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Probability of error deviations for the dependent sampling Monte Carlo methods: exponential bounds in the uniform norm*

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Abstract

Under Bernstein's condition a non-asymptotic exponential estimation for the probability of deviations of a sum of independent random fields in uniform norm is proposed. Application of this result to the problem of the error estimation for the dependent sampling Monte Carlo method is presented. It is shown that in the domain of moderately large deviations the suggested estimations have optimal asymptotics.

1. Introduction

In many Monte Carlo simulations, expectations of a random variable, or integrals of a function are evaluated not as a fixed value but as functions depending on a parameter which comes in, for example, as a parameter of the integrand. Then, it is a challenging problem to organize the algorithm so that the calculated function would be as smooth as possible, i.e., the scattering of statistical errors for different parameter values should be somehow correlated. This implies, we have to analyze the statistical errors in a functional space. In Monte Carlo methods, a popular algorithm of this kind is the dependent sampling method, see, for example [3]. This method is used not only in calculations of expectations and integrals, but also in the Monte Carlo methods for solving integral equations and boundary value problems for PDEs, see [7]. In these methods, the parameters may enter by different ways, for instance, through the boundary conditions, or via the boundary parametrization. We mention also other applications of the large and moderate deviation estimations beyond the dependent sampling Monte Carlo, for instance, in nonlinear detectors [1], and other empirical processes [2].

Let (Ω, A, P) be a probability space, $F(t) = F(t, \omega)$, $t \in T$ a random function with an index set T, such that the expectation $EF(t)$ is finite: $EF(t) = \int_\Omega F(t, \omega)\, P(d\omega) < \infty$ for each $t \in T$. Then the well known Monte Carlo dependent sampling method's estimation of the function $f(t) = EF(t)$ reads (see [3], [4], [5], for example)

$$f(t) \simeq \frac{1}{n} \sum_{i=1}^{n} F_i(t), \tag{1.1}$$

where $F_i(t) = F(t, \omega_i)$, $i = 1, 2, \ldots, n$ are n independent samples of the random field $F(t)$. Despite of the independency of summands in (1.1), this method is called dependent sampling method, since for a fixed i, the values of $F_i(t) = F(t, \omega_i)$ and of $F_i(s) = F(s, \omega_i)$ for different t and s are not necessarily independent.

For clarity, let us give the definition of independent sampling Monte Carlo method. Let us assume that the set $T = \{t_1, t_2, \ldots t_m\}$ consists of a finite number of elements. Let $\{\omega_{ij}, i = 1, \ldots, n;\ j = 1, \ldots, m\}$ be independent samples from Ω, with the same probability distribution $P(d\omega)$. Then the independent sampling method's estimation of $f(t_j) = EF(t_j)$ $(j = 1, \dots, m)$ reads (e.g., see [5]):

$$f(t_j) \simeq \frac{1}{n} \sum_{i=1}^{n} F(t, \omega_{ij}), \quad j = 1, \dots, m.$$

The dependent sampling method is computationally more efficient compared to the independent method, and what is practically important, provides more ßmoothäpproximation of $f(t)$. Therefore, this method is widely used in applications of Monte Carlo methods (e.g., see [5], [6], [8], [9]).

The aim of this paper is the estimation of the probability of deviations of the error of dependent sampling method:

$$P\left\{ \sup_{t \in T} \left| f(t) - \frac{1}{n} \sum_{i=1}^{n} F_i(t) \right| \geq x \right\} = P\left\{ \sup_{t \in T} \left| \frac{1}{n} \sum_{i=1}^{n} (F_i(t) - EF(t)) \right| \geq x \right\}.$$

The problem of estimation of the probability of the error deviations for the dependent sampling method have been considered in the literature (e.g., see [8], [9] and references therein). But these results have asymptotic character (in n) and they deal with the parametric set with finite number of elements. In this paper we are dealing with the set T of general nature and propose a non asymptotic estimation for the probability of deviations of the error in the dependent sampling method. Our result is essentially based on a result due to Ostrovsky [10] where the theory of random variables of subgaussian type is used.

2. A generalization of Bernstein's inequality for sums of random fields

Considering the centered random field $\xi(t) = F(t) - EF(t)$ we can reformulate our problem as the estimation of $P\{\sup_{t \in T} |\frac{1}{n} \sum\limits_{i=1}^{n} \xi_i(t)| \geq x\}$ for independent samples $\xi_1(t), \xi_2(t), \dots \xi_n(t)$ of $\xi(t)$.

So, let $\xi_1(t), \xi_2(t), ..., \xi_n(t)$, $t \in T$ be independent samples of a centered random field $\xi(t)$, $t \in T$, with an arbitrary parametric set T. Denote

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(t), \quad p_n(x) \equiv P\Big\{ \sup_{t \in T} |S_n(t)| \geq x \Big\}. \tag{2.2}$$

In this paper we will obtain the following type of exponential inequalities for $p_n(x)$:

$$p_n(x) \leq \exp(-\phi_n(x)), \quad x \geq 0, \tag{2.3}$$

where $\phi_n(x)$ is a convex function of x, parametrically depending on n, such that $\phi_n(x) \to \infty$ as $x \to \infty$. From the convexity of $\phi_n(x)$ it follows that $\phi_n(x) \geq c_0 + c_1 x \quad (c_1 > 0)$, for x large enough. Therefore, for validity of an estimation of type (2.3) the following condition on tails of one point distributions of $\xi(t)$ should be assumed:

$$\sup_{t \in T} P\{|\xi(t)| \geq x\} \leq c_2 \exp(-c_3 x)$$

for some positive constants c_2, c_3 which is, in turn, equivalent to the following *Generalized Kramer's condition*: there exists some positive constant γ such that

$$\sup_{t\in T} E\exp\{\gamma|\xi(t)|\} < \infty. \tag{2.4}$$

Estimation of type (2.3) given in this paper is a generalization of Bernstein's inequality for a sum of random variables.

2.1 Bernstein's inequality for sums of random variables

A real-valued random variable ξ is said to satisfy *Bernstein's condition* if there exist positive constants σ and b such that

$$E|\xi|^k \le \sigma^2 b^{k-2}\frac{k!}{2}, \quad k = 2, 3, \dots. \tag{2.5}$$

In the following assertion the Bernstein type inequality for probability of deviations of sums of random variables is given. For the sake of closeness we will give the proof of this assertion. Another form of this type of inequality can be found, for example, in [11] (see p. 90) and [12] (see p. 52).

Let $\xi_1, \xi_2, \dots, \xi_n$ be independent samples of a centered random variable ξ (i.e., $E\xi = 0$) satisfying Bernstein's condition (2.5), then

$$P\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i \ge x\right\} \le \exp\left\{-\frac{n\sigma^2}{2b^2}\left(\sqrt{1+\frac{2bx}{\sigma^2\sqrt{n}}}-1\right)^2\right\}, \quad \forall x \ge 0. \tag{2.6}$$

Proof. If $0 \le \lambda \le 1/b$, it follows from (2.5) that

$$E\{\exp(\lambda\xi)\} = 1 + \sum_{k=2}^\infty \frac{\lambda^k E\xi^k}{k!} \le 1 + \frac{1}{2}\sum_{i=2}^\infty \lambda^k\sigma^2 b^{k-2} = 1 + \frac{\sigma^2\lambda^2}{2(1-b\lambda)} \le \exp\left(\frac{\sigma^2\lambda^2}{2(1-b\lambda)}\right).$$

By the independence of summands, this estimate yields

$$E\left\{\exp\left(\frac{\lambda}{\sqrt{n}}\sum_{i=1}^n \xi_i\right)\right\} \le \exp\left(n\frac{\sigma^2(\lambda/\sqrt{n})^2}{2(1-b\lambda/\sqrt{n})}\right) = \exp\left(\frac{\sigma^2\lambda^2}{2(1-b\lambda/\sqrt{n})}\right), \tag{2.7}$$

for $0 \le \lambda \le \frac{\sqrt{n}}{b}$.

Using Chebyshev's inequality

$$P\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i \ge x\right\} \le e^{-\lambda x} E\left\{\exp\left(\frac{\lambda}{\sqrt{n}}\sum_{i=1}^n \xi_i\right)\right\}, \quad x > 0$$

and the inequality (2.7) we get

$$P\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n \xi_i \ge x\right\} \le \inf_{0<\lambda<\frac{\sqrt{n}}{b}} e^{-\lambda x} E\left\{\exp\left(\frac{\lambda}{\sqrt{n}}\sum_{i=1}^n \xi_i\right)\right\} \le$$

$$\inf_{0<\lambda<\frac{\sqrt{n}}{b}} \exp\left(-\lambda x + \frac{\sigma^2\lambda^2}{2(1-b\lambda/\sqrt{n})}\right) = \exp\left(\inf_{0<\lambda<\frac{\sqrt{n}}{b}}\left\{-\lambda x + \left(\frac{\sigma^2\lambda^2}{2(1-b\lambda/\sqrt{n})}\right).\right\}\right)$$

Now, in order to complete the proof of inequality (2.6) it is enough to apply the following equality which can be established by standard arguments

$$\inf_{0\le\mu\le 1/\beta}\left(-x\mu+\frac{\mu^2}{2(1-\beta\mu)}\right)=-\frac{1}{2\beta^2}\left(\sqrt{1+2\beta x}-1\right)^2.$$

Remark 1. Often, instead of inequality (2.6) the following slightly weaker but simpler form of Bernstein's inequality is used (e.g., see [11], p. 90):

$$P\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\ge x\right\}\le\exp\left\{-\frac{x^2}{2\sigma^2}\left(1+\frac{bx}{\sigma^2\sqrt{n}}\right)^{-1}\right\},\quad \forall x\ge 0. \tag{2.8}$$

An immediate consequence of (2.6) is the following inequality:

$$P\left\{\mid\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\xi_i\mid\ge x\right\}\le 2\exp\left\{-\frac{n\sigma^2}{2b^2}\left(\sqrt{1+\frac{2bx}{\sigma^2\sqrt{n}}}-1\right)^2\right\}. \tag{2.9}$$

2.2 Formulation of results

The aim of this section is the generalization of inequality (2.9) for a sum of independent samples of a random field. In order to formulate an estimation for $p_n(x)$ we need some definitions. Let $\xi(t)$, $t\in T$ be a centered random field with a parametric set T. Define a pseudometric $\rho_1(t,s)$ (i.e., $\rho_1(t,s)=0$ does not necessarily imply $t=s$) on T by

$$\rho_1(t,s)\equiv\|\xi(t)-\xi(s)\|_{(1)},$$

where for a random variable ξ the norm $\|\xi\|_{(1)}$ is defined by

$$\|\xi\|_{(1)}\equiv\sup_{k\ge 2}\left(\frac{2\,E|\xi|^k}{k!}\right)^{1/k}.$$

Let us denote by $H_1(\epsilon)$ Kolmogorov's metric ϵ entropy, i.e. the natural logarithm of N_ϵ, the minimal integer such that T can be covered by N_ϵ balls of radius ϵ. In what follows we will assume that the random field $\xi(t)$ is separable in the metric space (T,ρ_1). The definition of separable random fields can be found in [13], p. 203.
For a fixed positive constant β, let $\psi_\beta(\mu)$ be a function defined on the interval $0\le\mu<1/\beta$ by

$$\psi_\beta(x)=\frac{\mu^2}{2(1-\beta\mu)}.$$

Denote by $\psi_\beta^*(x)$ the Legendre transformation of $\psi_\beta(\mu)$:

$$\psi_\beta^*(x)=\sup_{0\le\mu\le 1/\beta}\left(x\mu-\psi_\beta(\mu)\right)=\frac{1}{2\beta^2}\left(\sqrt{1+2\beta x}-1\right)^2,\quad x\ge 0.$$

In what follows we will use the notation

$$[x]_1=max\{1,x\},\ x\ge 0.$$

The following assertion is the main result of this paper. It is a direct generalization of Bernstein's inequality (2.9) for a sum of independent samples of a random field.

Theorem 1. Let $\xi(t)$ be a centered random field on a parametric set T such that:
(i) $\xi(t)$ is separable on the metric space (T, ρ_1);
(ii) there exist positive constants σ and b such that for each $t \in T$ the random variable $\xi = \xi(t)$ satisfies Bernstein's condition (2.5);
(iii) the metric space (T, ρ_1) is precompact and $\int_0^1 H_1(\epsilon)\, d\epsilon < \infty$;
Then

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp \left\{ -n\, \psi_\beta^* \left(\frac{x(1-p)}{\sigma\sqrt{n}} \right) + \frac{1}{\sigma p} \int_0^{\sigma p} H_1(\epsilon)\, d\epsilon \right\} \tag{2.10}$$

for each $\beta \geq [b/\sigma]_1$ and $x \geq 0$.

Remark 2. It is easy to verify (provided $b \geq \sigma$, $\beta = b/\sigma$), that the inequality (2.9) can be derived by (2.10) if we take into account that a random variable can be considered as a random field given in a specific one element parametric set. Therefore $H_1(\epsilon) = 0$ for each $\epsilon > 0$ and letting in (2.10) $p \to 0$ one obtains the inequality (2.9).

The following assertion is an immediate consequence of Theorem 1.

Corollary 1. Let $\xi(t)$ be a random field satisfying all the conditions of Theorem 1. Assume that there exist positive constants C_1 and κ such that

$$H_1(\epsilon) \leq C_1 + \kappa\, |\ln \epsilon| \tag{2.11}$$

for each $\epsilon > 0$. Then for each $\beta \geq [b/\sigma]_1$ and each $x \geq 0$

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp \left\{ -n\, \psi_\beta^* \left(\frac{x(1-p)}{\sigma\sqrt{n}} \right) + C_1 + \kappa\, [|\ln(\sigma p)| + 1] \right\}. \tag{2.12}$$

Remark 3. Bogdosarov and Ostrovsky (see [14]) have suggested the following inequality:

$$p_n(x) \leq 2 \inf_{p \in (0,1)} \exp \left\{ -n\, \phi^* \left(\frac{x(1-p)}{\sigma\sqrt{n}} \right) + \frac{C_2 + \kappa\, |\ln(\sigma p)|}{1-p} \right\}, \quad x \geq 0, \tag{2.13}$$

where $\phi^*(x)$ is the Legendre transformation of the function

$$\phi(\lambda) \equiv \sup_{t \in T} \max_{z = \pm 1}\, \ln E \exp\{z \lambda \xi(t)\},$$

and it is assumed that Kolmogorov's ϵ entropy $H(\epsilon)$ of the metric space (T, d) with pseudometric

$$d(t,s) \equiv \sup_{\lambda > 0} \frac{1}{\lambda} \phi^{(-1)} \left(\sup_{t \in T} \max_{z = \pm 1}\, \ln E \exp\{z \lambda (\xi(t) - \xi(s))\} \right)$$

satisfies the condition $H(\epsilon) \leq C_1 + \kappa |\ln \epsilon|$, $\epsilon > 0$. Here $\phi^{(-1)}$ is the inverse function to ϕ.

The main advantage of our estimation (2.10) compared to (2.13) is that the function $\psi_\beta^*(x)$ on the right hand side of (2.10) is given explicitly while in general case, the determination of the function $\phi^*(x)$ on the right hand side of (2.13) is an independent and difficult task.

2.3 Proof of Theorem 1

To prove the theorem 1 we need in the following assertion:
Lemma 1. Let ξ be a random variable such that $E\exp\{\gamma|\xi|\} < \infty$ and $E|\xi|^2 \le \sigma^2$ for some positive constants γ and σ, then
(i) $\|\xi\|_{(1)} \le \frac{1}{\gamma}[2(E\exp\{\gamma|\xi|\}-1)]_1^{1/2}$; and
(ii) Bernstein's condition (2.5) is valid with

$$b = \frac{1}{\gamma}\left[\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^2\sigma^2}\right]_1 .$$

Proof. Since

$$\frac{\gamma^k E|\xi|^k}{k!} \le E\exp\{\gamma|\xi|\} - 1, \quad k = 2, 3, \dots$$

Hence

$$\|\xi\|_{(1)} = \sup_{k\ge 2}\left(\frac{2E|\xi|^k}{k!}\right)^{1/k} \le \sup_{k\ge 2}\left(\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^k}\right)^{1/k} = \frac{1}{\gamma}[2(E\exp\{\gamma|\xi|\}-1)]_1^{1/2},$$

which completes the proof of (i).

To prove (ii) we note that

$$b = \frac{1}{\gamma}\sup_{i\ge 3}\left\{\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^2\sigma^2}\right\}^{\frac{1}{i-2}}$$

and therefore

$$\sigma^2 b^{k-2} \ge \sigma^2\frac{1}{\gamma^{k-2}}\frac{2(E\exp\{\gamma|\xi|\}-1)}{\gamma^2\sigma^2} \ge \frac{2E|\xi|^k}{k!}, \quad k = 2, 3, \dots$$

which completes the proof of Lemma 1.

Proof of Theorem 1. To prove the theorem we need a result due to Ostrovsky (see [10]). Let $\psi : [0,\Lambda) \to R_+ = [0,\infty)$ be a convex and continuous function ($\Lambda \le \infty$), such that

$$0 < \lim_{\mu\to 0}\frac{\psi(\mu)}{\mu^2} < \infty, \quad \lim_{\mu\to\Lambda-0}\frac{\psi(\mu)}{\mu} = \infty.$$

Let $\psi^*(x) \equiv \sup_{\mu\ge 0}(\mu x - \psi(\mu))$ be the Legendre transformation of ψ. Let (T,ρ) be a pre-compact pseudometric space, $H(\epsilon)$ is Kolmogorov's ϵ entropy of (T,ρ). Assume that $\int_0^1 H(\epsilon)\,d\epsilon < \infty$.

Theorem 2. (see [10]). Let $\eta(t)$, $t \in T$ be a centered and separable on (T,ρ) random field such that

$$\ln E\exp\{\lambda\eta(t)\} \le \psi(\sigma|\lambda|), \quad \lambda \in R^1,\ t \in T;$$

$$\ln E\exp\{\lambda(\eta(t)-\eta(s))\} \le \psi(\sigma|\lambda|\rho(t,s)), \quad \lambda \in R^1,\ t,s \in T$$

for some $\sigma > 0$. Then

$$P\{\sup_{t\in T}|\eta(t)| \geq \sigma x\} \leq 2\exp\left\{-\psi^*(x(1-p)) + \sum_{k=1}^{\infty}(1-p)p^{k-1}H(p^k)\right\} \tag{2.14}$$

for each $x \geq 0$ and $p \in (0,1)$.

Now let us continue the proof of Theorem 1. From independency of $\xi_1(t), \xi_2(t), \ldots, \xi_n(t)$ it follows that (cf. (2.7)):

$$E\exp\{\lambda S_n(t)\} \leq \exp\left\{\frac{\sigma^2\lambda^2}{2(1-b|\lambda|/\sqrt{n})}\right\}, \quad |\lambda| \leq \frac{\sqrt{n}}{b}.$$

Taking into account $\beta \geq b/\sigma$ we have

$$E\exp\{\lambda S_n(t)\} \leq \exp\left\{\frac{\sigma^2\lambda^2}{2(1-\beta\frac{|\lambda|\sigma}{\sqrt{n}})}\right\} = \exp\{\psi_n(\sigma|\lambda|)\}, \quad \lambda \in R^1, \tag{2.15}$$

where $\psi_n(\mu) \equiv n\psi_\beta(\mu/\sqrt{n})$. Here and below we assume that $\psi_\beta(\mu) = \infty$ if $\mu \geq 1/\beta$. By definition of the norm $\|\cdot\|_{(1)}$ and pseudometric $\rho_1(t,s)$ we have

$$E|\xi(t)-\xi(s)|^k \leq \frac{k!\rho_1^k(t,s)}{2}, \quad k = 2, 3, \ldots.$$

Therefore

$$E\exp\{\lambda(S_n(t)-S_n(s))\} = \left(E\exp\left\{\frac{\lambda}{\sqrt{n}}(\xi(t)-\xi(s))\right\}\right)^n \leq \left(1+\sum_{k=2}^{\infty}\frac{|\lambda|^k\rho_1^k(t,s)}{2n^{k/2}}\right)^n$$

$$= (1+\psi_1(\mu))^n \leq \exp\{n\psi_1(\mu)\}, \ \text{ where } \mu = \frac{|\lambda|\rho_1(t,s)}{\sqrt{n}},$$

and taking into account that $\beta \geq 1$ (which implies $\psi_1(\mu) \leq \psi_\beta(\mu)$ for each $\mu \geq 0$) we have

$$E\exp\{\lambda(S_n(t)-S_n(s))\} \leq \exp\left\{n\psi_\beta(\frac{|\lambda|\rho_1(t,s)}{\sqrt{n}})\right\} = \exp\{\psi_n(|\lambda|\rho_1(t,s))\}, \quad \lambda \in R^1 \tag{2.16}$$

Thus, if we put $\eta(t) \equiv S_n(t)$, $\psi(\lambda) \equiv \psi_n(\lambda)$ and $\rho(t,s) \equiv \rho_1(t,s)/\sigma$, then it follows from (2.15)- (2.16) that all the assumptions of Theorem 2 are fulfilled. Therefore it follows from (2.14) that

$$p_n(x) \leq 2\cdot\inf_{p\in(0,1)}\exp\left\{-n\,\psi_\beta^*\left(\frac{x(1-p)}{\sigma\sqrt{n}}\right) + \sum_{k=1}^{\infty}(1-p)p^{k-1}H(p^k)\right\}.$$

From the fact that the Kolmogorov ϵ entropy is a monotonically decreasing function of ϵ it follows that

$$\sum_{k=1}^{\infty}(1-p)p^{k-1}H(p^k) \leq \frac{1}{p}\int_0^p H(\epsilon)\,d\epsilon \leq \frac{1}{\sigma p}\int_0^{\sigma p} H_1(\epsilon)\,d\epsilon$$

which completes the proof.

3. Random fields with parametric set $T \subset R^k$

Let us consider the case when $T \subset R^k$ is a bounded (therefore T is precompact) subset of the k dimensional Euclidean space R^k. Denote by $\|\cdot\|$ the norm $\|t\| = \max_{i=\overline{1,k}} |t_i|$. Let $F(t), t \in T$ be a random field, which is assumed to be separable on the metric space (T, ρ), where $\rho(t,s) = \|t-s\|$. Then the following assertion holds

Theorem 3. Let $F_1(t)$, $F_2(t), \ldots, F_n(t)$ be independent samples of the random field $F(t)$. Assume that
(i) there exists a positive constant γ, such that

$$A \equiv \sup_{t\in T} E\exp\{\gamma|F(t)|\} < \infty$$

(ii) there exist constants $\gamma_0 > 0$, $\alpha \in (0,1]$ and a positive random variable η_0 satisfying the condition $A_0 \equiv E\exp\{\gamma_0\eta_0\} < \infty$ such that

$$P\left\{|F(t) - F(s)| \le \eta_0 \|t-s\|^\alpha\right\} = 1 \text{ for each } t, s \in T;$$

(iii) there exists positive σ such that

$$\sup_{t\in T} E(F(t) - EF(t))^2 \le \sigma^2;$$

then for each $x \ge 0$

$$P\left\{\sup_{t\in T} \left|\frac{1}{\sqrt{n}}\sum_{i=1}^n (F_i(t) - EF(t))\right| \ge x\right\} \le \tag{3.17}$$
$$2 \cdot \inf_{p\in(0,1)} \exp\left\{-\frac{n}{2\beta_0^2}\left(\sqrt{1+\frac{2\beta_0 x(1-p)}{\sigma\sqrt{n}}} - 1\right)^2 + k\ln(1 + D\,C_3^{1/\alpha}) + \frac{k}{\alpha}(1 + |\ln(\sigma p)|)\right\},$$

where

$$\beta_0 = [b/\sigma]_1,\ b = \frac{1}{\gamma}\left[\frac{2(A^2-1)}{\gamma^2\sigma^2}\right]_1,\quad D = \sup_{t,s\in T}\|t-s\|,\ C_3 = \frac{\ln A_0}{\gamma_0} + \frac{1}{\gamma_0}[2(A_0-1)]_1^{1/2}.$$

Proof. Define the centered random field $\xi(t) = F(t) - EF(t)$, $t \in T$. Taking into account $|\xi(t)| \le |F(t)| + |EF(t)|$ we have

$$E\exp\{\gamma|\xi(t)|\} \le E\exp\{\gamma(|F(t)| + |EF(t)|)\} \le A^2.$$

From this inequality and by Lemma 1 it follwos that the random field $\xi(t)$ satisfies Bernstein's condition (2.5) with $b = \frac{1}{\gamma}\left[\frac{2(A^2-1)}{\gamma^2\sigma^2}\right]_1$.
Now let us estimate the ϵ entropy $H_1(\epsilon)$ of the metric space (T, ρ_1) where $\rho_1(t,s) = \|\xi(t) - \xi(s)\|_{(1)}$. From

$$|\xi(t) - \xi(s)| \le |F(t) - F(s)| + |EF(t) - EF(s)|$$

and by the assumptions of the Theorem it follows that

$$\|\xi(t)-\xi(s)\|_{(1)} \le (\|\eta_0\|_{(1)} + E\eta_0)\|t-s\|^\alpha.$$

Using Lemma 1 and Jensen's inequality we have

$$\|\eta_0\|_{(1)} \le \frac{1}{\gamma_0}[2(A_0-1)]_1^{1/2} \text{ and } \exp\{\gamma_0 E\eta_0\} \le E\exp\{\gamma_0\eta_0\} = A_0,$$

respectively. These inequalities show that

$$\rho_1(t,s) = \|\xi(t)-\xi(s)\|_{(1)} \le C_3\|t-s\|^\alpha.$$

Hence for each $\epsilon > 0$

$$H_1(\epsilon) \le H(\delta), \quad \text{where } \delta = (\epsilon/C_3)^{1/\alpha}.$$

Therefore taking into account $H(\delta) \le k\ln(1+D/\delta)$ we have

$$H_1(\epsilon) \le k\ln\left(1 + D\frac{C_3^{1/\alpha}}{\epsilon^{1/\alpha}}\right) \le k\left(\ln(1+D\,C_3^{1/\alpha}) + \frac{1}{\alpha}|\ln\epsilon|\right).$$

Here we applied the following simple inequality $\ln(1+a/x) \le \ln(1+a) + |\ln x|$ for each positive a and x.
Hence the inequality (3.17) follows from that of Corollary 1 if we put $C_1 = k\ln(1+D\,C_3^{1/\alpha})$ and $\kappa = k/\alpha$. This completes the proof.

4. Asymptotic behaviour of the estimation (2.12) for moderately large deviations

In this section we study asymptotic behaviour of the right- and the left-hand sides of the inequality (2.12) in the domain of moderately large deviations. For moderately large deviations (e.g., see [11], p. 123)

$$x_n \to \infty, \text{ and } \frac{x_n}{\sqrt{n}} \to 0 \text{ as } n \to \infty. \tag{4.18}$$

Remark 4. Let us explain the importance of moderately large deviations. We consider the equality

$$P\left\{\sup_{t\in T}|\frac{1}{n}\sum_{i=1}^n (F_i(t) - EF(t))| \ge \epsilon_n\right\} = p_n(x_n), \quad x_n = \epsilon_n\sqrt{n}.$$

From one side, it makes a sense to consider such ϵ_n that satisfies the condition $\epsilon_n \to 0$ as $n \to \infty$ (since ϵ is the measure of the error in the dependent sampling method). Therefore $\epsilon_n = x_n/\sqrt{n} \to 0$ as $n \to \infty$. From the other side it is meaningful to consider such ϵ_n which ensures the convergence of the probability $p_n(x_n)$ to zero (since this probability characterizes the confidence of the estimation based on the dependent sampling method). Therefore it should be assumed that $x_n \to \infty$ as $n \to \infty$.

In the domain of moderately large deviations, the asymptotic behaviour of probabilities of deviations is quite similar to that of Gaussian distributions. In this section we will use the following known result of the theory of large deviations (e.g., see [15]).

Theorem 4. Let ξ_1, $\xi_2, \ldots$ be a sequence of independent identically distributed centered random variables satisfying Bernstein's condition (2.5) and $x_n, n = 1, 2, \ldots$ a sequence satisfying the condition (4.18). Assume that $E\xi_i^2 = \sigma^2$, then

$$P\left\{ \mid \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i \mid \geq x_n \right\} = \exp\left\{ -\frac{x_n^2}{2\sigma^2}(1+\delta_n) \right\},$$

where $\delta_n, n = 1, 2, \ldots$ is a sequence satisfying the condition $\delta_n \to 0$ as $n \to \infty$.

Now let us consider the estimation (2.12). For the brevity of notations let us rewrite the inequality (2.12) in the form (2.3).
Let $\xi(t), t \in T$ be a random field satisfying all the conditions of Corollary 1, and the following condition

$$\exists t_0 \in T \text{ such that } \sigma^2 = E\xi^2(t_0). \tag{4.19}$$

Then it follows from Theorem 4 that

$$p_n(x_n) \geq P\left\{ \mid \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i(t_0) \mid \geq x_n \right\} = \exp\left\{ -\frac{x_n^2}{2\sigma^2}(1+\delta_n) \right\}. \tag{4.20}$$

From the definition of $\psi_\beta^*(x)$ it follows that

$$n\,\psi_\beta^* \left(\frac{x_n(1-p)}{\sigma\sqrt{n}} \right) = \frac{x_n^2}{2\sigma^2}(1+\delta_n')(1-p)^2, \tag{4.21}$$

where $\delta_n', n = 1, 2, \ldots$ is a sequence satisfying the condition $\delta_n' \to 0$ as $n \to \infty$.
Due to $x_n \to \infty$ as $n \to \infty$ we can choose a sequence $\{p_n\} \subset (0, 1)$ such that $p_n \to 0$ and $|\ln p_n|/x_n^2 \to 0$ as $n \to \infty$. Therefore, taking into account (4.21) we have

$$\phi_n(x_n) \geq \frac{x_n^2}{2\sigma^2}(1+\delta_n''),$$

where $\delta_n'', n = 1, 2, \ldots$ is a sequence satisfying the condition $\delta_n'' \to 0$ as $n \to \infty$. From this inequality and (4.20) we come to the following conclusion

$$\exp\left\{ -\frac{x_n^2}{2\sigma^2}(1+\delta_n) \right\} \leq p_n(x_n) \leq \exp\{-\phi_n(x_n)\} \leq \exp\left\{ -\frac{x_n^2}{2\sigma^2}(1+\delta_n'') \right\}.$$

Thus we establish that in the range of moderately large deviations under the conditions of Corollary 1 and the condition (4.19) the asymptotic behaviour of the right-hand side of (2.12) is optimal in the sense that

$$\lim_{n\to\infty} \frac{\phi_n(x_n)}{|\ln\, p_n(x_n)|} = 1. \tag{4.22}$$

5. Conclusions

An exponential estimations of the type (2.3) for the probability (2.2) of deviations a sum of independent random fields is established. Application of this result to the problem of the error estimations for the dependent sampling Monte Carlo method is discussed. Proposed estimations have the following features:

- the parametric set of a random field might be arbitrary;
- all the estimations are non asymptotic, i.e. they are valid for an arbitrary number of summands n;
- for moderately large deviations the estimations have optimal log-asymptotics in the sense that (4.22) holds.

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