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Entrainment of modulation frequency: A case study

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Abstract

We consider a system of autonomous ODE's which is S^1-equivariant and has a family of asymptotically stable modulated wave solutions with wave frequency α_0 and modulation frequency β_0. This system will be perturbed, where the applied nonautonomous force also represents a modulated wave, but with wave frequency α and modulations frequency β. The strength of this perturbation is not necessarily small. Our goal is to look for conditions such that the perturbed system exhibits an approximate entrainment of the modulation frequency β on any given finite time interval, where the approximation error can be controlled by the wave frequency.

1 Introduction

A classical problem in the theory of synchronization is the problem of frequency entrainment of an oscillatory system by an external force. In the simplest case, the oscillatory system is an autonomous differential system possessing an exponentially orbitally stable limit cycle with frequency β_0, and the external force is periodic with frequency β. If we characterize the strength of the forcing by the parameter γ, then the goal is to find regions in the (β, γ)-plane such that the perturbed system has a stable periodic solution with frequency β. These regions are called resonance horns or Arnold-tongues. The boundaries of these cone-like areas are determined by bifurcations of periodic solutions.

A more complicated scenario has been investigated by [Afraimovich and Shilnikov, 1974b; Afraimovich and Shilnikov, 1974a]. They considered the case that the unperturbed autonomous system has a homoclinic orbit instead of a limit cycle. Here, the application of a small periodic force leads to regions in the parameter space corresponding to synchronized as well as to chaotic regimes.

In this paper we study the case that the unperturbed system is an autonomous system possessing a modulated wave solution with wave frequency α_0 and modulation frequency β_0, and that the applied force with strength γ is also a modulated wave with wave frequency α and modulation frequency β. The case that $|\alpha - \alpha_0|, |\beta - \beta_0|$ and γ are small has been treated in [Recke and Peterhof, 1999]. In their paper they have proved that in the (γ, α, β)- parameter space there is a cone-like region with $(0, \alpha_0, \beta_0)$ as vortex such that to each point of this set there corresponds a perturbed system possessing a stable modulated wave solution with wave frequency α and modulation frequency β, that is, there is a frequency entrainment between corresponding frequencies.

The goal of this paper is to investigate the case $|\beta - \beta_0| \ll 1, \alpha \gg 1$. Moreover, we do not assume γ to be small.

The motivation to study such problems comes from the challenge in communication networks to increase strongly the data transmission rate. A promising class of devices to realize this goal are multisection semiconductor lasers with distributed feedback. A well-established model to describe such lasers is the traveling wave model (see, e.g., [Bandelow *et al.*, 1993; Radziunas and Wünsche, 2004] which consists of a hyperbolic system of partial differential equations for the optical field which is nonlinearly coupled with the system of ordinary differential equations for the carrier densities. This model exhibits two crucial properties, namely the S^1-equivariance and the linearity of the differential system for the optical field.

In our paper we consider a system of ordinary differential equations consisting of two coupled subsystems and exhibiting the same fundamental properties as the travelling wave model mentioned before. We additionally assume that one subsystem is under the forcing of a modulated wave with wave frequency α and modulation frequency β. After transforming this system into some normal form taking into account that α is a large parameter, we are able, to any given strength γ of the forcing, to establish the entrainment of the modulation frequency β of some truncated system. Concerning the complete model we can prove an approximate frequency entrainment of the modulation frequency on any given finite time interval, where the entrainment error can be controlled by the wave frequency α of the forcing.

The paper is organized as follows. In Sec. 2 we introduce the unperturbed system and formulate the corresponding assumptions. In Sec. 3 we describe the perturbed system and characterize the perturbation. Section 4 contains nearly identical transformations leading to some normal form of our perturbed system. In Sec. 5 we introduce the truncated nonautonomous system and establish the existence of frequency entrainment. In the final section we study the behavior of the full system and estimate the error between the solution of the truncated system and the solution of the full system starting at the same time at neighboring initial points.

2 The unperturbed system

We consider the system of autonomous differential equations

$$\begin{aligned} \frac{dx}{dt} &= f(x) + |y|^2 g(x), \\ \frac{dy}{dt} &= h(x)y \end{aligned} \tag{2.1}$$

under the smoothness assumption

(A$_1$). The functions $f, g : \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{C}$ are k-times ($k \geq 2$) continuously differentiable.

We note that system (2.1) is equivariant with respect to the $\mathbf{S}^1$-representation

$$(\psi, x, y) \in \mathbb{R}/2\pi \times \mathbb{R}^n \times \mathbb{C} \mapsto (x, e^{i\psi} y) \in \mathbb{R}^n \times \mathbb{C}.$$

Hence, if $(x(t), y(t))$ is a solution of (2.1), then, for each $\psi \in \mathbb{R}$, also $(x(t), e^{i\psi} y(t))$ is a solution of (2.1).

By introducing polar coordinates $y = re^{i\vartheta}$ with $r \in \mathbb{R}^+, \vartheta \in \mathbb{R}$ we get from (2.1)

$$\begin{aligned} \frac{dx}{dt} &= f(x) + r^2 g(x), \\ \frac{dr}{dt} &= \operatorname{Re} h(x)\, r, \end{aligned} \tag{2.2}$$

$$\frac{d\vartheta}{dt} = \operatorname{Im} h(x), \tag{2.3}$$

where Re and Im denote the real and the imaginary parts, respectively. Thus, the equation for the phase ϑ is decoupled from the equations for x and r, moreover, system (2.2) is no more equivariant.

Concerning the decoupled system (2.2) we suppose:

(A$_2$). System (2.2) has a periodic solution $p : \mathbb{R} \to \mathbb{R}^{n+1}$ with period $T_0 = \frac{2\pi}{\beta_0}$, $\beta_0 > 0$, where 1 is a simple Floquet multiplier and all other multipliers of p are located in the interior of the unit circle.

For the sequel we represent p in the form

$$p(t) \equiv p_0(\beta_0 t) := (x_0(\beta_0 t),\ r_0(\beta_0)), \tag{2.4}$$

where $x_0 : \mathbb{R} \to \mathbb{R}^n$ and $r_0 : \mathbb{R} \to \mathbb{R}^+$ are periodic with minimal period 2π.

We denote by

$$\mathcal{O}_0 := \{z \in \mathbb{R}^{n+1} : z = p_0(\tau),\ 0 \le \tau \le 2\pi\}$$

the closed orbit in $\mathbb{R}^{n+1}$ generated by the periodic solution p_0 of system (2.2). According to hypothesis (A_2), $\mathcal{O}_0$ is a nontrivial exponentially stable limit cycle.

It is obvious that the function ϑ defined by

$$\vartheta(t, t_0) := \int_{t_0}^{t} \operatorname{Im} h(x_0(\beta_0 \sigma))\, d\sigma \tag{2.5}$$

solves the phase equation (2.3). Using the notation

$$\alpha_0 := \frac{\beta_0}{2\pi} \int_0^{\frac{2\pi}{\beta_0}} \operatorname{Im} h(x_0(\beta_0 \sigma))\, d\sigma = \frac{1}{T_0} \int_0^{T_0} \operatorname{Im} h(x_0(\beta_0 \sigma))\, d\sigma$$

we can verify that $\vartheta_0(\beta_0 t) := \vartheta(t,t_0) - \alpha_0(t-t_0)$ is periodic in $\beta_0 t$ with period 2π. Thus, the solution ϑ of (2.3) can be represented in the form

$$\vartheta(t,t_0) = \vartheta_0(\beta_0 t) + \alpha_0(t-t_0).$$

Therefore, the full system (2.1) possesses the solution

$$\hat{w}_0(t) := (\hat{x}(t), \hat{y}(t)) := (x_0(\beta_0 t), e^{i\alpha_0(t-t_0)}\, e^{i\vartheta_0(\beta_0 t)} r_0(\beta_0 t)). \tag{2.6}$$

Each component of the solution (2.6) consists of a product of two functions with the frequencies β_0 and α_0, respectively. Therefore, the solution $(\hat{x}(t), \hat{y}(t))$ is called a modulated wave solution with wave frequency α_0 and modulation frequency β_0 (for this terminology see [Rand, 1982]).
The modulated wave solution (2.6) is called asymptotically stable if the related periodic solution (2.4) is asymptotically stable. Thus, under the hypothesis (A_2), $(\hat{x}(t), \hat{y}(t))$ is an asymptotically stable modulated wave solution of (2.1).

Remark 2.1 *In the case that α_0/β_0 is irrational, the modulated wave solution $\hat{w}_0$ represents a quasiperiodic solution of system* (2.1)*, otherwise it is a periodic solution. The S^1-equivariance of system* (2.1) *implies that together with $\hat{w}_0$ there exists a one-parameter family of modulated wave solutions $\hat{w}_0^{\psi}(t) := (\hat{x}(t), e^{i\psi}\hat{y}(t))$ which generates in $\mathbb{R}^n \times \mathbb{C}$ an exponentially attracting invariant set $\mathcal{M}_0$ which is diffeomorphic to a two-torus in $\mathbb{R}^n \times \mathbb{C}$ and consists either of periodic or of quasiperiodic solutions.*

3 The perturbed system

In the sequel we will study the influence of an external force of modulated wave type on system (2.1), where we assume that the perturbed system has the form

$$\begin{aligned} \frac{dx}{dt} &= f(x) + |y|^2 g(x), \\ \frac{dy}{dt} &= h(x)y + \gamma e^{i\alpha t} a(\beta t). \end{aligned} \tag{3.1}$$

Concerning the function a we suppose

(A_3). The function $a : \mathbb{R} \to \mathbb{C}$ is k-times ($k \geq 2$) continuously differentiable and periodic with primitive period 2π.

Thus, the external force

$$\gamma e^{i\alpha t} a(\beta t) \tag{3.2}$$

is a modulated wave with wave frequency α and modulation frequency β. Throughout the following we suppose that α, β and γ are positive constants.

We note that in contrast to the unperturbed system (2.1), system (3.1) is not equivariant with respect to the S^1-representation, and it is not possible by introducing polar coordinates for y to separate the equation for the phase from the other variables.

Our goal is to investigate the influence of the external force (3.2) on the asymptotically stable modulated wave solution (2.6) of system (2.1).

In the case

$$0 < \gamma \ll 1,\ |\alpha - \alpha_0| \ll 1,\ |\beta - \beta_0| \ll 1$$

this perturbation problem has been considered by [Recke and Peterhof, 1999]. Under the assumption that the parameter tuple (γ, α, β) varies in some cone-like open sets with $(0, \alpha_0, \beta_0)$ as vortex, the existence of an asymptotically stable modulated wave solution of the form

$$(\tilde{x}(\beta t), e^{i\alpha t}\tilde{y}(\beta t)) \text{ with } \tilde{x}(\tau) = \tilde{x}(\tau + 2\pi) \text{ and } \tilde{y}(\tau) = \tilde{y}(\tau + 2\pi) \tag{3.3}$$

to equation (3.1) has been proved. Note that the wave frequencies of (3.2) and (3.3) as well as the modulation frequencies coincide. That means, frequency entrainment occurs between "corresponding" frequencies for sufficiently small γ.

In the present paper we consider system (3.1) under the assumption $(A_1) - (A_3)$, where we emphasize that γ must not be small. Our goal is to look for conditions such that a solution of the perturbed system (3.1) starting near the modulated wave solution $\hat{w}_0$ stays for a prescribed time interval in a given (small) neighborhood of $\hat{w}_0$.

In the following Sec. we will transform system (3.1) for $(x, y) \in \mathcal{G}_0$ into some normal form which can be understood as a small perturbation of the unperturbed system (2.1) provided α is large.

4 Averaging transformations

In what follows we suppose that γ is any given positive constant. By means of the transformation

$$x = x_1,\ y = y_1 - i\frac{\gamma}{\alpha}e^{i\alpha t}a(\beta t), \tag{4.1}$$

which is in the compact region $\mathcal{G}_\kappa$ a nearly identical transformation for large α, we get from (4.1) and (3.1)

$$\begin{aligned} \frac{dx_1}{dt} &= f(x_1) + |y_1|^2 g(x_1) + \frac{\gamma^2}{\alpha^2}|a(\beta t)|^2 g(x_1) \\ &\quad + 2\frac{\gamma}{\alpha}\mathrm{Re}\left\{ie^{-i\alpha t}\bar{a}(\beta t)y_1\right\}g(x_1), \\ \frac{dy_1}{dt} &= h(x_1)y_1 - \frac{\gamma}{\alpha}ie^{i\alpha t}\Big(h(x_1)a(\beta t) - \beta a'(\beta t)\Big), \end{aligned} \tag{4.2}$$

where a' denotes the differentiation of $a(\tau)$ with respect to τ and $\bar{a}(\tau)$ is the conjugate complex value of $a(\tau)$.
The goal of the next transformation is to shift for large α the influence of the highly oscillating terms $e^{-i\alpha t}$ and $e^{i\alpha t}$ to terms which are of higher order in α^{-1}.

Applying the transformation

$$\begin{aligned} x_1 &= x_2 - 2\frac{\gamma}{\alpha^2}\,\mathrm{Re}\,\Big\{e^{-i\alpha t}\overline{a}(\beta t)y_2\Big\}g(x_2), \\ y_1 &= y_2 - \frac{\gamma}{\alpha^2}e^{i\alpha t}\Big[h(x_2)a(\beta t) - \beta a'(\beta t)\Big], \end{aligned} \tag{4.3}$$

which is also near the identity in $\mathcal{G}_0$ for sufficiently large α, we obtain from (4.3) and (4.2)

$$\begin{aligned} &\Big(E_n - 2\frac{\gamma}{\alpha^2}\,\mathrm{Re}\,\Big\{e^{-i\alpha t}\overline{a}(\beta t)y_2\Big\}g'(x_2)\Big)\frac{dx_2}{dt} \\ &-2\frac{\gamma}{\alpha^2}\,\mathrm{Re}\,\Big\{e^{-i\alpha t}\overline{a}(\beta t)\frac{dy_2}{dt}\Big\}g(x_2) \\ &= f\Big(x_2 - 2\frac{\gamma}{\alpha^2}\,\mathrm{Re}\,\Big\{e^{-i\alpha t}\overline{a}(\beta t)y_2\Big\}g(x_2)\Big) \\ &+\Big[\Big(y_2 - \frac{\gamma}{\alpha^2}e^{i\alpha t}\Big[h(x_2)a(\beta t) - \beta a'(\beta t)\Big]\Big) \\ &\times\Big(\overline{y}_2 - \frac{\gamma}{\alpha^2}e^{-i\alpha t}\Big[\overline{h}(x_2)\overline{a}(\beta t) - \beta\overline{a}'(\beta t)\Big]\Big) + \frac{\gamma^2}{\alpha^2}|a(\beta t)|^2 \\ &+2\frac{\gamma}{\alpha}\,\mathrm{Re}\,\Big\{ie^{-i\alpha t}\overline{a}(\beta t)\Big(y_2 - \frac{\gamma}{\alpha^2}e^{i\alpha t}\Big[h(x_2)a(\beta t) - \beta a'(\beta t)\Big]\Big)\Big\}\Big] \\ &\times g\Big(x_2 - 2\frac{\gamma}{\alpha^2}\,\mathrm{Re}\,\Big\{e^{-i\alpha t}\overline{a}(\beta t)y_2\Big\}g(x_2)\Big) \\ &-2\frac{\gamma}{\alpha}\,\mathrm{Re}\,\Big\{ie^{-i\alpha t}\overline{a}(\beta t)y_2\Big\}g(x_2) + 2\beta\frac{\gamma}{\alpha^2}\,\mathrm{Re}\,\Big\{e^{-i\alpha t}\overline{a}'(\beta t)y_2\Big\}g(x_2) \\ &=: u_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta), \end{aligned} \tag{4.4}$$

where E_n is the $n \times n$ - unit matrix,

$$
\begin{aligned}
& \frac{dy_2}{dt} - \frac{\gamma}{\alpha^2} e^{i\alpha t} a(\beta t) h'(x_2)\ \frac{dx_2}{dt} = \frac{\gamma}{\alpha} i e^{i\alpha t} \Big(h(x_2) a(\beta t) - \beta a'(\beta t) \Big) \\
+ \quad & \frac{\gamma}{\alpha^2} e^{i\alpha t} \Big(h(x_2) a'(\beta t)\beta - a''(\beta t)\beta^2 \Big) \\
+ \quad & h\Big(x_2 - 2\frac{\gamma}{\alpha^2} \mathrm{Re}\Big\{ e^{-i\alpha t} \overline{a}(\beta t) y_2 \Big\} g(x_2) \Big) \\
& \times \Big(y_2 - \frac{\gamma}{\alpha^2} e^{i\alpha t} \Big[h(x_2) a(\beta t) - \beta a'(\beta t) \Big] \Big) \\
- \quad & \frac{\gamma}{\alpha} i e^{i\alpha t} \Big[h\Big(x_2 - 2\frac{\gamma}{\alpha^2} e^{i\alpha t} \mathrm{Re}\Big\{ e^{-i\alpha t} \overline{a}(\beta t) y_2 \Big\} g(x_2) \Big) a(\beta t) - \beta a'(\beta t) \Big] \\
:= \quad & v_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta).
\end{aligned} \tag{4.5}
$$

By (4.4) and (4.5) we have the representations

$$
\begin{aligned}
& u_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta) \equiv f(x_2) + |y_2|^2 g(x_2) + \frac{\gamma^2}{\alpha^2} |a(\beta t)|^2 g(x_2) \\
& -2\frac{\gamma}{\alpha^2} \Big[\mathrm{Re}\ \Big\{ e^{-i\alpha t} \bar{a}(\beta t) y_2 \Big\} f'(x_2) g(x_2) \\
& + |y_2|^2 \mathrm{Re}\ \Big\{ e^{-i\alpha t} \bar{a}(\beta t) y_2 \Big\} g'(x_2) g(x_2) \\
& - \mathrm{Re}\ \Big\{ e^{-i\alpha t} \Big(\overline{h}(x_2) \overline{a}(\beta t) - \beta \overline{a}'(\beta t) y_2 \Big) \Big\} g(x_2) \\
& - \beta \mathrm{Re}\ \Big\{ e^{-i\alpha t} \overline{a}(\beta t) \overline{a}'(\beta t) y_2 \Big\} g(x_2) \Big] + \alpha^{-3} \tilde{u}_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta) \\
& =: f(x_2) + |y_2|^2 g(x_2) + \frac{\gamma^2}{\alpha^2} |a(\beta t)|^2 g(x_2) + \frac{\gamma}{\alpha^2} \mathrm{Re}\ \Big\{ e^{-i\alpha t} r_2(x_2, y_2, \beta t) \Big\} \\
& + \alpha^{-3} \tilde{u}_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta),
\end{aligned} \tag{4.6}
$$

where $\tilde{u}_2$ is continuous in all variables and uniformly bounded for $(x_2, y_2) \in \mathcal{G}_0, |\beta - \beta_0| \le \beta_0/2, t \in \mathbb{R}$, $\alpha \ge \bar{\alpha}$, where $\bar{\alpha}$ is any positive number, r_2 is continuously differentiable with respect to all variable and uniformly bounded for $(x_2, y_2) \in \mathcal{G}_\kappa, t \in \mathbb{R}$.

$$
\begin{aligned}
&v_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta) \equiv h(x_2) y_2 \\
&-\frac{\gamma}{\alpha^2}\Big[e^{i\alpha t} h(x_2)\Big(h(x_2) a(\beta t) - \beta a'(\beta t) y_2\Big) \\
&+2h'(x_2) \mathrm{Re}\ \Big\{ e^{-i\alpha t}\Big(\overline{a}(\beta t) y_2 \Big\} g(x_2) y_2 \\
&-e^{i\alpha t}\Big(h(x_2)\beta a'(\beta t) - \beta^2 a''(\beta t)\Big)\Big] \qquad\qquad (4.7)\\
&+\alpha^{-3} u_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta) := h(x_2) y_2 + \frac{\gamma}{\alpha^2} e^{i\alpha t} s_2(x_2, y_2, \beta t) \\
&+2\frac{\gamma}{\alpha^2} \mathrm{Re}\ \Big\{ e^{-i\alpha t} \varrho_2(x_2, y_2, \beta t)\Big\} + \alpha^{-3} \tilde{v}_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta),
\end{aligned}
$$

where the function $\tilde{v}_2$ has the same smoothness and boundedness properties as the function $\tilde{u}_2$, and the functions s_2 and ϱ_2 as the function r_2.

If we use the representation

$$
e^{-i\alpha t}\overline{a}(\beta t) = \mathrm{Re}\ \Big\{e^{-i\alpha t}\overline{a}(\beta t)\Big\} + i\ \mathrm{Im}\ \Big\{e^{-i\alpha t}\overline{a}(\beta t)\Big\},\ y_2 =: \nu_2 + i\ \omega_2,
$$

then we may rewrite (4.4) and (4.5) in the form

$$
\begin{aligned}
&\Big(E_n - \frac{2\gamma}{\alpha^2}\ \mathrm{Re}\ \Big\{ e^{-i\alpha t}\overline{a}(\beta t)(\nu_2 + i\omega_2)\Big\} g'(x_2)\Big) \frac{dx_2}{dt} \\
&-\frac{2\gamma}{\alpha^2}\Big(\mathrm{Re}\Big\{ e^{-i\alpha t}\overline{a}(\beta t)\Big\}\frac{d\nu_2}{dt} - \mathrm{Im}\ \Big\{ e^{-i\alpha t}\overline{a}(\beta t)\Big\}\frac{d\omega_2}{dt}\Big) g(x_2) \\
&= u_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta), \qquad\qquad (4.8)\\
&\frac{d\nu_2}{dt} - \frac{2\gamma}{\alpha^2}\ \mathrm{Re}\ \Big\{ e^{i\alpha t} a(\beta t) h'(x_2)\Big\}\frac{dx_2}{dt} = \ \mathrm{Re}\ v_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta), \\
&\frac{d\omega_2}{dt} - \frac{2\gamma}{\alpha^2}\ \mathrm{Im}\ \Big\{ e^{i\alpha t} a(\beta t) h'(x_2)\Big\}\frac{dx_2}{dt} = \ \mathrm{Im}\ v_2(x_2, y_2, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta).
\end{aligned}
$$

By means of the $(n+2) \times (n+2)$ matrix

$$
\begin{aligned}
&A := \qquad\qquad (4.9)\\
&-\frac{2\gamma}{\alpha^2}\begin{pmatrix} \mathrm{Re}\ \Big\{ e^{-i\alpha t}\overline{a}(\beta t) y_2\Big\} g'(x_2) & \mathrm{Re}\Big\{ e^{-i\alpha t}\overline{a}(\beta t)\Big\} g(x_2) & -\mathrm{Im}\Big\{ e^{-i\alpha t}\overline{a}(\beta t)\Big\} g(x_2) \\ \mathrm{Re}\ \Big\{ e^{i\alpha t} a(\beta t) h'(x_2)\Big\} & 0 & 0 \\ \mathrm{Im}\ \Big\{ e^{i\alpha t} a(\beta t) h'(x_2)\Big\} & 0 & 0 \end{pmatrix}
\end{aligned}
$$

the relation (4.8) can be represented in the form

$$(E_{n+2}+A)\begin{pmatrix} \frac{dx_2}{dt} \\ \frac{d\nu_2}{dt} \\ \frac{d\omega_2}{dt} \end{pmatrix} = \begin{pmatrix} u_2(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta) \\ \mathrm{Re}\ v_2(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta) \\ \mathrm{Im}\ v_2(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta) \end{pmatrix}. \tag{4.10}$$

For sufficiently large α and $(x_2,y_2)\in\mathcal{G}_\kappa$ we have $||A||<1$. Thus, it holds

$$(E_{n+2}+A)^{-1} = E_{n+2} - A + A^2 - \ldots + \ldots \tag{4.11}$$

Therefore, we get from (4.9)-(4.11)

$$\begin{aligned} \frac{dx_2}{dt} &= u_2(...) + \frac{2\gamma}{\alpha^2}\Big[\mathrm{Re}\ \Big\{e^{-i\alpha t}\overline{a}(\beta t)y_2\Big\}g'(x_2)u_2(...) \\ &+\mathrm{Re}\ \Big\{e^{-i\alpha t}\overline{a}(\beta t)u_2\Big\}g(x_2)\Big] + \alpha^{-3}\hat{u}_2(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta), \\ \frac{dy_2}{dt} &= v_2(...) + \frac{2\gamma}{\alpha^2}e^{i\alpha t}a(\beta t)h'(x_2)v_2(...) \\ &+\alpha^{-3}\hat{v}_2(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta), \end{aligned} \tag{4.12}$$

where $\hat{u}_2$ and $\hat{v}_2$ have the same properties as the functions $\tilde{u}_2$ and $\tilde{v}_2$.
Taking into account the representations (4.6) and (4.7) we get from (4.12)

$$\begin{aligned} \frac{dx_2}{dt} &= f(x_2) + |y_2|^2 g(x_2) + \frac{\gamma^2}{\alpha^2}|a(\beta t)|^2 g(x_2) \\ &+ \frac{2\gamma}{\alpha^2}\mathrm{Re}\Big\{e^{-i\alpha t}\tilde{r}_2(x_2,y_2,\beta t)\Big\} + \alpha^{-3}u_2^*(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta), \\ \frac{dy_2}{dt} &= h(x_2)y_2 + \frac{2\gamma}{\alpha^2}\Big[e^{i\alpha t}\tilde{s}_2(x_2,y_2,\beta t) + \mathrm{Re}\Big\{e^{-i\alpha t}\tilde{\varrho}_2(x_2,y_2,\beta t)\Big\}\Big] \\ &+ \alpha^{-3}v_2^*(x_2,y_2,\beta t,e^{i\alpha t},\alpha^{-1},\beta), \end{aligned} \tag{4.13}$$

where the functions $\tilde{r}_2$, $\tilde{s}_2$ and $\tilde{\varrho}_2$ have the same properties as the functions r_2, s_2, and ϱ_2 respectively, and the functions u_2^* and v_2^* as the functions $\tilde{u}_2$ and $\tilde{v}_2$, respectively.

It can be easily verified that by means of the nearly identical transformation

$$\begin{aligned} x_2 &= x_3 + 2\frac{\gamma}{\alpha^3}\ \mathrm{Re}\ \Big\{ie^{-i\alpha t}\tilde{r}_2(x_2,y_2,\beta t)\Big\}, \\ y_2 &= y_3 - 2\frac{\gamma}{\alpha^3}\Big[ie^{i\alpha t}\tilde{s}_2(x_2,y_2,\beta t) - \mathrm{Re}\Big\{ie^{-i\alpha t}\tilde{\varrho}_2(x_2,y_2,\beta t)\Big\}\Big], \end{aligned}$$

system (4.13) takes the form

$$
\begin{aligned}
\frac{dx_3}{dt} &= f(x_3) + |y_3|^2 g(x_3) + \frac{\gamma^2}{\alpha^2}|a(\beta t)|^2 g(x_3) \\
&\quad + \alpha^{-3}\tilde{u}_3(x_3, y_3, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta), \\
\frac{dy_3}{dt} &= h(x_3)y_3 + \alpha^{-3}\tilde{v}_3(x_3, y_3, \beta t, e^{i\alpha t}, \alpha^{-1}, \beta),
\end{aligned}
\tag{4.14}
$$

where the functions $\tilde{u}_3$ and $\tilde{v}_3$ have the same properties as the functions $\tilde{u}_2$ and $\tilde{v}_2$, respectively. System (4.14) can be considered as some normalized form of our perturbed system (3.1).
For the following we introduce the positive parameter ε by

$$
\varepsilon := \frac{1}{\alpha}. \tag{4.15}
$$

Moreover, in order to reduce the number of indices, we rename the coordinates (x_3, y_3) as (ξ, η). Hence, system (4.14) takes the form

$$
\begin{aligned}
\frac{d\xi}{dt} &= f(\xi) + |\eta|^2 g(\xi) + \varepsilon^2\gamma^2|a(\beta t)|^2 g(\xi) + \varepsilon^3\tilde{u}_3(\xi, \eta, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta), \\
\frac{d\eta}{dt} &= h(\xi)\eta + \varepsilon^3\tilde{v}_3(\xi, \eta, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta).
\end{aligned}
\tag{4.16}
$$

Here, we want to reformulate our problem to be studied. For $\varepsilon = 0$, system (4.16) has the asymptotically stable modulated wave $\hat{w}_0(t)$ with wave frequency α_0 and modulation frequency β_0 . By introducing polar coordinates $\eta = re^{i\vartheta}$ system (4.16) takes the form

$$
\begin{aligned}
\frac{d\xi}{dt} &= f(\xi) + r^2 g(\xi) + \varepsilon^2\gamma^2|a(\beta t)|^2 g(\xi) + \varepsilon^3\hat{u}_3(\xi, re^{i\vartheta}, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta) \\
\frac{dr}{dt} &= \mathrm{Re}\ h(\xi)r + \varepsilon^3\hat{v}_3^{(r)}(\xi, re^{i\vartheta}, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta),
\end{aligned}
\tag{4.17}
$$

$$
\frac{d\vartheta}{dt} = \mathrm{Im}\ h(\xi) + \varepsilon^3\hat{v}_3^{(\vartheta)}(\xi, re^{i\vartheta}, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta). \tag{4.18}
$$

The problem under consideration can be formulated as follows: Is it possible for sufficiently small ε to find solutions of the full system (4.17), (4.18) which stay for any prescribed finite time interval in a given small neighborhood of $\mathcal{M}_0$ such that their (ξ, r) components are nearly periodic with 'frequency' β? In a first step we will not consider the full system (4.16) but some truncated system.

5 The truncated system

If we drop all terms in (4.16) multiplied by ε we get the unperturbed system (2.1). The lowest order of the perturbation terms in (4.16) is ε^2. If we omit all terms of order $O(\varepsilon^3)$, we get the system

$$\begin{aligned} \frac{d\xi}{dt} &= f(\xi) + |\eta|^2 g(\xi) + \varepsilon^2\gamma^2 |a(\beta t)|^2 g(\xi), \\ \frac{d\eta}{dt} &= h(\xi)\eta, \end{aligned} \tag{5.1}$$

which we call the truncated system. This system can be considered as a small perturbation of the unperturbed system

$$\begin{aligned} \frac{d\xi}{dt} &= f(\xi) + |\eta|^2 g(\xi), \\ \frac{d\eta}{dt} &= h(\xi)\eta. \end{aligned}$$

Different to the perturbed system (3.1), where the forcing term represents a modulated wave and is not necessarily small, the perturbation in system (5.1) is small and periodic. Moreover, in contrast to (3.1), the perturbation term in (5.1) breaks only the autonomy but not the S^1-equivariance. Thus, by introducing polar coordinates

$$\eta = r\, e^{i\vartheta} \qquad r \in \mathbb{R}^+, \vartheta \in S^1 \tag{5.2}$$

we can decouple the equation for the phase ϑ from the system for the variables ξ and r. Indeed, putting (5.2) into (5.1) we get

$$\begin{aligned} \frac{d\xi}{dt} &= f(\xi) + r^2 g(\xi) + \varepsilon^2\gamma^2 |a(\beta t)|^2 g(\xi), \\ \frac{dr}{dt} &= \operatorname{Re} h(\xi) r, \end{aligned} \tag{5.3}$$

$$\frac{d\vartheta}{dt} = \operatorname{Im} h(\xi). \tag{5.4}$$

In case of the perturbed system (3.1), such a decomposition is possible only for $\gamma = 0$, that means for the unperturbed system (2.1).

Setting $\varepsilon = 0$ in (5.3) we obtain the unperturbed autonomous system

$$\begin{aligned} \frac{d\xi}{dt} &= f(\xi) + r^2 g(\xi), \\ \frac{dr}{dt} &= \operatorname{Re} h(\xi) r. \end{aligned} \tag{5.5}$$

By hypothesis (A_2), this system has an exponentially stable periodic solution $p_0(\beta_0 t) = (x_0(\beta_0 t), r_0(\beta_0 t))$ with frequency β_0. Since the forcing term $\varepsilon^2\gamma^2 a(\beta t)g(\xi)$ in (5.3) is periodic in t with frequency β, system (5.3) represents the classical problem of harmonic synchronization of a hyperbolic periodic solution of an autonomous system by means of a small periodic forcing.

Let p^0 be the point on the closed orbit $\mathcal{O}_0$ defined by $p^0 := p_0(0)$. (We recall that $\mathcal{O}_0$ is generated by the periodic solution $p_0(\beta_0 t)$ of system (5.5)). Let $\mathcal{S}$ be a smooth surface in the phase space R^{n+1} intersecting $\mathcal{O}_0$ transversally in p^0. We denote by $\varphi(t; t_0, \xi^*, \varepsilon, \beta)$ the solution of the perturbed system (5.3) for fixed γ satisfying $\varphi(t_0; t_0, \xi^*, \varepsilon, \beta) = \xi^*$. According to hypothesis (A_2) we have $\varphi(t; 0, p^0, 0, \hat{\beta}) \equiv p_0(\beta_0 t)$ for any $\hat{\beta}$. Let $\mathcal{R}_{\varepsilon_0}$ be the rectangle in $\mathbb{R}^+ \times \mathbb{R}$ defined by $\mathcal{R}_{\varepsilon_0} := \{(\varepsilon, t_0) \in \mathbb{R}^+ \times \mathbb{R} : 0 \leq \varepsilon \leq \varepsilon_0, -\varepsilon_0 \leq t_0 \leq \varepsilon_0\}$.

The following theorem is a reformulated and adapted result due to [Farkas, 1994], p. 316.

Theorem 5.1 *Suppose the assumptions* $(A_1) - (A_3)$ *are satisfied. Then there exist a sufficiently small positive number* ε_0 *and functions* $\pi \in C^k(\mathcal{R}_{\varepsilon_0}, \mathbb{R}^{n+1}), \tilde{\beta} \in C^k(\mathcal{R}_{\varepsilon_0}, \mathbb{R}^+)$ *satisfying*

$$\pi(0, t_0) \equiv 0, \quad \tilde{\beta}(0, t_0) \equiv \beta_0 \tag{5.6}$$

such that to any $(\varepsilon, t_0) \in \mathcal{R}_{\varepsilon_0}$ *system* (5.3) *has for* $\beta = \tilde{\beta}(\varepsilon, t_0)$ *a periodic solution* $\varphi(t; t_0, p^0 + \pi(\varepsilon, t_0), \varepsilon, \tilde{\beta}(\varepsilon, t_0)) \equiv p_\varepsilon(\tilde{\beta}(\varepsilon, t_0)\, t, t_0, \tilde{p}^0_\varepsilon)$ *with frequency* $\tilde{\beta}(\varepsilon, t_0)$ *and passing for* $t = t_0$ *the point* $p^0 + \pi(\varepsilon, t_0) =: \tilde{p}^0_\varepsilon \in \mathcal{S}$.

Remark 5.2 *From Theorem* 5.1 *and taking into account* (4.15) *we can conclude that to any suffciently small* ε, *that is for* $\alpha \gg 1$, *there exists a periodic solution* $p_\varepsilon(\tilde{\beta}(\varepsilon, t_0)\, t, t_0, \tilde{p}^0_\varepsilon)$ *whose frequency* $\beta = \tilde{\beta}(\varepsilon, t_0)$ *and starting point* $\tilde{p}^0_\varepsilon$ *for* $t = t_0$ *satisfy the relations*

$$|\tilde{p}^0_\varepsilon - p^0| \ll 1, \quad |\beta - \beta_0| \ll 1. \tag{5.7}$$

In order to determine the stability of the periodic solution $p_\varepsilon(\tilde{\beta}(\varepsilon, t_0)t, t_0, \tilde{p}^0_\varepsilon)$ we consider the variational system to (5.3) with respect to $p_\varepsilon(\tilde{\beta}(\varepsilon, t_0)t, t_0, \tilde{p}^0_\varepsilon)$ and with $\beta = \tilde{\beta}(\varepsilon, t_0)$. To simplify notation we use for system (5.3) the representation

$$\frac{dz}{dt} = F(z) + \varepsilon^2 G(z, \beta t), \tag{5.8}$$

where $z = (\xi, r)$,

$$F(z) := \begin{pmatrix} f(\xi) + r^2 g(\xi) \\ Re\, h(\xi) r \end{pmatrix}, \qquad G(z, \beta t) := \begin{pmatrix} \gamma^2 |a(\beta t)|^2 g(\xi) \\ 0 \end{pmatrix}. \tag{5.9}$$

Using the matrix

$$A_\varepsilon(\tilde{\beta}(\varepsilon, t_0)t, t_0) := F'_z(p_\varepsilon(\tilde{\beta}(\varepsilon, t_0)t, t_0, \tilde{p}^0_\varepsilon) + \varepsilon^2 G'_z(p_\varepsilon(\tilde{\beta}(\varepsilon, t_0)t, t_0, \tilde{p}^0_\varepsilon), \tilde{\beta}(\varepsilon, t_0)t), \tag{5.10}$$

then the variational system of (5.8) with respect to $p_\varepsilon(\tilde\beta(\varepsilon,t_0)t,t_0,\tilde p_\varepsilon^0)$ and with $\beta=\tilde\beta(\varepsilon,t_0)$ can be written as

$$\frac{d\zeta}{dt}=A_\varepsilon(\tilde\beta(\varepsilon,t_0)t,t_0)\,\zeta. \tag{5.11}$$

We denote by $\Phi_\varepsilon(\tilde\beta(\varepsilon,t_0)t,t_0)$ the fundamental matrix of (5.11) satisfying $\Phi_\varepsilon(\tilde\beta(\varepsilon,t_0)t_0,t_0)=E_{n+1}$. The monodromy matrix M_ε of (5.11) is defined by $M_\varepsilon:=\Phi_\varepsilon(\tilde\beta(\varepsilon,t_0)t_0+2\pi,t_0)$. The eigenvalues of M_ε are the characteristic multipliers of the periodic solution $p_\varepsilon(\tilde\beta(\varepsilon,t_0)\,t,t_0,\tilde p_\varepsilon^0)$. According to assumption (A_2), the matrix M_0 has the simple eigenvalue 1, while all other eigenvalues are located in the interior of the unit circle. Under the hypotheses (A_1)-(A_3) there are a sufficiently small number ε_1, $\varepsilon_1\le\varepsilon_0$, and a unique differentiable function λ mapping $\mathcal{R}_{\varepsilon_1}$ into $\mathbb{R}$ with the properties

- (i). $\lambda(\varepsilon,t_0)$is an eigenvalue ofM_ε. (ii). $\lambda(0,t_0)\equiv 1$.

For the following we assume

(A_4) $\qquad \lambda''_{\varepsilon\varepsilon}(0,0)<0.$

Concerning the stability of the periodic solution $p_\varepsilon(\tilde\beta(\varepsilon,t_0)\,t,t_0,\tilde p_\varepsilon^0)$ the following result is valid.

Theorem 5.3 *Assume the hypotheses $(A_1)-(A_4)$ to be valid. Then, for $(\varepsilon,t_0)\in R_{\varepsilon_1}$, the periodic solution $p_\varepsilon(\tilde\beta(\varepsilon,t_0)\,t,t_0,\tilde p_\varepsilon^0)$ is asymptotically stable.*

The proof can be obtained by a modification of the proof of a similar result in [Farkas, 1994], pp 345/345.

To be able to formulate a result about frequency entrainment we introduce for $0\le\varepsilon\le\varepsilon_1$ the functions

$$\overline{\beta}(\varepsilon):=\max_{-\varepsilon_1\le t_0\le\varepsilon_1}\tilde\beta(\varepsilon,t_0),\quad \underline{\beta}(\varepsilon):=\min_{-\varepsilon_1\le t_0\le\varepsilon_1}\tilde\beta(\varepsilon,t_0).$$

By (5.6) we have

$$\lim_{\varepsilon\to0}\overline{\beta}(\varepsilon)=\lim_{\varepsilon\to0}\underline{\beta}(\varepsilon)=\beta_0.$$

From Theorem 5.1 and Theorem 5.3 we get

Theorem 5.4 *Assume the hypotheses $(A_1)-(A_4)$ to be satisfied. Then, to any small positive number κ_0, there exists a sufficiently small positive number ε_2, $\varepsilon_2\le\varepsilon_1$, such that to any given (ε,β) obeying $0<\varepsilon\le\varepsilon_2$, $\beta\in(\underline{\beta}(\varepsilon),\overline{\beta}(\varepsilon))$ there is an asymptotically stable periodic solutions $p_\varepsilon(\beta t,t_0,p_\varepsilon^0)$ with frequency β of system (5.3) whose corresponding closed curve $\mathcal{O}_\varepsilon$ in the phase space is located in a κ_0-neighborhood of $\mathcal{O}_0$.*

If we represent the periodic solution $p_\varepsilon(\beta t, t_0, p_\varepsilon^0)$ of system (5.3) in the form

$$p_\varepsilon(\beta t, t_0, p_\varepsilon^0) = (\hat{\xi}_\varepsilon(\beta t, t_0, \hat{\xi}_\varepsilon^0), \hat{r}_\varepsilon(\beta t, t_0, \hat{r}_\varepsilon^0)),$$

then we can conclude in the same way as for system (2.1) that the truncated system (5.1) has the asymptotically stable modulated wave solution

$$\hat{w}_\varepsilon(t, t_0, \hat{w}^0) := \left(\hat{\xi}_\varepsilon(\beta t, t_0, \hat{\xi}_\varepsilon^0), e^{i\alpha_\varepsilon(t-t_0)}\, e^{i\hat{\vartheta}_\varepsilon(t,t_0)}\hat{r}_\varepsilon(\beta t, t_0, \hat{r}_\varepsilon^0)\right)$$

with wave frequency α_ε and modulation frequency β, where

$$\alpha_\varepsilon := \frac{\beta}{2\pi}\int_0^{\frac{2\pi}{\beta}} Im\, h(\xi_\varepsilon(\beta\sigma, t_0, \xi_\varepsilon^0))\, d\sigma,\ \vartheta_\varepsilon(t, t_0) := \int_{t_0}^{t} Im h(\hat{\xi}_\varepsilon(\beta\sigma, t_0, \hat{\xi}_\varepsilon^0))d\sigma,$$

$$\hat{\vartheta}_\varepsilon(t, t_0) := \vartheta_\varepsilon(t, t_0) - \alpha_\varepsilon(t - t_0).$$

Remark 5.5 *In the case that α_ε/β is irrational, the modulated wave solution $\hat{w}_\varepsilon$ represents a quasiperiodic solution of system* (5.1)*, otherwise it is a periodic solution. The S^1-equivariance of system* (5.1) *implies that together with $\hat{w}_\varepsilon$ there exists a family of modulated wave solutions which generates in $\mathbb{R}^n \times \mathbb{C} \times S^1$ an exponentially attracting invariant set $\mathcal{M}_\varepsilon$ which is diffeomorphic to a two-torus in $\mathbb{R}^n \times \mathbb{C}$.*

We cannot expect that $\hat{w}_\varepsilon(t, t_0, \hat{w}^0)$ is also a solution of the full system (4.16). But by Remark 5.5 we can conclude that the exponentially attracting invariant set $\mathcal{M}_\varepsilon$ persists for sufficiently small ε as an exponentially attracting invariant set $\tilde{\mathcal{M}}_\varepsilon$ for the full system (4.16). In the next Sec. we will consider the solution $w_\varepsilon(t, t_0, \hat{w}^0)$ of (4.16) starting for $t = t_0$ at the same point $\hat{w}^0$ as $\hat{w}_\varepsilon(t, t_0, \hat{w}^0)$ in order to give an asymptotic estimate of the difference $|\hat{w}_\varepsilon(t, t_0, \hat{w}^0) - w_\varepsilon(t, t_0, \hat{w}^0)|$ on some time-interval.

6 Behavior of the solutions of the full system

Let $w_\varepsilon(t, t_0, \hat{w}^0)$ be the solution of the full system

$$\begin{aligned} \frac{d\xi}{dt} &= f(\xi) + |\eta|^2 g(\xi) + \varepsilon^2\gamma^2|a(\beta t)|^2 g(\xi) + \varepsilon^3\tilde{u}_3(\xi, \eta, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta), \\ \frac{d\eta}{dt} &= h(\xi)\eta + \varepsilon^3\tilde{v}_3(\xi, \eta, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta) \end{aligned} \tag{6.1}$$

satisfying $w_\varepsilon(t_0, t_0, \hat{w}^0) = \hat{w}^0$. In the following we will estimate the difference $|\hat{w}_\varepsilon(t, t_0, \hat{w}^0) - w_\varepsilon(t, t_0, \hat{w}^0)|$. To this end we introduce polar coordinates by $\eta = re^{i\vartheta}$ such that system (6.1) takes the form

$$\begin{aligned}\frac{d\xi}{dt} &= f(\xi)+r^2g(\xi)+\varepsilon^2\gamma^2|a(\beta t)|^2g(\xi)+\varepsilon^3\hat{u}_3(\xi,re^{i\vartheta},\beta t,e^{i\varepsilon^{-1}t},\varepsilon,\beta)\\ \frac{dr}{dt} &= \text{Re}\,h(\xi)r+\varepsilon^3\hat{v}_3^{(r)}(\xi,re^{i\vartheta},\beta t,e^{i\varepsilon^{-1}t},\varepsilon,\beta),\end{aligned} \tag{6.2}$$

$$\frac{d\vartheta}{dt} = \text{Im}\,h(\xi)+\varepsilon^3\hat{v}_3^{(\vartheta)}(\xi,re^{i\vartheta},\beta t,e^{i\varepsilon^{-1}t},\varepsilon,\beta). \tag{6.3}$$

Equations (6.2) and (6.3) represent two systems which are coupled by terms of order $O(\varepsilon^3)$, that is, they are weakly coupled. If we omit all terms of order $O(\varepsilon^3)$ in (6.2) we get the truncated system to system (6.2)

$$\begin{aligned}\frac{d\xi}{dt} &= f(\xi)+r^2g(\xi)+\varepsilon^2\gamma^2|a(\beta t)|^2g(\xi),\\ \frac{dr}{dt} &= \text{Re}\,h(\xi)r.\end{aligned} \tag{6.4}$$

For the sequel we rewrite the truncated system (6.4) as in (5.8) in the form

$$\frac{dz}{dt}=F(z)+\varepsilon^2G(z,\beta t),\quad z\in R^{n+1}, \tag{6.5}$$

where the functions F and G are defined in (5.9). The full system (6.2) will be represented as

$$\frac{d\hat{z}}{dt}=F(\hat{z})+\varepsilon^2G(\hat{z},\beta t)+\varepsilon^3H(\hat{z},\beta t,e^{i\varepsilon^{-1}t},\varepsilon,\beta),\quad \hat{z}\in R^{n+1}, \tag{6.6}$$

where H has the same smoothness and boundedness properties as the function $(\tilde{u}_2,\tilde{v}_2)$.

The function $z=p_\varepsilon(\beta t,t_0,p_\varepsilon^0)$ is a solution of (6.5). We denote by $\hat{z}(t,t_0,p_\varepsilon^0)$ the solution of the full system (6.6) satisfying $\hat{z}(t_0,t_0,p_\varepsilon^0)=p_\varepsilon^0$. To estimate the difference $|p_\varepsilon(\beta t,t_0,p_\varepsilon^0)-\hat{z}(t_0,t_0,p_\varepsilon^0)|$ we introduce in (6.6) new coordinates ζ by

$$\hat{z}=p_\varepsilon(\beta t,t_0,p_\varepsilon^0)+\varepsilon\zeta. \tag{6.7}$$

Taking into account that $p_\varepsilon(\beta t,t_0,p_\varepsilon^0)$ is a periodic solution of system (6.5) and using in analogy to (5.10) the notation

$$A_\varepsilon(\beta t,t_0,p_\varepsilon^0):=F_z'(p_\varepsilon(\beta t,t_0,p_\varepsilon^0))+\varepsilon^2G_z'(p_\varepsilon(\beta t,t_0,\tilde{p}_\varepsilon^0),\beta t), \tag{6.8}$$

we get from (6.7) and (6.6)

$$\begin{aligned}\frac{d\zeta}{dt} &= A_\varepsilon(\beta t,t_0,p_\varepsilon^0)\,\zeta+\varepsilon F''(p_\varepsilon(\beta t,t_0,p_\varepsilon^0))\,\zeta\zeta\\ &\quad+\varepsilon^2\tilde{H}(\zeta,\beta t,e^{i\varepsilon^{-1}t},\varepsilon,\beta),\end{aligned} \tag{6.9}$$

where F'' is the Hessian of F at $p_\varepsilon(\beta t,t_0,p_\varepsilon^0)$, and $\tilde{H}(\zeta,\beta t,e^{i\varepsilon^{-1}t},\varepsilon,\beta)$ is continuous in all variables and uniformly bounded for $|\zeta| \leq c$, $|\beta - \beta_0| \leq \beta_0$, $0 < \varepsilon \leq \varepsilon_2$, $t \in \mathbb{R}$, where c is any given positive constant.

Our next goal is to estimate the time interval such that the solution $\zeta_\varepsilon(t, t_0, 0, \beta)$ of (6.9) satisfying

$$\zeta_\varepsilon(t_0, t_0, 0, \beta) = 0 \tag{6.10}$$

stays in the compact ball

$$\mathcal{K}_\delta := \{\zeta \in R^{n+1} : |\zeta| \leq \delta\},$$

where δ is any given positive number. The Cauchy problem (6.9), (6.10) is equivalent to the integral equation

$$\begin{aligned} \zeta_\varepsilon(t) &= \varepsilon\, \Phi_\varepsilon(\beta t, t_0) \int_{t_0}^{t} (\Phi_\varepsilon(\beta s, t_0))^{-1} \\ &\times \Big(F''(p_\varepsilon(\beta s, t_0)\zeta_\varepsilon(s)\zeta_\varepsilon(s) + \varepsilon \tilde{H}(\zeta_\varepsilon(s), \beta s, e^{i\varepsilon^{-1}}, \varepsilon, \beta)\Big) ds, \end{aligned} \tag{6.11}$$

where $\Phi_\varepsilon(\beta t, t_0)$ is the fundamental matrix of the linear system

$$\frac{d\tilde{\zeta}}{dt} = A_\varepsilon(\beta t, t_0, p_\varepsilon^0)\, \tilde{\zeta} \tag{6.12}$$

satisfying $\Phi_\varepsilon(\beta t_0, t_0) = E_{n+1}$, and the function $\tilde{H}(\zeta, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta)$ satisfies

$$|\tilde{H}(\zeta, \beta t, e^{i\varepsilon^{-1}t}, \varepsilon, \beta)| \leq c_1 \quad \text{for} \quad \zeta \in \mathcal{K}_\delta, |\beta - \beta_0| \leq \beta_0/2, t \in R, 0 < \varepsilon \leq \varepsilon_2. \tag{6.13}$$

Since $A_\varepsilon(\beta t, t_0, p_\varepsilon^0))$ is 2π-periodic in βt, the fundamental matrix $\Phi_\varepsilon(\beta t, t_0)$ of (6.12) can be represented in the form

$$\Phi_\varepsilon(\beta t, t_0) a = P_\varepsilon(\beta t, t_0) e^{B_\varepsilon \beta t}, \tag{6.14}$$

where $P_\varepsilon(\beta t, t_0)$ is a regular matrix for all t satisfying $P_\varepsilon(\beta t_0, t_0) = E_{n+1}$, additionally P_ε is continuous and 2π-periodic in βt, and the eigenvalues of B_ε are the characteristic exponents of the periodic solution $p_\varepsilon(\beta t, t_0, p_\varepsilon^0)$.

If we substitute (6.14) into (6.11) we obtain

$$\begin{aligned} &\zeta_\varepsilon(t) = \varepsilon\, P_\varepsilon(\beta t, t_0) \int_{t_0}^{t} e^{B_\varepsilon \beta (t-s)} (P_\varepsilon(\beta s, t_0))^{-1} \\ &\times \Big(F''(p_\varepsilon(\beta t, t_0, p_\varepsilon^0))\zeta_\varepsilon(s)\zeta_\varepsilon(s) + \varepsilon \tilde{H}(\zeta_\varepsilon(s), \beta s, e^{i\varepsilon^{-1}s}, \varepsilon, \beta)\Big) ds. \end{aligned} \tag{6.15}$$

From the hypotheses (A_2) and (A_4) it follows that for sufficiently small ε the matrix B_ε has only eigenvalue with negative real parts. If we denote by $||\cdot||$ the norm of a matrix which is compatible with the Euclidean norm, then we have

$$||e^{B_\varepsilon \beta (t-s)}|| \leq c_2 \quad \text{for} \quad t \geq s, \tag{6.16}$$

where c_2 is some positive constant. Furthermore, under our assumptions there is a positive constant c_3 such that

$$||(F''(p_\varepsilon(\beta t, t_0, p_\varepsilon^0))|| \le c_3, ||P_\varepsilon(\beta t, t_0)|| \le c_3, ||(P_\varepsilon(\beta t, t_0)^{-1}|| \le c_3 \quad \text{for} \quad t \in R. \tag{6.17}$$

If we assume that t_{max} is the largest positive value of $t \ge t_0$ such that $\zeta_\varepsilon(t, t_0, 0, \beta)$ belongs to $\mathcal{K}_\delta$, then we obtain from (6.15) by taking into account (6.13), (6.16), (6.17)

$$|\zeta_\varepsilon(t, t_0, 0, \beta)| \le \varepsilon c_3^2(t_{max} - t_0)(c_2\delta^2 + \varepsilon c_1). \tag{6.18}$$

In order to ensure the inequality $|\zeta_\varepsilon(t, t_0, 0, \beta)| \le \delta$, the constant t_{max} has to satisfy the condition

$$t_{max} \le t_0 + \frac{\delta}{\varepsilon c_3^2(c_2\delta^2 + \varepsilon c_1)} = O(\varepsilon^{-1}). \tag{6.19}$$

Taking into account (6.7) we have the following result.

Theorem 6.1 *Assume the hypotheses $(A_1) - (A_4)$ are valid, then the following asymptotic estimate holds true*

$$|p_\varepsilon(\beta t, t_0, p_\varepsilon^0) - \hat{z}(t, t_0, p_\varepsilon^0| = |\varepsilon\zeta_\varepsilon(t, t_0, 0, \beta)| = O(\varepsilon) \quad \textit{for} \quad t_0 \le t \le O\Big(\frac{1}{\varepsilon}\Big). \tag{6.20}$$

Using the relation (6.20) we get from (6.3)

Lemma 6.2 *Let $\tilde{\vartheta}_\varepsilon(t, t_0)$ be the solution of* (6.3) *satisfying $\tilde{\vartheta}_\varepsilon(t_0, t_0) = 0$, let T be any given number satisfying $T > t_0$. Then, under the assumptions of Theorem* 6.1, *the following asymptotic estimate holds true*

$$|\tilde{\vartheta}_\varepsilon(t, t_0) - \vartheta(t, t_0)| = O(\varepsilon) \quad \textit{for} \quad t_0 \le t \le T, \tag{6.21}$$

where $\vartheta(t, t_0)$ is defined in (2.5).

From Theorem 6.1 and Lemma 6.2 we get

Theorem 6.3 *Assume the hypotheses $(A_1) - (A_4)$ are valid. Then the following asymptotic estimate holds true*

$$|w_\varepsilon(t, t_0, \hat{w}_\varepsilon^0) - \hat{w}_\varepsilon(t, t_0, \hat{w}_\varepsilon^0)| = O(\varepsilon) \quad \textit{for} \quad t_0 \le t \le T.$$

Remark 6.4 *Since $\tilde{\mathcal{M}}_\varepsilon$ is an exponentially attracting invariant set, we can conclude that the estimate remains valid, when the start point w_ε^0 of the solution $w_\varepsilon(t, t_0, w_\varepsilon^0)$ is near $\hat{w}_\varepsilon^0$.*

Remark 6.5 *The result of Theorem* 6.3 *can be interpreted as approximated synchronization on some finite interval, where the synchronization error can be controlled by the wave frequency $\alpha = \varepsilon^{-1}$. Especially, this result implies an approximate entrainment of the modulation frequency β.*

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