partial t}\zeta(x) - v \rangle \geq 0$ for all pairs (z, v) such that $z \in \partial\varrho(v)$ and for a.a. $x \in \Omega$. In particular, for $v = 0$ one gets the statement about maximum dissipation:

$$\left\langle \frac{\partial \zeta}{\partial t}, \omega \right\rangle = \max_{z \in \partial\varrho(0)} \left\langle \frac{\partial \zeta}{\partial t}, z \right\rangle . \tag{2.20}$$

This says that, for the considered damage rate $\frac{\partial}{\partial t}\zeta$, the driving stress ω makes the dissipation caused by the damage maximal among all other admissible driving stresses, i.e. those from $\partial\zeta(0)$. In plasticity theory, this maximum-dissipation principle can alternatively be expressed as a normality in the sense that the *rate of plastic deformation belongs to the cone of outward normals to the elasticity domain*; see also [24, Sect.2.4.4]. Here, the "elasticity" domain is simply $\partial\zeta(0) = [-d, +\infty)$ so that (2.20) merely says that the damage rate $\frac{\partial}{\partial t}\zeta(x) \leq 0$ vanishes if $-\omega(x) < d$, as indeed is expressed in (2.15).

3 The non-degenerate case: incomplete damage

We first deal with the simpler situation that, beside a p-growth for $p > 3$, the elastic response after a complete damage described by the stored energy φ_0 is coercive similarly as φ_1, namely

$$\varepsilon_0|F|^p - C \leq \varphi_0(F) \leq C\big(1 + |F|^p\big) \qquad \& \qquad |\varphi_0'(F)| \leq C\big(1 + |F|^p\big), \tag{3.1a}$$
$$\varepsilon_1|F|^p - C \leq \varphi_1(F) \leq C\big(1 + |F|^p\big) \qquad \& \qquad \big|\varphi_1'(F)\big| \leq C\big(1 + |F|^p\big) \tag{3.1b}$$

with some $\varepsilon_0, \varepsilon_1 > 0$ and $C < +\infty$. Note that the growth of φ_0' and φ_1' in (3.1) can equally be replaced by

$$|\varphi_0'(F)| \leq C\big(1 + \varphi_0(F)\big) \qquad \& \qquad |\varphi_1'(F)| \leq C\big(1 + \varphi_1(F)\big). \tag{3.2}$$

This situation occurs quite naturally in materials composed from two components (e.g. fibers in a matrix) from which only one (here it would mean the fibers) can undergo a damage.

3.1 Energetical formulation

To proceed further, we must define the *reaction force* $\mathfrak{r} = \mathfrak{r}(t)$ to the impacting body B as well as the *power of external load* due to the varying Dirichlet data $w_{\rm D} = w_{\rm D}(t)$ (resulted possibly from the transformation from Section 2.3). As R involves only ζ but not u, it seems in many situations reasonable to accept the hypothesis that $u(t)$ is the *global minimizer* of $G(t,\cdot,\zeta(t))$, although some counterexample of nonadequacy of this concept in particular situations can easily be imagined, too. By the definition (2.10) of G, this concept means that the deformation $u = u(t)$ solves, for $\zeta(t) \geq 0$ and for $w_{\rm D}(t)$ fixed, the following minimization problem

$$\left.\begin{array}{ll}\text{Minimize} & V\big(u,\zeta(t)\big)\\ \text{subject to} & u(\bar\Omega)\cap B=\emptyset\ ,\\ & u\in W^{1,p}(\Omega;\mathbb{R}^3)\ ,\quad u|_{\Gamma_0}=w_{\rm D}(t).\end{array}\right\} \tag{3.3}$$

As already said, large deformations theoretically allow for situations when an inner point $x \in \Omega$ can touch, after deformation, the boundary ∂B and therefore the reaction force can be distributed even inside Ω, although in reasonable situations it will be supported rather on the boundary $\partial\Omega$.

We will use the notation $\mathcal{M}(\bar\Omega)$ for the set of Radon measures on the closed set $\bar\Omega$. By Riesz theorem, $\mathcal{M}(\bar\Omega)$ is (isometrically isomorphic with) the dual to the space of continuous functions $C(\bar\Omega)$ on $\bar\Omega$.

Proposition 3.1 (Schuricht [33], here a special case.) *Let $\zeta = \zeta(t) \geq 0$ and $w_{\rm D} = w_{\rm D}(t)$ be fixed in (3.3), and (2.16) and (3.1a) hold with $p > 3$. Then (3.3) has at least one solution $u \in W^{1,p}(\Omega;\mathbb{R}^3)$. Moreover, fixing this solution u, there is a positive measure $\mathfrak{m} \in \mathcal{M}(\bar\Omega)$ supported on the contact zone $\Omega_{\rm c} = \Omega_{\rm c}(u) := \{x \in \bar\Omega;\ u(x) \in \bar B\}$ such that the following (weakly-formulated) Euler-Lagrange equation*

$$\int_\Omega \varphi'_F\big(\nabla u(x),\zeta(x)\big) : \nabla v(x)\,{\rm d}x = \int_{\bar\Omega} \nu\big(u(x)\big)\cdot v(x)\,\mathfrak{m}({\rm d}x) \tag{3.4}$$

holds for any $v \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ such that $v|_{\Gamma_0} = 0$, where $\nu = \nu(x)$ is the unit outward normal to the boundary ∂B at the point $x \in \bar B$, and "$\cdot$" and "$:$" denote the the scalar product between vectors and matrices, respectively.

Comments to the proof. The proof in [33] covers nonsmooth boundary of B and uses Clarke's generalized-gradient calculus. Our situation is a rather simple special case.

Note that $\zeta \geq 0$ ensures a uniform coercivity of $\varphi(\cdot,\zeta)$ because obviously $\varphi(F,\zeta) \geq \varepsilon_0|F|^p - C$, cf. (2.16e) and (3.1). Moreover, that the signed distance from B (see [33]) being nonpositive is equivalent to our condition $u(\bar\Omega)\cap B = \emptyset$. Then existence of u was proved in [33, Theorem 3.7].

Furthermore, existence of $\mathfrak{m}$ and the identity (3.4) for $v \in W^{1,\infty}(\Omega;\mathbb{R}^3)$ with $v|_{\Gamma_0}$ follows directly from [33, Theorem 4.3 & Remark 4.6(2)]; the important facts are that B a C^1-domain implies $\nu(\cdot)$ a continuous function on ∂B and the signed distance is differentiable near ∂B.

Note also that $p>3$, ensures $u\in C(\bar\Omega)$ so that the vector-valued mapping $x\mapsto \nu\big(u(x)\big)$ is continuous on $\Omega_{\rm c}$ where $\mathfrak{m}$ is supported. $\square$

The sought reaction force $\mathfrak{r}$ is then a vector-valued measure from $\mathcal{M}(\bar\Omega;\mathbb{R}^3)$ defined by

$$\mathfrak{r}(x)=\mathfrak{m}(x)\nu\big(u(x)\big), \tag{3.5}$$

note that $\nu\big(u(\cdot)\big)$ is well defined on the support of $\mathfrak{m}$ and elsewhere it is unimportant hence the measure $\mathfrak{r}$ is indeed well defined by (3.5).

Following [13], we will now derive formally the energy balance. An important issue is to have a suitable prolongation of the boundary data $w_{\rm D}$ on the whole domain Ω. For this, we take a selection of the inverse to the trace operator $v\mapsto v|_{\Gamma_0}:W^{1,\infty}(\Omega;\mathbb{R}^3)\to W^{1,\infty}(\Gamma_0;\mathbb{R}^3)$ as a bounded linear operator, let us denote it by $\mathfrak{T}:W^{1,\infty}(\Gamma_0;\mathbb{R}^3)\to W^{1,\infty}(\Omega;\mathbb{R}^3)$, and require additionally that $\mathfrak{T}$ vanishes away from Γ_0 in the sense that

$$\forall v\in W^{1,\infty}(\Gamma_0;\mathbb{R}^3):\qquad {\rm dist}(x,\Gamma_0)\ge\eta\quad\Rightarrow\quad \mathfrak{T}v(x)=0. \tag{3.6}$$

We can choose $\eta>0$ arbitrarily for (3.6) to hold but later we will fix η suitably, cf. Lemma 3.6. Now we construct a prolongation $u_{\rm D}$ of $w_{\rm D}$ by applying the operator $\mathfrak{T}$ on the displacement (not on the deformation), i.e. by the formula

$$u_{\rm D}(t)={\rm Id}+\mathfrak{T}(w_{\rm D}(t)-{\rm Id})=\mathfrak{T}w_{\rm D}(t)+{\rm Id}-\mathfrak{T}{\rm Id} \tag{3.7}$$

where ${\rm Id}:\mathbb{R}^3\to\mathbb{R}^3$ denotes the identity (corresponding to zero displacement) considered as restricted on Ω or on Γ_0. We will, for a moment, assume that we can take $\eta>0$ in such a way that

$$\forall t\in[0,T]:\qquad \Omega_{\rm c}\big(u(t)\big)\subset\Omega_\eta:=\{x\in\Omega;\ {\rm dist}(x,\Gamma_0)\ge\eta\}. \tag{3.8}$$

We will later show that it is indeed possible and fix $\eta>0$ in dependence on the data $(V,R,w_{\rm D},q_0)$, cf. (3.27). In this way, we ensure

$$u_{\rm D}(t)\big|_{\Gamma_0}=w_{\rm D}(t),\quad u_{\rm D}(t)\big|_{\Omega_\eta}={\rm Id},\quad u_{\rm D}\in W^{1,1}\big(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3)\big). \tag{3.9}$$

Now, putting formally $v=\frac{\partial}{\partial t}u-\frac{\partial}{\partial t}u_{\rm D}$ to (3.4) with (3.5), as one can because then $v|_{\Gamma_0}=\frac{\partial}{\partial t}u|_{\Gamma_0}-\frac{\partial}{\partial t}u_{\rm D}|_{\Gamma_0}=\frac{\partial}{\partial t}w_{\rm D}-\frac{\partial}{\partial t}w_{\rm D}=0$, one obtains, for a.a. $t\in[0,T]$,

$$\int_\Omega\varphi_F'\big(\nabla u,\zeta\big):\nabla\frac{\partial u}{\partial t}\,{\rm d}x=\int_\Omega\varphi_F'\big(\nabla u,\zeta\big):\nabla\frac{\partial u_{\rm D}}{\partial t}\,{\rm d}x+\int_{\bar\Omega}\frac{\partial(u-u_{\rm D})}{\partial t}\cdot\mathfrak{r}({\rm d}x). \tag{3.10}$$

Then, in view of (2.2), again formally,

$$\begin{aligned}\frac{\rm d}{{\rm d}t}V(u(t),\zeta(t))&=\int_\Omega\varphi_F'\big(\nabla u,\zeta\big):\nabla\frac{\partial u}{\partial t}+\varphi_z'\big(\nabla u,\zeta\big)\frac{\partial\zeta}{\partial t}\\&\qquad+\kappa|\nabla\zeta|^{r-2}\nabla\zeta\cdot\nabla\frac{\partial\zeta}{\partial t}\,{\rm d}x\\&=\int_\Omega\varphi_z'\big(\nabla u,\zeta\big)\frac{\partial\zeta}{\partial t}+\kappa|\nabla\zeta|^{r-2}\nabla\zeta\cdot\nabla\frac{\partial\zeta}{\partial t}\\&\qquad+\varphi_F'\big(\nabla u,\zeta\big):\nabla\frac{\partial u_{\rm D}}{\partial t}\,{\rm d}x+\int_{\bar\Omega}\frac{\partial(u-u_{\rm D})}{\partial t}\cdot\mathfrak{r}({\rm d}x).\end{aligned} \tag{3.11}$$

Using (2.12b) in the weak formulation tested formally by $\frac{\partial}{\partial t}\zeta$ together with the imposed homogeneous Neumann boundary conditions for ζ, one gets

$$\int_\Omega \varphi'_z(\nabla u,\zeta)\frac{\partial\zeta}{\partial t}+\kappa|\nabla\zeta|^{r-2}\nabla\zeta\cdot\nabla\frac{\partial\zeta}{\partial t}$$
$$=-\int_\Omega \partial\varrho\Big(\frac{\partial\zeta}{\partial t}\Big)\frac{\partial\zeta}{\partial t}\,\mathrm{d}x=-\int_\Omega \varrho\Big(\frac{\partial\zeta}{\partial t}\Big)\,\mathrm{d}x=-R\Big(\frac{\partial q}{\partial t}\Big) \tag{3.12}$$

due to the degree-1 homogeneity of ϱ, see definition (2.5). If $u_{\rm D}$ is taken as suggested in (3.7) with (3.8) satisfied, the term $\int_{\bar\Omega}\frac{\partial}{\partial t}u_{\rm D}\cdot\mathfrak{r}(\mathrm{d}x)$ vanishes because either $\frac{\partial}{\partial t}u_{\rm D}=\frac{\partial}{\partial t}\mathrm{Id}=0$ on Ω_η or $\mathfrak{r}=0$ on $\bar\Omega\setminus\Omega_\eta$. If the deformation rate $\frac{\partial}{\partial t}u$ as well as the reaction force $\mathfrak{r}$ are enough regular, then also $\int_{\bar\Omega}\frac{\partial}{\partial t}u\cdot\mathfrak{r}(\mathrm{d}x)$ vanishes because if $u(x)\in\partial B$, then either $\mathfrak{r}\neq 0$ but then $\frac{\partial}{\partial t}u$ is tangential to B while $\mathfrak{r}$ is normal hence both vectors are orthogonal to each other and $\frac{\partial}{\partial t}u\cdot\mathfrak{r}=0$, or $\frac{\partial}{\partial t}u$ not is tangential to B but it must have a positive normal component and then $\mathfrak{r}=0$, which is also the case if $u(x)\notin\partial B$. Altogether, the last term in (3.11) vanishes. Putting (3.12) into (3.11), integrating it over a time interval $[s,t]$, realizing that simply $V(u(t),\zeta(t))=G(t,q(t))$ because $q(t)$ is admissible for (3.3), and expressing the dissipated energy $\int_s^t R(\frac{\partial}{\partial t}q(\theta))\mathrm{d}\theta$ as the total variation without referring explicitly to the time derivative $\frac{\partial}{\partial t}\zeta$, i.e.

$$\mathrm{Var}_R(q;s,t):=\sup\sum_{i=1}^{j}R\big(q(t_i)-q(t_{i-1})\big) \tag{3.13}$$

with the supremum taken over all $j\in\mathbb{N}$ and over all partitions of $[s,t]$ in the form $s=t_0<t_1<...<t_{j-1}<t_j=t$, we eventually obtain

$$G(t,q(t))+\mathrm{Var}_R(q;s,t)=G(s,q(s))+\int_s^t P(\theta,q(\theta))\,\mathrm{d}\theta \tag{3.14a}$$

$$\text{with}\quad P(t,q)\equiv P(t,u,\zeta):=\int_\Omega\varphi'_F\big(\nabla u(x),\zeta(x)\big){:}\nabla\frac{\partial u_{\rm D}}{\partial t}(t,x)\,\mathrm{d}x. \tag{3.14b}$$

Remark 3.2 The particular terms in (3.14a) thus represent

- stored energy at time t,
- the energy dissipated by damage during the time interval $[s,t]$,
- stored energy at the initial time s, and
- work done by external loadings during the time interval $[s,t]$.

The global-minimization hypothesis adopted already in (3.3) is related with a *stability* property, i.e.

$$\forall\tilde q\in Q:\qquad G\big(t,q(t)\big)\le G(t,\tilde q)+R\big(\tilde q-q(t)\big). \tag{3.15}$$

Note that $G(t,\cdot)$ can take the value $+\infty$ on Q. The philosophy of (3.15) is that the gain of Gibbs' energy $G(t,q(t))-G(t,\tilde q)$ at any other state $\tilde q$ is not larger than the dissipation $R(\tilde q-q(t))$; cf. [30] for discussion. Now, following [28] (see also [29, 30]), we introduce a definition of an energetic solution to the considered problem. By $\mathrm{B}([0,T];X)$ we denote the Banach space of bounded X-valued mappings defined everywhere on $[0,T]$.

Definition 3.3 The process $q : [0,T] \to Q$ is called an *energetic solution* to the problem given by the quadruple $(V, R, w_{\mathrm{D}}, q_0)$ if

(i) $q = (u, \zeta) \in \mathrm{B}([0,T]; W^{1,p}(\Omega;\mathbb{R}^3)) \times \mathrm{BV}(0,T; L^1(\Omega))$,

(ii) it is stable in the sense that (3.15) holds for all $t \in [0,T]$,

(iii) the energy balance (3.14a) holds for any $0 \le s < t \le T$ and for u_{D} related (iv) to w_{D} by (3.7), in particular $t \mapsto P(t, q(t))$ is in $L^1(0,T)$, and

$q(0) = q_0$.

Of course, this definition works only if $\mathfrak{T}$ used in (3.7) satisfies (3.6) for $\eta > 0$ such that (3.8) holds.

In our special situation with R defined via (2.5) and (2.6), the R-variation takes the simple form

$$\mathrm{Var}_R(q; s, t) = \int_\Omega d\big(\zeta(s,x) - \zeta(t,x)\big)\,\mathrm{d}x, \tag{3.16}$$

whenever for a.e. $x \in \Omega$ the functions $\zeta(\cdot, x)$ are nonincreasing on $[s,t]$ while in all other cases the R-variation will be $+\infty$. Moreover, if the Piola-Kirchhoff stress φ'_F is enough regular near Γ_0, one can choose u_{D} not only to be supported in a small neighbourhood of Γ_0 as (3.8) implicitly says but even pass to a limit with this neighbourhood towards Γ_0. Then the last term in (3.14a) converges to

$$\int_s^t \int_{\Gamma_0} \varphi'_F(\nabla u, \zeta) : \Big(\frac{\partial u_{\mathrm{D}}}{\partial \theta} \otimes \nu(x)\Big)\,\mathrm{d}S\mathrm{d}\theta = \int_s^t \int_{\Gamma_0} \sigma(x) \cdot \frac{\partial w_{\mathrm{D}}}{\partial \theta}\,\mathrm{d}S\mathrm{d}\theta \tag{3.17}$$

where $\sigma := \varphi'_F(\nabla u, \zeta)\nu$ is the normal stress and ν denotes the outward unit normal to Ω. Hence this term indeed represents the power of the loading through the Dirichlet boundary conditions w_{D} on Γ_0. More rigorously, the normal stress σ can be defined as the linear bounded functional on $W^{1-1/p,p}(\Gamma_0;\mathbb{R}^3)$ by the formula

$$\big\langle \sigma, v|_{\Gamma_0} \big\rangle = \int_\Omega \varphi'_F\big(\nabla u(x), \zeta(x)\big) : \nabla v(x)\,\mathrm{d}x - \int_{\bar\Omega} v(x) \cdot \mathfrak{r}(\mathrm{d}x). \tag{3.18}$$

In view of (3.4) and (3.5), the right-hand side of (3.18) is independent of the particular extension v of $v|_{\Gamma_0}$ into Ω and thus σ is well defined by (3.18).

3.2 Discretization in time

We will prove the existence of a solution process q quite constructively by a discretization in time, using the implicit Euler scheme. To construct approximate solutions, we consider a time step $\tau > 0$, assuming T/τ integer and also that $\tau \to 0$ in such a way that the equidistant partitions will be nested; for example, the reader can think about a sequence of time steps $\tau = 2^{-k}T$ for $k \in \mathbb{N}$. For $\tau > 0$ fixed, this equi-distant partition of the interval $[0,T]$ leads to the following recursive increment formula: we put $q_\tau^0 = q_0$ a given initial condition, and, for $k = 1, ..., T/\tau$ we define q_τ^k, an approximation of a solution at time $t_\tau^k = \tau k$, to be any solution of the minimization problem

$$\left.\begin{array}{ll} \text{Minimize} & G(t_\tau^k, q) + R(q - q_\tau^{k-1}) \\ \text{subject to} & q \equiv (u, \zeta) \in Q. \end{array}\right\} \tag{3.19}$$

It is worth realizing that, in view of the definitions (2.1), (2.2), (2.5), (2.6), and (2.10), the minimization problem (3.19) takes the more specific form

$$\left.\begin{array}{ll}\text{Minimize} & \displaystyle\int_\Omega \varphi_0(\nabla u) + \psi(\zeta)\varphi_1(\nabla u) + \frac{\kappa}{r}|\nabla\zeta|^r - d(\zeta - \zeta_\tau^{k-1})\,\mathrm{d}x \\ \text{subject to} & u(\bar\Omega)\cap B = \emptyset, \quad u|_{\Gamma_0} = w_{\mathrm{D}}(t_\tau^k), \\ & 0 \le \zeta \le \zeta_\tau^{k-1} \quad \text{a.e. on } \Omega, \\ & u \in W^{1,p}(\Omega;\mathbb{R}^3), \quad \zeta \in W^{1,r}(\Omega). \end{array}\right\} \tag{3.20}$$

In particular, (2.16f) ensures the continuity of $t \mapsto w_{\mathrm{D}}(t) \in W^{1,\infty}(\Gamma_0;\mathbb{R}^3)$, so that the values $w_{\mathrm{D}}(t_\tau^k)$ are well-defined. The chosen solution to (3.19) will be denoted by q_τ^k, and then we assemble the piecewise constant interpolation $q_\tau \in L^\infty(0,T;Q)$ so that $q_\tau|_{((k-1)\tau,k\tau]} = q_\tau^k$ for $k = 1,...,T/\tau$. Likewise, $u_{\mathrm{D},\tau}$ (resp. $w_{\mathrm{D},\tau}$) denotes the piecewise constant interpolation of u_{D} (resp. w_{D}). For the right-hand side of (3.28) below, we assume the prolongation $[(u_{\mathrm{D},\tau},\zeta_\tau)](t) = (u_{\mathrm{D},\tau}^0,\zeta_\tau^0)$ for $t<0$.

Lemma 3.4 (Existence of q_τ.) *Let the assumptions (2.16), (2.17) and (3.1) be valid. Then the approximate solution q_τ does exist.*

Proof. Existence of a solution $q_\tau^k \in Q$ to (3.19) follows recursively for $k = 1,...,T/\tau$ by the direct method of the calculus of variations, cf [8]. Closedness of the set of admissible pairs (u,ζ) for (3.19), i.e. those $(u,\zeta) \in Q$ for which $0 \le \zeta \le \zeta_\tau^{k-1}$, $u|_{\Gamma_0} = w_{\mathrm{D}}(t_\tau^k)$, and $u(\bar\Omega)\cap B = \emptyset$ holds because B is closed and $W^{1,p}(\Omega;\mathbb{R}^3) \subset C(\bar\Omega;\mathbb{R}^3)$ compactly because $p>3$. Coercivity of the minimized functional $G(t_\tau^k,\cdot)$ is a consequence of (3.1), $\psi \ge 0$, and $\kappa > 0$, while $R \ge 0$. It suffices to prove weak lower semicontinuity of both V and R. As to R, it is obvious. The only nontrivial issue is V.

The weak lower semicontinuity of the convex term $\kappa\|\nabla\zeta\|^r_{L^r(\Omega;\mathbb{R}^3)}$ in V is obvious. As for the term $\|\varphi(\nabla u,\zeta)\|_{L^1(\Omega)}$, let us consider a sequence (u_k,ζ_k) converging weakly in $W^{1,p}(\Omega;\mathbb{R}^3)\times W^{1,r}(\Omega)$ to (u,ζ) such that $0 \le \zeta_k \le 1$. By the compact embedding $W^{1,r}(\Omega) \subset L^\infty(\Omega)$ (recall that $r>3$ is assumed) we have $\zeta_k \to \zeta$ in $L^\infty(\Omega)$ and, by uniform continuity of ψ, also $\psi(\zeta_k) \to \psi(\zeta)$ in any $L^\infty(\Omega)$. Since $\nabla u_k \to \nabla u$ in $L^p(\Omega;\mathbb{R}^{3\times3})$ weakly, by (3.1b) the sequence $\{\varphi_1(\nabla u_k)\}_{k\in\mathbb{N}}$ is bounded in $L^1(\Omega)$ and we can conclude $\int_\Omega \big(\psi(\zeta_k) - \psi(\zeta)\big)\phi_1(\nabla u_k)\,\mathrm{d}x \to 0$. Thus we have

$$\begin{aligned}&\liminf_{k\to\infty}\int_\Omega \phi(\nabla u_k,\zeta_k)\,\mathrm{d}x = \liminf_{k\to\infty}\int_\Omega \phi_0(\nabla u_k) + \psi(\zeta_k)\phi_1(\nabla u_k)\,\mathrm{d}x \\ &= \liminf_{k\to\infty}\int_\Omega \phi_0(\nabla u_k) + \psi(\zeta)\phi_1(\nabla u_k)\,\mathrm{d}x + \lim_{k\to\infty}\int_\Omega \big(\psi(\zeta_k) - \psi(\zeta)\big)\phi_1(\nabla u_k)\,\mathrm{d}x \\ &\ge \int_\Omega \phi_0(\nabla u) + \psi(\zeta)\phi_1(\nabla u)\,\mathrm{d}x \ + \ 0 \end{aligned} \tag{3.21}$$

where, using (2.16c) and (3.1), we applied classical Acerbi and Fusco's results [1] with the integrand $(x,F) \mapsto \varphi_0(F) + \psi(\zeta(x))\varphi_1(F)$ showing sequential weak lower-semicontinuity of $u \mapsto \int_\Omega \phi_0(\nabla u) + \psi(\zeta(x))\varphi_1(\nabla u)\mathrm{d}x$ on $W^{1,p}(\Omega;\mathbb{R}^3)$. $\square$

Lemma 3.5 (A-PRIORI ESTIMATES OF q_τ.) *Let the assumptions (2.16), (2.17), and (3.1) be valid. Then the approximate solution q_τ satisfies stability, i.e.,*

$$\forall \tilde q \in Q: \qquad G_\tau\big(t, q_\tau(t)\big) \le G_\tau(t, \tilde q) + R\big(q_\tau(t) - \tilde q\big) \tag{3.22}$$

for all $t \in (0, T]$, where G_τ is the piecewise constant approximation of G defined by (2.10) but with $w_{\mathrm{D},\tau}$ in place of w_{D}. Further, there exist constants C_1 and C_2 which are independent of the time step τ such the following a-priori estimates hold:

$$\|u_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))} \le C_1, \quad \text{and} \tag{3.23}$$

$$\|\zeta_\tau\|_{\mathrm{BV}([0,T];L^1(\Omega))\cap L^\infty(0,T;W^{1,r}(\Omega))} \le C_2. \tag{3.24}$$

Proof. As to the discrete stability condition, as in [30, Thm.3.4], by using successively that q_τ^k is a minimizer (cf. (3.19)) and the triangle inequality (2.7) for R, we obtain

$$\begin{aligned} G_\tau(t_\tau^k, q_\tau^k) &\le G_\tau(t_\tau^k, \tilde q) + R(\tilde q - q_\tau^{k-1}) - R(q_\tau^k - q_\tau^{k-1}) \\ &\le G_\tau(t_\tau^k, \tilde q) + R(\tilde q - q_\tau^k) \end{aligned} \tag{3.25}$$

for any $k = 1, ..., K = T/\tau$. In view of the definition of q_τ and G_τ, it just means (3.22).

Let us consider some fixed $\bar u \in C([0,T]; W^{1,p}(\Omega;\mathbb{R}^3))$ such that $\bar u(t)|_{\Gamma_0} = w_{\mathrm{D}}(t)$ for all $t \in [0,T]$ and $\bar u(t, \bar\Omega) \cap B = \emptyset$, which is always possible due to (2.16g). It is important that this $\bar u$ is independent of τ and also of the particular extension u_{D} of w_{D} which we will fix later. The first test, proposed in [13], is by $\tilde q = (\bar u(t_\tau^k), 0)$ which is obviously admissible for (3.19). This gives

$$V(q_\tau^k) + R(q_\tau^k - q_\tau^{k-1}) \le V(\bar u(t_\tau^k), 0) + R((\bar u(t_\tau^k), 0) - q_\tau^{k-1}). \tag{3.26}$$

Taking into account $0 \le R(\cdot) \le d\mathrm{meas}(\Omega)$ for our arguments, the definition (2.2) of V, and the assumptions (2.16e) and (3.1a), this results to

$$\varepsilon_0 \|\nabla u_\tau^k\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})} + \kappa \|\nabla \zeta_\tau^k\|^r_{L^r(\Omega;\mathbb{R}^3)} \le \big(1{+}C{+}d\big)\mathrm{meas}(\Omega) + C\big\|\nabla \bar u(t_\tau^k)\big\|^p_{L^p(\Omega;\mathbb{R}^{3\times3})}.$$

This gives immediately (3.23) and the second part of (3.24).

The BV-estimate in (3.24) follows simply because $\zeta_\tau(\cdot, x) : [0,T] \to [0,1]$ is ultimately monotone for a.a. $x \in \Omega$, hence $\mathrm{Var}_R(q_\tau, 0, T) = d\int_\Omega (1 - \zeta_\tau(T, x))\mathrm{d}x \le d\,\mathrm{meas}(\Omega)$, when (2.17b) is taken into consideration. $\square$

Lemma 3.6 (APPROXIMATE ENERGY INEQUALITY FOR q_τ.) *Let the assumptions (2.16), (2.17), and (3.1) be valid. Then there is $\eta > 0$ depending on $(V, R, w_{\mathrm{D}}, q_0)$ such that, for u_{D} constructed by (3.7) with $\mathfrak{T}$ satisfying (3.6) and for all $\tau > 0$ sufficiently small, the approximate solution $q_\tau = (u_\tau, \zeta_\tau)$ satisfies*

$$\forall t \in [0,T]: \qquad \Omega_{\mathrm{c}}(u_\tau(t)) \subset \Omega_\eta \tag{3.27}$$

and the two-sided discrete energy estimate:

$$\begin{aligned} &\int_0^t \int_\Omega \varphi_F'\big(\nabla(u_\tau{+}u_{\mathrm{D}}{-}u_{\mathrm{D},\tau}), \zeta_\tau\big) : \nabla \frac{\partial u_{\mathrm{D}}}{\partial \theta}\,\mathrm{d}x\mathrm{d}\theta \\ &\quad \le G_\tau\big(t, q_\tau(t)\big) + \mathrm{Var}_R(q_\tau; 0, t) - G_\tau(0, q_0) \\ &\quad\quad \le \int_0^t \int_\Omega \varphi_F'\big(\nabla(u_\tau^{\mathrm{R}}{+}u_{\mathrm{D}}{-}u_{\mathrm{D},\tau}^{\mathrm{R}}), \zeta_\tau^{\mathrm{R}}\big) : \nabla \frac{\partial u_{\mathrm{D}}}{\partial \theta}\,\mathrm{d}x\mathrm{d}\theta \end{aligned} \tag{3.28}$$

holds with $t = t_\tau^k$ for any $k = 1, ..., T/\tau$, where $(\cdot)_\tau^{\rm R}$ denotes functions "retarded" by τ, i.e. $[u_\tau^{\rm R}](t) := u_\tau(t-\tau)$ and analogously for $\zeta_\tau^{\rm R}$ and $u_{{\rm D},\tau}^{\rm R}$. Moreover, the following a-priori estimate for the Gibbs energy holds:

$$||\mathfrak{G}_\tau||_{{\rm BV}([0,T])} \le C_3 \quad \textit{with} \ \ \mathfrak{G}_\tau(t) := G_\tau\big(t, q_\tau(t)\big) \tag{3.29}$$

where C_3 is independent of τ.

Proof. We are in the position to fix $\eta > 0$. As $p > 3$, we have the embedding $W^{1,p}(\Omega;\mathbb{R}^3) \subset C^{0,\alpha}(\Omega;\mathbb{R}^3)$ with $\alpha = (p-3)/p > 0$. By (3.23), we have $\|u_\tau\|_{L^\infty(0,T;C^{0,\alpha}(\Omega;\mathbb{R}^3))} \le M$ with M independent of τ. Then we choose $\eta > 0$ such that

$$\begin{aligned}&\forall t \in [0,T], \quad \forall v \in C^{0,\alpha}(\Omega;\mathbb{R}^3) :\\ &\|v\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)} \le M{+}1 \ \ \& \ \ v|_{\Gamma_0} = w_{\rm D}(t) \quad \Rightarrow \quad v(\bar\Omega \setminus \Omega_\eta) \cap B = \emptyset.\end{aligned} \tag{3.30}$$

More in detail, denoting $\delta := \min_{t\in[0,T]} {\rm dist}(w_{\rm D}(t,\Gamma_0), B)$ which is positive due to (2.16f)–(2.16g), we can take $\eta > 0$ such that $(M+1)\eta^\alpha \le \delta/2$ if the standard norm in $C^{0,\alpha}(\Omega;\mathbb{R}^3)$ is considered. Indeed, for any $\xi \in \bar\Omega \setminus \Omega_\eta$ and v as in (3.30) we have

$$\begin{aligned}{\rm dist}\big(v(\xi), B\big) &\ge {\rm dist}\big(v(x), B\big) - \big|v(\xi) - v(x)\big|\\ &\ge {\rm dist}\big(v(x), B\big) - (M{+}1)|\xi - x|^\alpha\\ &= {\rm dist}\big(w_{\rm D}(t,x), B\big) - (M{+}1)|\xi - x|^\alpha \ge \delta - (M{+}1)\eta^\alpha \ge \frac{\delta}{2}\end{aligned} \tag{3.31}$$

for some $x \in \Gamma_0$ such that $|x - \xi| \le \eta$ (note that, thanks to the definition (3.8) of Ω_η, such x does exists), therefore the conclusion in (3.30) follows. By this choice of $\eta > 0$ we get the effect that

$$q := q_\tau^{k-1} + \big(u_{\rm D}(t_\tau^k) - u_{\rm D}(t_\tau^{k-1}), 0\big) \tag{3.32}$$

is admissible for (3.19) provided the prolongation $u_{\rm D}$ is constructed by (3.7) with $\mathfrak{T}$ satisfying (3.6) and provided $\tau > 0$ is sufficiently small, say $\tau \le \tau_0$. Indeed, we have

$$\begin{aligned}\big\|u_{\rm D}(t_\tau^k) - u_{\rm D}(t_\tau^{k-1})\big\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)} &= \Big\| \int_{(k-1)\tau}^{k\tau} \frac{\partial u_{\rm D}(t)}{\partial t} {\rm d}t \Big\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)}\\ &\le \int_{(k-1)\tau}^{k\tau} \Big\| \frac{\partial u_{\rm D}(t)}{\partial t} \Big\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)} {\rm d}t \le 1\end{aligned} \tag{3.33}$$

for all k and all $\tau \le \tau_0$ if τ_0 is sufficiently small; the existence of a positive τ_0 with this property follows from the absolute continuity of the Lebesgue integral and from the fact that $u_{\rm D} \in W^{1,1}(0,T;C^{0,\alpha}(\Omega;\mathbb{R}^3))$. Then we can rely either on the fact that, for $x \in \Omega_\eta$, we have

$$v(x) = u_\tau^{k-1}(x) + \big(u_{\rm D}(t_\tau^k)(x) {-} u_{\rm D}(t_\tau^{k-1})(x)\big) = u_\tau^{k-1}(x) + x - x = u_\tau^{k-1}(x) \notin B,$$

or on the fact that, for $x \in \bar\Omega \setminus \Omega_\eta$, we have also $v(x) \notin B$ thanks to (3.30) because $v|_{\Gamma_0} = u_\tau^{k-1}|_{\Gamma_0} + w_{\rm D}(t_\tau^k) - w_{\rm D}(t_\tau^{k-1}) = w_{\rm D}(t_\tau^k)$ and

$$\|v\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)} \le \big\|u_\tau^{k-1}\big\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)} + \big\|u_{\rm D}(t_\tau^k) - u_{\rm D}(t_\tau^{k-1})\big\|_{C^{0,\alpha}(\Omega;\mathbb{R}^3)} \le M + 1$$

due to (3.33). Then the proof of the energy inequality follows in line of the abstract framework in [27, Sect.5.5]: testing (3.19) by q from (3.32), we obtain

$$
\begin{aligned}
&V(q_\tau^k)+R(q_\tau^k-q_\tau^{k-1})\le V\big(q_\tau^{k-1}+(u_{\rm D}(t_\tau^k)-u_{\rm D}(t_\tau^{k-1}),0)\big)\\
&\ =V\big(q_\tau^{k-1}+(u_{\rm D}(t_\tau^k)-u_{\rm D}(t_\tau^{k-1}),0)\big)-V(q_\tau^{k-1})+V(q_\tau^{k-1})\\
&\ =V(q_\tau^{k-1})+\int_{t_\tau^{k-1}}^{t_\tau^k}\frac{\rm d}{{\rm d}\theta}V\big(q_\tau^{k-1}+(u_{\rm D}(\theta))-u_{\rm D}(t_\tau^{k-1}),0)\big){\rm d}\theta\\
&\ =V(q_\tau^{k-1})+\int_{t_\tau^{k-1}}^{t_\tau^k}\int_\Omega\varphi_F'\big(\nabla(u_\tau^{k-1}{+}u_{\rm D}(\theta){-}u_{\rm D}(t_\tau^{k-1})),\zeta_\tau^{k-1}\big){:}\nabla\frac{\partial u_{\rm D}(\theta)}{\partial\theta}\,{\rm d}x{\rm d}\theta.
\end{aligned}
\tag{3.34}
$$

Summing it for $k=1,2,...$, we come to the second inequality in (3.28).

To get the first inequality in (3.28), by the stability (3.25) written for q_τ^{k-1} (for $k=1$, it is just (2.18) which follows from (2.16) and (2.17)), we can see that q_τ^{k-1} minimizes the functional $q\mapsto G_\tau(t_\tau^{k-1},q)+R(q-q_\tau^{k-1})$. Now, likewise (3.32), an admissible test q for this problem will be

$$
q:=q_\tau^k+\big(u_{\rm D}(t_\tau^{k-1})-u_{\rm D}(t_\tau^k),0\big)
\tag{3.35}
$$

provided the prolongation $u_{\rm D}$ is again constructed by (3.7) with $\mathfrak{T}$ satisfying (3.6) and provided $0<\tau\le\tau_0$ with τ_0 as before.

$$
\begin{aligned}
&V(q_\tau^{k-1})-R(q_\tau^k-q_\tau^{k-1})\le V\big(q_\tau^{k}+(u_{\rm D}(t_\tau^{k-1})-u_{\rm D}(t_\tau^{k}),0)\big)\\
&\ =V\big(q_\tau^{k}+(u_{\rm D}(t_\tau^{k-1})-u_{\rm D}(t_\tau^{k}),0)\big)-V(q_\tau^{k})+V(q_\tau^{k})\\
&\ =V(q_\tau^{k})-\int_{t_\tau^{k-1}}^{t_\tau^k}\frac{\rm d}{{\rm d}\theta}V\big(q_\tau^{k}+(u_{\rm D}(\theta))-u_{\rm D}(t_\tau^{k}),0)\big){\rm d}\theta\\
&\ =V(q_\tau^{k})-\int_{t_\tau^{k-1}}^{t_\tau^k}\int_\Omega\varphi_F'\big(\nabla(u_\tau^{k}{+}u_{\rm D}(\theta){-}u_{\rm D}(t_\tau^{k})),\zeta_\tau^{k}\big):\nabla\frac{\partial u_{\rm D}(\theta)}{\partial\theta}\,{\rm d}x{\rm d}\theta.
\end{aligned}
\tag{3.36}
$$

Summing it for $k=1,2,...$, we come to the first inequality in (3.28).

Combining (3.34) and (3.36), we get

$$
\begin{aligned}
\big|V(q_\tau^k)-V(q_\tau^{k-1})\big|\le R(q_\tau^k-q_\tau^{k-1})+\int_{t_\tau^{k-1}}^{t_\tau^k}\max_{i=0,1}\big\|\varphi_F'\big(\nabla(u_\tau^{k-i}+{}\\
+u_{\rm D}(\theta){-}u_{\rm D}(t_\tau^{k-i})),\zeta_\tau^{k-i}\big)\big\|_{L^1(\Omega;\mathbb{R}^{3\times3})}\Big\|\nabla\frac{\partial u_{\rm D}(\theta)}{\partial\theta}\Big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}{\rm d}\theta
\end{aligned}
\tag{3.37}
$$

for $k=1,...,T/\tau$. As $\mathfrak{G}_\tau(t)=V(q_\tau(t))$, (3.37) yields

$$
\big\|\mathfrak{G}_\tau\big\|_{{\rm BV}([0,T])}\le\big\|V(q_\tau(\cdot))\big\|_{L^\infty(0,T)}+{\rm Var}_R(q_\tau;0,T)+N\Big\|\frac{\partial w_{\rm D}}{\partial t}\Big\|_{L^1(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))}
$$

with a constant N estimated through $|\varphi_F'(F,z)|\le C(1+\psi(1))(1+|F|^p)$ with C occurring in (3.1) and through the $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))$-estimate of u_τ and $u_\tau^{\rm R}$, cf. (3.23) as well as of $w_{{\rm D},\tau}$ and and $w_{{\rm D},\tau}^{\rm R}$, cf. (3.9). Using (3.24) and the $W^{1,1}(0,T;W^{1,\infty}(\Omega;\mathbb{R}^3))$-estimate of $w_{\rm D}$, see (3.9), we eventually obtain the desired estimate (3.29). □

Remark 3.7 (*Shift of Dirichlet boundary conditions.*) For some investigations, it is advantageous to work with a "shifted" displacement $\hat{u}(t) := u(t) - u_{\rm D}(t)$. Then obviously $\hat{u}(t)|_{\Gamma_0} = (u(t) - u_{\rm D}(t))|_{\Gamma_0} = u(t)|_{\Gamma_0} - w_{\rm D}(t) = 0$. In terms of $\hat{q} := (\hat{u}, \zeta)$, the "shifted" G, V and P, denoted respectively by $\hat{G}$, $\hat{V}$ and $\hat{P}$, looks as

$$\hat{G}(t,\hat{q}) := \begin{cases} \hat{V}(t,\hat{u},\zeta) & \text{if } \hat{u}(\bar{\Omega}) \cap B = \emptyset,\ \ \hat{u}|_{\Gamma_0} = 0,\ \text{ and } \zeta \geq 0 \text{ a.e. on } \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.38}$$

with

$$\hat{V}(t,\hat{u},\zeta) := \int_\Omega \hat{\varphi}\big(t,x,\nabla\hat{u}(x),\zeta(x)\big) + \frac{\kappa}{r}|\nabla\zeta(x)|^r \,\mathrm{d}x \tag{3.39}$$
$$\text{where} \quad \hat{\varphi}(t,x,F,z) := \varphi(F{+}\nabla u_{\rm D}(t,x),z),$$

and

$$\hat{P}(t,\hat{q}) := \int_\Omega \hat{\varphi}'_F\big(t,x,\nabla\hat{u}(x),\zeta(x)\big) : \nabla\frac{\partial u_{\rm D}(t,x)}{\partial t}\,\mathrm{d}x. \tag{3.40}$$

Having in mind $\hat{q} := (\hat{u},\zeta)$, it holds both $\hat{G}(t,\hat{q}) = G(t,q)$ and $\hat{P}(t,\hat{q}) = P(t,q)$ with P defined in (3.14b). Note also that $\hat{V}$ now depend on t and that there is no need to modify R which depends on ζ only. Since $\hat{\varphi}'_t(t,x,F,z) = \varphi'_F(F{+}\nabla u_{\rm D}(t,x),z) : \nabla\frac{\partial}{\partial t}u_{\rm D}(t,x)$ and since the domain where $\hat{G}(t,\cdot) < +\infty$ is time independent, it now holds

$$\hat{P}(t,\hat{q}) = \hat{V}'_t(t,\hat{q}) = \hat{G}'_t(t,\hat{q}) \tag{3.41}$$

for any $\hat{q}$ such that $\hat{G}(t,\hat{q}) < +\infty$. It is not difficult to re-calculate (3.34) and (3.36) to see that (3.28) transforms to

$$\begin{aligned}\int_0^t \hat{P}(\theta,\hat{q}_\tau(\theta))\,\mathrm{d}\theta &= \int_0^t\int_\Omega \varphi'_F\big(\nabla(\hat{u}_\tau{+}u_{\rm D}),\zeta_\tau\big) : \nabla\frac{\partial u_{\rm D}}{\partial\theta}\,\mathrm{d}x\mathrm{d}\theta \\ &\leq \hat{G}_\tau\big(t,\hat{q}_\tau(t)\big) + \mathrm{Var}_R(\hat{q}_\tau;0,t) - \hat{G}_\tau(0,\hat{q}_0) \\ &\leq \int_0^t\int_\Omega \varphi'_F\big(\nabla(\hat{u}_\tau^{\rm R}{+}u_{\rm D}),\zeta_\tau^{\rm R}\big) : \nabla\frac{\partial u_{\rm D}}{\partial\theta}\,\mathrm{d}x\mathrm{d}\theta = \int_0^t \hat{P}(\theta,\hat{q}_\tau^{\rm R}(\theta))\,\mathrm{d}\theta\end{aligned} \tag{3.42}$$

for $t = t_\tau^k$ with $k = 1,...,T/\tau$, where $\hat{q}_0 = (u_0 - u_{\rm D}(0),\zeta_0)$.

3.3 Convergence of the approximate solution.

We first provide two assertions having their own interest and referring to the continuous problem, disregarding the considered time-discrete scheme. The first one states continuity in the Mosco's sense of the value of (3.3). Let us emphasize the technique we used to prove both Lemmas 3.8(ii) and 3.9 uses the assumption $r > 3$.

Lemma 3.8 (Stability of minimum of (3.3).) *Let $\mathfrak{v} = \mathfrak{v}(t,\zeta)$ denote the value of (3.3), i.e.*

$$\mathfrak{v}(t,\zeta) := \min_{\substack{u\in W^{1,p}(\Omega;\mathbb{R}^3)\\ u|_{\Gamma_0}=w_{\rm D}(t),\ u(\bar{\Omega})\cap B=\emptyset}} V(u,\zeta), \tag{3.43}$$

and let all assumptions of Lemma 3.4 hold. Then:

(i) *The mapping $(t,\zeta)\mapsto \mathfrak{v}(t,\zeta):[0,T]\times W^{1,r}(\Omega)\to\mathbb{R}$ is weakly lower semi-continuous.* (ii)

Moreover, $\mathfrak{v}$ is also strongly upper semicontinuous. In particular, for any $0\le\zeta\le 1$ fixed, $t\mapsto \mathfrak{v}(t,\zeta):[0,T]\to\mathbb{R}$ is continuous.

Proof. Weak lower semicontinuity: Take $t_k\to t$, $\zeta_k\to\zeta$ weakly in $W^{1,r}(\Omega)$, and $u_k\in W^{1,p}(\Omega;\mathbb{R}^3)$ as a minimizer for the respective problem (3.43) with (t_k,ζ_k) in place of (t,ζ); this minimizer does exists as a consequence of Lemma 3.4. By uniform coercivity of the problem (cf. the assumptions (2.16f), (2.16g), and (3.1)) the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$. Up to a subsequence, $u_k\to u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^3)$. By the weak lower semicontinuity of V which we used in Lemma 3.4 we have $\liminf_{k\to\infty}\mathfrak{v}(t_k,\zeta_k)=\liminf_{k\to\infty}V(u_k,\zeta_k)\ge V(u,\zeta)$. Now, passing to the limit in $u_k|_{\Gamma_0}=w_{\rm D}(t_k)$, we get $u|_{\Gamma_0}=w_{\rm D}(t)$. Also the constraint $u_k(\bar\Omega)\cap B=\emptyset$ is preserved in the limit for $u_k\to u$ in $C(\bar\Omega)$. Hence, u is admissible for the problem in (3.43) and thus $\mathfrak{v}(t,\zeta)\le V(u,\zeta)\le\liminf_{k\to\infty}\mathfrak{v}(t_k,\zeta_k)$.

Upper semicontinuity: Take $t_k\to t$, $\zeta_k\to\zeta$ strongly in $W^{1,r}(\Omega)$, and u as a minimizer for the problem in (3.43). Put $v_k:=u+u_{\rm D}(t_k)-u_{\rm D}(t)$. Obviously, $v_k|_{\Gamma_0}=u|_{\Gamma_0}+w_{\rm D}(t_k)-w_{\rm D}(t)=w_{\rm D}(t_k)$. Also, if $|t_k-t|\le\tau_0$, then $v_k(x)\notin B$ both for $x\in\bar\Omega\setminus\Omega_\eta$ and for $x\in\Omega_\eta$, cf. the arguments in the proof of Lemma 3.6 where also $\tau_0>0$ has been implicitly specified. Altogether, v_k is admissible for the minimization problem determining $\mathfrak{v}(t_k,\zeta_k)$, hence certainly $\mathfrak{v}(t_k,\zeta_k)\le V(v_k,\zeta_k)$. Moreover,

$$
\begin{aligned}
&\big\|u_{\rm D}(t_k)-u_{\rm D}(t)\big\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\\
&=\Big\|\int_t^{t_k}\frac{{\rm d}u_{\rm D}(\theta)}{{\rm d}\theta}{\rm d}\theta\Big\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\le\int_t^{t_k}\Big\|\frac{{\rm d}u_{\rm D}(\theta)}{{\rm d}\theta}\Big\|_{W^{1,p}(\Omega;\mathbb{R}^3)}{\rm d}\theta\\
&\le\big\|\mathfrak{T}\big\|_{\mathcal{L}(W^{1,\infty}(\Gamma_0;\mathbb{R}^3),W^{1,\infty}(\Omega;\mathbb{R}^3))}\sqrt[p]{{\rm meas}(\Omega)}\int_t^{t_k}\Big\|\frac{{\rm d}w_{\rm D}(\theta)}{{\rm d}\theta}\Big\|_{W^{1,\infty}(\Gamma_0;\mathbb{R}^3)}{\rm d}\theta.
\end{aligned}
\tag{3.44}
$$

In view of (2.16f) and the absolute continuity of the Lebesgue integral, $u_{\rm D}(t_k)\to u_{\rm D}(t)$ and hence also $v_k\to u$ in $W^{1,p}(\Omega;\mathbb{R}^3)$. Due to the growth conditions (3.1), $V:W^{1,p}(\Omega;\mathbb{R}^3)\times W^{1,r}(\Omega)\to\mathbb{R}$ is (norm×norm)-continuous as $r>3$ is assumed in (2.16h), so that by uniform continuity of $\psi:[0,1]\to\mathbb{R}$ and continuous embedding $W^{1,r}(\Omega)\subset C(\bar\Omega)$, we can pass to the limit in the term $\int_\Omega\psi(\zeta_k)\varphi_1(\nabla u_k){\rm d}x$, and also the term $\int_\Omega|\nabla\zeta_k|^r{\rm d}x$ converges because of the assumed strong convergence of $\nabla\zeta_k$ in $L^r(\Omega;\mathbb{R}^3)$. Thus $V(v_k,\zeta_k)\to V(u,\zeta)=\mathfrak{v}(t,\zeta)$. $\square$

Let us define the *stable set* at time t via

$$
S(t):=\big\{q\in Q;\quad\forall\tilde q\in Q:\ G(t,q)\le G(t,\tilde q)+R(\tilde q-q)\big\}.
\tag{3.45}
$$

The importance of the following property of R and the closed-graph property of the set-valued mapping $t\mapsto S(t)$ has essentially been proved in [23, Theorem 5.3]:

Lemma 3.9 *The dissipation potential R has the properties*

$$
\begin{aligned}
\forall\tilde q\in Q\quad\forall\{q_k\}_{k\in\mathbb{N}}\subset Q,\ q=\underset{k\to\infty}{\text{w-lim}}\,q_k\quad&\exists\{\tilde q_k\}_{k\in\mathbb{N}}\subset Q:\\
\tilde q=\underset{k\to\infty}{\text{w-lim}}\,\tilde q_k\quad\&\quad&\lim_{k\to\infty}R(q_k-\tilde q_k)=R(q-\tilde q).
\end{aligned}
\tag{3.46}
$$

Moreover, if $q_k \in S(t_k)$, $t_k \to t$ and $q_k \to q$ weakly, then
(i) $\quad q \in S(t)$, *and hence also* (ii)
$G(t_k, q_k) \to G(t, q)$ *for* $k \to \infty$.

Sketch of the proof. Considering $q_k = (u_k, \zeta_k)$ and $q = (u, \zeta)$ from (3.46), it holds $\zeta_k \to \zeta$ in $C(\bar\Omega)$ due to the compact embedding of $W^{1,r}(\Omega) \subset C(\bar\Omega)$, recall that we assume $r > 3$ in (2.16h). Then it suffices to take $\tilde q_k = (\tilde u_k, \tilde\zeta_k)$ in (3.46) with $\tilde\zeta_k$ shifted slightly up by a constant, namely

$$\tilde\zeta_k := \tilde\zeta + \operatorname*{ess\,sup}_{x\in\Omega} \big(\zeta_k(x) - \zeta(x)\big) \tag{3.47}$$

so that always $\zeta_k \le \tilde\zeta_k$ a.e. on Ω and simultaneously $\tilde\zeta_k \to \tilde\zeta$ in the norm topology of $W^{1,r}(\Omega)$. Then obviously (3.46) is satisfied.

Now, take $\tilde q = (\tilde u, \tilde\zeta) \in Q$ arbitrary and assume $q_k \in S(t_k)$, in particular

$$G\big(t_k, q_k\big) \le G(t_k, \tilde q_k) + R\big(\tilde q_k - q_k\big) \tag{3.48}$$

for $\tilde q = (\tilde u_k, \tilde\zeta_k)$ with $\tilde\zeta_k$ from (3.47) and with

$$\tilde u_k := \tilde u + u_{\rm D}(t_k) - u_{\rm D}(t). \tag{3.49}$$

Hence, for $|t_k - t| \le \tau_0$ with τ_0 from the proof of Lemma 3.6, $\tilde u_k$ is admissible (in the sense $\tilde u_k|_{\Gamma_0} = u_{\rm D}(t_k)$ and $\tilde u_k(\bar\Omega) \cap B = \emptyset$) if $\tilde u$ is admissible (in the sense $\tilde u|_{\Gamma_0} = u_{\rm D}(t)$ and $\tilde u(\bar\Omega) \cap B = \emptyset$). Moreover, by (3.44), $\tilde u_k \to \tilde u$ in the norm topology of $W^{1,p}(\Omega;\mathbb{R}^3)$. Therefore, $\lim_{k\to\infty} G(t_k, \tilde q_k) + R\big(\tilde q_k - q_k\big) = G(t, \tilde q) + R\big(\tilde q - q\big)$ provided $G(t, \tilde q) < +\infty$ and $R\big(\tilde q - q\big) < +\infty$, which is the case of our interest. Otherwise the stability condition (3.15) we want to prove is trivially fulfilled. Moreover, by the weak lower semicontinuity of V proved in Lemma 3.4, we also have $\liminf_{k\to\infty} G\big(t_k, q_k\big) \ge G\big(t, q\big)$. Altogether, passing to the limit in (3.48) yields

$$\begin{aligned} G\big(t, q\big) &\le \liminf_{k\to\infty} G\big(t_k, q_k\big) \\ &\le \lim_{k\to\infty} G(t_k, \tilde q_k) + R\big(\tilde q_k - q_k\big) = G(t, \tilde q) + R\big(\tilde q - q\big), \end{aligned} \tag{3.50}$$

which proves the stability condition (3.15) and thus (i).

As to (ii), it suffices to put $\tilde q = q$ into (3.50) to see that all "$\le$" in (3.50) must be equalities in this case and, by a contradiction argument, "liminf" must be "lim". □

Let us recall the concept of nets which generalizes the concept of sequences. A set Ξ is called *directed* by an ordering "$\preceq$" if, for any $\xi_1, \xi_2 \in \Xi$, there is $\xi_3 \in \Xi$ such that both $\xi_1 \preceq \xi_3$ and $\xi_2 \preceq \xi_3$. A subset A of a directed set Ξ is called cofinal if for any $\xi_1 \in \Xi$ there is $\xi_2 \in A$ such that $\xi_1 \preceq \xi_2$. Having a directed set Ξ and another set X, we say that $\{x_\xi\}_{\xi\in\Xi}$ is a *net* in X if there is a mapping $\Xi \to X : \xi \mapsto x_\xi$. If X is a topological space, we write $x = \lim_{\xi\in\Xi} x_\xi$ if, for any neighbourhood N of x there is $\xi_0 \in \Xi$ such that $x_\xi \in N$ whenever $\xi_0 \preceq \xi$, and then we say that the net $\{x_\xi\}_{\xi\in\Xi}$ *converges* to x (in the so-called Moore-Smith sense). Now, having a net $\{x_\xi\}_{\xi\in\Xi_0}$ indexed by a directed set Ξ_0

and another net $\{\tilde{x}_{\tilde{\xi}}\}_{\tilde{\xi}\in\Xi}$ in X, we say that this net is *finer* than the net $\{x_\xi\}_{\xi\in\Xi_0}$ if there is a mapping $j:\Xi\to\Xi_0$ such that, for any $\tilde{\xi}\in\Xi$, it holds $\tilde{x}_{\tilde{\xi}}=x_{j(\tilde{\xi})}$ and moreover, for any $\xi\in\Xi_0$ there is $\tilde{\xi}\in\Xi$ large enough so that $j(\tilde{\xi}_1)\succeq\xi$ whenever $\tilde{\xi}_1\succeq\tilde{\xi}$. For example, every non-decreasing mapping $j:\Xi\to\Xi_0$ such that $j(\Xi)$ is cofinal in Ξ_0 produces a finer net by putting $\tilde{x}_{\tilde{\xi}}=x_{j(\tilde{\xi})}$. Obviously, a finer net may have an index set of strictly greater cardinality than the original net. Compact sets are characterized by the property that every net possesses a finer net that converges. We use $\Xi_0\subset\mathbb{N}$ (ordered standardly, hence a net indexed by Ξ_0 is called a sequence or subsequence) and $\Xi\subset\{\text{finite subsets of }[0,T]\}$ ordered by inclusion. Note that Ξ is indeed directed by this way.

Proposition 3.10 (Convergence of q_τ.) *Let (2.16), (2.17), and (3.1) hold. Then there are a net $\{q_{\tau_\xi}\}_{\xi\in\Xi}$, finer than the sequence $\{q_\tau\}_{\tau=T/2^k,\ k\in\mathbb{N}}$ and such that $\lim_{\xi\in\Xi}\tau_\xi=0$, and a process $q=(u,\zeta):[0,T]\to W^{1,p}(\Omega;\mathbb{R}^3)\times W^{1,r}(\Omega)$ such that $u\in\mathrm{B}([0,T];W^{1,p}(\Omega;\mathbb{R}^3))$, $\zeta\in\mathrm{BV}(0,T;L^1(\Omega))\cap L^\infty(0,T;W^{1,r}(\Omega))$ and such that:*

$$\forall t\in[0,T]:\quad \lim_{\xi\in\Xi}u_{\tau_\xi}(t)=u(t)\ \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3),\tag{3.51}$$

$$\forall t\in[0,T]:\quad \lim_{\xi\in\Xi}\zeta_{\tau_\xi}(t)=\zeta(t)\ \text{ weakly* in } L^\infty(\Omega;\mathbb{R}^L)\cap W^{1,r}(\Omega),\tag{3.52}$$

$$\forall t\in[0,T]:\quad \lim_{\xi\in\Xi}G_{\tau_\xi}(t,q_{\tau_\xi}(t))=G(t,q(t)).\tag{3.53}$$

Moreover, every $q:[0,T]\to Q$ obtained as such a limit satisfies the points (i), (ii), (iv) in Definition 3.3, and ζ is measurable, $t\mapsto G(t,q(t))$ is a BV*-function (and thus measurable), and, instead of the energy equality (3.14a), we have only an inequality with $s=0$, see (3.61) below. If, in addition, the following weighted Lipschitz continuity of φ_0' and φ_1' holds:*

$$\forall i{=}0,1\ \exists\ell{\in}\mathbb{R}\ \forall F,\tilde{F}{\in}\mathbb{R}^{3\times3}:\quad \big|\varphi_i'(F)-\varphi_i'(\tilde{F})\big|\le\ell\big(1+|F|^p+|\tilde{F}|^p\big)\big|F{-}\tilde{F}\big|,\tag{3.54}$$

then the energy equality holds for all $s,t\in[0,T]$ with $s<t$, i.e.

$$G(t,q(t))+\mathrm{Var}_R(q;s,t)=G(s,q(s))-\int_s^t\!\!\int_\Omega\varphi_F'(\nabla u,\zeta):\nabla\frac{\partial u_{\rm D}}{\partial\theta}\mathrm{d}x\mathrm{d}\theta,\tag{3.55}$$

and hence q is an energetic solution according to Definition 3.3.

Proof. We follow the procedure from [13, Sect.4] and, for clarity, we divide it into five steps.

Step 1. (*Selection of subsequences.*) By the a-priori estimate (3.29) and Helly's selection principle for Banach-space valued functions (see Barbu and Precupanu [4]), we can select a subsequence and a function $\mathfrak{G}\in\mathrm{BV}([0,T])$ such that $\lim_{\tau\to0}G_\tau(t,q_\tau(t))=\mathfrak{G}(t)$ for all $t\in[0,T]$. Moreover, taking into account the first part of the a-priori estimate (3.24), by a generalized Helly selection principle, see [29, Theorem 6.1] or [23, Thm.3.1], we can further select a subsequence in such a way that, for some $\zeta\in\mathrm{BV}(0,T;L^1(\Omega))$, $\zeta_\tau(t)\to\zeta(t)$ weakly in $L^1(\Omega)$ for all $t\in[0,T]$; here we use also the a-priori bound (3.24) for $\zeta_\tau\in L^\infty(0,T;W^{1,r}(\Omega))$ combined with Helly's principle valued in the nonreflexive space $L^1(\Omega)$ as in [30, Corollary 2.8] modifying the proof of [29, Theorem 6.1]. The mentioned second part of the a-priori estimate (3.24) then allows us to say that also $\zeta\in L^\infty(0,T;W^{1,r}(\Omega))$ and even (3.52) holds for this subsequence.

Furthermore, we denote the stress $\sigma_\tau := \varphi_F'(\nabla(u_\tau^{\rm R}{+}u_{\rm D}{-}u_{{\rm D},\tau}^{\rm R}),\zeta_\tau^{\rm R})$ and the power of external load $\mathfrak{P}_\tau := \int_\Omega \sigma_\tau : \nabla\frac{\partial}{\partial t}u_{\rm D}\,{\rm d}x$. The sequence $\{\mathfrak{P}_\tau\}_{\tau>0}$ is bounded in $L^1(0,T)$ due to the estimate

$$\big\|\mathfrak{P}_\tau\big\|_{L^1(0,T)} \le \big\|\sigma_\tau\big\|_{L^\infty(0,T;L^1(\Omega;\mathbb{R}^{3\times3}))}\Big\|\nabla\frac{\partial u_{\rm D}}{\partial t}\Big\|_{L^1(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))} =: C_4 \tag{3.56}$$

which follows from (3.1) and (3.23) and from (2.16f). Choosing a further subsequence, we have $\sigma_\tau \to \sigma_*$ weakly* in $L^\infty(0,T;V^*)$ where V denotes a separable subspace of $L^\infty(\Omega;\mathbb{R}^{3\times3})$ where the Bochner measurable function $t\mapsto\nabla\frac{\partial}{\partial t}u_{\rm D}(t)$ is valued; recall the Pettis' theorem saying that Bochner measurable functions are just a.e. separably valued and weakly measurable. Then $\mathfrak{P}_\tau\to\mathfrak{P}_*$ weakly in $L^1(0,T)$ for $\mathfrak{P}_*\in L^1(0,T)$ defined by $\mathfrak{P}_*(t)=\langle\sigma_*(t),\nabla\frac{\partial}{\partial t}u_{\rm D}(t)\rangle$ with the duality between V^* and V. Denoting further

$$\mathfrak{P}(t) := \limsup_{\tau\to0}\,\mathfrak{P}_\tau(t) \tag{3.57}$$

for the already selected subsequence and for each $t\in[0,T]$, by Fatou's lemma we have that $\mathfrak{P}$ is measurable and bounded from below by $\mathfrak{P}_*$ a.e. on $[0,T]$; the important fact is that $\{\mathfrak{P}_\tau\}_{\tau>0}$ has a common integrable majorant, namely $t\mapsto \|\sigma_\tau\|_{L^\infty(0,T;L^1(\Omega;\mathbb{R}^{3\times3}))}\|\nabla\frac{\partial}{\partial t}u_{\rm D}(t)\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}$.

In fact, we have u_τ bounded not only in $L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^3))$ but even in the space of everywhere-defined bounded functions ${\rm B}([0,T];W^{1,p}(\Omega;\mathbb{R}^3))$. Hence, identifying naturally the function u_τ with the collection $\{u_\tau(t)\}_{t\in[0,T]}$, we can see that the already selected subsequence $\{u_\tau\}$ belongs to the hypercube $\{v\in W^{1,p}(\Omega;\mathbb{R}^3);\ \|v\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\le C_1\}^{[0,T]}$ with C_1 from (3.23). By Tikhonov theorem (and the Kuratowski-Zorn lemma), this hypercube is compact if equipped by the canonical product topology of the particular weak topologies on the ball $\{v\in W^{1,p}(\Omega;\mathbb{R}^3);\ \|v\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\le C_1\}$. Then there is a net $\{q_{\tau_\xi}\}_{\xi\in\Xi}=\{(u_{\tau_\xi},\zeta_{\tau_\xi})\}_{\xi\in\Xi}$, finer than the already selected subsequence $\{q_\tau\}$, indexed by an unspecified directed set Ξ and converging to some $u\cong\{u(t)\}_{t\in[0,T]}$. This just means (3.51). Of course, the convergence (3.52) is preserved.

Step 2. (*Stability of the limit process.*) In view of the definition of G_τ, we can write $G_\tau(t,q_\tau(t))=G(\vartheta_\tau(t),q_\tau(t))$ with $\vartheta_\tau(t):=\min_{k\in\mathbb{N}\cup\{0\}}\{t_\tau^k\ge t\}$ such that $\lim_{\tau\to0}\vartheta_\tau(t)=t$. Hence we have $q_\tau(t)\in S(\vartheta_\tau(t))$ for all $t\in[0,T]$. By Lemma 3.9(i) and by $q_{\tau_\xi}(t)\to q(t)$ for any $t\in[0,T]$, we can conclude that $q(t)\in S(t)$.

Step 3. (*Convergence of stored energies (3.53).*) Using Lemma 3.8(i), we now have $\liminf_{\xi\in\Xi}G_{\tau_\xi}(t,q_{\tau_\xi}(t))=\liminf_{\xi\in\Xi}\mathfrak{v}(\vartheta_{\tau_\xi}(t),\zeta_{\tau_\xi}(t))\ge\mathfrak{v}(t,\zeta(t))=G(t,q(t))$.

As in [13], the test of stability for $q(\vartheta_\tau(t))$ by $q:=(u(t)-u_{\rm D}(t)+u_{\rm D}(\vartheta_\tau(t)),\zeta(t))$ gives

$$\begin{aligned}G(\vartheta_\tau(t),q_\tau(t)) &\le G(\vartheta_\tau(t),u(t)-u_{\rm D}(t)+u_{\rm D}(\vartheta_\tau(t)),\zeta(t))\\&\quad+\int_\Omega\varrho(\zeta(t)-\zeta(\vartheta_\tau(t-\tau)))-\varrho(\zeta(\vartheta_\tau(t))-\zeta(\vartheta_\tau(t-\tau))\,{\rm d}x\\&= G(\vartheta_\tau(t),u(t)-u_{\rm D}(t)+u_{\rm D}(\vartheta_\tau(t)),\zeta(t))+d\int_\Omega\zeta(\vartheta_\tau(t))-\zeta(t)\,{\rm d}x\\&\le G(\vartheta_\tau(t),u(t)-u_{\rm D}(t)+u_{\rm D}(\vartheta_\tau(t)),\zeta(t)).\end{aligned}\tag{3.58}$$

Considering $\tau=\tau_\xi$, the last expression converges to $G(t,u(t),\zeta(t))$ by Lemma 3.8(ii). To be more specific, the proof of Lemma 3.8(ii) takes now corresponding modifications, in particular, one is to use (3.44) with $\vartheta_{\tau_\xi}(t)$ instead of t_k. Thus $\limsup_{\xi\in\Xi} G_{\tau_\xi}(t,q_{\tau_\xi}(t)) = \limsup_{\xi\in\Xi} \mathfrak{v}(\vartheta_{\tau_\xi}(t),\zeta_{\tau_\xi}(t)) \le \mathfrak{v}(t,\zeta(t)) = G(t,u(t),\zeta(t))$.

Altogether, $\lim_{\xi\in\Xi} G_{\tau_\xi}(t,q_{\tau_\xi}(t)) = G(t,u(t),\zeta(t))$ and, comparing it with what we got by Helly's selection principle, we can see that $\mathfrak{G}(t) = G(t,q(t))$ for all $t\in[0,T]$, which proves (3.53).

Step 4. (*Upper energy estimate.*) To exploit some results from [13] relying on differentiability of $G(\cdot,q)$, we will now work with the transformed quantities as outlined in Remark 3.7 to have, e.g., (3.41) at our disposal. Since $\hat\varphi$ defined by (3.39) satisfies all conditions we used for φ (suitably only generalized for Carathéodory functions reflecting (t,x)-dependence of $\hat\varphi$), we can rely on that all already established convergences holds in terms of $\hat q$, as well. One can then pass to the limit in the second inequality in (3.42), realizing that $\hat P(t,\hat q_\tau^{\rm R}(t)) = P(t,q_\tau^{\rm R}(t)) = \mathfrak{P}_\tau(t)$. From the pointwise convergence of $\zeta_{\tau_\xi}(\cdot)$ and from the definition (3.13) of $\mathrm{Var}_R(\cdot;0,t)$, we get $\liminf_{\xi\in\Xi}\mathrm{Var}_R(q_{\tau_\xi};0,t) \ge \mathrm{Var}_R(q;0,t)$; in fact, taking into account the formula (3.16), we can see that even $\lim_{\xi\in\Xi}\mathrm{Var}_R(q_{\tau_\xi};0,t) = \mathrm{Var}_R(q;0,t)$. Using also (3.53) and $\mathfrak{P}_\tau\to\mathfrak{P}_*$ weakly in $L^1(0,T)$, one gets

$$\hat G(t,\hat q(t)) + \mathrm{Var}_R(\hat q;0,t) = \hat G(0,\hat q_0) + \int_0^t \mathfrak{P}_*(\theta)\,\mathrm{d}\theta. \tag{3.59}$$

It has been proved in [13, Prop. 3.3] generalizing ideas of DalMaso, Francfort and Toader [9, Lemma 4.11], that, for a fixed t, it holds

$$\begin{aligned}&\hat G(t,\hat q_\tau(t)) \to \hat G(t,\hat q(t)) < +\infty \text{ and}\\ &\hat q_\tau(t)\to\hat q(t) \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^3)\times W^{1,r}(\Omega)\\ &\qquad\Longrightarrow\quad \lim_{\tau\to0}\hat G_t'(t,\hat q_\tau(t)) = \hat G_t'(t,\hat q(t)).\end{aligned} \tag{3.60}$$

From this, taking into account (3.16), i.e. $\hat G_t'(t,\hat q(t)) = \hat P(t,\hat q(t))$, and from (3.57), one can identify $\mathfrak{P} = \int_\Omega \varphi_F'\big(\nabla\hat u+\nabla u_{\rm D},\zeta\big) : \nabla\frac{\partial}{\partial t}u_{\rm D}\,\mathrm{d}x = \int_\Omega \varphi_F'\big(\nabla u,\zeta\big) : \nabla\frac{\partial}{\partial t}u_{\rm D}\,\mathrm{d}x$. Taking into account also $\mathfrak{P}_*\le\mathfrak{P}$ a.e. on $[0,T]$, in terms of the original "non-hatted" variables, (3.59) turns into

$$G(t,q(t)) + \mathrm{Var}_R(q;0,t) \le G(0,q_0) + \int_0^t\int_\Omega \varphi_F'\big(\nabla u,\zeta\big) : \nabla\frac{\partial u_{\rm D}}{\partial\theta}\,\mathrm{d}x\mathrm{d}\theta. \tag{3.61}$$

Step 5. (*Lower energy estimate.*) We consider now $\varepsilon>0$, $t\in[0,T]$, and a partition $0=t_0^\varepsilon<t_1^\varepsilon<\ldots<t_{k_\varepsilon}^\varepsilon\le t$ with $\max_{i=1,\ldots,k_\varepsilon}(t_i^\varepsilon-t_{i-1}^\varepsilon)\le\varepsilon$ and also $t-t_{k_\varepsilon}^\varepsilon\le\varepsilon$. Again, we apply the shift from Remark 3.7 and work with the transformed quantities as in Step 4. Realizing that we have got $\hat q(\cdot)$ defined everywhere on $[0,T]$, the already proved stability of $\hat q(t_{i-1}^\varepsilon)$ gives, when tested by $\hat q(t_i^\varepsilon)$, the estimate

$$\begin{aligned}\hat G(t_{i-1}^\varepsilon,\hat q(t_{i-1}^\varepsilon)) &\le \hat G(t_{i-1}^\varepsilon,\hat q(t_i^\varepsilon)) + R(\hat q(t_i^\varepsilon)-\hat q(t_{i-1}^\varepsilon))\\ &= \hat G(t_i^\varepsilon,\hat q(t_i^\varepsilon)) + R(\hat q(t_i^\varepsilon)-\hat q(t_{i-1}^\varepsilon)) - \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon}\hat P(\theta,\hat q(t_i^\varepsilon))\,\mathrm{d}\theta,\end{aligned} \tag{3.62}$$

where we use that $\hat q(t_i^\varepsilon)$ is an admissible test function and that (3.41) holds. In other words, here we essentially used again that the shifted variables behave better. Summing (3.62) for $i = 1, ..., k_\varepsilon$ and using (3.39), (3.40), and assuming that $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ are chosen so that $\frac{\partial}{\partial t} u_{\rm D}(t_i^\varepsilon) \in W^{1,\infty}(\Omega)$ are well defined, we obtain

$$
\begin{aligned}
&\hat{G}(t,\hat{q}(t)) + \mathrm{Var}_R(\hat{q};0,t) - G(0,\hat{q}_0) \geq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \hat{P}(\theta, q(t_i^\varepsilon))\,\mathrm{d}\theta \\
&\quad = \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(\theta), \zeta(t_i^\varepsilon)\big) : \nabla \frac{\partial u_{\rm D}}{\partial \theta}\,\mathrm{d}x\mathrm{d}\theta \\
&\quad = \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(t_i^\varepsilon), \zeta(t_i^\varepsilon)\big) : \nabla \frac{\partial u_{\rm D}(t_i^\varepsilon)}{\partial t}\,\mathrm{d}x\mathrm{d}\theta \\
&\quad + \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \Big(\varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(\theta), \zeta(t_i^\varepsilon)\big) \\
&\qquad\qquad\qquad\qquad - \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(t_i^\varepsilon), \zeta(t_i^\varepsilon)\big)\Big) : \nabla \frac{\partial u_{\rm D}}{\partial \theta}\,\mathrm{d}x\mathrm{d}\theta \\
&\quad + \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \int_\Omega \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(t_i^\varepsilon), \zeta(t_i^\varepsilon)\big) : \nabla \Big(\frac{\partial u_{\rm D}}{\partial \theta} - \frac{\partial u_{\rm D}(t_i^\varepsilon)}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}\theta \\
&\quad =: S_1^\varepsilon + S_2^\varepsilon + S_3^\varepsilon.
\end{aligned}
\tag{3.63}
$$

As to S_2^ε, using Hölder's inequality and the assumption (3.54), we can estimate

$$
\begin{aligned}
|S_2^\varepsilon| &\leq \sum_{i=1}^{k_\varepsilon} \int_{t_{i-1}^\varepsilon}^{t_i^\varepsilon} \big\|\varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(\theta), \zeta(t_i^\varepsilon)\big) \\
&\quad - \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(t_i^\varepsilon), \zeta(t_i^\varepsilon)\big)\big\|_{L^1(\Omega;\mathbb{R}^{3\times3})} \Big\|\nabla\frac{\partial u_{\rm D}}{\partial\theta}\Big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}\mathrm{d}\theta \\
&\leq \ell\big(1+\psi(1)\big)\Big(\mathrm{meas}(\Omega) + 2^p\big\|\nabla\hat{u}\big\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))}^p + 2^p\big\|\nabla u_{\rm D}\big\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^{3\times3}))}^p\Big) \\
&\quad \times\Big(\max_{i=1,\dots,k_\varepsilon}\ \max_{\theta\in[t_{i-1}^\varepsilon,t_i^\varepsilon]} \big\|\nabla u_{\rm D}(\theta) - \nabla u_{\rm D}(t_i^\varepsilon)\big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}\Big)\big\|\nabla u_{\rm D}\big\|_{W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))}.
\end{aligned}
$$

Since $\nabla u_{\rm D} \in W^{1,1}(0,T;L^\infty(\Omega;\mathbb{R}^{3\times3}))$, the "max max"-term tends to zero with $\varepsilon \to 0$, hence $\lim_{\varepsilon\to0+} S_2^\varepsilon = 0$. As to S_3^ε, by Fubini's theorem, we can estimate

$$
\begin{aligned}
|S_3^\varepsilon| &= \bigg|\sum_{i=1}^{k_\varepsilon} \int_\Omega \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon) + \nabla u_{\rm D}(t_i^\varepsilon), \zeta(t_i^\varepsilon)\big) \\
&\qquad : \nabla\Big(u_{\rm D}(t_i^\varepsilon) - u_{\rm D}(t_{i-1}^\varepsilon) - \frac{\partial u_{\rm D}(t_i^\varepsilon)}{\partial t}(t_i^\varepsilon - t_{i-1}^\varepsilon)\Big)\,\mathrm{d}x\bigg| \\
&\leq \big\|\varphi_F'\big(\nabla\hat{u} + \nabla u_{\rm D}, \zeta\big)\big\|_{L^\infty(0,T;L^1(\Omega))} \\
&\qquad \times \sum_{i=1}^{k_\varepsilon} \Big\|\nabla u_{\rm D}(t_i^\varepsilon) - \nabla u_{\rm D}(t_{i-1}^\varepsilon) - \nabla\frac{\partial u_{\rm D}(t_i^\varepsilon)}{\partial t}(t_i^\varepsilon - t_{i-1}^\varepsilon)\Big\|_{L^\infty(\Omega;\mathbb{R}^{3\times3})}.
\end{aligned}
$$

We have still a freedom to choose the partition $\{t_i^\varepsilon\}_{i=1}^{k_\varepsilon}$ in such a way that both the last sum approaches zero when $\varepsilon \to 0$ (hence $\lim_{\varepsilon\to0+} S_3^\varepsilon = 0$) and that the Riemann sum S_1^ε

approaches the corresponding Lebesgue integral, namely

$$S_1^\varepsilon = \sum_{i=1}^{k_\varepsilon}(t_i^\varepsilon - t_{i-1}^\varepsilon)\int_\Omega \varphi_F'\big(\nabla\hat{u}(t_i^\varepsilon)+\nabla u_{\rm D}(t_i^\varepsilon),\zeta(t_i^\varepsilon)\big):\nabla\frac{\partial u_{\rm D}(t_i^\varepsilon)}{\partial t}\,\mathrm{d}x$$
$$= \sum_{i=0}^{k_\varepsilon}(t_i^\varepsilon - t_{i-1}^\varepsilon)\hat{P}(t_i^\varepsilon,\hat{q}(t_i^\varepsilon)) \to \int_0^t \hat{P}(\theta,\hat{q}(\theta))\,\mathrm{d}\theta. \tag{3.64}$$

cf. [9, Lemma 4.12] or [13, Lemma 4.5]. This allows for a limit passage in (3.63) for $\varepsilon\to 0$, which gives

$$\hat{G}(t,\hat{q}(t)) + \mathrm{Var}_R(\hat{q};0,t) - \hat{G}(0,\hat{q}_0) \geq \int_0^t \hat{P}(\theta,\hat{q}(\theta))\,\mathrm{d}\theta. \tag{3.65}$$

In terms of the original "non-hatted" variables, we just get the opposite inequality in (3.61). Hence we proved that $q=(u,\zeta)$ is an energetic solution. □

Remark 3.11 In the preceding proof, we can also conclude that $\mathfrak{P}_* = \mathfrak{P}$ a.e. on $[0,T]$. From this proof, it can also be seen that (3.52) and (3.53) holds "sequentially" (i.e. for a selected subsequence). Besides, $\mathfrak{P}_\tau$ can be shown to converge strongly in $L^1(0,T)$ to $\mathfrak{P}$, see [13, Sect.3, Step 6]. Moreover, the measurability of u can be proved by considering all possible values $U(t)$ of u at a given t, establishing measurability of the set-valued mapping $t\mapsto U(t)$, and taking a measurable selection, which is indeed possible because we proved that $U(t)\neq\emptyset$; for elaboration of this idea see [22, Theorem 1.6.3].

4 The degenerate case: complete damage

Now we modify the assumption (3.1) on φ_0 and on φ_1 to

$$0 \leq \varphi_0(F) \leq C\big(1+|F|^p\big) \qquad \& \qquad |\varphi_0'(F)|^{1+\lambda} \leq C\big(1+\varphi_0(F)\big) \tag{4.1a}$$
$$\varepsilon_1|F|^p - C \leq \varphi_1(F) \leq C\big(1+|F|^p\big) \qquad \& \qquad \big|\varphi_1'(F)\big|^{1+\lambda} \leq C\big(1+|F|^p\big) \tag{4.1b}$$

again with $p>3$, $\varepsilon_1>0$, and now with some $0<\lambda\leq 1/(p-1)$. The second condition in (4.1a), cf. also (3.2), has been inspired by [3] where an analogous condition has been used for Kirchhoff's stress to handle polyconvex potentials with a fast growth. Here we use it just for the opposite to handle potentials with a small (possibly zero) growth because the first inequality in (4.1a) makes it possible that $F\mapsto\varphi(F,z)$ may lose coercivity if $z=0$. This obviously models the situation that the material is completely disintegrated and cannot resist all type of load (typically shear and tension) although, because we do not assume $\varphi_0\equiv 0$, it can still keep some elastic resistivity to certain types of loading (typically compression). Anyhow, this degeneration brings naturally substantial problems and we can expect substantially less results under these assumptions. In particular, we cannot control the deformation as in (3.23) and therefore also the choice of $\eta>0$ in Lemma 3.6 breaks down. In view of this, we cannot expect full results in the line of Section 3. On the other hand, one can expect that at least some energetics will be correctly recorded and also the damage parameter itself will keep a good sense. This is what actually matters in many applications.

We will make first a regularization in order to be able to exploit results we already obtained in Section 3, and then will pass to a limit. Let us emphasize that this regularization seems also suitable for a numerical treatment through the fully-implicit formula (3.19) if further discretization in space would be adopted. Taking a regularization parameter $\varepsilon > 0$, we consider

$$\varphi^\varepsilon(F,z) := \varphi(F,z) + \frac{\varepsilon}{p}|F|^p \tag{4.2}$$

and define the regularized (Gibbs-type) stored energy by

$$G^\varepsilon(t,q) := \begin{cases} V^\varepsilon(u,\zeta) & \text{if } u(\bar\Omega)\cap B = \emptyset,\ u|_{\Gamma_0} = w_{\rm D}(t), \text{ and } \zeta \ge 0 \text{ a.e. on } \Omega, \\ +\infty & \text{otherwise,} \end{cases} \tag{4.3}$$

where

$$\begin{aligned} V^\varepsilon(u,\zeta) &:= \int_\Omega \varphi^\varepsilon\big(\nabla u(x),\zeta(x)\big) + \frac{\kappa}{r}\big|\nabla\zeta(x)\big|^r \,\mathrm{d}x \\ &= \int_\Omega \varphi_0(\nabla u) + \psi(\zeta)\varphi_1(\nabla u) + \frac{\kappa}{r}\big|\nabla\zeta\big|^r + \frac{\varepsilon}{p}\big|\nabla u\big|^p \,\mathrm{d}x. \end{aligned} \tag{4.4}$$

Obviously, for $\varepsilon = 0$, the regularized stored energies V^ε and G^ε just coincides with the original V and G, respectively.

The regularized stored energy (4.2) satisfies, under the conditions (4.1), the growth condition

$$\begin{aligned} \big|[\varphi^\varepsilon]'_F(F,z)\big| &= \big|\varphi'_F(F,z) + \varepsilon|F|^{p-2}F\big| \\ &\le \big|\varphi'_0(F) + \psi(z)\varphi'_1(F)\big| + \varepsilon|F|^{p-1} \\ &\le C_{\lambda,\varepsilon,p}\big(1+\psi(1)\big)\big(1+|F|^p\big) \end{aligned} \tag{4.5}$$

for any $z \in [0,1]$, where $C_{\lambda,\varepsilon,p}$ depends on C's from (4.1) and on λ, ε, and p. Therefore, we can indeed use previous results from Section 3, namely Proposition 3.10 (without assuming (3.54)) which provides us a process $q^\varepsilon = (u^\varepsilon,\zeta^\varepsilon) : [0,T] \to W^{1,p}(\Omega;\mathbb{R}^3) \times W^{1,r}(\Omega)$ such that $u^\varepsilon \in \mathrm{B}([0,T];W^{1,p}(\Omega;\mathbb{R}^3))$, $\zeta^\varepsilon \in \mathrm{BV}(0,T;L^1(\Omega)) \cap L^\infty(0,T;W^{1,r}(\Omega))$ and such that, for all $t \in [0,T]$, it holds

$$\forall \tilde q \in Q: \quad G^\varepsilon\big(t,q^\varepsilon(t)\big) \le G^\varepsilon(t,\tilde q) + R\big(\tilde q - q^\varepsilon(t)\big) \quad \text{and} \tag{4.6a}$$

$$G^\varepsilon(t,q^\varepsilon(t)) + \mathrm{Var}_R(q^\varepsilon;0,t) \le G^\varepsilon(0,q_0) - \int_0^t \mathfrak{P}^\varepsilon(\theta)\,\mathrm{d}\theta \tag{4.6b}$$

$$\text{with } \mathfrak{P}^\varepsilon(\theta) := \int_\Omega [\varphi^\varepsilon]'_F(\nabla u^\varepsilon,\zeta^\varepsilon) : \nabla\frac{u_{\rm D}}{\partial\theta}\,\mathrm{d}x. \tag{4.6c}$$

Let us define, for a given damage profile ζ and loading $w_{\rm D}(t)$, the "substantial" stored energy as the Γ-limit of the collection $\{(t,\zeta) \mapsto \min G^\varepsilon(t,\cdot,\zeta)\}_{\varepsilon>0}$, i.e.

$$\mathfrak{g}(t,\zeta) := \liminf_{\substack{\varepsilon\to 0+,\ \tilde\zeta\to\zeta \\ \text{weakly in } W^{1,r}(\Omega)}} \ \min_{u\in W^{1,p}(\Omega;\mathbb{R}^3)} G^\varepsilon(t,u,\tilde\zeta). \tag{4.7}$$

Let us note that "min" in (4.7) does exist by the same arguments as we used in Lemma 3.4. Also note that $\mathfrak{g} > -\infty$ because we do not consider any external dead loading like gravity force.

Furthermore, let us define a "substantial" stress $\mathfrak{s} \in L^\infty(0,T;L^{1+\lambda}(\Omega;\mathbb{R}^{3\times3}))$ which is attainable weakly* in $L^\infty(0,T;L^{1+\lambda}(\Omega;\mathbb{R}^{3\times3}))$ by a selected subsequence of $\{[\varphi^\varepsilon]'_F(\nabla u^\varepsilon,\zeta^\varepsilon)\}_{\varepsilon>0}$. Let us emphasize that it makes no sense now to try to identify $\mathfrak{s}$ with $[\varphi^0]'_F(\nabla u,\zeta)$ for some u because the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ may (and expectedly will) blow up when the damage is complete in some parts and the material disintegrates. By the same reason, it is now natural to formulate the solution process in terms of $(\mathfrak{s},\zeta)$ instead of (u,ζ). Naturally, the stability and the energy equality (or, more precisely, now only an inequality) are now to be written in terms of the substantial stored energy $\mathfrak{g}$ and the substantial stress $\mathfrak{s}$ as

$$\mathfrak{g}(t,\zeta(t)) \le \mathfrak{g}(t,\tilde\zeta) + \int_\Omega \varrho(\tilde\zeta-\zeta(t))\,\mathrm{d}x \quad \text{for any } \tilde\zeta\in L^1(\Omega), \quad \text{and} \tag{4.8a}$$

$$\mathfrak{g}(t,\zeta(t)) + \mathrm{Var}_\rho(\zeta;0,t) \le \mathfrak{g}(0,1) - \int_0^t\int_\Omega \mathfrak{s} : \nabla\frac{\partial u_{\mathrm{D}}}{\partial t}\,\mathrm{d}x\mathrm{d}t \tag{4.8b}$$

where $\mathrm{Var}_\varrho(\zeta;0,t)$ means the total variation of $\theta \mapsto \int_\Omega \varrho(\zeta(\theta))\mathrm{d}x$ on the interval $[0,t]$, i.e. here $\int_\Omega d(1-\zeta(t))\mathrm{d}x$. In addition, we need now the following qualification of ψ:

$$\exists K_\psi < +\infty, \alpha > 0 \ \ \forall z \ge 0\ \forall a \ge 1: \quad \psi(az) - \psi(z) \le K_\psi(a^\alpha - 1)\psi(z). \tag{4.9}$$

Let us note that (4.9) is satisfied for the most frequently used case $\psi(z) = z^\alpha$ with $K_\psi = 1$. In fact, even a weaker condition than (4.9) can be designed for ψ with non-polynomial decay to 0.

Proposition 4.1 *Let (2.16), (2.17), (4.1), and (4.9) hold. Then there exist* $(\mathfrak{s},\zeta) \in L^\infty(0,T;L^{1+\lambda}(\Omega;\mathbb{R}^{3\times3})) \times \big(\mathrm{BV}(0,T;L^1(\Omega)) \cap L^\infty(0,T;W^{1,r}(\Omega))\big)$ *satisfying (4.8) and* $\mathfrak{s}$ *being a substantial stress, i.e. attainable by a subsequence of* $\{[\varphi^\varepsilon]'_F(\nabla u^\varepsilon,\zeta^\varepsilon)\}_{\varepsilon>0}$.

Proof. Our strategy is to make a limit passage in (4.6) for $\varepsilon \to 0$.

First, we note that the constant C_2 in (3.24) is independent of ε, hence, by Helly's theorem, we can make the limit passage (up to a subsequence) $\zeta^\varepsilon(t) \to \zeta(t)$ weakly in $W^{1,r}(\Omega)$ for all $t \in [0,T]$, and thus also

$$\mathrm{Var}_R(q^\varepsilon;0,t) = \int_\Omega d\big(1-\zeta^\varepsilon(t)\big)\,\mathrm{d}x \to \int_\Omega d\big(1-\zeta(t)\big)\,\mathrm{d}x = \mathrm{Var}_\varrho(\zeta;0,t). \tag{4.10}$$

Moreover, like in (4.5) and using also (3.1), we can estimate

$$\begin{aligned}
\big\|[\varphi^\varepsilon]'_F(\nabla u^\varepsilon,\zeta^\varepsilon)\big\|^{1+\lambda}_{L^{1+\lambda}(\Omega;\mathbb{R}^{3\times3})} &= \int_\Omega \big|[\varphi^\varepsilon]'_F(\nabla u^\varepsilon,\zeta^\varepsilon)\big|^{1+\lambda}\mathrm{d}x\\
&\le 3^\lambda\int_\Omega \big|\varphi_0'(\nabla u^\varepsilon)\big|^{1+\lambda} + \psi(\zeta^\varepsilon)^{1+\lambda}\big|\varphi_1'(\nabla u^\varepsilon)\big|^{1+\lambda} + \varepsilon p|\nabla u^\varepsilon|^{(p-1)(1+\lambda)}\mathrm{d}x\\
&\le 3^\lambda\int_\Omega C\big(1+\varphi_0(\nabla u^\varepsilon)\big) + \psi(\zeta^\varepsilon)^{1+\lambda}C\Big(1+\frac{\varphi_1(\nabla u^\varepsilon)+C}{\varepsilon_1}\Big)\\
&\qquad\qquad + \varepsilon p|\nabla u^\varepsilon|^{(p-1)(1+\lambda)}\mathrm{d}x\\
&\le C_{\lambda,\varepsilon_1,p}\int_\Omega 1 + \varphi^\varepsilon(\nabla u^\varepsilon,\zeta^\varepsilon)\,\mathrm{d}x
\end{aligned} \tag{4.11}$$

for some constant $C_{\lambda,\varepsilon_1,p}$ depending on its indices as indicated, provided $\zeta^\varepsilon \in [0,1]$ a.e. and $(p-1)(1+\lambda) \le p$, i.e. $\lambda \le 1/(p-1)$. From this and from the uniform bound of the energy $V(q(t))$ independently of ε, cf. (3.29), we can see that $\{[\varphi^\varepsilon]'_F(\nabla u^\varepsilon, \zeta^\varepsilon)\}_{\varepsilon>0}$ is bounded and we can further select another subsequence converging to some $\mathfrak{s}$ weakly* in $L^\infty(0,T;L^{1+\lambda}(\Omega;\mathbb{R}^{3\times3}))$. At the original time $t=0$, in view of (2.17b,c) we have

$$\begin{aligned} G^\varepsilon(0,q_0) = G^\varepsilon(0,\mathrm{I},1) &= \int_\Omega \varphi(\mathrm{I},1) + \varepsilon|\mathrm{I}|^p \mathrm{d}x \\ &= \varepsilon \mathrm{meas}(\Omega)|\mathrm{I}|^p \to 0 = G(0,q_0) = \mathfrak{g}(0,1). \end{aligned} \tag{4.12}$$

The lower semicontinuity of $\mathfrak{g}(t,\cdot)$ we need is

$$\begin{aligned} \liminf_{\substack{\zeta^\varepsilon\to\zeta \text{ weakly} \\ \text{in } W^{1,r}(\Omega)}} \mathfrak{g}(t,\zeta^\varepsilon) &= \liminf_{\substack{\zeta^\varepsilon\to\zeta \text{ weakly} \\ \text{in } W^{1,r}(\Omega)}} \Bigg(\liminf_{\substack{\varepsilon\to0,\ \ \tilde\zeta\to\zeta^\varepsilon \\ \text{weakly in } W^{1,r}(\Omega)}} \ \min_{u\in W^{1,p}(\Omega;\mathbb{R}^3)} G^\varepsilon(t,u,\tilde\zeta) \Bigg) \\ &\ge \liminf_{\substack{\varepsilon\to0,\ \ \tilde\zeta\to\zeta \\ \text{weakly in } W^{1,r}(\Omega)}} \ \min_{u\in W^{1,p}(\Omega;\mathbb{R}^3)} G^\varepsilon(t,u,\tilde\zeta) = \mathfrak{g}(t,\zeta); \end{aligned} \tag{4.13}$$

in fact, this lower semicontinuity is a general property of Γ-limits. Then, using both (4.12) and (4.13) for $\zeta^\varepsilon = \zeta^\varepsilon(t)$, and realizing that certainly $G^\varepsilon(t,q^\varepsilon) \ge \mathfrak{g}(t,\zeta^\varepsilon)$ (in fact, even the equality holds), we can make a limit passage in (4.6b) to obtain (4.8b).

To obtain (4.8a), we must use a suitable sequence, let us denote it by $\{\tilde q^\varepsilon\}_{\varepsilon>0}$ with $\tilde q^\varepsilon = (\tilde u^\varepsilon, \tilde\zeta^\varepsilon)$, such that

$$\lim_{\varepsilon\to0+} G^\varepsilon(t,\tilde q^\varepsilon) = \mathfrak{g}(t,\tilde\zeta). \tag{4.14}$$

Such a sequence does exist simply due to the definition of "liminf" used in (4.7) and, in addition, $\tilde\zeta^\varepsilon \to \tilde\zeta$ and $\tilde u^\varepsilon$ minimizes $G^\varepsilon(t,\cdot,\tilde\zeta^\varepsilon)$. Unfortunately, we still cannot pass to the limit in (4.6a) (if $\tilde q$ is replaced by $\tilde q^\varepsilon$) because R is not upper semicontinuous, and we also cannot make the shift above like in (3.47) because we now do not have any a-priori bound for u^ε and thus cannot control the term $\psi(\tilde\zeta)\varphi_1(\nabla\tilde u^\varepsilon)$ needed to prove the convergence (4.14). Fortunately, we can make a "multiplicative modification" of $\tilde\zeta^\varepsilon$ as follows. Let us assume that $\tilde\zeta(x) \ge \zeta(t,x)$ for a.a. $x \in \Omega$, otherwise the integral in (4.8a) equals to $+\infty$ and (4.8a) is trivially satisfied. Then we put

$$\bar q^\varepsilon := (\tilde u^\varepsilon, \bar\zeta^\varepsilon), \qquad \bar\zeta^\varepsilon(x) := \frac{\tilde\zeta^\varepsilon(x)}{\rho^\varepsilon}, \qquad \rho^\varepsilon := \min\Big(1, \min_{x\in\bar\Omega} \frac{\tilde\zeta^\varepsilon(x)}{\zeta^\varepsilon(x)}\Big). \tag{4.15}$$

We have always $\bar\zeta^\varepsilon \ge \tilde\zeta^\varepsilon$ and $\bar\zeta^\varepsilon \ge \zeta^\varepsilon(t,\cdot)$ a.e. on Ω, and also $\lim_{\varepsilon\to0+}\rho^\varepsilon = 1$ because $\tilde\zeta \ge \zeta(t,\cdot)$ and because $\tilde\zeta^\varepsilon \to \tilde\zeta$ and $\zeta^\varepsilon(t,\cdot) \to \zeta(t,\cdot)$ in $C(\bar\Omega)$. Therefore, we have again $\lim_{\varepsilon\to0+}\bar\zeta^\varepsilon = \tilde\zeta$ weakly in $W^{1,r}(\Omega)$ but now, in addition, also

$$\lim_{\varepsilon\to0+} R(\bar q^\varepsilon - q^\varepsilon(t)) = \int_\Omega \varrho(\tilde\zeta - \zeta(t))\mathrm{d}x \tag{4.16}$$

because of $\bar\zeta^\varepsilon \ge \zeta^\varepsilon(t,\cdot)$. Also the convergence (4.14) is preserved when $\tilde q^\varepsilon$ is replaced by $\bar q^\varepsilon$ because

$$\begin{aligned} 0 &\le G^\varepsilon(t,\bar q^\varepsilon) - G^\varepsilon(t,\tilde q^\varepsilon) \\ &= \int_\Omega \big(\psi(\bar\zeta^\varepsilon) - \psi(\tilde\zeta^\varepsilon)\big)\varphi_1(\tilde u^\varepsilon) + \Big(\frac{1}{(\rho^\varepsilon)^r} - 1\Big)\frac{\kappa}{r}\big|\nabla\tilde\zeta^\varepsilon\big|^r \mathrm{d}x \\ &\le K_\psi\Big(\frac{1}{(\rho^\varepsilon)^\alpha} - 1\Big)\big\|\psi(\tilde\zeta^\varepsilon)\varphi_1(\tilde u^\varepsilon)\big\|_{L^1(\Omega)} + \Big(\frac{1}{(\rho^\varepsilon)^r} - 1\Big)\frac{\kappa}{r}\big\|\nabla\tilde\zeta^\varepsilon\big\|^r_{L^r(\Omega;\mathbb{R}^3)} \end{aligned} \tag{4.17}$$

where K_ψ and α come from (4.9). Due to (4.14), both $\|\psi(\tilde{\zeta}^\varepsilon)\varphi_1(\tilde{u}^\varepsilon)\|_{L^1(\Omega)}$ and $\|\nabla\tilde{\zeta}^\varepsilon\|_{L^r(\Omega;\mathbb{R}^3)}$ are bounded independently of ε, hence the right-hand side in (4.17) converges to 0 because $\rho^\varepsilon \to 1$. Thus, merging (4.17) and (4.14), we get

$$\lim_{\varepsilon\to 0+} G^\varepsilon(t,\bar{q}^\varepsilon) = \mathfrak{g}(t,\tilde{\zeta}). \tag{4.18}$$

Now, putting $\bar{q}^\varepsilon$ into (4.6a) and using (4.13), (4.16), (4.18), we can pass to the limit in the right-hand side of (4.6a) and estimate from below the limes inferior of the left-hand side of (4.6a) to obtain eventually (4.8a). $\square$

Remark 4.2 It should be emphasized that investigations in Sect. 4 are only a basic scenario leaving most crucial questions open. In particular, more specific characterization of $\mathfrak{g}$ and $\mathfrak{s}$ would be desirable at least in special cases. The conjecture is that $\mathfrak{g}(t,\zeta) = \inf_{u\in W^{1,p}(\Omega;\mathbb{R}^3)} G^0(t,u,\zeta)$ under some conditions. Further questions concern equality in (4.8b) or the relation $\int_\Omega \mathfrak{s} : \nabla\frac{\partial}{\partial t}u_{\rm D}\,\mathrm{d}x = \mathfrak{g}_t'(t,\zeta(t))$.

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