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Mean–square approximation for stochastic differential equations with small noises

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Abstract

New approach to construction of mean-square numerical methods for solution of stochastic differential equations with small noises is proposed. The approach is based on expanding of the exact solution of the system with small noises by powers of time increment and regrouping of expansion terms according to powers of time increment and small parameter. The theorem on mean-square estimate of method errors is proved. Various efficient numerical schemes are derived for a general system with small noises and for systems with small additive and small colored noises. The proposed methods are tested by calculation of Lyapunov exponents and simulation of a laser Langevin equation with multiplicative noises.

Contents

1 Introduction

The stochastic approach has found wide application in physics.[1,2] Usually analytic solutions for stochastic dynamical systems are hardly available. In this case the importance of numerical methods is obvious. In previous works various mean-square and weak numerical methods were derived for a general system of stochastic differential equations and for some specific systems such as systems with additive and colored noises, etc. (see Refs. 3-7). In the general case some difficulties arise to realize numerical methods for stochastic differential equations. For instance, there are no efficient Runge-Kutta schemes. As to mean-square approximation there are no sufficiently constructive methods to simulate multiple Ito integrals.

But often fluctuations, which affect a physical system, are sufficiently small. Fortunately, as shown in the paper, for a stochastic system with small noises it is possible to construct special numerical methods which are more effective and easier than in the general case.

Herein for the first time numerical integration of stochastic differential equations with small noises is systematically considered. In the paper mean-square approximation is investigated. Weak methods will be the subject of a separate paper.

The system of Ito stochastic differential equations with small noises may be written in the form

$$dX = a(t,X)\,dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X)\,dW_r\ ,\ \ X(t_o) = X_o,\ \ t \in [t_o, T] \qquad (1.1)$$

where X , $a(t,X)$ and $\sigma_r(t,X)$ are n-dimensional vectors, W_r are uncorrelated standard Wiener processes and ε is a small parameter.

If the parameter ε tends to zero, we have a deterministic system for which various effective numerical methods exist. One can believe that if parameter ε is sufficiently small, i.e., the system (1.1) is sufficiently close to deterministic one, it is also possible to obtain effective methods taking into account that ε is small.

In the paper for the system (1.1) the approach to construction of effective numerical methods is proposed, the theorem on estimate of mean-square error of a method on the whole interval is proved, various mean-square methods with low errors are constructed including explicit, implicit and Runge-Kutta schemes. Derived methods are efficient as to simulation of needed random variables. In the case of a general system, i.e., $\varepsilon = 1$, only time order 1/2 schemes (i.e., mean-square error of the method on the whole interval is equal to $O(h^{1/2})$) may be efficient. And first order methods already require calculation of complicated multiple Ito integrals which is difficult and laborious problem.[6] Using the approach, developed in the paper, one can obtain for the system (1.1) numerical schemes with mean-square errors on the whole interval which have a form

$$O(h^{r_o-1} + \sum_{r\in S} h^r \varepsilon^{J(r)}),$$

S is a set of positive integers and semi-integers which are less than positive integer $r_o - 1$, $J(r)$ is a decreasing function with natural values. In the paper various schemes with $r_o = 2,3,4,5$ are derived. Because of the small parameter ε both the sum $\sum h^r \varepsilon^{J(r)}$ and the method error may be sufficiently low. So, the approach gives an opportunity to construct numerical methods with low errors for solution of the system (1.1) which are efficient with respect to both simulation of needed random variables and calculation expenses. As we believe, these methods would be useful for many physical applications.

Our approach is based on expanding of the exact solution of the system (1.1) by powers of time increment h and regrouping of expansion terms according to their factors $h^i\varepsilon^j$ (h-ε approach). It may be seemed as more natural to expand the exact solution firstly by powers of small parameter ε and then by powers of time increment h (ε-h approach). But ε-h approach suffers from grave shortcoming because of divergence of methods constructed in this way. For details see Subsection 3.4.

The organization of the paper is as follows. In Section 2 we consider some preliminary examples and construct one-step approximation of the solution of the system (1.1). The theorem on mean-square estimate of method error is proved in Section 3. Section 4 is devoted to a stochastic system with small noises in Stratonovich interpretation. By the results of Sections 3 and 4 we derive various efficient numerical schemes which are presented in Section 5. In Sections 6 and 7 one can find numerical methods for systems with small additive noises and small colored noises. Although the concept of construction of special methods for a system with small noises is not difficult, the derivations of appropriate schemes are sufficiently laborious. So, in Sections 5-7 we write down more important and useful methods without detailed derivations. Numerical tests of the proposed methods are given in Section 8.

2 One-step approximation

Let us introduce a discretization Δ_N of the interval $[t_o, T]$: $\Delta_N = \{t_i : 0, 1, ..., N; t_o < t_1 < ... < t_N = T\}$; the time increment $h = t_{i+1} - t_i$; the approximation X_k or $\bar{X}(t_k)$ of the exact solution $X(t_k)$ of the system (1.1); the mean value $E\xi$ of a random variable ξ ; operators

$$\begin{aligned} L &= L_1 + \varepsilon^2 L_2, \;\; L_1 = \frac{\partial}{\partial t} + (a, \frac{\partial}{\partial x}) = \frac{\partial}{\partial t} + \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i} \\ L_2 &= \frac{1}{2}\sum_{r=1}^{q}(\sigma_r, \frac{\partial}{\partial x})^2 = \frac{1}{2}\sum_{r=1}^{q}\sum_{i,j=1}^{n} \sigma_r^i \sigma_r^j \frac{\partial^2}{\partial x^i \partial x^j}, \\ \Lambda_r &= (\sigma_r, \frac{\partial}{\partial x}) = \sum_{i=1}^{n} \sigma_r^i \frac{\partial}{\partial x^i}; \end{aligned}$$

Ito integrals

$$I_{i_1,...,i_j}(F,t,h) = \int_t^{t+h} dW_{i_j}(\theta) \int_t^{\theta} dW_{i_{j-1}}(\theta_1) \int_t^{\theta_1} ... \int_t^{\theta_{j-2}} F(\theta_{j-1})\, dW_{i_1}(\theta_{j-1})$$

where $i_1, ..., i_j$ are from the set of numbers $\{0, 1, ..., q\}$ and $dW_o(\theta_r)$ designates $d\theta_r$, $F(\theta)$ is some deterministic (for simplicity continuous) function;

$I_{i_1,i_2,\dots,i_j}(t,h) \equiv I_{i_1,i_2,\dots,i_j}(1,t,h)$,
where $1(\theta)$ is the function which is exactly equal to one.

We assume that restrictions on the coefficients of the system (1.1) are so that they ensure the existence and uniqueness of the solution on the whole time interval $[t_o, T]$. Moreover, for construction of high order methods the coefficients must be sufficiently smooth functions. Note that an initial value X_o of the system (1.1) may be equal to a deterministic value or a random variable which does not depend on the Wiener process.

2.1 Preliminary consideration

The simplest numerical method for solution of the system (1.1) is the Euler one

$$\bar{X}(t+h) = X(t) + \varepsilon \sum_{r=1}^{q} \sigma_r(t, X(t))\, I_r(t,h) + a(t, X(t))h \qquad (2.1)$$

The remainder ρ of the approximation (2.1) is equal to[6,7]

$$\begin{aligned}\rho \quad &= X(t+h) - \bar{X}(t+h) = \varepsilon^2 \sum_{s,r=1}^{q} I_{sr}(\Lambda_s \sigma_r, t, h) + \\ &+\varepsilon \sum_{r=1}^{q} I_{or}(L_1\sigma_r, t, h) + \varepsilon^3 \sum_{r=1}^{q} I_{or}(L_2\sigma_r, t, h) + \\ &+\varepsilon \sum_{r=1}^{q} I_{ro}(\Lambda_r a, t, h) + I_{oo}(L_1 a, t, h) + \varepsilon^2 I_{oo}(L_2 a, t, h)\end{aligned} \qquad (2.2)$$

It is known[6,7] that the Ito integrals have the properties

$$\begin{aligned}&E\, I_{i_1,\dots,i_j} = 0 \;\; if\ one\ of\ the\ indices\ i_k \neq 0, \\ &E\, I_{i_1,\dots,i_j} = O(h^j) \;\; if\ all\ indices\ i_k = 0, \\ &(E\,(I_{i_1,\dots,i_j})^2)^{1/2} = O(h^r), \quad r = l_1 + l_2/2,\end{aligned} \qquad (2.3)$$

l_1 is the number of zero indices i_k and l_2 is the number of non-zero indices i_k.

By the properties (2.3) the remainder ρ of (2.2) can be estimated as

$$\begin{aligned}&E\,\rho = O(h^2) \\ &(E\rho^2)^{1/2} = O(h^2 + \varepsilon h^{3/2} + \varepsilon^2 h) = O(h^2 + \varepsilon^2 h)\end{aligned} \qquad (2.4)$$

The term $\varepsilon h^{3/2}$ of the second estimate is omitted because it is not greater than $(h^2 + \varepsilon^2 h)/2$. The one-step time-increment accuracy order of the Euler method is certainly equal to one, but the principal term of the error (2.4) with respect to h has the small factor ε^2.

Let us consider the method with the one-step order two[6,7]

$$\bar{X}(t+h) = X(t) + \varepsilon \sum_{r=1}^{q} \sigma_r(t, X(t)) I_r(t,h) + a(t, X(t))h +$$

$$+\varepsilon^2 \sum_{i,r=1}^{q} \Lambda_r\sigma_i(t,X(t))I_{ri}(t,h) + \varepsilon \sum_{r=1}^{q} L_1\sigma_r(t,X(t))I_{or}(t,h) +$$

$$+\varepsilon^3 \sum_{r=1}^{q} L_2\sigma_r(t,X(t))I_{or}(t,h) + \varepsilon \sum_{r=1}^{q} \Lambda_r a(t,X(t))I_{ro}(t,h) +$$

$$+\varepsilon^3 \sum_{s,i,r=1}^{q} \Lambda_s\Lambda_i\sigma_r(t,X(t))I_{sir}(t,h) + L_1 a(t,X(t))h^2/2 +$$

$$+\varepsilon^2 L_2 a(t,X(t))h^2/2 \tag{2.5}$$

The remainder ρ of this approximation has the form

$$\begin{aligned}
\rho \quad &= \varepsilon^4 \sum_{r,i,s,j=1}^{q} I_{risj}(\Lambda_r\Lambda_i\Lambda_s\sigma_j,t,h) + \varepsilon^2 \sum_{i,r=1}^{q} I_{oir}(L_1\Lambda_i\sigma_r,t,h) + \\
&+\varepsilon^4 \sum_{i,r=1}^{q} I_{oir}(L_2\Lambda_i\sigma_r,t,h) + \varepsilon^2 \sum_{i,r=1}^{q} I_{ior}(\Lambda_i L_1\sigma_r,t,h) + \\
&+\varepsilon^4 \sum_{i,r=1}^{q} I_{ior}(\Lambda_i L_2\sigma_r,t,h) + \varepsilon^2 \sum_{i,r=1}^{q} I_{iro}(\Lambda_i\Lambda_r a,t,h) + \\
&+\varepsilon^3 \sum_{r,i,s=1}^{q} I_{osir}(L_1\Lambda_s\Lambda_i\sigma_r,t,h) + \varepsilon^5 \sum_{r,i,s=1}^{q} I_{osir}(L_2\Lambda_s\Lambda_i\sigma_r,t,h) + \\
&+\varepsilon \sum_{r=1}^{q} I_{oor}(L_1^2\sigma_r,t,h) + \varepsilon^3 \sum_{r=1}^{q} I_{oor}((L_1L_2+L_2L_1)\sigma_r,t,h) + \\
&+\varepsilon^5 \sum_{r=1}^{q} I_{oor}(L_2^2\sigma_r,t,h) + \varepsilon \sum_{r=1}^{q} I_{oro}(L_1\Lambda_r a,t,h) + \\
&+\varepsilon^3 \sum_{r=1}^{q} I_{oro}(L_2\Lambda_r a,t,h) + \varepsilon \sum_{r=1}^{q} I_{roo}(\Lambda_r L_1 a,t,h) + \qquad (2.6) \\
&+\varepsilon^3 \sum_{r=1}^{q} I_{roo}(\Lambda_r L_2 a,t,h) + I_{ooo}(L_1^2 a,t,h) + \\
&+\varepsilon^2 I_{ooo}((L_1L_2+L_2L_1)a,t,h) + \varepsilon^4 I_{ooo}(L_2^2 a,t,h).
\end{aligned}$$

The Ito integrals I_{ri} and I_{sir} of the method (2.5) cannot be easily simulated. But these integrals are multiplied by ε^α. That is why, they may be omitted and error of the approximation would be still not large. On this way we obtain the reduced method

$$\bar{X}(t+h) = X(t) + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X(t))I_r(t,h) + a(t,X(t))h +$$

$$+\varepsilon \sum_{r=1}^{q} L_1\sigma_r(t,X(t))I_{or}(t,h) + \varepsilon \sum_{r=1}^{q} \Lambda_r a(t,X(t))I_{ro}(t,h) +$$

$$+L_1 a(t,X(t))h^2/2, \tag{2.7}$$

the remainder ρ_1 of which is equal to

$$\rho_1 = \rho + \varepsilon^2 \sum_{i,r=1}^{q} \Lambda_r\sigma_i(t,X(t))I_{ri}(t,h) + \varepsilon^3 \sum_{r=1}^{q} L_2\sigma_r(t,X(t))I_{or}(t,h) +$$

$$+\varepsilon^3\sum_{s,i,r=1}^{q}\Lambda_s\Lambda_i\sigma_r(t,X(t))I_{sir}(t,h)+\varepsilon^2L_2a(t,X(t))h^2/2, \quad (2.8)$$

where ρ is taken from (2.6).

Let us regroup the terms of the approximation (2.7) and its remainder according to their factors $h^i\varepsilon^j$

$$\bar{X}(t+h)=X(t)+h^{1/2}\varepsilon\sum_{r=1}^{q}\sigma_r(t,X(t))I_r(0,1)+ha(t,X(t))+$$
$$+h^{3/2}\varepsilon\left\{\sum_{r=1}^{q}L_1\sigma_r(t,X(t))I_{or}(0,1)+\sum_{r=1}^{q}\Lambda_ra(t,X(t))I_{ro}(0,1)\right\}+$$
$$+h^2L_1a(t,X(t))/2, \quad (2.9)$$

$$\rho_1 = h\varepsilon^2\sum_{i,r=1}^{q}\Lambda_r\sigma_i(t,X(t))I_{ri}(0,1)+h^{3/2}\varepsilon^3\{\sum_{r=1}^{q}L_2\sigma_r(t,X(t))I_{or}(0,1)+$$
$$+\sum_{s,i,r=1}^{q}\Lambda_s\Lambda_i\sigma_r(t,X(t))I_{sir}(0,1)\}+h^2\varepsilon^2L_2a(t,X(t))/2+$$
$$+h^2\{\varepsilon^2[\sum_{i,r=1}^{q}I_{oir}(L_1\Lambda_i\sigma_r,0,1)+\sum_{i,r=1}^{q}I_{ior}(\Lambda_iL_1\sigma_r,0,1)+$$
$$+\sum_{i,r=1}^{q}I_{iro}(\Lambda_i\Lambda_ra,0,1)]+\varepsilon^4[\sum_{r,i,s,j=1}^{q}I_{risj}(\Lambda_r\Lambda_i\Lambda_s\sigma_j,0,1)+$$
$$+\sum_{i,r=1}^{q}I_{oir}(L_2\Lambda_i\sigma_r,0,1)+\sum_{i,r=1}^{q}I_{ior}(\Lambda_iL_2\sigma_r,0,1)]\}+$$
$$+h^{5/2}\{\varepsilon[\sum_{r=1}^{q}I_{oor}(L_1^2\sigma_r,0,1)+\sum_{r=1}^{q}I_{oro}(L_1\Lambda_ra,0,1)+$$
$$+\sum_{r=1}^{q}I_{roo}(\Lambda_rL_1a,0,1)]+\varepsilon^3[\sum_{r,i,s=1}^{q}I_{osir}(L_1\Lambda_s\Lambda_i\sigma_r,0,1)+$$
$$+\sum_{r=1}^{q}I_{oor}((L_1L_2+L_2L_1)\sigma_r,0,1)+\sum_{r=1}^{q}I_{oro}(L_2\Lambda_ra,0,1)+$$
$$+\sum_{r=1}^{q}I_{roo}(\Lambda_rL_2a,0,1)]+\varepsilon^5[\sum_{r,i,s=1}^{q}I_{osir}(L_2\Lambda_s\Lambda_i\sigma_r,0,1)+$$
$$+\sum_{r=1}^{q}I_{oor}(L_2^2\sigma_r,0,1)]\}+h^3\{I_{ooo}(L_1^2a,0,1)+$$
$$+\varepsilon^2I_{ooo}((L_1L_2+L_2L_1)a,0,1)+\varepsilon^4I_{ooo}(L_2^2a,0,1)\} \quad (2.10)$$

where, as it can be done without ambiguity, the previous notation $I_{i_1,\ldots,i_k}(F,t,h)$ is used, while it is obtained from the previously defined $I_{i_1,\ldots,i_k}(F,t,h)$ by change of variables: new θ is equal to $(\theta-t)/h$.

Now one can obtain

$$E\,\rho_1=O(h^3+\varepsilon^2h^2) \quad (2.11)$$

$$(E\rho_1^2)^{1/2} = O(h^3 + \varepsilon h^{5/2} + \varepsilon^2 h^2 + \varepsilon^3 h^{3/2} + \varepsilon^2 h) = O(h^3 + \varepsilon^2 h)$$

Of course, the approximation (2.7) has lower time increment order (due to the term $\varepsilon^2 \sum_{i,r=1}^{q} \Lambda_r \sigma_i I_{ri}$ the one-step order is equal to one) than the approximation (2.5). But the error of the method (2.7) has small factor ε^2 at h. Thus, we obtain the method (2.7) one-step mean-square error of which is sufficiently small and which is efficient as to simulation of needed random variables. Note that we exclude from (2.5) not only the terms with complicated Ito integrals, but also the terms with equal or higher smallness orders with respect to h and ε together than orders of the terms with complicated integrals.

By the preliminary consideration we have demonstrated the idea which is the base of the paper. In contrast to the general case smallness of terms of an approximation for a system with small noise and of its remainder depends not only on time increment h, but also on small parameter ε. This circumstance, as shown above, allows us to construct new numerical methods by excluding complicated terms, for instance multiple Ito integrals, from a method and including them in its remainder. New methods are efficient as to simulation of needed random variables and have low mean-square errors in the sense of product $\varepsilon^i h^j$. Moreover, such methods contain much less terms with operators than the corresponding schemes for a general system.

2.2 One-step approximation

Let us assume that one-step approximation $\bar{X}_{t,x}(t+h)$ of the exact solution $X_{t,x}(t+h)$ of the system (1.1) ($X(t) = \bar{X}(t) = x, t_o \le t < t+h \le T$) is constructed so that it depends on t, x, h, ε and $\{W_1(\vartheta) - W_1(t), ..., W_q(\vartheta) - W_q(t); t \le \vartheta \le t+h\}$:

$$\bar{X}_{t,x}(t+h) = x + A(t, x, h, \varepsilon; W_i(\vartheta) - W_i(t), i = 1, ..., q, t \le \vartheta \le t+h) \qquad (2.12)$$

If we suppose that the approximation (2.12) is obtained by the Wagner-Platen expansion,[4,6,7] then the function A may be written in the following form

$$A(t, x, h, \varepsilon; W_i(\vartheta) - W_i(t), i = 1, ..., q, t \le \vartheta \le t+h) = \qquad (2.13)$$

$$= \sum_{i=1}^{r_o - 1} \sum_{j=0}^{\bar{K}(i)-1} \bar{a}_{ij} h^i \varepsilon^{2j} + \sum_{i=1}^{r_1} \sum_{j=1}^{\bar{L}(i)-1} \bar{b}_{ij} h^{i-1/2} \varepsilon^{2j-1} + \sum_{i=1}^{r_2} \sum_{j=1}^{\bar{M}(i)-1} \bar{c}_{ij} h^i \varepsilon^{2j}$$

where $r_o \ge 2$ is a natural number; $1 \le r_1 \le r_o$ is a natural number too; the integer $r_2 < r_o$ can be zero, in this case the third sum $\sum_{i=1}^{r_2} \sum_{j=1}^{\bar{M}(i)-1} \bar{c}_{ij} h^i \varepsilon^{2j}$ is neglected by definition; $1 \le \bar{K}(i) \le i$, $2 \le \bar{L}(i) \le i+1$, $2 \le \bar{M}(i) \le i+1$ are functions with natural values; $\bar{a}_{ij}$ depend on t and x; $\bar{b}_{ij}$ and $\bar{c}_{ij}$ depend on t, x and $\{W_1(\vartheta) - W_1(t), ..., W_q(\vartheta) - W_q(t); t \le \vartheta \le t+h\}$, $E\,\bar{b}_{ij} = E\,\bar{c}_{ij} = 0$. The concrete expressions for $\bar{a}_{ij}$, $\bar{b}_{ij}$ and $\bar{c}_{ij}$ are followed from construction of the Wagner-Platen expansion.[4,6,7] The expression (2.13) generalizes the examples (2.1) and (2.7) of the previous subsection. For instance, as followed from (2.9), in the case of the approximation (2.7) $\bar{b}_{11} = \sum_{r=1}^{q} \sigma_r(t, X(t)) I_r(0,1)$.

To estimate one-step error of an approximation one must thoroughly analyze its remainder. The remainder ρ of the approximation (2.12)-(2.13) may be written in the form

$$\rho=\sum_{i=2}^{r_o}\sum_{j=K(i)}^{i-1} a_{ij}h^i\varepsilon^{2j}+\sum_{i=2}^{r_o}\sum_{j=L(i)}^{i} b_{ij}h^{i-1/2}\varepsilon^{2j-1}+$$

$$+\sum_{i=1}^{r_o-1}\sum_{j=M(i)}^{i} c_{ij}h^i\varepsilon^{2j}, \tag{2.14}$$

where $K(i)=\bar{K}(i)$ if $i<r_o$ and $K(r_o)=0$; $L(i)=\bar{L}(i)$ if $i\le r_1$ and $L(i)=1$ if $i>r_1$; $M(i)=\bar{M}(i)$ if $i\le r_2$ and $M(i)=1$ if $i>r_2$; $K(i),L(i),M(i)$ are the functions with natural values. If $K(i)=i$ (correspondingly $L(i)=i+1$, $M(i)=i+1$), the sum $\sum_{j=K(i)}^{i-1} a_{ij}h^i\varepsilon^{2j}$ (correspondingly the sum $\sum_{j=L(i)}^{i} b_{ij}h^{i-1/2}\varepsilon^{2j-1}$, the sum $\sum_{j=M(i)}^{i} c_{ij}h^i\varepsilon^{2j}$) is neglected by definition. It must be mentioned that a_{ij}, b_{ij} and c_{ij} contain integrals and depend on values of functions in the interval $(t,t+h)$. The expressions (2.2) and (2.8) are examples of such a remainder. For instance, as followed from (2.10), in the case of approximation (2.7) $r_o=3$, $a_{32}=I_{ooo}(L_2^2a,0,1)$, $b_{22}=\sum_{r=1}^{q} L_2\sigma_r(t,X(t))I_{or}(0,1)+\sum_{s,i,r=1}^{q}\Lambda_s\Lambda_i\sigma_r(t,X(t))I_{sir}(0,1)$,
$c_{21}=\sum_{i,r=1}^{q} I_{oir}(L_1\Lambda_i\sigma_r,0,1)+\sum_{i,r=1}^{q} I_{ior}(\Lambda_iL_1\sigma_r,0,1)+\sum_{i,r=1}^{q} I_{iro}(\Lambda_i\Lambda_r a,0,1)$.

Now let us estimate mean and mean-square values of the remainder ρ (2.14), which will be useful below,

$$E\rho=O(\sum_{i=1}^{r_o}\sum_{j=K(i)}^{i-1} h^i\varepsilon^{2j})=O(h^{r_o}+\sum_{l\in S_1} h^l\varepsilon^{2K(l)}) \tag{2.15}$$

where S_1 is either empty set or the bounded set of positive integers l $(1\le l<r_o)$, $K(l)<l$.
Note that such a set S_1 can contain superfluous, unessential numbers l, corresponding terms for which $h^l\varepsilon^{2K(l)}$ are less than all others. Such numbers can be excluded from the set S_1. For instance, if $l_1>l_2$ and $K(l_1)\ge K(l_2)$, then $h^{l_1}\varepsilon^{2K(l_1)}$ is always less than $h^{l_2}\varepsilon^{2K(l_2)}$ and l_1 is excluded from S_1. So, S_1 must contain only such numbers l that $K(l)$, $l\in S_1$, would be the decreasing function with natural values. The mean-square value of the remainder ρ is estimated as

$$[E\rho^2]^{1/2}=O(\sum_{i=1}^{r_o}\sum_{j=K(i)}^{i-1} h^i\varepsilon^{2j}+\sum_{i=1}^{r_o}\sum_{j=L(i)}^{i} h^{i-1/2}\varepsilon^{2j-1}+$$

$$+\sum_{i=1}^{r_o-1}\sum_{j=M(i)}^{i} h^i\varepsilon^{2j})=O(h^{r_o}+\sum_{l\in S_2} h^l\varepsilon^{J(l)}) \tag{2.16}$$

where S_2 is either empty set or the bounded set of positive integers m and semi-integers k $(1\le m<r_o, 1/2\le k\le r_o-1/2)$, the function $J(l)$ with natural values is defined as $J(m)=min(2K(m),2M(m))$, $J(k)=2L(k+1/2)-1$. Note that from the set S_2 both repeated and unessential, as described above, numbers l can be excluded, for instance see (2.4) and (2.11), and the function $J(l)$, $l\in S_2$, would be the decreasing

one. Usually, estimates of remainders of concrete schemes are simple and contain only two terms.

It must be mentioned that just as in the general case[6,7] numerical methods for solution of the system (1.1) can be constructed not only on the basis of the formula (2.13), but estimates of one-step approximation always have the form (2.15)-(2.16).

3 The theorem on mean-square estimate on the whole interval

The necessity of a theorem on relation between properties of one-step approximation and estimate of mean-square error of the corresponding method, similar to the mean-square convergence theorem in the general case,[3,4,6-8] is obvious.

Note that both in the formulation of Theorem 1 and its proof the same letter K is used for various constants.

Theorem 1. *If the following inequalities are fulfilled*

$$|E\,(X_{t,x}(t+h)-\bar{X}_{t,x}(t+h))| \le K\,(1+|x|^2)^{1/2}[h^{r_o}+\sum_{l\in S_1} h^l \varepsilon^{2K(l)}] \qquad (3.1)$$

$$[E\,|X_{t,x}(t+h)-\bar{X}_{t,x}(t+h)|^2]^{1/2} \le K\,(1+|x|^2)^{1/2}[h^{r_o}+\sum_{l\in S_2} h^l \varepsilon^{J(l)}] \qquad (3.2)$$

where $\bar{X}_{t,x}(t+h)$ is an approximation of the exact solution $X_{t,x}(t+h)$ of the system (1.1) with initial condition $X(t)=\bar{X}(t)=x$; S_1 is either empty set or bounded set of positive integers l which are less than natural number r_o; S_2 is either empty set or bounded set of positive integers and semi-integers l which are less than r_o; $K(l)$ and $J(l)$ are decreasing functions with natural values. Then

$$[E\,|X_{t_o,X_o}(t_k)-\bar{X}_{t_o,X_o}(t_k)|^2]^{1/2} \le$$
$$\le K\,(1+E|X_o|^2)^{1/2}[h^{r_o-1}+\sum_{l\in S_1} h^{l-1}\varepsilon^{2K(l)}+\sum_{l\in S_2} h^{l-1/2}\varepsilon^{J(l)}], \qquad (3.3)$$

where K does not depend on discretization step h and parameter ε, i.e., the method, corresponding to the one-step approximation $\bar{X}(t+h)$, gives mean-square error evaluated by

$$O(h^{r_o-1}+\sum_{l\in S_1} h^{l-1}\varepsilon^{2K(l)}+\sum_{l\in S_2} h^{l-1/2}\varepsilon^{J(l)}) \qquad (3.4)$$

on the whole interval.

3.1 Proof of Theorem 1

The proof of Theorem 1 is similar to the proof of the mean-square convergence theorem for a general system.[6,8] To prove Theorem 1 we need three lemmas which are

formulated below. One can find proofs of these lemmas in the monograph.[6]
Lemma 1. *The following representation takes place*

$$X_{t,x}(t+h) - X_{t,y}(t+h) = x - y - Z, \tag{3.5}$$

for which

$$E|X_{t,x}(t+h) - X_{t,y}(t+h)|^2 \le |x-y|^2(1+Kh), \tag{3.6}$$

$$EZ^2 \le K|x-y|^2 h. \tag{3.7}$$

Lemma 2. *For any time discretization Δ_N the inequality*

$$E|\bar{X}_k|^2 \le K(1+E|X_o|^2) \tag{3.8}$$

is fulfilled.
Lemma 3. *If*

$$u_{k+1} \le (1+Ah)u_k + \sum_{i=1}^{l} B_i h^{p_i},$$

where $h = T/N$, $A \ge 0$, $B_i \ge 0$, $p_i \ge 1$, $u_k \ge 0$, $k = 0, 1, ..., N$. Then

$$u_k \le e^{AT} u_o + \frac{1}{A}(e^{AT} - 1)\sum_{i=1}^{l} B_i h^{p_i - 1}.$$

Proof of Theorem 1.
Let us remind that non-decreasing family of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ and Wiener process $W(t)$ are defined on probability space $(\Sigma, \mathcal{F}, P)$, Wiener process $W(t)$ conforms to the family $\mathcal{F}_t$ and increment $W(s) - W(t)$ does not depend on $\mathcal{F}_t$ for $s \ge t$.

It is obvious that

$$X_{t_o,X_o}(t_{k+1}) - \bar{X}_{t_o,X_o}(t_{k+1}) = X_{t_k,X(t_k)}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}) = \tag{3.9}$$
$$= (X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1})) + (X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1})).$$

The first difference in the right-hand side of (3.9) is connected with the error accumulated to step k. The second difference is the one-step error at step $(k+1)$. From (3.9) one can obtain

$$E|X_{t_o,X_o}(t_{k+1}) - \bar{X}_{t_o,X_o}(t_{k+1})|^2 = EE((X_{t_k,X(t_k)}(t_{k+1}) -$$
$$-X_{t_k,\bar{X}_k}(t_{k+1}))^2|\mathcal{F}_{t_k}) + EE((X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}))^2|\mathcal{F}_{t_k}) +$$
$$2EE((X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}))(X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}))|\mathcal{F}_{t_k}) \tag{3.10}$$

According to the conditional variant of Lemma 1 we have

$$EE((X_{t_k,X(t_k)}(t_{k+1}) - X_{t_k,\bar{X}_k}(t_{k+1}))^2|\mathcal{F}_{t_k}) \le E|X(t_k) - \bar{X}_k|^2(1+Kh) \tag{3.11}$$

By the conditional variant of the inequality (3.2) and by Lemma 2 we obtain

$$EE((X_{t_k,\bar{X}_k}(t_{k+1}) - \bar{X}_{t_k,\bar{X}_k}(t_{k+1}))^2|\mathcal{F}_{t_k}) \le \tag{3.12}$$

$$\leq K(1+E|X_o|^2)[h^{r_o}+\sum_{l\in S_2} h^l\varepsilon^{J(l)}]^2$$

Let us rewrite the difference $X_{t_k,X(t_k)}(t_{k+1})-X_{t_k,\bar{X}_k}(t_{k+1})$ of the last term of (3.10) by Lemma 1. Then we have two terms, each of which is estimated separately. By the conditional variant of (3.1) and by Lemma 2 the first of these terms is estimated as

$$|EE((X(t_k)-\bar{X}_k)(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})|\mathcal{F}_{t_k})|\leq \tag{3.13}$$

$$\leq (E|X(t_k)-\bar{X}_k|^2)^{1/2}K(1+E|X_o|^2)^{1/2}[h^{r_o}+\sum_{l\in S_1} h^l\varepsilon^{2K(l)}]$$

By Lemma 1 and inequality (3.2) for the second term we obtain

$$|E(Z(X_{t_k,\bar{X}_k}(t_{k+1})-\bar{X}_{t_k,\bar{X}_k}(t_{k+1})))|\leq \tag{3.14}$$

$$\leq K(E|X(t_k)-\bar{X}_k|^2)^{1/2}(1+E|X_o|^2)^{1/2}h^{1/2}[h^{r_o}+\sum_{l\in S_2} h^l\varepsilon^{J(l)}]$$

Let us denote $\mu_k^2=E|X(t_k)-\bar{X}_k|^2$. Substituting (3.11)-(3.14) in (3.10) we have

$$\mu_{k+1}^2\leq \mu_k^2(1+Kh)+K(1+E|X_o|^2)([h^{r_o}+\sum_{l\in S_2} h^l\varepsilon^{J(l)}]^2+$$

$$+[h^{r_o-1/2}+\sum_{l\in S_1} h^{l-1/2}\varepsilon^{2K(l)}]^2). \tag{3.15}$$

Then by Lemma 3 and taking into account that $\mu_o^2=0$ we come to the inequality (3.3). Theorem 1 is proved.

3.2 Remarks

According to Theorem 1 the Euler method (2.1) has 1/2 time order, and its mean-square error on the whole interval is estimated by $O(h+\varepsilon^2h^{1/2})$; the method (2.7) has the mean-square error estimated by $O(h^2+\varepsilon^2h^{1/2})$, and its time order is also equal to 1/2. While the methods (2.1) and (2.7) have only 1/2 time order, their errors on the whole interval are sufficiently low because of small factor ε^2 at $h^{1/2}$.

From Theorem 1 it follows that if r_o is greater than one and the set S_1 does not contain a number l, which is not greater than one, and the set S_2 does not contain a number l, which is not greater than 1/2, then the corresponding method converges. However, the primary meaning of Theorem 1 is not that it gives convergence order of a method, but is that it gives a method error on the whole interval in terms of h and ε.

It must be mentioned that the mean-square convergence theorem for a general system,[6,8] i.e., $\varepsilon=1$, may be obtained as a corollary of Theorem 1. If parameter ε is equal to one, it is possible to rewrite the inequalities (3.1) and (3.2) in the form

$$|E(X_{t,x}(t+h)-\bar{X}_{t,x}(t+h))|=O(h^{p_1}),$$

$$[E|X_{t,x}(t+h)-\bar{X}_{t,x}(t+h)|^2]^{1/2}=O(h^{p_2}),$$

where p_1 is the minimum of the set $\{r_o\} \cup S_1$ and p_2 is the minimum of the set $\{r_o\} \cup S_2$. Then by Theorem 1 we have

$$[E|X_{t_o,X_o}(t_k) - \bar{X}_{t_o,X_o}(t_k)|^2]^{1/2} = O(h^{p_1-1} + h^{p_2-1/2}).$$

If we assume, as done in Refs. 6,8, that $p_1 \geq p_2 + 1/2$ and $p_2 \geq 1/2$, then time order of the corresponding method is equal to $p_2 - 1/2$. So, we obtain the mean-square convergence theorem for a general system which was previously proved.[6,8]

3.3 Selection of time increment h depending on parameter ε

Let us choose time increment h so that $h \leq C\varepsilon^{\alpha}$. Then error of a method can be estimated by powers of small parameter ε

$$[E|X_{t_o,X_o}(t_k) - \bar{X}_{t_o,X_o}(t_k)|^2]^{1/2} = O(\varepsilon^{\beta}),$$

where $\beta = min\{\alpha(r_o - 1), min_{l \in S_2}(\alpha(l - 1/2) + J(l)), min_{l \in S_1}(\alpha(l - 1) + 2K(l))\}$. The parameter α and a method may be so that some term of this method is less than the method error $O(\varepsilon^{\beta})$. If it does not lead to divergence of the method (see Subsection 3.2), such a term may be omitted and, in spite of this, smallness order of the method error does not change with respect to ε.

Let us analyze the method (2.7). If $h \leq C\varepsilon^{\alpha}$, the mean-square error of the method (2.7) on the whole interval is estimated by $O(\varepsilon^{2\alpha} + \varepsilon^{2+\alpha/2})$. Let us choose α be equal to one. In this case the method error is estimated by $O(\varepsilon^2)$, smallness order of the terms $\varepsilon L_1 \sigma_r I_{or}$ and $\varepsilon \Lambda_r a I_{ro}$ is equal to $O(\varepsilon^{5/2})$ and their omission gives $O(\varepsilon^2)$ to the mean-square error on the whole interval. So, in the case of $\alpha = 1$ these terms may be omitted and it does not lead to increasing of the error. Thus, we obtain a new method

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + a_k h + L_1 a_k h^2/2, \tag{3.16}$$

$$(ER^2)^{1/2} = O(h^2 + \varepsilon h + \varepsilon^2 h^{1/2}),$$

where $\sigma_{r_k} = \sigma_r(t_k, X_k)$, $a_k = a(t_k, X_k)$, $(I_r)_k = I_r(t_k, h)$, $(ER^2)^{1/2}$ is the mean-square error of the method on the whole interval. It is clear that if $h \leq C\varepsilon$ or $h \geq C\varepsilon^2$, errors of the methods (2.7) and (3.16) have the same order with respect to ε. But, for example, if $h = C\varepsilon^{3/2}$, the method (3.16) has the lower order with respect to ε than the method (2.7).

3.4 $h - \varepsilon$ approach versus $\varepsilon - h$ approach

In the paper we construct numerical methods by $h - \varepsilon$ approach, for instance, see the methods (2.1), (2.7) and (3.16). According to $h - \varepsilon$ approach at first we expand the exact solution $X(t)$ of the system (1.1) by powers of time increment h and obtain an expansion which is similar to the Wagner-Platen one.[4,6,7] Then we regroup terms of the expansion with respect to their $h^i\varepsilon^j$ factors and decide which terms must be included in a method. Such a decision depends on mean-square error of a method, which we want to reach, and on calculation complicacy of an expansion term, especially on complicacy of simulation of needed random variables.

$\varepsilon - h$ approach is based on another idea. At first the exact solution of the system (1.1) is expanded by powers of small parameter ε, for instance,

$$\bar{X}(t) = X^o(t) + \varepsilon X^1(t), \tag{3.17}$$

$$R = \bar{X}(t) - X(t) = O(\varepsilon^2)$$

where $X^o(t)$ and $X^1(t)$ are found as solutions of the original system under $\varepsilon = 0$ and its system of the first approximation

$$dX^o = a(t, X^o)\,dt, \quad X^o(0) = X_o \tag{3.18}$$

$$dX^1 = a'_x(t, X^o)X^1\,dt + \sum_{r=1}^{q} \sigma_r(t, X^o)\,dW_r, \quad X^1(0) = 0 \tag{3.19}$$

The system (3.18) is the system of deterministic differential equations for which, as is generally known, efficient high order numerical methods exist, for example,

$$X^o_{k+1} = X^o_k + a_k h + [a\,a'_x + a'_t]_k h^2/2, \tag{3.20}$$

$$X^o_o = X_o, R_o = O(h^2),$$

where $a_k = a(t_k, X^o_k)$, nxn-matrix $[a'_x]_k$ is equal to $\partial a(t_k, X^o_k)/\partial x$, n-vector $[a'_t]_k$ is equal to $\partial a(t_k, X^o_k)/\partial t$, R_o is the error of the method on the whole interval.The system (3.19) is the system of stochastic differential equations with additive noises.[6,7] The Euler method for the system (3.19) has the form

$$X^1_{k+1} = X^1_k + \sum_{r=1}^{q}[\sigma_r I_r]_k + [a'_x X^1]_k h \tag{3.21}$$

$$X^1_o = 0, \quad [E(R_1)^2]^{1/2} = O(h),$$

where $\sigma_{r_k} = \sigma_r(t_k, X^o_k)$, $[a'_x]_k = \partial a(t_k, X^o_k)/\partial x$, R_1 is the error of the method on the whole interval. So, we obtain the method (3.17),(3.20),(3.21) for numerical solution of the system (1.1) with the mean-square error $O(h^2 + \varepsilon^2)$ on the whole interval.

One can see that $h - \varepsilon$ approach and $\varepsilon - h$ approach are essentially different. If time increment h tends to zero, a method, constructed by $\varepsilon - h$ approach, does not converge to the exact solution and converges to $X_o(t) + \varepsilon X_1(t)$. In contrast to $\varepsilon - h$ approach $h - \varepsilon$ approach gives a method which always converges to the exact solution of the system (1.1) in the case of $h \to 0$. Our aim is to derive numerical methods for solution of the system (1.1) with small, but fixed parameter $\varepsilon > 0$. That is why $h - \varepsilon$ approach is more preferable than $\varepsilon - h$ one.

4 Stratonovich stochastic differential equations with small noises

For some physical applications Stratonovich integration of a stochastic system is preferable.[1,2] It is known that the stochastic system in Stratonovich sense (marked by "*")

$$dX = a(t,X)\,dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X) * dW_r \ , \ \ X(t_o) = X_o \tag{4.1}$$

is equivalent to the following system of the Ito stochastic differential equations

$$dX = [a(t,X) + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} \frac{\partial \sigma_r}{\partial x}(t,X)\sigma_r(t,X)]dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t,X) dW_r \tag{4.2}$$

$$X(t_o) = X_o.$$

In the general case numerical methods, constructed for the Ito system, are easily rewritten for the Stratonovich system by adding the term $\frac{\varepsilon^2}{2}\sum_{r=1}^{q}\frac{\partial\sigma_r}{\partial x}\sigma_r$ to the drift.[6,7] However, in the case of small noises the additional term is multiplied by small factor ε^2 and, thus, it is usually less than the coefficient $a(t,X)$. So, the Stratonovich system with small noises (4.1) is distinguished from the Ito system $dX = a(t,X)dt + \varepsilon\sum_{r=1}^{q}\sigma_r(t,X)dW_r$ by the small component in the shift, and constructing a numerical method for the system (4.2) one must take into account smallness order of the additional term.

4.1 Examples

By $h-\varepsilon$ approach one can derive numerical methods for the system (4.2) as in Subsection 2.1 for the system (1.1). Here instead of the operator L_2 we introduce the operator $\tilde{L}_2$:

$$\tilde{L}_2 = \frac{1}{2}\sum_{r=1}^{q}(\sigma_r, \frac{\partial}{\partial x})^2 + \frac{1}{2}\sum_{r=1}^{q}(\frac{\partial \sigma_r}{\partial x}\sigma_r, \frac{\partial}{\partial x})$$

The Euler method is rewritten for the Stratonovich system (4.1) or for the equivalent Ito system (4.2) as

$$X_{k+1} = X_k + \varepsilon\sum_{r=1}^{q}(\sigma_r I_r)_k + [a_k + \frac{\varepsilon^2}{2}\sum_{r=1}^{q}\frac{\partial\sigma_r}{\partial x}(t_k,X_k)\sigma_{r_k}]h, \tag{4.3}$$

$$E\rho = O(h^2), \ \ (E\rho^2)^{1/2} = O(h^2 + \varepsilon^2 h)$$

where $\sigma_{r_k} = \sigma_r(t_k, X_k)$, $a_k = a(t_k, X_k)$ and ρ is the one-step error of the method.

The method, which is similar to (2.7), for the system (4.1) has the form

$$X_{k+1} = X_k + \varepsilon\sum_{r=1}^{q}(\sigma_r I_r)_k + [a_k + \frac{\varepsilon^2}{2}\sum_{r=1}^{q}\frac{\partial\sigma_r}{\partial x}(t_k,X_k)\sigma_{r_k}]h +$$

$$+\varepsilon\sum_{r=1}^{q}(L_1\sigma_r I_{or})_k+\varepsilon\sum_{r=1}^{q}(\Lambda_r a I_{ro})_k+L_1 a_k h^2/2, \tag{4.4}$$

$$E\rho=O(h^3+\varepsilon^2h^2),$$

$$(E\rho^2)^{1/2}=O(h^3+\varepsilon h^{5/2}+\varepsilon^2h^2+\varepsilon^3h^{3/2}+\varepsilon^2h)=O(h^3+\varepsilon^2h)$$

One can see that the methods (4.3), (4.4) for the Stratonovich system (1.1) differ from the methods (2.1), (2.7) by additional terms $\frac{\varepsilon^2}{2}\sum_{r=1}^{q}\frac{\partial\sigma_r}{\partial x}\sigma_r h$.

4.2 One-step approximation

In the case of the Stratonovich system (4.1) the function A of the approximation (2.12) may be written in the same form as (2.13), but $K(i)$ is not greater than $i+1$ ($K(i)\le i+1$). The expressions (4.3) and (4.4) are examples of such an approximation. In the case of the Stratonovich system the remainder ρ of the approximation (2.12)-(2.13) has the form

$$\rho=\sum_{i=1}^{r_o}\sum_{j=K(i)}^{i}a_{ij}h^i\varepsilon^{2j}+\sum_{i=1}^{r_o}\sum_{j=L(i)}^{i}b_{ij}h^{i-1/2}\varepsilon^{2j-1}+$$

$$+\sum_{i=1}^{r_o-1}\sum_{j=M(i)}^{i}c_{ij}h^i\varepsilon^{2j}, \tag{4.5}$$

Then we have

$$E\rho=O(\sum_{i=1}^{r_o}\sum_{j=K(i)}^{i}h^i\varepsilon^{2j})=O(h^{r_o}+\sum_{r\in S_1}h^l\varepsilon^{2K(l)}) \tag{4.6}$$

where S_1 is the bounded set of positive integers l ($1\le l<r_o$), $K(l)\le l$;

$$[E\rho^2]^{1/2}=O(h^{r_o}+\sum_{l\in S_2}h^l\varepsilon^{J(l)}) \tag{4.7}$$

where S_2 is the bounded set of positive integers and semi-integers, $J(l)\le 2l$.
In the Stratonovich case in contrast to the Ito case the first sums of the approximation (2.13) and its remainder have the following additional terms

$$\frac{\varepsilon^2}{2}\sum_{r=1}^{q}(L_1+\varepsilon^2\tilde{L}_2)^i\frac{\partial\sigma_r}{\partial x}\sigma_r\frac{h^{i+1}}{(i+1)!};$$

the second and the third sums include the additional terms

$$\varepsilon^{2+n}\sum_{r=1}^{q}\Lambda_{i_1}...\Lambda_{i_k}\frac{\partial\sigma_r}{\partial x}\sigma_r I_{i_1,...,i_k}/2,$$

where $i_j\in\{0,1,...,q\}$, $n\ge 1$ is the number of non-zero indices i_j, $\Lambda_o=(L_1+\varepsilon^2\tilde{L}_2)$. Besides that, the operator L_2 in the approximation (2.13) and in its remainder is replaced by the operator $\tilde{L}_2$.

4.3 The theorem on mean-square estimate

Similar to Theorem 1 the theorem on mean-square estimate for the Stratonovich system (4.1) may be formulated and proved.

Theorem 2. *If for an approximation $\bar{X}_{t,x}(t+h)$ of the exact solution $X_{t,x}(t+h)$ of the system (4.1) the inequalities (3.1) and (3.2) are fulfilled (the sets S_1, S_2 and the decreasing functions with natural values $K(l)$, $J(l)$ are defined as in Subsection 4.2), the following inequality takes place*

$$[E\,|X_{t_o,X_o}(t_k)-\bar{X}_{t_o,X_o}(t_k)|^2]^{1/2}\leq$$

$$\leq K\,(1+E|X_o|^2)^{1/2}[h^{r_o-1}+\sum_{l\in S_1}h^{l-1}\varepsilon^{2K(l)}+\sum_{l\in S_2}h^{l-1/2}\varepsilon^{J(l)}], \tag{4.8}$$

where K does not depend on discretization step h and parameter ε.

4.4 Remarks

According to Theorem 2 the mean-square error of the method (4.3) on the whole interval is estimated by $O(h+\varepsilon^2 h^{1/2})$. The scheme (4.4) on the whole interval has the mean-square error $O(h^2+\varepsilon^2 h^{1/2})$. Note that additional terms in (4.3) and (4.4) are sufficiently small, but they cannot be omitted because the omission would lead to divergence of the methods. It must be also mentioned that on the other hand under small parameter ε this omission may not lead to a large error that, of course, is the consequence of closeness of the Stratonovich and Ito systems with small noises which has been marked above. Thus, in the case of small parameter ε the difference between methods for the Ito and Stratonovich systems is less than in the case of a general system. For instance, the method (2.7) for the Ito system with small noises (1.1) and the method (4.4) for the Stratonovich system with small noises (4.1) have the same mean-square errors but they are distinguished only by the term $h\frac{\varepsilon^2}{2}\sum_{r=1}^{q}\frac{\partial\sigma_r}{\partial x}\sigma_r$.

5 Some methods for general systems with small noises

Let us involve the notation: ρ is one-step error of a method and R is a method error on the whole interval. Method errors R on the whole interval are obtained by Theorem 1 and Theorem 2.

Our aim is to construct methods with low mean-square errors (provided that ε is a small parameter) and with simply simulated random variables. Herein we restrict ourselves to the methods which contain the following Ito integrals

$$I_r = h^{1/2}\xi_r,$$

$$I_{ro} = h^{3/2}[\eta_r/\sqrt{3}+\xi_r]/2$$

$$I_{or} = hI_r - I_{ro},$$

$$J_r = \int_0^h \vartheta W_r(\vartheta) d\vartheta = h^{5/2}[\xi_r/3 + \eta_r/(4\sqrt{3}) + \zeta_r/(12\sqrt{5})],$$

$$I_{roo} = hI_{ro} - J_r,$$

$$I_{oro} = 2J_r - hI_{ro},$$

$$I_{oor} = h^2 I_r/2 - J_r, \tag{5.1}$$

where ξ_r, η_r, ζ_r are independent normally distributed $N(0,1)$ random variables with zero mean and unit standard derivation. The needed random variables (Ito integrals) of the methods (2.1), (2.7), (3.16), (4.3), (4.4) and of methods of Sections 5-7 can be simulated at each step according to the formulas (5.1).

5.1 Taylor-type numerical methods

1.Methods $O(h + ...)$ and $O(h^2 + ...)$

These methods have been written above. The Euler schemes (2.1) and (4.3) for the Ito and Stratonovich systems have mean-square errors $O(h + \varepsilon^2 h^{1/2})$. The mean-square errors of the methods (2.7) and (4.4) are equal to $O(h^2 + \varepsilon^2 h^{1/2})$. The mean-square error of the algorithm (3.16) for the Ito system is estimated by $O(h^2 + \varepsilon h + \varepsilon^2 h^{1/2})$.

2.Methods $O(h^3 + ...)$

For the Ito system (1.1) we derive the method

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + a_k h + \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a I_{ro})_k +$$

$$+ L_1 a_k h^2/2 + L_1^2 a_k h^3/6, \tag{5.2}$$

$$E\rho = O(h^4 + \varepsilon^2 h^2), \ (E\rho^2)^{1/2} = O(h^4 + \varepsilon^2 h),$$

$$[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{1/2}).$$

For the Stratonovich system (4.1) we obtain

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\frac{\partial \sigma_r}{\partial x} \sigma_r]_k h, \tag{5.3}$$

$$[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is taken from (5.2).

3.Methods $O(h^4 + ...)$

For the Ito system (1.1) we have

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + a_k h + \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a I_{ro})_k +$$

$$+ L_1 a_k h^2/2 + \varepsilon \sum_{r=1}^{q} (L_1^2 \sigma_r I_{oor})_k + \varepsilon \sum_{r=1}^{q} (L_1 \Lambda_r a I_{oro})_k + \tag{5.4}$$

$$+\varepsilon \sum_{r=1}^{q} (\Lambda_r L_1 a I_{roo})_k + L_1^2 a_k h^3/6 + L_1^3 a_k h^4/24,$$

$$E\rho = O(h^5 + \varepsilon^2 h^2), \; (E\rho^2)^{1/2} = O(h^5 + \varepsilon^2 h),$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{1/2}).$$

If we choose h so that $h \leq C\varepsilon^{1/2}$, we obtain for the method (5.4) $[ER^2]^{1/2} = O(\varepsilon^2)$.

For the Stratonovich system (4.1) the method, which is similar to (5.4), has the form

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\frac{\partial \sigma_r}{\partial x} \sigma_r]_k h, \tag{5.5}$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is taken from (5.4). The random variables of (5.4) and (5.5) are simulated according to (5.1).

In some cases, using a special properties of a concrete system, the derived methods may be improved. For instance, let us consider the commutative case, i.e., $\Lambda_i \sigma_r = \Lambda_r \sigma_i$, or a system with one noise ($q = 1$). For such systems we obtain
(in the Ito case)

$$\hat{X}_{k+1} = X_{k+1} + \varepsilon^2 \sum_{i=1}^{q-1} \sum_{r=i+1}^{q} (\Lambda_i \sigma_r I_i I_r)_k + \varepsilon^2 \sum_{i=1}^{q} (\Lambda_i \sigma_i (I_i^2 - h)/2)_k + \tag{5.6}$$

$$+\varepsilon^2 L_2 a_k h^2/2,$$

$$E\rho = O(h^5 + \varepsilon^2 h^3), \; (E\rho^2)^{1/2} = O(h^5 + \varepsilon^2 h^2 + \varepsilon^3 h^{3/2}),$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{3/2} + \varepsilon^3 h),$$

where X_{k+1} is taken from (5.4)
(in the Stratonovich case)

$$\hat{X}_{k+1} = \tilde{X}_{k+1} + \varepsilon^2 \sum_{i=1}^{q-1} \sum_{r=i+1}^{q} (\Lambda_i \sigma_r I_i I_r)_k + \varepsilon^2 \sum_{i=1}^{q} (\Lambda_i \sigma_i (I_i^2 - h)/2)_k + \tag{5.7}$$

$$+\varepsilon^2 L_1 \left\{ \sum_{r=1}^{q} \frac{\partial \sigma_r}{\partial x} \sigma_r \right\}_k h^2/4 + \varepsilon^2 \tilde{L}_2 a_k h^2/2,$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{3/2} + \varepsilon^3 h),$$

where $\tilde{X}_{k+1}$ is taken from (5.5).
Here we use the well-known rule[(6)]

$$I_{ir} = I_i I_r - I_{ri}, \; r \neq i,$$

$$I_{ii} = (I_i^2 - h)/2.$$

Note that for the system with one noise ($q = 1$) the term $\varepsilon^2 \sum_{i=1}^{q-1} \sum_{r=i+1}^{q} (\Lambda_i \sigma_r I_i I_r)_k$ is neglected. One can see that errors of the methods (5.6), (5.7) are less than errors of the schemes (5.4), (5.5). Moreover, the methods (5.6), (5.7) have the first order of convergence, while the time-step orders of the methods (5.4), (5.5) are equal to 1/2.

As it is possible to obtain schemes $O(h^5)$, $O(h^6)$ and so on for deterministic systems, methods with mean-square errors $O(h^5 + ...)$, $O(h^6 + ...)$, etc. for a system with small noises can be also derived. In the same way as above terms with complicated Ito integrals,of course, multiplied by ε^α, would be omitted in methods and included in their remainders. This remark is true for all groups of methods which are presented in the paper.

5.2 Runge-Kutta methods

To reduce calculations of derivatives in the methods of Subsection 5.1 we propose Runge-Kutta schemes.

1.Methods $O(h^2 + ...)$

For the Ito system we obtain

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + (\bar{a}_k + a_k)h/2 + \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \tag{5.8}$$

$$+\varepsilon \sum_{r=1}^{q} (\Lambda_r a (I_{ro} - I_r h/2))_k,$$

$$E\rho = O(h^3), \ (E\rho^2)^{1/2} = O(h^3 + \varepsilon^2 h),$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon^2 h^{1/2}),$$

where $\bar{a}_k = a(t+h, X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + a_k h)$ and the needed Ito integrals are simulated as in (5.1).

For the Stratonovich system we have

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\frac{\partial \sigma_r}{\partial x} \sigma_r]_k h, \tag{5.9}$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is taken from (5.8).

2.Methods $O(h^3 + ...)$

By the idea of attracting a subsidiary deterministic system[6] the methods (5.2), (5.3) can be simplified. Let us involve the subsidiary system

$$dx/dt = a(t, x), \ x(t_k) = X_k \tag{5.10}$$

The deterministic system (5.10) may be solved by a third order Runge-Kutta rule, for instance,

$$K_1 = ha(t_k, X_k),\ K_2 = ha(t_k + h/2, X_k + K_1/2),$$
$$K_3 = ha(t_{k+1}, X_k - K_1 + 2K_2)$$
$$x_{k+1} = X_k + [K_1 + 4K_2 + K_3]/6. \tag{5.11}$$

Then one can obtain the following Runge-Kutta method for solving the stochastic system (1.1)

$$X_{k+1} = X_k + [K_1 + 4K_2 + K_3]/6 + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k +$$
$$+ \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a I_{ro})_k, \tag{5.12}$$
$$[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{1/2}),$$

where K_1, K_2, K_3 are from (5.11).

For the Stratonovich system (4.1) we have

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\frac{\partial \sigma_r}{\partial x} \sigma_r]_k h, \tag{5.13}$$

$$[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is taken from (5.12). The needed random variables of the methods (5.12), (5.13) are simulated as in (5.1).

3.Methods $O(h^4 + ...)$

The system (5.10) may be solved by a deterministic fourth order Runge-Kutta rule and the following Runge-Kutta method for the Ito system (1.1) is obtained

$$K_1 = ha(t_k, X_k), K_2 = ha(t_k + h/2, X_k + K_1/2),$$
$$K_3 = ha(t_k + h/2, X_k + K_2/2), K_4 = ha(t_{k+1}, X_k + K_3),$$
$$X_{k+1} = X_k + [K_1 + 2K_2 + 2K_3 + K_4]/6 + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k +$$
$$+ \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a I_{ro})_k + \varepsilon \sum_{r=1}^{q} (L_1^2 \sigma_r I_{oor})_k + \tag{5.14}$$
$$+ \varepsilon \sum_{r=1}^{q} (L_1 \Lambda_r a I_{oro})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r L_1 a I_{roo})_k,$$
$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{1/2}).$$

For the Stratonovich system the method (5.14) is modified as

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\frac{\partial \sigma_r}{\partial x} \sigma_r]_k h, \tag{5.15}$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is from (5.14). The random variables of (5.14), (5.15) are the same as in (5.1). In the commutative case the methods (5.14), (5.15) can be improved as in Subsection 5.1.3.

The methods (5.14) and (5.15) may be simplified in the following way (Ito system)

$$X_{k+1} = X_k + [K_1 + 2K_2 + 2K_3 + K_4]/6 + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k, \tag{5.16}$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2}),$$

where K_i are calculated as in (5.14);
(Stratonovich system)

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\frac{\partial \sigma_r}{\partial x} \sigma_r]_k h, \tag{5.17}$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is from (5.16). The mean-square errors of these methods with respect to ε are greater than the errors of the schemes (5.14), (5.15) under $C_1 \varepsilon^{1/3} < h < C_2 \dot{\varepsilon}^2$ and otherwise they have the same order.

5.3 Implicit methods

Implicit methods are useful for stiff stochastic systems.

1.Methods $O(h + ...)$

The implicit Euler schemes for the system (1.1) are written in the form[6,7]

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + \alpha h a_k + (1 - \alpha) h a_{k+1}, \tag{5.18}$$

$$0 \le \alpha \le 1, \;\; [ER^2]^{1/2} = O(h + \varepsilon^2 h^{1/2}).$$

The similar methods for the Stratonovich system (4.1) have the form

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\alpha (\frac{\partial \sigma_r}{\partial x} \sigma_r)_k + (1 - \alpha)(\frac{\partial \sigma_r}{\partial x} \sigma_r)_{k+1}] h, \tag{5.19}$$

$$[ER^2]^{1/2} = O(h + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is taken from (5.18).

2.*Methods* $O(h^2 + ...)$

By the ideas of the monograph[6] the two parametric family (α and β) of implicit schemes for the Ito system (1.1) is constructed

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + \alpha h a_k + (1-\alpha) h a_{k+1} +$$

$$+ \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a (I_{ro} - (1-\alpha) I_r h))_k + \tag{5.20}$$

$$+\beta(2\alpha - 1) L_1 a_k h^2/2 + (1-\beta)(2\alpha - 1) L_1 a_{k+1} h^2/2,$$

$$0 \leq \alpha \leq 1, \ \ 0 \leq \beta \leq 1,$$

$$E\rho = O(h^3 + \varepsilon^2 h^2), \ \ (E\rho^2)^{1/2} = O(h^3 + \varepsilon^2 h),$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon^2 h^{1/2}),$$

For the Stratonovich system we have

$$\tilde{X}_{k+1} = X_{k+1} + \frac{\varepsilon^2}{2} \sum_{r=1}^{q} [\alpha(\frac{\partial \sigma_r}{\partial x} \sigma_r)_k + (1-\alpha)(\frac{\partial \sigma_r}{\partial x} \sigma_r)_{k+1}] h, \tag{5.21}$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon^2 h^{1/2}),$$

where X_{k+1} is taken from (5.20). The needed random variables I_r, I_{ro}, I_{or} are simulated as in (5.1).

If $\alpha = 1/2$, we obtain the trapezoidal method which is the simplest of the family (5.20)

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + h[a_k + a_{k+1}]/2 +$$

$$+ \varepsilon \sum_{r=1}^{q} (L_1 \sigma_r I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a (I_{ro} - I_r h/2))_k, \tag{5.22}$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon^2 h^{1/2}).$$

In the commutative case or in the case of one noise the methods (5.20)-(5.22) may be improved as the method (5.4) in Subsection 5.1.

5.4 Remark

Obviously, a lot of other methods may be derived. Firstly, by adding or omitting some terms one can obtain methods that are similar to above but have other mean-square errors. Secondary, it is possible to derive other types of methods, for instance, implicit Runge-Kutta methods. In this Section, however, we have restricted ourselves to the set of more common and, in our opinion, useful methods and have illustrated the proposed approach to numerical solution of a stochastic system with small noises.

6 Numerical methods for a system with small additive noises

One of the important particular cases of the systems (1.1) and (4.1) is the system with additive noises

$$dX = a(t,X)\,dt + \varepsilon \sum_{r=1}^{q} \sigma_r(t)\,dW_r. \tag{6.1}$$

Note that in this case the Stratonovich system coincides with the Ito system.

6.1 Taylor-type explicit methods

1.The Euler method

The Euler method for the system (6.1) coincides with the scheme (2.1). However, in the case of additive noises it has the first time order and the following mean-square error[6,7]

$$(ER^2)^{1/2} = O(h).$$

2.Methods $O(h^2 + ...)$

The method (2.7) for the system (6.1) has the form

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q}(\sigma_r I_r)_k + a_k h + \varepsilon \sum_{r=1}^{q}(\frac{d\sigma_r}{dt} I_{or})_k + \varepsilon \sum_{r=1}^{q}(\Lambda_r a I_{ro})_k +$$

$$+ L_1 a_k h^2/2 \tag{6.2}$$

$$E\rho = O(h^3 + \varepsilon^2 h^2),\ (E\rho^2)^{1/2} = O(h^3 + \varepsilon^2 h^2),$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon^2 h).$$

If we lightly modify this method so that

$$\tilde{X}_{k+1} = X_{k+1} + \varepsilon^2 L_2 a_k h^2/2 \tag{6.3}$$

where X_{k+1} is taken from (6.2), the mean-square error R is equal to $O(h^2+\varepsilon^2 h^{3/2})$. The needed random variables of the methods (6.2) and (6.3) are simulated as in (5.1). The scheme (6.3) coincides with the well-known [6,7] explicit method for a general system with additive noises ($\varepsilon = 1$). Note that omission of the terms with order $\varepsilon h^{3/2}$ in the scheme (6.2) leads to the method with $[ER^2]^{1/2} = O(h^2 + \varepsilon h)$.

3.Method $O(h^3 + ...)$

Let us rewrite the method (5.2) for the system with additive noises (6.1)

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q}(\sigma_r I_r)_k + a_k h + \varepsilon \sum_{r=1}^{q}(\frac{d\sigma_r}{dt} I_{or})_k + \varepsilon \sum_{r=1}^{q}(\Lambda_r a I_{ro})_k +$$

$$+ (L_1 + \varepsilon^2 L_2) a_k h^2/2 + \varepsilon \sum_{r=1}^{q}(\frac{d^2\sigma_r}{dt^2} I_{oor})_k + \varepsilon \sum_{r=1}^{q}(L_1 \Lambda_r a I_{oro})_k + \tag{6.4}$$

$$+\varepsilon\sum_{r=1}^{q}(\Lambda_r L_1 a I_{roo})_k + L_1^2 a_k h^3/6$$

$$E\rho = O(h^4+\varepsilon^2 h^3),\ (E\rho^2)^{1/2} = O(h^4+\varepsilon^2 h^2),$$
$$[ER^2]^{1/2} = O(h^3+\varepsilon^2 h^{3/2}).$$

The needed random variables are simulated as in (5.1).

4.Method $O(h^4+...)$

From the method (5.4) we obtain the following scheme for the system (6.1)

$$\tilde{X}_{k+1} = X_{k+1} + L_1^3 a_k h^4/24, \tag{6.5}$$
$$[ER^2]^{1/2} = O(h^4+\varepsilon^2 h^{3/2}).$$

where X_{k+1} is taken from (6.4).
For systems under $\varepsilon = 1$ the methods (6.4), (6.5) have the order $h^{3/2}$ and they are not preferable in comparison with more simple scheme (6.3).

6.2 Runge-Kutta methods

1.Method $O(h^2+...)$

In the case of the system with additive noises (6.1) the Runge-Kutta scheme (5.8) has the form

$$X_{k+1} = X_k + \varepsilon\sum_{r=1}^{q}(\sigma_r I_r)_k + (a_k+\bar{a}_k)h/2 + \varepsilon\sum_{r=1}^{q}(\frac{d\sigma_r}{dt}I_{or})_k + \tag{6.6}$$

$$+\varepsilon\sum_{r=1}^{q}[\Lambda_r a(I_{ro} - I_r h/2)]_k$$

$$E\rho = O(h^3),\ (E\rho^2)^{1/2} = O(h^3+\varepsilon^2 h^2),$$
$$[ER^2]^{1/2} = O(h^2+\varepsilon^2 h^{3/2}),$$

where $\bar{a}_k = a(t+h, X_k + \varepsilon\sum_{r=1}^{q}(\sigma_r I_r)_k + a_k h)$.
The scheme (6.6) coincides with the Runge-Kutta method for a general system with additive noises that was proposed in the monograph.[(6)]

2.Method $O(h^3+...)$

Similar to the method (5.12) one can obtain the following Runge-Kutta method for the system (6.1)

$$X_{k+1} = X_k + [K_1 + 4K_2 + K_3]/6 + \varepsilon\sum_{r=1}^{q}(\sigma_r I_r)_k + \varepsilon\sum_{r=1}^{q}(\frac{d\sigma_r}{dt}I_{or})_k +$$

$$+\varepsilon\sum_{r=1}^{q}(\Lambda_r a I_{ro})_k + \varepsilon^2 L_2 a_k h^2/2 + \varepsilon\sum_{r=1}^{q}(\frac{d^2\sigma_r}{dt^2}I_{oor})_k + \tag{6.7}$$

$$+\varepsilon \sum_{r=1}^{q} (L_1 \Lambda_r a I_{oro})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r L_1 a I_{roo})_k$$

where

$$K_1 = ha(t_k, X_k), \ K_2 = ha(t_k + h/2, X_k + K_1/2),$$
$$K_3 = ha(t_{k+1}, X_k - K_1 + 2K_2).$$

The remainder R of this method on the whole interval is estimated as $[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{3/2})$. The needed random variables are simulated as in (5.1).

3.Method $O(h^4 + ...)$

For the system with additive noises the method (5.14) gives

$$X_{k+1} = X_k + [K_1 + 2K_2 + 2K_3 + K_4]/6 + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k +$$
$$+ \varepsilon \sum_{r=1}^{q} (\frac{d\sigma_r}{dt} I_{or})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r a I_{ro})_k + \varepsilon^2 L_2 a_k h^2/2+ \qquad (6.8)$$
$$+\varepsilon \sum_{r=1}^{q} (\frac{d^2\sigma_r}{dt^2} I_{oor})_k + \varepsilon \sum_{r=1}^{q} (L_1 \Lambda_r a I_{oro})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r L_1 a I_{roo})_k,$$
$$[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{3/2})$$

where

$$K_1 = ha(t_k, X_k), \ K_2 = ha(t_k + h/2, X_k + K_1/2),$$
$$K_3 = ha(t_k + h/2, X_k + K_2/2), \ K_4 = ha(t_{k+1}, X_k + K_3), \qquad (6.9)$$

the needed Ito integrals are calculated as in (5.1).

The method (6.8) can be simplified in the same way as the scheme (5.14) in Subsection 5.2.3. In the case of additive noises the resulting simplified method coincides with the scheme (5.16) but has the mean-square error on the whole interval which is equal to

$$[ER^2]^{1/2} = O(h^4 + \varepsilon h) \qquad (6.10)$$

6.3 Implicit methods

The implicit Euler schemes for the system (6.1) are the same as (5.18). But the mean-square error of such schemes for the system with additive noises is equal to $[ER^2]^{1/2} = O(h)$.

Just as other methods of Section 5 have been modified for the system (6.1), the family of implicit schemes (5.20) can be also rewritten

$$X_{k+1} = X_k + \varepsilon \sum_{r=1}^{q} (\sigma_r I_r)_k + \alpha h a_k + (1 - \alpha) h a_{k+1} +$$

$$+\varepsilon\sum_{r=1}^{q}(\frac{d\sigma_r}{dt}I_{or})_k+\varepsilon\sum_{r=1}^{q}(\Lambda_r a(I_{ro}-(1-\alpha)I_r h))_k+ \tag{6.11}$$

$$+\beta(2\alpha-1)(L_1+\varepsilon^2L_2)a_k h^2/2+(1-\beta)(2\alpha-1)(L_1+\varepsilon^2L_2)a_{k+1}h^2/2,$$

$$0\le\alpha\le 1,\ \ 0\le\beta\le 1,$$

$$[ER^2]^{1/2}=O(h^2+\varepsilon^2h^{3/2}),$$

This method coincides with 3/2 order implicit scheme for a general system with additive noises that was proposed in the monograph.[6]

The simplest method among the family (6.11) is the trapezoidal scheme

$$X_{k+1}=X_k+\varepsilon\sum_{r=1}^{q}(\sigma_r I_r)_k+h[a_k+a_{k+1}]/2+$$

$$+\varepsilon\sum_{r=1}^{q}(\frac{d\sigma_r}{dt}I_{or})_k+\varepsilon\sum_{r=1}^{q}(\Lambda_r a(I_{ro}-I_r h/2))_k, \tag{6.12}$$

$$[ER^2]^{1/2}=O(h^2+\varepsilon^2h^{3/2}).$$

The needed random variables of the methods (6.11), (6.12) can be simulated as in (5.1).

7 Numerical methods for a system with small colored noises

It is known that for some physical applications colored noises are more preferable than white ones. In Refs. 9,10 various special numerical methods for solution of a system with colored noises were derived. Here we present schemes for a system with small colored noises. Thanks to small parameter ε they are simpler and have less errors than in the case of a general system with colored noises.

A system with small colored noises may be written in the form

$$dY=f(t,Y)\,dt\ +\ \varepsilon G(t,Y)Z\,dt, \tag{7.1}$$

$$dZ=A(t)Z\,dt\ +\ \sum_{r=1}^{q}b_r(t)\,dW_r,$$

$$Y(t_o)=Y_o,\ Z(t_o)=Z_o,\ t\in[t_o,T],$$

where Y and $f(t,Y)$ are l-dimensional vectors, Z and $b_r(t)$ are m-dimensional vectors, $A(t)$ is mxm-matrix and $G(t,Y)$ is lxm-matrix, W_r are uncorrelated standard Wiener processes and ε is small parameter.

Let us introduce new variable U

$$U=\varepsilon Z \tag{7.2}$$

Then the system (7.1) is rewritten in the convenient form

$$dY=f(t,Y)\,dt\ +\ G(t,Y)U\,dt, \tag{7.3}$$

$$dU = A(t)U\,dt + \varepsilon\sum_{r=1}^{q} b_r(t)\,dW_r,$$

$$Y(t_o) = Y_o,\; U(t_o) = \varepsilon Z_o.$$

The system (7.3) is the particular case of the system with small additive noises (6.1). However, the system (7.3) is simpler than (6.1), because it is linear with respect to U and the first equation of (7.3) does not contain Wiener differentials. These properties allow us to construct special numerical methods for the system (7.3) which are simpler and have less errors than the corresponding schemes for the system (6.1).

Operators L_1, L_2 and Λ_r for the system (7.3) have the form

$$L_1 = \frac{\partial}{\partial t} + (f(t,Y) + G(t,Y)U, \frac{\partial}{\partial y}) + (A(t)U, \frac{\partial}{\partial u}),$$

$$L_2 = \frac{1}{2}\sum_{r=1}^{q}\sum_{i,j=1}^{m} b_r^i b_r^j \frac{\partial^2}{\partial u^i \partial u^j},\; \Lambda_r = (b_r, \frac{\partial}{\partial u}). \tag{7.4}$$

7.1 Taylor-type explicit methods

1.Method $O(h)$

The Euler method for the system (7.3) has the well-known form[9,10] and its error is

$$[ER^2]^{1/2} = O(h).$$

2.Methods $O(h^2 + ...)$

From the method (6.2) we have for the system with small colored noises (7.3)

$$Y_{k+1} = Y_k + [f + GU]_k h + \varepsilon G_k \sum_{r=1}^{q} (b_r I_{ro})_k +$$

$$+ [f_t' + G_t' U + (f + GU)_y'(f + GU) + GAU]_k h^2/2, \tag{7.5}$$

$$U_{k+1} = U_k + \varepsilon\sum_{r=1}^{q}(b_r I_r)_k + A_k U_k h + \varepsilon\sum_{r=1}^{q}[\frac{db_r}{dt} I_{or}]_k +$$

$$+\varepsilon\sum_{r=1}^{q}(A b_r I_{ro})_k + [A_t' U + A^2 U]_k h^2/2,$$

$$[ER^2]^{1/2} = O(h^2).$$

The autonomous version of the method (7.5) coincides with the second order explicit method for a general system with colored noises ($\varepsilon = 1$) that was proposed in Ref. 9.

If we omit terms with order $h^{3/2}$ in the scheme (7.5), we obtain the following simpler method

$$Y_{k+1} = Y_k + [f + GU]_k h + [f_t' + G_t' U + (f + GU)_y'(f + GU) + \tag{7.6}$$

$$+GAU]_k h^2/2,$$

$$U_{k+1} = U_k + \varepsilon \sum_{r=1}^{q} (b_r I_r)_k + A_k U_k h + [A'_t U + A^2 U]_k h^2/2,$$

$$[ER^2]^{1/2} = O(h^2 + \varepsilon h).$$

If one chooses time increment h so that $h = O(\varepsilon)$, the errors of both the method (7.5) and the method (7.6) would be estimated by $O(\varepsilon^2)$. The needed Ito integrals of the schemes (7.5), (7.6) are simulated as in (5.1).

3.Methods $O(h^3 + ...)$

The method (6.4) for the system (7.3) has the form

$$\tilde{Y}_{k+1} = Y_{k+1} + \varepsilon \sum_{r=1}^{q} ([(Gb_r)'_t + (Gb_r)'_y (f + GU)] I_{oro})_k +$$

$$+ \varepsilon \sum_{r=1}^{q} (\Lambda_r L_1 [f + GU] I_{roo})_k + L_1^2 [f + GU]_k h^3/6 + \tag{7.7}$$

$$+ \varepsilon^2 \sum_{r=1}^{q} ((Gb_r)'_y Gb_r)_k h^3/6,$$

$$\tilde{U}_{k+1} = U_{k+1} + \varepsilon \sum_{r=1}^{q} (\frac{d^2 b_r}{dt^2} I_{oor})_k + \varepsilon \sum_{r=1}^{q} ((Ab_r)'_t I_{oro})_k +$$

$$+ \varepsilon \sum_{r=1}^{q} [(A'_t b_r + A^2 b_r) I_{roo}]_k + (A''_{tt} U + A A'_t U + A^3 U + 2 A'_t A U)_k h^3/6,$$

$$[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{5/2}),$$

where Y_{k+1} and U_{k+1} are taken from (7.5) and needed Ito integrals are calculated as in (5.1). The autonomous version of the method (7.7) coincides with the 5/2 mean-square order method of Ref. 9. If one chooses time increment h so that $h = O(\varepsilon)$, the error of the method (7.7) is estimated by $O(\varepsilon^3)$. The same result may be obtained by the simpler method

$$\tilde{Y}_{k+1} = Y_{k+1} + L_1^2 [f + GU]_k h^3/6 \tag{7.8}$$

$$\tilde{U}_{k+1} = U_{k+1} + (A''_{tt} U + A A'_t U + A^3 U + 2 A'_t A U)_k h^3/6,$$

$$[ER^2]^{1/2} = O(h^3 + \varepsilon h^2),$$

Y_{k+1} and U_{k+1} are taken from (7.5). However, if one chooses $h = O(\varepsilon^2)$, the method (7.7) gives $O(\varepsilon^6)$ and the method (7.8) gives $O(\varepsilon^5)$.

4.Method $O(h^4 + ...)$

On the base of the method (6.5) we obtain the following method for the system with small colored noises (7.3)

$$\hat{Y}_{k+1} = \tilde{Y}_{k+1} + L_1^3 [f + GU]_k h^4/24, \tag{7.9}$$

$$\hat{U}_{k+1} = \tilde{U}_{k+1} + \{(\frac{\partial}{\partial t} + (Au, \frac{\partial}{\partial u}))[A''_{tt}U + AA'_tU + A^3U +$$
$$+2A'_tAU]\}_k h^4/24,$$
$$[ER^2]^{1/2} = O(h^4 + \varepsilon h^3 + \varepsilon^2 h^{5/2}),$$

$\tilde{Y}_{k+1}$ and $\tilde{U}_{k+1}$ are taken from (7.7).

The method (7.9) may be improved up to $[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{5/2})$ by adding terms with the order $\varepsilon h^{7/2}$.

7.2 Runge-Kutta methods

1.Method $O(h^2)$

Let us rewrite the Runge-Kutta method (6.6) for the system (7.3)

$$Y_{k+1} = Y_k + ([f + GU]_k + [\bar{f} + \bar{G}\bar{U}]_k)h/2+$$
$$+ \varepsilon G_k \sum_{r=1}^{q}(b_r(I_{ro} - I_r h/2)_k, \tag{7.10}$$
$$U_{k+1} = U_k + \varepsilon \sum_{r=1}^{q}(b_r I_r)_k + (A_k U_k + \bar{A}_k \bar{U}_k)h/2 + \varepsilon \sum_{r=1}^{q}[\frac{db_r}{dt} I_{or}]_k +$$
$$+\varepsilon \sum_{r=1}^{q}(Ab_r(I_{ro} - I_r h/2)_k,$$
$$[ER^2]^{1/2} = O(h^2).$$

where $\bar{f}_k = f(t + h, \bar{Y}_k)$, $\bar{G}_k = G(t + h, \bar{Y}_k)$, $\bar{Y}_k = Y_k + [f + GU]_k h$, $\bar{U}_k = U_k + \varepsilon \sum_{r=1}^{q}(b_r I_r)_k + A_k U_k h$.

The autonomous version of the method (7.10) coincides with the second order Runge-Kutta method of Ref. 9.

2.Method $O(h^3 + ...)$

For the system with small colored noises the method (6.7) becomes

$$Y_{k+1} = Y_k + [K_1 + 4K_2 + K_3]h/6 + \varepsilon G_k \sum_{r=1}^{q}(b_r I_{ro})_k + \tag{7.11}$$
$$\varepsilon \sum_{r=1}^{q}([(Gb_r)'_t + (Gb_r)'_y(f + GU)]I_{oro})_k + \varepsilon \sum_{r=1}^{q}(\Lambda_r L_1[f + GU]I_{roo})_k +$$
$$+\varepsilon^2 \sum_{r=1}^{q}((Gb_r)'_y Gb_r)_k h^3/6,$$
$$U_{k+1} = U_k + \varepsilon \sum_{r=1}^{q}(b_r I_r)_k + A_k U_k h + \varepsilon \sum_{r=1}^{q}[\frac{db_r}{dt} I_{or}]_k + \varepsilon \sum_{r=1}^{q}(Ab_r I_{ro})_k +$$

$$+[A_t'U + A^2U]_k h^2/2 + \varepsilon \sum_{r=1}^{q} (\frac{d^2 b_r}{dt^2} I_{oor})_k + \varepsilon \sum_{r=1}^{q} ((Ab_r)_t' I_{oro})_k +$$

$$+\varepsilon \sum_{r=1}^{q} [(A_t' b_r + A^2 b_r) I_{roo}]_k + (A_{tt}'' U + AA_t'U + A^3U + 2A_t'AU)_k h^3/6,$$

$$[ER^2]^{1/2} = O(h^3 + \varepsilon^2 h^{5/2}),$$

where

$$K_1 = hF(t_k, Y_k, U_k),\ K_2 = hF(t_k + h/2, Y_k + K_1/2, U_k + hA_kU_k/2)$$

$$K_3 = hF(t_{k+1}, Y_k - K_1 + 2K_2, U_k - hA_kU_k + 2hA_{k+1/2}(U_k + hA_kU_k/2))$$

$$F(t,Y,U) = f(t,Y) + G(t,Y)U,\ A_{k+1/2} = A(t_k + h/2). \tag{7.12}$$

The needed random variables are simulated as in (5.1).

3.Methods $O(h^4 + ...)$

For the system (7.3) the method (6.8) may be rewritten in the form

$$K_1 = hF(t_k, Y_k, U_k),\ K_2 = hF(t_k + h/2, Y_k + K_1/2, U_k + hA_kU_k/2)$$

$$K_3 = hF(t_k + h/2, Y_k + K_2/2, U_k + hA_{k+1/2}(U_k + hA_kU_k/2)/2)$$

$$K_4 = hF(t_{k+1}, Y_k + K_3, U_k + hA_{k+1/2}[U_k + hA_{k+1/2}(U_k + hA_kU_k/2)/2])$$

$$Y_{k+1} = Y_k + [K_1 + 2K_2 + 2K_3 + K_4]h/6 + \varepsilon G_k \sum_{r=1}^{q} (b_r I_{ro})_k +$$

$$\varepsilon \sum_{r=1}^{q} ([(Gb_r)_t' + (Gb_r)_y'(f + GU)] I_{oro})_k + \varepsilon \sum_{r=1}^{q} (\Lambda_r L_1 [f + GU] I_{roo})_k + \tag{7.13}$$

$$+\varepsilon^2 \sum_{r=1}^{q} ((Gb_r)_y' Gb_r)_k h^3/6,$$

$$[ER^2]^{1/2} = O(h^4 + \varepsilon h^3 + \varepsilon^2 h^{5/2}),$$

where

$$F(t,Y,U) = f(t,Y) + G(t,Y)U,\ A_{k+1/2} = A(t_k + h/2),$$

U_k is calculated as $\hat{U}_{k+1}$ in (7.9) and the needed random variables are simulated as in (5.1).

Just as (5.14), the method (7.13) may be simplified

$$Y_{k+1} = Y_k + [K_1 + 2K_2 + 2K_3 + K_4]h/6, \tag{7.14}$$

$$U_{k+1} = U_k + \varepsilon \sum_{r=1}^{q} (b_r I_r)_k + A_kU_kh + [A_t'U + A^2U]_k h^2/2 +$$

$$+(A_{tt}''U + AA_t'U + A^3U + 2A_t'AU)_k h^3/6 +$$

$$+\left\{(\frac{\partial}{\partial t}+(Au,\frac{\partial}{\partial u}))[A''_{tt}U+AA'_tU+A^3U+2A'_tAU]\right\}_k h^4/24,$$

$$[ER^2]^{1/2}=O(h^4+\varepsilon h),$$

where K_i are taken from (7.13). Note that if $h=O(\varepsilon^{1/4})$, the errors of both methods (7.13) and (7.14) are estimated by $O(\varepsilon)$. However, if $h=O(\varepsilon)$, the method (7.13) gives $[ER^2]^{1/2}=O(\varepsilon^4)$ and the method (7.14) gives only $O(\varepsilon^2)$.

7.3 Implicit methods

1.Methods $O(h)$

From the family of Euler methods (5.18) we obtain

$$Y_{k+1}=Y_k+\alpha[f+GU]_kh+(1-\alpha)[f+GU]_{k+1}h, \tag{7.15}$$

$$U_{k+1}=U_k+\varepsilon\sum_{r=1}^{q}(b_rI_r)_k+\alpha(AU)_kh+(1-\alpha)(AU)_{k+1}h,$$

$$0\le\alpha\le 1,$$

$$[ER^2]^{1/2}=O(h).$$

2.Methods $O(h^2)$

The family of implicit schemes (6.11) is rewritten for the system (7.3) in the form

$$Y_{k+1}=Y_k+\alpha[f+GU]_kh+(1-\alpha)[f+GU]_{k+1}h+ \tag{7.16}$$

$$+\varepsilon G_k\sum_{r=1}^{q}(b_r(I_{ro}-(1-\alpha)I_rh))_k+\beta(2\alpha-1)L_1[f+GU]_kh^2/2+$$

$$+(1-\beta)(2\alpha-1)L_1[f+GU]_{k+1}h^2/2,$$

$$U_{k+1}=U_k+\varepsilon\sum_{r=1}^{q}(b_rI_r)_k+\alpha(AU)_kh+(1-\alpha)(AU)_{k+1}h+$$

$$+\varepsilon\sum_{r=1}^{q}[\frac{db_r}{dt}I_{or}]_k+\varepsilon\sum_{r=1}^{q}(Ab_r(I_{ro}-(1-\alpha)I_rh))_k+$$

$$+\beta(2\alpha-1)[A'_tU+A^2U]_kh^2/2+(1-\beta)(2\alpha-1)[A'_tU+A^2U]_{k+1}h^2/2,$$

$$0\le\alpha\le 1,\ 0\le\beta\le 1,$$

$$[ER^2]^{1/2}=O(h^2).$$

The needed Ito integrals are calculated as in (5.1). In the case of $\alpha=1/2$ we obtain the simplest method of the family (7.16)

$$Y_{k+1}=Y_k+([f+GU]_k+[f+GU]_{k+1})h/2+\varepsilon G_k\sum_{r=1}^{q}(b_r(I_{ro}-I_rh/2))_k \tag{7.17}$$

$$U_{k+1} = U_k + \varepsilon \sum_{r=1}^{q} (b_r I_r)_k + [(AU)_k + (AU)_{k+1}]h/2 +$$

$$+\varepsilon \sum_{r=1}^{q} [\frac{db_r}{dt} I_{or}]_k + \varepsilon \sum_{r=1}^{q} (Ab_r(I_{ro} - I_r h/2))_k,$$

$$[ER^2]^{1/2} = O(h^2).$$

The autonomous variants of the methods (7.15)-(7.17) coincide with the corresponding implicit schemes of Ref. 9.

8 Numerical tests

8.1 Simulation of Lyapunov exponent of a linear system with small noises

The stability problem of a stochastic system is of great importance from physical and engineering points of view. It is known(11,12) that one can investigate stability of a dynamical stochastic system by Lyapunov exponents. The negativeness of upper Lyapunov exponents is an indication of system stability. Usually, it is impossible to derive analytical expressions for Lyapunov exponents. In this case numerical approaches are useful. For the first time an algorithm of numerical computation of Lyapunov exponents was proposed by D.Talay.(13) Here we use another method to calculate Lyapunov exponent of a linear system with small noises.

Let us consider the following two-dimensional linear Ito stochastic system

$$dX = AX + \varepsilon \sum_{r=1}^{q} B_r X dW_r(t), \tag{8.1}$$

where X is two-dimensional vector, A and B_r are constant 2x2 matrices, W_r are independent standard Wiener processes, $\varepsilon > 0$ is a small parameter. In ergodic case the unique Lyapunov exponent λ of the system (8.1) exists(11)

$$\lambda = \lim_{t\to\infty} \frac{1}{t} E(ln|X(t)|) = \lim_{t\to\infty} \frac{1}{t} ln|X(t)|, \tag{8.2}$$

$X(t)$, $t \geq 0$, is non-trivial solution of the system (8.1). The last equality of (8.2) holds with the probability one. Non-trivial solution of the system (8.1) is asymptotically stable with probability one if and only if the Lyapunov exponent λ is negative.(11)

In Ref. 14 the expansion of Lyapunov exponent of the system (8.1) by powers of small parameter ε was obtained. In the case of

$$A = \begin{bmatrix} a & c \\ -c & a \end{bmatrix}, \quad B_r = \begin{bmatrix} b_r & d_r \\ -d_r & b_r \end{bmatrix}$$

$$r = 1, q \tag{8.3}$$

the Lyapunov exponent of the system (8.1) is exactly equal to[14]

$$\lambda = a + \frac{\varepsilon^2}{2}\sum_{r=1}^{q}[(d_r)^2 - (b_r)^2]. \tag{8.4}$$

To test the numerical schemes of the present paper we choose the case (8.3) of the system (8.1) with two independent noises ($q = 2$).

We calculate the function $\lambda(t)$

$$\lambda(t) = \frac{1}{t}ln|X(t)| \approx \frac{1}{t}ln|\bar{X}(t)| \tag{8.5}$$

which in the limit of large time ($t \to \infty$) tends to the Lyapunov exponent λ. The approximation $\bar{X}(t)$ of the exact solution $X(t)$ of the system (8.1) is simulated by three mean-square schemes: 1) the first order method,[3,6,7] 2) the simplified version of the Runge-Kutta scheme (5.8) with the mean-square error $O(h^2 + \varepsilon h + \varepsilon^2 h^{1/2})$ and 3) the Runge-Kutta scheme (5.16) with the error $O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2})$.

Since the system (8.1) with the matrices defined by (8.3) is the commutative (i.e., $\Lambda_i(B_r X) = \Lambda_r(B_i X)$) two-dimensional system with two multiplicative noises, the first order method is the highest mean-square order scheme with easily simulated needed random variables for numerical solution of this system among known general methods.[6,7] This method for the system (8.1)-(8.3) has the form[3,6,7]

$$\begin{aligned}
X^1_{k+1} &= X^1_k + \varepsilon\sum_{r=1}^{2}[b_r X^1_k + d_r X^2_k]\xi_{r_k}h^{1/2} + [aX^1_k + cX^2_k]h+\\
&+\varepsilon^2 h\{\sum_{r=1}^{2}[(b_r)^2 X^1_k + 2b_r d_r X^2_k - (d_r)^2 X^1_k]((\xi_{r_k})^2 - 1)/2+\\
&\quad + [b_1 b_2 X^1_k + b_1 d_2 X^2_k - d_1 d_2 X^1_k + d_1 b_2 X^2_k]\xi_{1_k}\xi_{2_k}\},\\
X^2_{k+1} &= X^2_k + \varepsilon\sum_{r=1}^{2}[-d_r X^1_k + b_r X^2_k]\xi_{r_k}h^{1/2} + [-cX^1_k + aX^2_k]h+\\
&+\varepsilon^2 h\{\sum_{r=1}^{2}[-2d_r b_r X^1_k - (d_r)^2 X^2_k + (b_r)^2 X^2_k]((\xi_{r_k})^2 - 1)/2+\\
&\quad +[-d_1 b_2 X^1_k - d_1 d_2 X^2_k - b_1 d_2 X^1_k + b_1 b_2 X^2_k]\xi_{1_k}\xi_{2_k}\},\\
&(ER^2)^{1/2} = O(h),
\end{aligned} \tag{8.6}$$

where $\xi_r, r = 1, 2$, are independent random variables with standard normal distribution $N(0, 1)$.

The simplified version of the Runge-Kutta scheme (5.8) for the system (8.1)-(8.3) is written as

$$X^1_{k+1} = X^1_k + \varepsilon\sum_{r=1}^{2}[b_r X^1_k + d_r X^2_k]\xi_{r_k}h^{1/2} + [K^1_1 + K^1_2]/2,$$

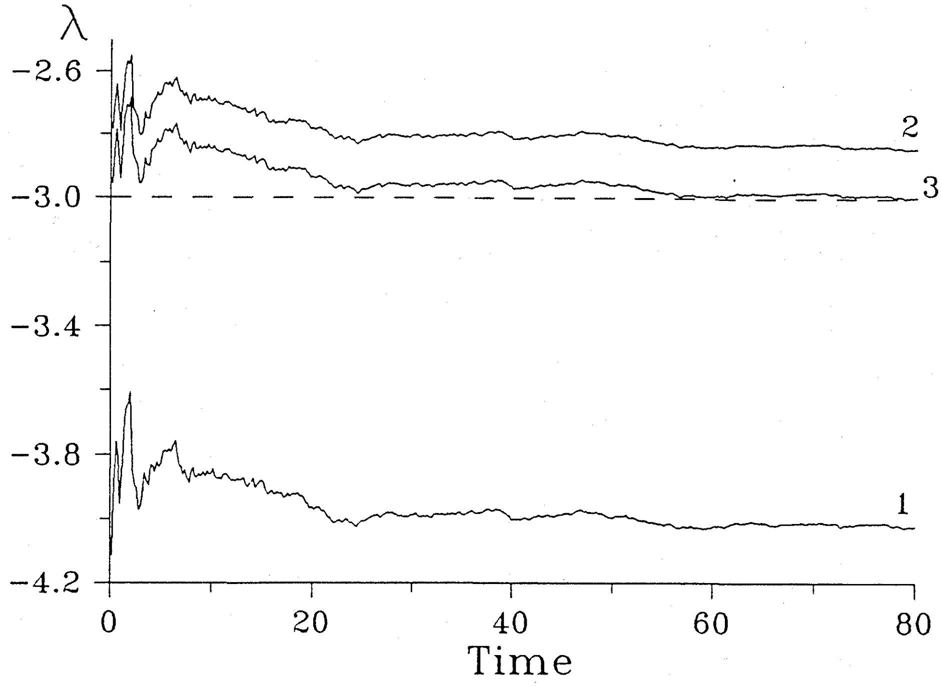

Figure 1: *Lyapunov exponent. Time dependence of the function* $\lambda(t)$ *for* $a = -3$, $c = 1$, $b_1 = b_2 = 1$, $d_1 = 1$, $d_2 = -1$, $\varepsilon = 0.1$, $X^1(0) = 0$, $X^2(0) = 1$ *and time step* $h = 0.3$. *The solution of the system (8.1)-(8.3) is approximated by (1) the method (8.6), (2) the Runge-Kutta method (8.7), (3) the Runge-Kutta method (8.8). Dashed line is the exact value of the Lyapunov exponent* λ $(\lambda = -3)$.

$$X^2_{k+1} = X^2_k + \varepsilon \sum_{r=1}^{2} [-d_r X^1_k + b_r X^2_k] \xi_{r_k} h^{1/2} + [K^2_1 + K^2_2]/2, \tag{8.7}$$

$$K^1_1 = h[aX^1_k + cX^2_k], \; K^1_2 = h[a(X^1_k + K^1_1) + c(X^2_k + K^2_1)],$$

$$K^2_1 = h[-cX^1_k + aX^2_k], \; K^2_2 = h[-c(X^1_k + K^1_1) + a(X^2_k + K^2_1)],$$

$$(ER^2)^{1/2} = O(h^2 + \varepsilon h + \varepsilon^2 h^{1/2}),$$

$\xi_r, r = 1, 2$, are independent random variables with standard normal distribution $N(0, 1)$. According to the Runge-Kutta scheme (5.16) we obtain the following algorithm for the system (8.1)-(8.3)

$$X^1_{k+1} = X^1_k + \varepsilon \sum_{r=1}^{2} [b_r X^1_k + d_r X^2_k] \xi_{r_k} h^{1/2} + [K^1_1 + 2K^1_2 + 2K^1_3 + K^1_4]/6,$$

$$X^2_{k+1} = X^2_k + \varepsilon \sum_{r=1}^{2} [-d_r X^1_k + b_r X^2_k] \xi_{r_k} h^{1/2} + [K^2_1 + 2K^2_2 + 2K^2_3 + K^2_4]/6, \tag{8.8}$$

$$K^1_1 = h[aX^1_k + cX^2_k], \; K^2_1 = h[-cX^1_k + aX^2_k],$$

$$K^1_2 = h[a(X^1_k + K^1_1/2) + c(X^2_k + K^2_1/2)],$$

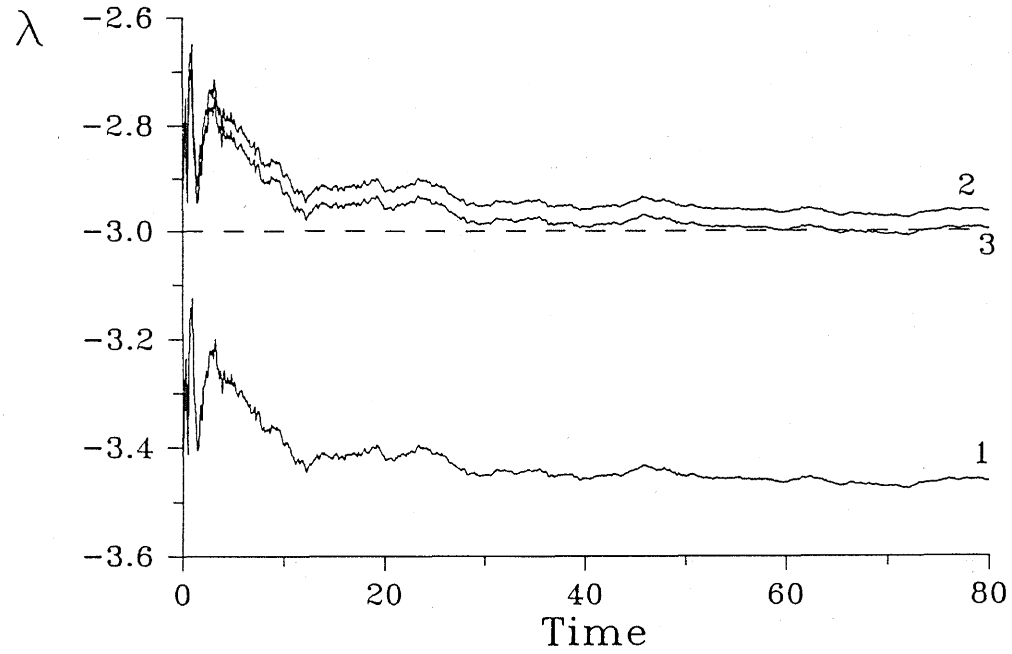

Figure 2: *Lyapunov exponent. Time dependence of the function $\lambda(t)$ for $h = 0.1$, other parameters are the same as in Fig. 1. The solution of the system (8.1)-(8.3) is approximated by (1) the method (8.6), (2) the Runge-Kutta method (8.7), (3) the Runge-Kutta method (8.8). Dashed line is the exact value of the Lyapunov exponent λ ($\lambda = -3$).*

$$K_2^2 = h[-c(X_k^1 + K_1^1/2) + a(X_k^2 + K_1^2/2)],$$
$$K_3^1 = h[a(X_k^1 + K_2^1/2) + c(X_k^2 + K_2^2/2)],$$
$$K_3^2 = h[-c(X_k^1 + K_2^1/2) + a(X_k^2 + K_2^2/2)],$$
$$K_4^1 = h[a(X_k^1 + K_3^1) + c(X_k^2 + K_3^2)],$$
$$K_4^2 = h[-c(X_k^1 + K_3^1) + a(X_k^2 + K_3^2)],$$
$$(ER^2)^{1/2} = O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2}),$$

ξ_r, $r = 1, 2$, are independent normally distributed $N(0,1)$ random variables. To simulate Gaussian random numbers we use the procedure GASDEV.[15]

If one chooses $h = O(\varepsilon^{1/2})$ (see Fig.1), the method (8.6) gives the mean-square error $O(\varepsilon^{1/2})$ on the whole interval, the Runge-Kutta method (8.7) - $O(\varepsilon)$ and the Runge-Kutta method (8.8) - $O(\varepsilon^{3/2})$. In the case of $h = O(\varepsilon)$ (see Fig. 2) the mean-square errors of these methods are estimated as: the scheme (8.6) - $O(\varepsilon)$, the scheme (8.7) - $O(\varepsilon^2 + \varepsilon^2 + \varepsilon^{5/2}) = O(\varepsilon^2)$ and the method (8.8) - $O(\varepsilon^4 + \varepsilon^2 + \varepsilon^{5/2}) = O(\varepsilon^2)$. For $h = O(\varepsilon^2)$ (see Fig. 3) we have: (8.6) - $O(\varepsilon^2)$ and (8.7), (8.8) -$O(\varepsilon^3)$. Analyzing Figures 1-3 one can conclude that 1)the proposed methods for a system with small noises are

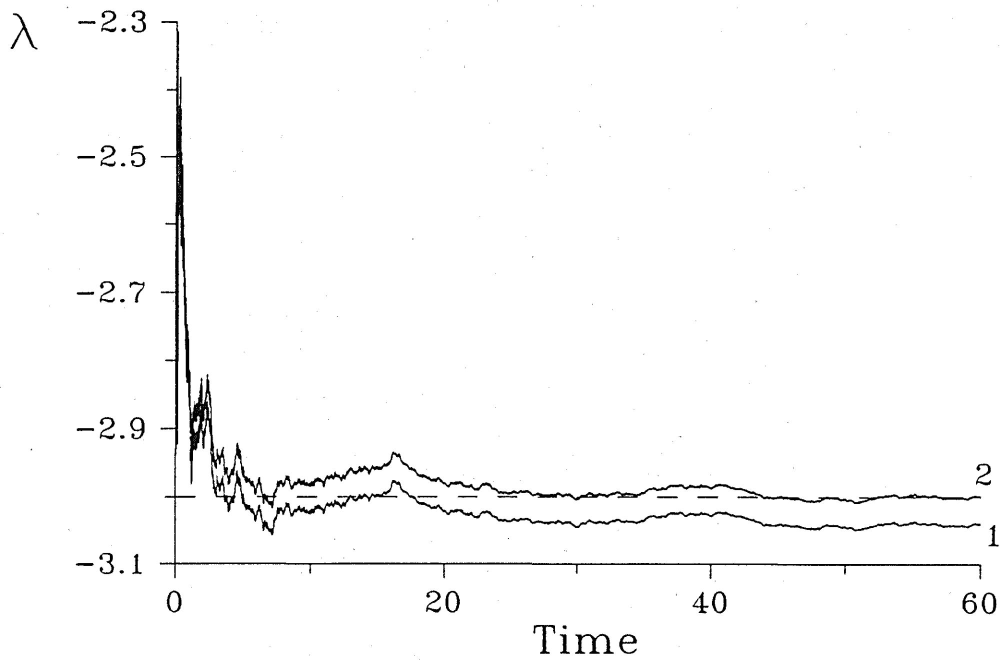

Figure 3: *Lyapunov exponent. Time dependence of the function* $\lambda(t)$ *for* $h = 0.01$, *other parameters are the same as in Fig. 1. The solution of the system (8.1)-(8.3) is approximated by (1) the method (8.6), (2) the Runge-Kutta methods (8.7) and (8.8). Dashed line is the exact value of the Lyapunov exponent* λ ($\lambda = -3$).

correct; 2)in the case of small noises new methods may have less errors than ordinary methods and permit to save CPU time.

It must be mentioned that if ε is equal to one, the method (8.6) always gives better results than the schemes (8.7), (8.8) since the method (8.6) has higher order with respect to h than the schemes (8.7), (8.8). However, for the commutative system (8.1)-(8.3) the methods (8.7) and (8.8) may be easily improved (see the scheme (5.6) as an example) up to $O(h^2 + \varepsilon^2 h^{3/2} + \varepsilon^3 h)$ and $O(h^4 + \varepsilon^2 h^{3/2} + \varepsilon^3 h)$ correspondingly.

8.2 Laser Langevin equation with multiplicative noises

Our second example is devoted to trajectory simulation of the following laser Langevin equation[16,17]

$$db/dt = [(\alpha + i\beta) - (A + iB)|b|^2]b + \Gamma(t) \tag{8.9}$$

where α, β and Γ fluctuate according to

$$\alpha = \alpha_o + \Gamma_\alpha(t),\ \beta = \beta_o + \Gamma_\beta(t),\ \Gamma(t) = \Gamma_1(t) + i\Gamma_2(t),$$

$$< \Gamma_\alpha >=< \Gamma_\beta >=< \Gamma_i >= 0, < \Gamma_\alpha(t)\,\Gamma_\alpha(t') >= Q_\alpha \delta(t-t'), \tag{8.10}$$

$$< \Gamma_\beta(t)\,\Gamma_\beta(t') >= Q_\beta \delta(t-t'), < \Gamma_\alpha(t)\,\Gamma_\beta(t') >= Q_{\alpha\beta}\delta(t-t'),$$

$$< \Gamma_i(t)\,\Gamma_j(t') >= Q\delta_{ij}\delta(t-t'), < \Gamma_\alpha(t)\,\Gamma_i(t') >=< \Gamma_\beta(t)\,\Gamma_i(t') >= 0,$$
$$i,j = 1,2.$$

Let us suppose that fluctuations are small. According to the notation of the paper the system (8.9)-(8.10) may be rewritten in the form

$$b \equiv X = X^1 + iX^2, \tag{8.11}$$

$$dX^1 = [\alpha_o X^1 - \beta_o X^2 - (AX^1 - BX^2)XX^*]dt+$$
$$+\varepsilon\{\sum_{i=1}^{2}[\alpha_i X^1 - \beta_i X^2] * dW_i + \sigma dW_3\}, \tag{8.12}$$
$$dX^2 = [\beta_o X^1 + \alpha_o X^2 - (BX^1 + AX^2)XX^*]dt+$$
$$+\varepsilon\{\sum_{i=1}^{2}[\beta_i X^1 + \alpha_i X^2] * dW_i + \sigma dW_4\},$$

where

$$\varepsilon^2[(\alpha_1)^2 + (\alpha_2)^2] = Q_\alpha, \;\; \varepsilon^2[(\beta_1)^2 + (\beta_2)^2] = Q_\beta,$$
$$\varepsilon^2[\alpha_1\beta_1 + \alpha_2\beta_2] = Q_{\alpha\beta}, \;\; \varepsilon\sigma = \sqrt{Q}.$$

Under $\varepsilon = 0$ the system (8.12) becomes deterministic. In the case of $\alpha/A > 0$ it has asymptotically stable limit cycle $(X^1)^2 + (X^2)^2 = \alpha_o/A$. The radius $\rho = |X|$ satisfies the equation

$$d\rho/dt = (\alpha - A\rho^2)\rho$$

and does not depend on the detuning parameters β_o and B. But difference equations, which are the result of applying numerical methods to the system (8.12), essentially depend on these parameters, and growing of $|\beta - B|$ leads to vanishing of stable cycle. Therefore, to solve the system (8.12) one must use high order schemes or choose quite small time step. Since the system (8.12) contains multiplicative noises and does not belong to class of systems with commutative noises, the Euler method is the highest order scheme among known mean-square methods with easily simulated random variables.[6,7] The Euler method has the mean-square error $O(h + \varepsilon^2 h^{1/2})$ and in the case of large $|\beta_o - B|$ too small step h is required. On the other hand, for instance, the method (5.17) with the mean-square error $O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2})$ allows to obtain sufficiently accurate approximations of solutions of the system (8.12) and, particularly, to simulate phase trajectories.

For the convenience of the reader we write down the Runge-Kutta method (5.17) for the system (8.12)

$$K_1^1 = h[\alpha_o X_k^1 - \beta_o X_k^2 - (AX_k^1 - BX_k^2)X_k X_k^*],$$
$$K_1^2 = h[\beta_o X_k^1 + \alpha_o X_k^2 - (BX_k^1 + AX_k^2)X_k X_k^*],$$
$$\tilde{X}_k^i = X_k^i + K_1^i/2, \;\; i = 1,2,$$
$$K_2^1 = h[\alpha_o \tilde{X}_k^1 - \beta_o \tilde{X}_k^2 - (A\tilde{X}_k^1 - B\tilde{X}_k^2)\tilde{X}_k \tilde{X}_k^*],$$

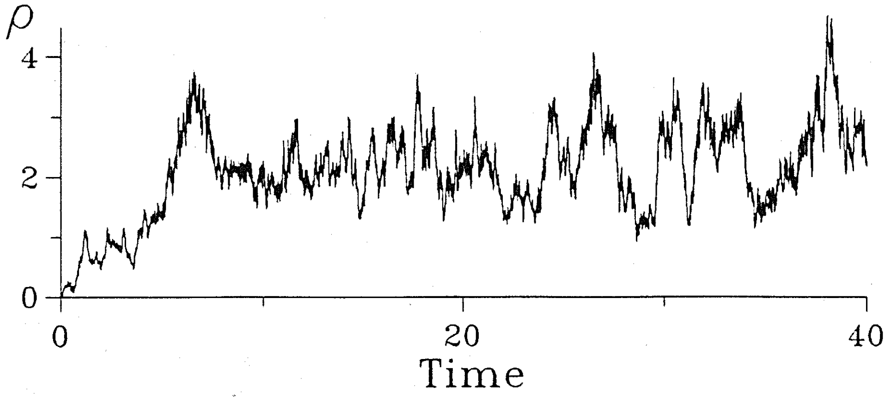

Figure 4: *Laser Langevin equation. Time dependence of the radius $\rho = |X_k|$ for $\alpha_o = 0.5$, $\beta_o = 1$, $A = 0.1$, $B = 0.4$, $\varepsilon = 0.3$, $\alpha_i = \beta_i = \sigma = 1$, $i = 1, 2$, $X^1(0) = X^2(0) = 0$ and time step $h = 0.01$. The solution X_k of the system (8.12) is approximated by the Euler method and by the Runge-Kutta method (8.13).*

$$K_2^2 = h[\beta_o \check{X}_k^1 + \alpha_o \check{X}_k^2 - (B\check{X}_k^1 + A\check{X}_k^2)\check{X}_k\check{X}_k^*],$$

$$\hat{X}_k^i = X_k^i + K_2^i/2, \; i = 1, 2,$$

$$K_3^1 = h[\alpha_o \hat{X}_k^1 - \beta_o \hat{X}_k^2 - (A\hat{X}_k^1 - B\hat{X}_k^2)\hat{X}_k\hat{X}_k^*],$$

$$K_3^2 = h[\beta_o \hat{X}_k^1 + \alpha_o \hat{X}_k^2 - (B\hat{X}_k^1 + A\hat{X}_k^2)\hat{X}_k\hat{X}_k^*],$$

$$\check{X}_k^i = X_k^i + K_3^i, \; i = 1, 2,$$

$$K_4^1 = h[\alpha_o \check{X}_k^1 - \beta_o \check{X}_k^2 - (A\check{X}_k^1 - B\check{X}_k^2)\check{X}_k\check{X}_k^*],$$

$$K_4^2 = h[\beta_o \check{X}_k^1 + \alpha_o \check{X}_k^2 - (B\check{X}_k^1 + A\check{X}_k^2)\check{X}_k\check{X}_k^*],$$

$$\begin{aligned} X_{k+1}^1 &= X_k^1 + [K_1^1 + 2K_2^1 + 2K_3^1 + K_4^1]/6 + \\ &+\varepsilon h^{1/2}[\sum_{i=1}^{2}(\alpha_i X_k^1 - \beta_i X_k^2)\xi_{i_k} + \sigma\eta_{1_k}] + \\ &+ \varepsilon^2 h \sum_{i=1}^{2}[((\alpha_i)^2 - (\beta_i)^2)X_k^1 - 2\alpha_i\beta_i X_k^2]/2, \\ X_{k+1}^2 &= X_k^2 + [K_1^2 + 2K_2^2 + 2K_3^2 + K_4^2]/6 + \\ &+\varepsilon h^{1/2}[\sum_{i=1}^{2}(\beta_i X_k^1 + \alpha_i X_k^2)\xi_{i_k} + \sigma\eta_{2_k}] + \\ &+\varepsilon^2 h \sum_{i=1}^{2}[((\alpha_i)^2 - (\beta_i)^2)X_k^2 + 2\alpha_i\beta_i X_k^1]/2, \\ &[ER^2]^{1/2} = O(h^4 + \varepsilon h + \varepsilon^2 h^{1/2}), \end{aligned} \tag{8.13}$$

where ξ_i and η_i are independent random variables with normal distribution $N(0, 1)$.

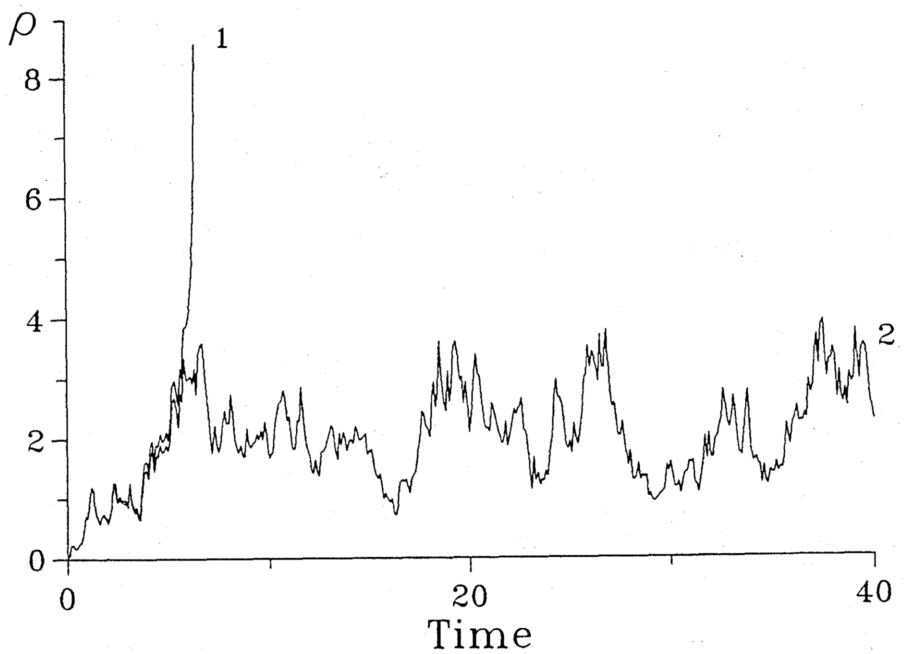

Figure 5: *Laser Langevin equation. Time dependence of the radius* $\rho = |X_k|$ *for time step* $h = 0.1$, *other parameters are the same as in Fig. 4. The solution* X_k *of the system (8.12) is approximated by (1) the Euler method, (2) the Runge-Kutta method (8.13).*

The radius $\rho = |X_k|$ is plotted in Fig. 4 and 5. Fig. 4 demonstrates the radius ρ calculated by Euler scheme and the Runge-Kutta scheme (8.13) with the time step $h = 0.01$. In this case both methods give the same results. As seen in Fig. 5 if one chooses greater time step ($h = 0.1$), the Runge-Kutta scheme (8.13) gives quite well results (compare with Fig. 4), but the Euler method becomes unstable. In both cases we use the same sample paths for the Wiener processes.

Note that the Runge-Kutta method (8.13) may be improved up to $[ER^2]^{1/2} = O(h^4 + \varepsilon^2 h^{1/2})$ (see the method (5.15)).

9 Conclusions

Differential equations with small noises is an important case of a stochastic system. In the paper the approach to construction of efficient mean-square methods with low errors for a system with small noises is developed. Thanks to a small parameter ε new methods may be easier, require less computer time and have less errors than general schemes.[6,7] Special attention has been paid to constructing methods with efficiently simulated random variables. An accuracy and convergence of a method on the whole interval are analyzed by the theorem on estimate of mean-square errors that have been proved in the paper. Herein the explicit, implicit and Runge-Kutta methods with the mean-square errors from $O(h + \varepsilon^2 h^{1/2})$ up to $O(h^4 + \varepsilon^2 h^{1/2})$ are proposed for general Ito and Stratonovich systems with small noises. Moreover, systems with small additive noises and systems with small colored noises are considered. The appropriate methods for these systems have been also derived: for systems with small additive noises - schemes with mean-square errors from $O(h^2+\varepsilon^2 h)$ up to $O(h^4+\varepsilon^2 h^{3/2})$ and for systems with small colored noises - schemes with mean-square errors from $O(h^2 + \varepsilon h)$ up to $O(h^4 + \varepsilon h^3 + \varepsilon^2 h^{5/2})$. Obviously, by the proposed approach it is possible to derive a lot of other numerical schemes for a system with small noises.

Mean-square methods are useful for direct simulation of stochastic trajectories, which, for instance, may give an information on qualitative behaviour of a stochastic model. However, for practical applications weak methods[6,7] are more important. Firstly, they are sufficient for calculation of mean values and solving problems of mathematical physics by Monte-Carlo technique. Secondary, they are simpler than mean-square methods. Weak methods, constructed for a system with small noises, may be a useful tool for numerical solution of partial differential equations with a small parameter at high derivative. Note that mean-square methods are the basis for construction of weak ones. Weak methods for a system with small noises and their applications to problems of mathematical physics will be the subject of our next paper.

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