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Functional central limit theorem for the occupation time of the origin for branching random walks in $d \geq 3$

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Abstract

We show that the centred occupation time process of the origin of a system of critical binary branching random walks in dimension $d \geq 3$, started off either from a Poisson field or in equilibrium, when suitably normalised, converges to a Brownian motion in $d \geq 4$. In $d = 3$, the limit process is fractional Brownian motion with Hurst parameter $3/4$ when starting in equilibrium, and a related Gaussian process when starting from a Poisson field.

Keywords: Branching random walk, occupation time, functional central limit theorem

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1 Introduction and main result

We study the fluctuation behaviour of the occupation time in a single point of a system of critical binary branching random walks (BRW). BRW consists of particles which move independently on $\mathbb{Z}^d$ in continuous time according to a given random walk kernel a. Additionally, each particle has an exponentially distributed life time with parameter $\rho > 0$. At the end of its life, a particle leaves either two or zero offspring at its current location, each possibility occuring with probability $1/2$. The behaviour of different particles alive at the same time is independent. We denote by $\xi_t(x)$ the number of particles present at location x at time t. We assume that the transition rate matrix $a(x, y) = a(0, y - x)$ governing the individual motion of particles is symmetric, irreducible and has finite second moments, which implies

$$(Q_{ij})_{i,j=1,\dots,d} = \left(\sum_x a(0, x) x_i x_j\right)_{i,j} \text{ is finite and invertible.} \tag{1.1}$$

We have $\sum_x a(0, x)x = 0$ by symmetry, and we can assume without loss of generality that a is stochastic, i.e. $\sum_x a(0, x) = 1$.

It is well known that BRW in $d \leq 2$, starting from any initial condition with bounded local density, suffers local extinction, i.e. $\xi_t(x) \to 0$ in probability as $t \to \infty$ for any $x \in \mathbb{Z}^d$. On the other hand, in $d \geq 3$, there exists a one-parameter family of extremal invariant probability measures Λ_ϑ, $\vartheta \geq 0$, parametrised by the expected density: $\int \xi(x)\, \Lambda_\vartheta(d\xi) = \vartheta$. Each Λ_ϑ is shift-invariant, and $\{\xi_t(x) : x \in \mathbb{Z}^d, t \geq 0\}$ under Λ_ϑ is ergodic with respect to space- and time-shifts.

Let us denote the distribution of a Poisson field on $\mathbb{Z}^d$ with homogeneous intensity ϑ by $\mathcal{H}(\vartheta)$, i.e. under $\mathcal{H}(\vartheta)$, the rv's $\xi(x)$, $x \in \mathbb{Z}^d$, are i.i.d. Poisson(ϑ). If $\mathscr{L}(\xi_0) = \mathcal{H}(\vartheta)$, we have $\mathscr{L}(\xi_t) \to \Lambda_\vartheta$ weakly as $t \to \infty$.

Let $\mathscr{L}(\xi_0) \in \{\mathcal{H}(\vartheta), \Lambda_\vartheta\}$. By ergodicity the occupation time of any point $x \in \mathbb{Z}^d$ satisfies

$$\frac{1}{T}\int_0^T \xi_t(x)\,dt \to \vartheta \quad \text{almost surely as } T \to \infty.$$

Thus, a natural question concerns the random fluctuations of the occupation time around its asymptotic limit. This is the content of our main result:

Theorem 1 *1.) If $(\xi_s)_{s\geq 0}$ is started in the (unique extremal) equilibrium distribution Λ_ϑ with intensity $\vartheta > 0$, then the processes*

$$X_t^N := \frac{1}{h_d(N)}\int_0^{Nt} (\xi_s(0) - \vartheta)\,ds, \quad t \geq 0$$

converge towards a Brownian motion in $d \geq 4$ and to a fractional Brownian motion with Hurst parameter $3/4$ in $d = 3$ as $N \to \infty$, where the norming is given by

$$h_d(t) = \begin{cases} t^{3/4}, & d = 3 \\ \sqrt{t\log t}, & d = 4 \\ \sqrt{t}, & d \geq 5. \end{cases}$$

The covariance of the limiting process X is given by

$$\mathrm{Cov}(X_s, X_t) = \begin{cases} \frac{\sqrt{2}}{3\pi^{3/2}}(\det Q)^{-1/2}\vartheta\rho\left[t^{3/2} + s^{3/2} - |t-s|^{3/2}\right], & d = 3 \\ (2\pi)^{-2}(\det Q)^{-1/2}\vartheta\rho \times (s \wedge t), & d = 4 \\ \left[2\int_0^\infty du\, a_u(0,0) + \rho\int_0^\infty du\, u a_u(0,0)\right]\vartheta \times (s \wedge t), & d \geq 5. \end{cases}$$

2.) The same conclusions hold if $\mathscr{L}(\xi_0) = \mathcal{H}(\vartheta)$, and $d \geq 4$. In the case $\mathscr{L}(\xi_0) = \mathcal{H}(\vartheta)$ and $d = 3$, the processes X^N converge towards a Gaussian process X with covariance given by

$$\mathrm{Cov}(X_s, X_t) = \frac{2\sqrt{2}}{3\pi^{3/2}}(\det Q)^{-1/2}\vartheta\rho\left[t^{3/2} + s^{3/2} - \frac{1}{2}|t-s|^{3/2} - \frac{1}{2}(t+s)^{3/2}\right]. \tag{1.2}$$

The normalisations h_d are dictated by the requirement of a non-trivial covariance function for the limit process, and this in turn is determined by the decay properties of the transition probabilities of the underlying random walk a, see the calculations in Section 4.2. Note that with $\rho = 0$, BRW becomes a system of independent random walks, and has the family $\mathcal{H}(\vartheta)$, $\vartheta \geq 0$ of shift-invariant extremal equilibria. In the situation $\rho = 0$, we see from Theorem 1 that the limit process X is trivial in $d \leq 4$ and a Brownian motion in $d \geq 5$. This is in keeping with the 'metatheorem' that the introduction of branching shifts 'critical dimensions' by 2: In a system of independent random walks, the occupation time requires normalisation by $t^{3/4}$ in $d = 1$, $\sqrt{t\log t}$ in $d = 2$ and $\sqrt{t}$ in $d \geq 3$ in order to obtain a non-trivial limit (see [CG84]).

While for non-branching random walks, the non-classical norming is due to recurrence properties of the individual particles, the behaviour in our case is governed by the recurrence properties of *families*: The equilibrium of a BRW can be decomposed into a Poisson system of ‘clans’ of particles with a common ancestor (see e.g. [Zäh02]), and such a clan will visit the origin infinitely often if and only if $d \leq 4$. It is remarkable that the correlations introduced by the branching are strong enough that in $d = 3$, the limit process itself depends on the initial condition, not only on its density. Even though ξ_t, starting from $\mathcal{H}(\vartheta)$, converges in distribution to Λ_ϑ, the ‘building up’ of equilibrium is reflected in the different covariance structure of the renormalised occupation time process.

Note that the centred Gaussian process (X_t) with covariance given by (1.2) can be represented as $X_t = (B_t^{(3/4)} + B_{t-}^{(3/4)})/\sqrt{2}$, where $(B_t^{(3/4)})_{t\in\mathbb{R}}$ is a fractional Brownian motion with Hurst parameter 3/4 and $B_0^{(3/4)} = 0$ (see [BGTar]). It remains an intruiging question to explain this representation from the point of view of branching particle systems.

Corresponding functional central limit theorems for the occupation time of *reversible* interacting particle systems are well known, see e.g. [Kip87], [QJS02], or more generally [KV86] for central limit theorems for additive functionals of reversible Markov processes. In the non-reversible situation of a branching system, non-functional versions of central limit theorems have been obtained in [DGW01]. One might argue that we have traded reversibility for infinite divisibility, which opens the possibility of rather explicit calculations. This is indeed true to some extent: we obtain the Gaussianity of the limit process using a relatively general martingale decomposition inspired by [QJS02], but we have to resort to fourth moment calculations in order to prove tightness. This is feasible, although cumbersome, because of the independence of families founded by different particles. While in principle moment formulas for BRW are well known, we found ourselves compelled to develop a formalism to represent arbitrary space-time moments of BRW in terms of integrals over tree-indexed random walks (in the spirit of [Dyn88], who elaborated an analogous scheme for super-Brownian motion).

A program similar to ours has been carried out by Bojdecki, Gorostiza and Talarczyk in [BGT04a] and [BGT04b] in a somewhat different scenario with completely different techniques: They consider critical binary branching particles in $\mathbb{R}^d$, where the individual particle moves according to a symmetric α-stable process, with $\alpha \in (0, 2]$, and obtain the following results: for $\alpha < d < 2\alpha$, starting from a homogeneous Poisson process, the occupation time requires a non-classical norming and converges to sub-fractional Brownian motion, whereas the limit process is Brownian for $d \geq 2\alpha$, (with a logarithmic correction to the norming in the boundary case $d = 2\alpha$).

Our set-up is different in the following respect: we consider the lattice instead of continuous space, and we focus on the occupation time of a single point, whereas Bojdecki, Gorostiza and Talarczyk consider $\mathcal{S}'(\mathbb{R}^d)$-valued processes. As to the techniques: Bojdecki et al rely on computations of Laplace functionals and Fourier analysis, while in our case the discreteness of space allows to use martingale decompositions of the occupation time, and to employ techniques from the field of interacting particle systems (similar to [Kip87], [QJS02]) . Our scenario, namely individual motion with a finite second moment, combined with critical binary branching, corresponds to the case $\alpha = 2$. This invites to conjecture that if we used an individual motion which is in the domain of attraction of an α-stable law (with general $\alpha \in (0, 2]$), we would find the same α-dependence of regimes as Bojdecki et al. On the other hand, our Theorem 1, part 1.) suggests that in the scenario of [BGT04a], starting off from an extremal equilibrium for the branching system instead of a Poisson process, the limit process should be a fractional Brownian motion.

The rest of this paper is organised as follows: We collect some well-known facts about random walks and branching random walks in Section 2. Convergence and asymptotic Gaussianity of finite dimensional distributions is proved in Section 3: in the case $d \geq 4$ we decompose the occupation time into a martingale plus an asymptotically negligible remainder term (Subsection 3.1), in the case $d = 3$ we 'distill' a white noise out of the fluctuations of the particle system and represent the occupation time as an integral with respect to this noise (Subsection 3.2). In order to prove tightness, we use moment estimates; in Section 4 we develop representations of moments of branching random walks in terms of integrals over tree-indexed random walks. These are used in Section 5 to complete the proof of Theorem 1. Finally, we collect some auxiliary calculations as well as a list of all the relevant tree types appearing in our computations in an appendix.

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2 Preliminaries

2.1 Formulas related to random walks

The underlying motion process has generator

$$Lf(x) = \sum_{y \in \mathbb{Z}^d} a(x,y)(f(y) - f(x)).$$

Denote the transition probabilities by $a_t(x, y)$. They solve the backward equation

$$\frac{\partial}{\partial t} a_t(x,y) = \big(La_t(\cdot, y)\big)(x).$$

We denote the transition semigroup by $S_t f(x) := \sum_y a_t(x,y) f(y)$. Let

$$g(x,y) := \int_0^\infty a_t(x,y)\, dt$$

be the Green's function and

$$g_\lambda := \int_0^\infty e^{-\lambda t} a_t(x,y)\, dt$$

the resolvent. We denote the Green operator by

$$\mathcal{G} f(x) := \sum_y g(x,y) f(y).$$

The function $x \mapsto g(x, 0)$ is a solution of $-L\phi = \delta_0$ and $x \mapsto g_\lambda(x, 0)$ a solution of $\lambda\phi - L\phi = \delta_0$. Define

$$u_t(x,y) := \int_0^t a_s(x,y)\, ds,$$

the Green's function of a random walk killed at time t. The function $(t, x) \mapsto u_t(x, 0)$ solves

$$(\partial_t - L)\phi = \delta_0, \quad \phi_0(x) \equiv 0.$$

The Dirichlet form of the underlying random walk is

$$\sum_{x,y \in \mathbb{Z}^d} a(x,y)\big(\phi(y) - \phi(x)\big)^2 = 2\langle \phi, (-L)\phi \rangle \quad \text{for } \phi \in \ell_2(\mathbb{Z}^d).$$

Note that our assumptions on a imply the following form of the local CLT, cf. Proposition A.2.

$$a_t(0,0) = (2\pi t)^{-d/2}\det(Q)^{-1/2} + o(t^{-d/2}) \quad \text{as } t \to \infty. \tag{2.3}$$

Furthermore recall that $||g(\cdot,0)||_2^2 = \int_0^\infty ds \int_0^\infty dt\, a_{s+t}(0,0)$, so that $g(\cdot,0) \in \ell_2(\mathbb{Z}^d)$ in case $d \geq 5$, whereas $||g_{1/N}(\cdot,0)||_2^2 \sim C \log N$ in case $d = 4$.

2.2 Basic results on branching random walk

A convenient choice of the state space for branching random walk (as well as many other 'spatially homogeneous' particle systems), going back to Liggett & Spitzer ([LS81]), is

$$\mathfrak{X} = \Big\{\mu \text{ an integer-valued measure on } \mathbb{Z}^d : \sum\nolimits_{x\in\mathbb{Z}^d} \gamma(x)\mu(x) < \infty\Big\},$$

where γ is a strictly positive function on $\mathbb{Z}^d$ satisfying $\sum_{y\in\mathbb{Z}^d} a(x,y)\gamma(y) \leq M\gamma(x)$ for some constant $M > 0$. Note that the dependence of $\mathfrak{X}$ on the particular choice of γ is irrelevant for our purposes, as any random $(\xi_x)_{x\in\mathbb{Z}^d}$ satisfying $\sup_x \mathbb{E}\xi_x < \infty$ automatically has $\mathbb{P}(\xi \in \mathfrak{X}) = 1$ irrespective of γ. A formal construction of the BRW $(\xi_t)_{t\geq 0}$ as an $\mathfrak{X}$-valued Markov process can be found e.g. in Section 1 of [Gre91]. The generator is given by

$$\mathscr{L}F(\xi) = \sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} \xi(x)a(x,y)\big(F(\xi^{x,y}) - F(\xi)\big) + \sum_{x\in\mathbb{Z}^d} \frac{\rho}{2}\xi(x)\big(F(\xi^{x,+}) + F(\xi^{x,-}) - 2F(\xi)\big) \tag{2.4}$$

with $\xi^{x,y} = \xi - \delta_x + \delta_y$, $\xi^{x,+} = \xi + \delta_x$ and $\xi^{x,+} = \xi - \delta_x$.

It is well known that a branching random walk ξ with initial condition $\xi_0 \in \mathfrak{X}$ can be constructed as the unique solution to

$$\begin{aligned}\xi_t(x) \quad &= \quad \xi_0(x) + \sum_{y\neq x}\Big[\int_0^t \mathbf{1}(\xi_{s-}(y) \geq n)\bar{N}^{y,x}(ds\,dn) - \int_0^t \mathbf{1}(\xi_{s-}(x) \geq n)\bar{N}^{x,y}(ds\,dn)\Big] \\ &\quad + \int_0^t \mathbf{1}(\xi_{s-}(x) \geq n)\bar{N}^{x,+}(ds\,dn) - \int_0^t \mathbf{1}(\xi_{s-}(x) \geq n)\bar{N}^{x,-}(ds\,dn)\end{aligned}$$

for all $x \in \mathbb{Z}^d$, $t \geq 0$. Here, $\bar{N}^{x,y}$, $x \neq y$, $\bar{N}^{x,+}, \bar{N}^{x,-}$, $x \in \mathbb{Z}^d$, are independent Poisson processes on $[0,\infty) \times \mathbb{N}$, independent of ξ_0, $\bar{N}^{x,y}$ has intensity measure $a(x,y)dt \otimes d\ell$, $\bar{N}^{x,+}$, $\bar{N}^{x,-}$ have intensity measure $(\rho/2)dt \otimes d\ell$ (dt is Lebesgue measure, ℓ is counting measure). For fixed ξ_0, (ξ_t) is adapted to the filtration generated by these Poisson processes. See e.g. [Bir03], Lemma 2.1 and Remark 2.3. Define

$$N_t^{x,y} := \int_0^t \mathbf{1}(\xi_{s-}(x) \geq n)\bar{N}^{x,y}(ds\,dn),\ N_t^{x,\pm} := \int_0^t \mathbf{1}(\xi_{s-}(x) \geq n)\bar{N}^{x,\pm}(ds\,dn) \tag{2.5}$$

(with the obvious interpretations: $N^{x,+}$ counts the number of births at x, $N^{x,-}$ counts the number of deaths at x, $N^{x,y}$ counts how many times a particle jumps from x to y). Thus we can rewrite

$$\xi_t(x) = \xi_0(x) + N_t^{x,+} - N_t^{x,-} + \sum_{y\neq x}\big(N_t^{y,x} - N_t^{x,y}\big), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{2.6}$$

Lemma 2.1 *Assume that* $\sup_x \mathbb{E}[\xi_0(x)^2] < \infty$. *The compensated processes*

$$\tilde{N}_t^{x,y} := N_t^{x,y} - a(x,y)\int_0^t \xi_s(x)\,ds,\ \tilde{N}_t^{x,\pm} := N_t^{x,\pm} - \frac{\rho}{2}\int_0^t \xi_s(x)\,ds \tag{2.7}$$

are pairwise orthogonal, square integrable martingales with compensators given by

$$\langle \tilde{N}^{x,y}\rangle_t = a(x,y)\int_0^t \xi_s(x)\,ds,\ \langle \tilde{N}^{x,+}\rangle_t = \langle \tilde{N}^{x,-}\rangle_t = \frac{\rho}{2}\int_0^t \xi_s(x)\,ds. \tag{2.8}$$

Proof Immediate from the independence properties of the driving Poisson processes $\bar{N}$. □

For $f_t \in \ell_1(\mathbb{Z}^d)$ put

$$F_t(\xi) := \langle f_t, \xi - \vartheta\lambda\rangle = \sum_{x\in\mathbb{Z}^d} f_t(x)(\xi(x) - \vartheta). \tag{2.9}$$

Note that this sum is well defined if $\sup_x \mathbb{E}\,|\xi(x) - \vartheta| < \infty$.

Lemma 2.2 *Let* $f : [0,\infty)\times\mathbb{Z}^d \to \mathbb{R}$ *satisfy* $sup_{t\le T}\big(||f_t||_1 + ||\partial_t f_t||_1 + ||Lf_t||_1\big) < \infty$, *and let* F_t *be defined by (2.9). Then we have*

$$\big(\partial_t + \mathscr{L}\big)F_t(\xi) = \big\langle (\partial_t + L)f_t, \xi - \vartheta\ell\big\rangle,$$

for $t \in [0,T]$, *and the martingale* $M_t := F_t(\xi_t) - F_0(\xi(0)) - \int_0^t(\partial_s + \mathscr{L})F_s(\xi_s)\,ds$, $0 \le t \le T$, *can be represented as*

$$M_t = \sum_{x\in\mathbb{Z}^d}\int_0^t f_s(x)\big(d\tilde{N}_s^{x,+} - d\tilde{N}_s^{x,-}\big) + \sum_{x,y\in\mathbb{Z}^d}\int_0^t (f_s(y) - f_s(x))d\tilde{N}_s^{x,y}. \tag{2.10}$$

Proof Note that $\Phi_x(\xi) := \xi(x)$ satisfies $\mathscr{L}\Phi_x(\xi) = \sum_y(\xi(y) - \xi(x))a(y,x)$, hence

$$\begin{aligned}\mathscr{L}F_t(\xi) &= \sum_{x\in\mathbb{Z}^d} f_t(x)\sum_{y\in\mathbb{Z}^d}(\xi(y) - \vartheta - \xi(x) + \vartheta)a(y,x)\\ &= \sum_{y\in\mathbb{Z}^d}(\xi(y) - \vartheta)\sum_{x\in\mathbb{Z}^d} a(y,x)(f_t(x) - f_t(y)) = \langle Lf_t, \xi - \vartheta\ell\rangle.\end{aligned}$$

By the linearity of the function $F_t(\cdot)$ we can express (using integration by parts)

$$\begin{aligned}F_t(\xi_t) &= F_0(\xi_0) + \sum_{x\in\mathbb{Z}^d}\int_0^t f_s(x)\,dN_s^{x,+} - \sum_{x\in\mathbb{Z}^d}\int_0^t f_s(x)\,dN_s^{x,-}\\ &\quad + \sum_{x,y\in\mathbb{Z}^d}\int_0^t (f_s(y) - f_s(x))\,dN_s^{x,y} + \sum_{x\in\mathbb{Z}^d}\int_0^t \partial_s f_s(x)(\xi_s(x) - \vartheta)\,ds\\ &= \sum_{x\in\mathbb{Z}^d}\int_0^t f_s(x)\big(d\tilde{N}_s^{x,+} - d\tilde{N}_s^{x,-}\big) + \sum_{x\in\mathbb{Z}^d}\int_0^t f_s(x)\underbrace{(\xi_s(x)\,ds - \xi_s(x)\,ds)}_{=0}\\ &\quad + \sum_{x,y\in\mathbb{Z}^d}\int_0^t (f_s(y) - f_s(x))\,d\tilde{N}_s^{x,y} + \sum_{x,y\in\mathbb{Z}^d}\int_0^t (f_s(y) - f_s(x))\xi_s(x)a(x,y)\,ds\\ &\quad + \sum_{x\in\mathbb{Z}^d}\int_0^t \partial_s f_s(x)(\xi_s(x) - \vartheta)\,ds + F_0(\xi_0)\\ &= \sum_{x\in\mathbb{Z}^d}\int_0^t f_s(x)\big(d\tilde{N}_s^{x,+} - d\tilde{N}_s^{x,-}\big) + \sum_{x,y\in\mathbb{Z}^d}\int_0^t (f_s(y) - f_s(x))\,d\tilde{N}_s^{x,y}\\ &\quad + \int_0^t \langle(\partial_s + L)f_s, \xi_s - \vartheta\lambda\rangle\,ds + F_0(\xi_0).\end{aligned}$$

This completes the proof. □

We recall some well-known properties of critical (finite variance) branching random walk in $d \geq 3$. Let $\hat{a}(x,y) = \frac{1}{2}(a(x,y) + a(y,x))$ be the symmetrised transition kernel. (In our case $\hat{a} = a$.) For a proof of the following results, see e.g. [Gre91] and the references given there.

Theorem 2 *Assume that $\hat{a}$ is transient. Then for each $\vartheta \geq 0$ there exists exactly one extremal invariant probability measure $\Lambda_\vartheta \in \mathcal{P}(\mathcal{N}(\mathbb{Z}^d))$ with $\int \xi(0)\Lambda_\vartheta(d\xi) = \vartheta$. Each Λ_ϑ is translation invariant.*

Theorem 3 *If $\mathscr{L}(\xi_0) \in \{\mathcal{H}(\vartheta), \Lambda_\vartheta\}$,*

$$\frac{1}{t}\int_0^t f(\xi_s)\, ds \underset{t\to\infty}{\longrightarrow} \int f(\xi)\, \Lambda_\vartheta(d\xi) \quad \textit{almost surely and in } L_1$$

(for polynomially bounded, local functions f).

3 Finite dimensional distributions

Proposition 3.1 *Let $\mathscr{L}(\xi_0)$ be either Λ_ϑ or $\mathcal{H}(\vartheta)$. As $N \to \infty$, the processes X^N defined in Theorem 1 converge in finite dimensional distributions to a Gaussian process X (whose covariance structure depends on d, ϑ and the choice of the initial condition, as specified in Theorem 1).*

The rest of this section is devoted to the proof of Proposition 3.1 in the various cases.

3.1 The case $d \geq 4$

Our strategy is as follows: similarly to the technique applied in [QJS02] we are looking for a function $G(\xi)$ that satisfies $LG(\xi) = (\xi(0) - \vartheta) +$ "small error" in order to obtain a representation of the form

centered occupation time = martingale + "small error term".

We then use a general functional central limit theorem to treat the martingale term, while we use second moment estimates to show that the error term becomes small. Put

$$G_\lambda(\xi) = \sum_{x\in\mathbb{Z}^d} g_\lambda(x,0)(\xi(x) - \vartheta)$$

where g_λ is the resolvent of the underlying random walk. By Lemma 2.2 we have

$$(\lambda\,\mathrm{Id} - \mathscr{L})G_\lambda(\xi) = \xi(0) - \vartheta. \tag{3.1}$$

Again by Lemma 2.2,

$$\begin{aligned} M_t^\lambda &:= G_\lambda(\xi_t) - G_\lambda(\xi_0) - \int_0^t \mathscr{L}G_\lambda(\xi_s)\, ds & (3.2)\\ &= \sum_{x,y\in\mathbb{Z}^d} \big(g_\lambda(y,0) - g_\lambda(x,0)\big)\tilde{N}_t^{x,y} + \sum_{x\in\mathbb{Z}^d} g_\lambda(x,0)(\tilde{N}_t^{x,+} - \tilde{N}_t^{x,-}) & (3.3)\end{aligned}$$

is a martingale. Using (3.1) we obtain a representation

$$\int_0^t (\xi_s(0) - \vartheta)\, ds = -G_\lambda(\xi_t) + G_\lambda(\xi_0) + \lambda \int_0^t G_\lambda(\xi_s)\, ds + M_t^\lambda =: R_t^\lambda + M_t^\lambda. \qquad (3.4)$$

We choose $\lambda = 1/N$ and we study the terms $h_d(N)^{-1} R_{Nt}^{1/N}$ and $h_d(N)^{-1} M_{Nt}^{1/N}$ separately in two steps.

Martingale part: Using Lemma 2.1 we have

$$\langle M^{1/N} \rangle_t = \sum_{x,y\in\mathbb{Z}^d} \big(g_{1/N}(y,0) - g_{1/N}(x,0)\big)^2 \int_0^t a(x,y)\xi_s(x)\, ds + \sum_{x\in\mathbb{Z}^d} g_{1/N}(x,0)^2 \int_0^t \rho\xi_s(x)\, ds. \qquad (3.5)$$

Case 1: $(d > 4)$ The martingale $N^{-1/2} M_{Nt}^{1/N}$ has globally bounded jumps $(g_\lambda(x,0) \leq g(x,0) \leq ||g||_\infty < \infty)$, furthermore the jump size tends to 0 as $N \to \infty$. (3.5) yields for any fixed $t > 0$

$$\begin{aligned} \langle N^{-1/2} M_{N\cdot}^{1/N} \rangle_t &= \sum_{x,y\in\mathbb{Z}^d} a(x,y)\big(g_{1/N}(y,0) - g_{1/N}(x,0)\big)^2 \frac{1}{N} \int_0^{Nt} \xi_s(x)\, ds \\ &\quad + \rho \sum_{x\in\mathbb{Z}^d} g_{1/N}(x,0)^2 \frac{1}{N} \int_0^{Nt} \xi_s(x)\, ds \\ &\xrightarrow[N\to\infty]{P} \mathrm{const} \cdot t \end{aligned}$$

because each summand $(1/N)\int_0^{Nt} \xi_s(x)\, ds$ converges to $\vartheta\, t$ almost surely and in L_1 by Theorem 3. This and the shift-invariance of Λ_ϑ, resp. $\mathcal{H}(\vartheta)$, proves that the r.h.s. converges in L^1, so in particular it converges in probability (note that the Green's function g is in $\ell^2(\mathbb{Z}^d)$ for $d > 4$).

Using Proposition A.1 we conclude that $\big(N^{-1/2} M_{Nt}^{1/N}\big)_{t\geq 0}$ converges in distribution to the law of a Brownian motion.

Case 2: $(d = 4)$ Here we have to slightly modify our approach because the Green's function is no longer in $\ell^2(\mathbb{Z}^4)$. Instead we note that

$$\frac{1}{\log N} \sum_{x\in\mathbb{Z}^d} g_{1/N}(x,0)^2 \xrightarrow[N\to\infty]{} const. > 0$$

and that

$$\begin{aligned} &\frac{1}{\log N} \sum_{x,y\in\mathbb{Z}^d} a(x,y)(g_{1/N}(y,0) - g_{1/N}(x,0))^2 \\ &= \frac{2}{\log N} \big\langle g_{1/N}(\cdot,0), (-L) g_{1/N}(\cdot,0) \big\rangle = \frac{2}{\log N} \big\langle g_{1/N}(\cdot,0), \delta_0 - \frac{1}{N} g_{1/N}(\cdot,0) \big\rangle \\ &\leq \frac{2}{\log N} g_{1/N}(0,0) \leq \frac{2}{\log N} g(0,0) \xrightarrow[N\to\infty]{} 0 \end{aligned}$$

We then argue analogously to the case above that $(\frac{1}{\sqrt{N\log N}} M_{Nt}^{1/N})_{t\geq 0}$ converges in distribution to a Brownian motion.

Error part: Let us first consider Λ_ϑ as initial condition. We estimate $\mathbb{E}^{\Lambda_\vartheta}[(G_\lambda(\xi_0))^2]$ in order to treat the remainder term. By Corollary 4.5 we have

$$\begin{aligned}
\mathbb{E}^{\Lambda_\vartheta}[(G_\lambda(\xi_0))^2] &= \sum_{x,y\in\mathbb{Z}^d} g_\lambda(x,0)g_\lambda(y,0)\mathrm{Cov}^{\Lambda_\vartheta}(\xi_0(x),\xi_0(y)) \\
&= \vartheta\int_0^\infty dt\, e^{-\lambda t}\int_0^\infty ds\, e^{-\lambda s}\sum_{x\in\mathbb{Z}^d} a_t(x,0)a_s(x,0) \\
&\quad +\frac{\vartheta\rho}{2}\int_0^\infty dt\, e^{-\lambda t}\int_0^\infty ds\, e^{-\lambda s}\int_0^\infty du \sum_{x,y\in\mathbb{Z}^d} a_t(x,0)a_s(y,0)a_u(x,y) \\
&= \vartheta\int_0^\infty dt\int_0^\infty ds\, e^{-\lambda(t+s)}\left\{a_{t+s}(0,0)+\frac{\rho}{2}\int_0^\infty du\, a_{t+s+u}(0,0)\right\} \\
&= \vartheta\int_0^\infty dr\, e^{-\lambda r}r\left\{a_r(0,0)+\frac{\rho}{2}\int_r^\infty dv\, a_v(0,0)\right\}.
\end{aligned}$$

For $d>4$ we estimate using (2.3)

$$\begin{aligned}
\mathbb{E}^{\Lambda_\vartheta}[(G_\lambda(\xi_0))^2] &\le C\left(1+\int_1^\infty dr\, e^{-\lambda r}r(r^{-d/2}+r^{-d/2+1})\right) \\
&\le 2C\left(1+\int_1^\infty dr\, e^{-\lambda r}r^{-d/2+2}\right)
\end{aligned}$$

to find that

$$\mathbb{E}^{\Lambda_\vartheta}\left[\left(N^{-1/2}G_{1/N}(\xi_0)\right)^2\right]\le\frac{C}{N}+C\int_1^\infty e^{-r/N}r^{-d/2+2}\frac{dr}{N}=\frac{C}{N}+\frac{C'}{N^{d/2-2}}\underset{N\to\infty}{\longrightarrow}0.$$

The case $d=4$ can be treated analogously

$$\mathbb{E}^{\Lambda_\vartheta}\left[\left((N\log N)^{-1/2}G_\lambda(\xi_0)\right)^2\right]\le\frac{C}{N\log N}+\frac{C}{\log N}\int_1^\infty e^{-s/N}\frac{ds}{N}\underset{N\to\infty}{\longrightarrow}0.$$

Thus the second term of R_t in (3.4) converges to 0 in L^2 after norming with $h_d(N)$, so in particular it converges to 0 in probability. By the time-stationarity of (ξ_t) started from Λ_ϑ we see that also the normed first term in (3.4) converges to 0 in probability. Finally, the remaining integral term can be estimated in the following way:

$$\mathbb{E}^{\Lambda_\vartheta}\left|\frac{1}{h_d(N)}\frac{1}{N}\int_0^{Nt}G_{1/N}(\xi_s)\,ds\right|\le t\mathbb{E}^{\Lambda_\vartheta}|h_d(N)^{-1}G_{1/N}(\xi_0)|\to 0.$$

Putting things together we conclude that $(h_d(N)^{-1}R^{1/N}_{Nt})_t\to 0$ as $N\to\infty$ in the sense of finite-dimensional distributions.

Now consider Poisson initial conditions. Note that we have $0\le\mathrm{Cov}^{\mathcal{H}(\vartheta)}(\xi_t(x),\xi_t(y))\le\mathrm{Cov}^{\Lambda(\vartheta)}(\xi_0(x),\xi_0(y))$ for all $x,y\in\mathbb{Z}^d$, $t\ge 0$ (see Corollary 4.5). Thus we have

$$\sup_{t\ge 0}\mathbb{E}^{\mathcal{H}(\vartheta)}[(G_\lambda(\xi_t))^2]\le\mathbb{E}^{\Lambda_\vartheta}[(G_\lambda(\xi_0))^2],$$

and then we argue as before.

3.2 The case $d = 3$

The decomposition (3.4) of the occupation time in a martingale term and a remainder term as for the case $d > 3$ can not be used in the case $d = 3$: First, $N^{-3/4}G_{1/N}(\xi)$ does not become small in L^2, second, as the limit process cannot be a Brownian motion, the Rebolledo-type arguments we used above would not help anyway.

Our approach, again inspired by [QJS02], is to instead "distill" a white noise out of the space-time fluctuations of the ergodic branching random walk system, and to express the normalised occupation time process as a linear functional of this approximate white noise. Technically, for a (momentarily fixed) time horizon T, we decompose the occupation time in a term M_T^T and a remainder term, where M_T^T is the final value of a martingale $(M_t^T)_{t\le T}$.

Recall $u_t(x,0) = \int_0^t a_s(x,0)\,ds$, define

$$U_t^T(\xi) = \sum_{x\in\mathbb{Z}^3} u_{T-t}(x,0)\big(\xi(x)-\vartheta\big).$$

Now

$$M_t^T := U_t^T(\xi_t) - U_0^T(\xi_0) - \int_0^t (\partial_s + \mathscr{L})U_s^T(\xi_s)\,ds \tag{3.6}$$

is a martingale, and as $(\partial_t - L)u_t(\cdot,0) = \delta(\cdot,0)$, we obtain, using Lemma 2.2, the following decomposition of the occupation time:

$$\int_0^T (\xi_s(0)-\vartheta)\,ds = M_T^T + U_0^T(\xi_0).$$

Being interested in $N^{-3/4}\int_0^{NT}(\xi_s(0)-\vartheta)\,ds$, we find ourselves obliged to study $N^{-3/4}M_{NT}^{NT}$ and $N^{-3/4}U_0^{NT}(\xi_0)$.

Lemma 3.2 *Let $\mathscr{L}(\xi_0) \in \{\Lambda_\vartheta, \mathcal{H}(\vartheta)\}$. The processes*

$$\big(N^{-3/4}M_{NT}^{NT}\big)_{T\ge 0} \quad \text{and} \quad \big(N^{-3/4}U_0^{NT}(\xi_0)\big)_{T\ge 0}$$

converge jointly in the sense of finite dimensional distributions to independent Gaussian limits.

Proof We first consider $\big(N^{-3/4}U_0^{NT}(\xi_0)\big)_{T\ge0}$. If we start from a Poisson field, i.e. $\mathscr{L}(\xi_0) = \mathcal{H}(\vartheta)$, $N^{-3/4}U^{NT}$ will converge in finite dimensional distributions to the zero process: The norming with $N^{-3/4}$ is too strong in this case, as can be seen e.g. from

$$N^{-3/2}\mathbb{E}^{\mathcal{H}(\vartheta)}\Big[\big(U_0^{NT}(\xi_0)\big)^2\Big] \;=\; N^{-3/2}\vartheta\sum_{x\in\mathbb{Z}^3} u_{NT}(x,0)^2 = O(N^{-1}).$$

On the other hand, if $\mathscr{L}(\xi_0) = \Lambda_\vartheta$, the norming will be adequate, and the processes $N^{-3/4}U^{NT}$ will have a non-trivial Gaussian limit. Heuristically, if we could simply replace $a_t(0,x)$ by its local CLT analogue, we would find

$$\begin{aligned} N^{-3/4}U_0^{NT}(\xi_0) &= N^{-3/4}\sum_{x\in\mathbb{Z}^3}[\xi_0(x)-\vartheta]\int_0^{NT} a_s(0,x)\,ds \\ &\approx N^{-3/4}\sum_{x\in\mathbb{Z}^3}[\xi_0(x)-\vartheta]\int_0^{NT}(2\pi s)^{-3/2}(\det Q)^{-1/2}\exp\Big(-\frac{x^TQ^{-1}x}{2s}\Big)\,ds \\ &= N^{-5/4}\sum_{x\in\mathbb{Z}^3}[\xi_0(x)-\vartheta]\varphi_T(x/\sqrt{N}), \end{aligned} \tag{3.7}$$

where $\varphi_T(x) = \int_0^T (2\pi r)^{-3/2} (\det Q)^{-1/2} \exp\left(-\frac{x^T Q^{-1} x}{2r}\right) dr$. If furthermore φ_T were a Schwartz function, we could conclude using Theorem 1 in [Zäh02]. The method of proof used there can be adapted to our situation, technical details are given in Lemma B.1.

Now let us consider M^{NT}. Using Lemma 2.2 we can write (we abbreviate $u_s(x) := u_s(x,0)$)

$$\begin{aligned} M_t^T &= \sum_{x,y\in\mathbb{Z}^3} \int_0^t \Big(u_{T-s}(y) - u_{T-s}(x)\Big) d\tilde{N}_s^{x,y} \\ &\quad + \sum_{x\in\mathbb{Z}^3} \int_0^t u_{T-s}(x) d\tilde{N}_s^{x,+} - \sum_{x\in\mathbb{Z}^3} \int_0^t u_{T-s}(x) d\tilde{N}_s^{x,-}. \end{aligned}$$

Now we replace t and T by NT and multiply by $N^{-3/4}$ which yields

$$N^{-3/4} M_{NT}^{NT} = Z_1(N,T) + Z_2(N,T) - Z_3(N,T),$$

where

$$\begin{aligned} Z_1(N,T) &= N^{-3/4} \sum_{x,y\in\mathbb{Z}^3} \int_0^{NT} \Big(u_{NT-s}(y) - u_{NT-s}(x)\Big) d\tilde{N}_s^{x,y} \\ Z_2(N,T) &= N^{-3/4} \sum_{x\in\mathbb{Z}^3} \int_0^{NT} u_{NT-s}(x) d\tilde{N}_s^{x,+} \\ Z_3(N,T) &= N^{-3/4} \sum_{x\in\mathbb{Z}^3} \int_0^{NT} u_{NT-s}(x) d\tilde{N}_s^{x,-}. \end{aligned}$$

We proceed in two steps. In the first step we investigate $Z_1(N,T)$ and in the second step we consider $Z_2(N,T)$ and $Z_3(N,T)$.

Step 1: The term $Z_1(N,T)$ converges to zero in probability, since the second moment converges to zero:

$$\begin{aligned} \mathbb{E}\left[(Z_1(N,T))^2\right] &= \vartheta N^{-3/2} \sum_{x,y\in\mathbb{Z}^3} a(x,y) \int_0^{NT} \Big(u_{NT-s}(y) - u_{NT-s}(x)\Big)^2 ds \\ &= \vartheta N^{-3/2} \sum_{x,y\in\mathbb{Z}^3} a(x,y) \int_0^{NT} \Big(u_s(y) - u_s(x)\Big)^2 ds \\ &= 2\vartheta N^{-3/2} \int_0^{NT} \big\langle u_s, (-Lu_s)\big\rangle\, ds \\ &= 2\vartheta N^{-3/2} \int_0^{NT} \big\langle u_s, \delta_0 - a_s(\cdot,0)\big\rangle\, ds \\ &\leq 2\vartheta N^{-3/2} \int_0^{NT} u_s(0)\, ds \leq \vartheta N^{-1/2} T g(0,0) \to 0. \end{aligned}$$

Step 2: Now we consider the remaining terms $Z_2(N,T)$ and $Z_3(N,T)$. We define a random field $Y_{N,T}$ on $L^2([0,T]\times\mathbb{R}^3)$ via

$$\langle Y_{N,T}, \varphi\rangle := N^{1/4} \int_{\mathbb{R}^3} dz \int_0^T d\tilde{N}_{Ns}^{\sqrt{N}\lfloor z\rfloor_N,+} \varphi(s,z) = N^{1/4} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T d\tilde{N}_{Ns}^{\sqrt{N}x,+} \int_{x+\Lambda_N} dz\, \varphi(s,z),$$

where $\lfloor z \rfloor_N$ is determined by $\lfloor z \rfloor_N \in \mathbb{Z}^3/\sqrt{N}$ and $z \in \lfloor z \rfloor_N + \Lambda_N$, with $\Lambda_N = \big(-\frac{1}{2\sqrt{N}}, \frac{1}{2\sqrt{N}} \big]^3$. Thus we can write

$$Z_2(N,T) = \langle Y_{N,T}, v_{N,T} \rangle,$$

where

$$v_{N,T}(s,z) = N^{1/2} \sum_{x \in \mathbb{Z}^3/\sqrt{N}} u_{N(T-s)}(\sqrt{N}x) \mathbf{1}_{x+\Lambda_N}(z).$$

Next we wish to show that $Y_{N,T}$ converges towards a white noise Y_T on $[0,T] \times \mathbb{R}^3$ (with covariance measure given by $\rho\vartheta/2$ times Lebesgue measure). Furthermore, for large N, the CLT suggests that $v_{N,T}$ should be similar to

$$v_T(s,z) := \int_0^{T-s} p_r(z,0)\, dr,$$

where $p_r(x,y) := (2\pi r)^{-3/2} (\det Q)^{-1/2} \exp\Big(-\frac{(y-x)^T Q^{-1} (y-x)}{2r} \Big)$. Thus we expect $Z_2(N,T) \approx \langle Y_T, v_T \rangle$, which shows the Gaussian nature. We proceed in two parts to justify this heuristics:

Part 1: Here we show that $\langle Y_{N,T}, \varphi \rangle \to \langle Y_T, \varphi \rangle$ as $N \to \infty$ when $\varphi \in L^2_{\frac{1}{2}\vartheta\rho}([0,T] \times \mathbb{R}^3)$. The index $\frac{1}{2}\vartheta\rho$ indicates that this is the L^2-space corresponding to $\frac{1}{2}\vartheta\rho$ times the Lebesgue measure on $[0,T] \times \mathbb{R}^3$. We write $\|\varphi\|_2$ for the norm of φ in this space. Y_T is a space-time white noise based on $\frac{1}{2}\vartheta\rho$ times the Lebesgue measure on $[0,T] \times \mathbb{R}^3$. That is a random field $Y_T = \langle Y_T, \varphi \rangle$ with $\varphi \in L^2_{\frac{1}{2}\vartheta\rho}([0,T] \times \mathbb{R}^3)$, such that Y_T is a linear isometry from $L^2_{\frac{1}{2}\vartheta\rho}([0,T] \times \mathbb{R}^3)$ to the space of Gaussian random variables equipped with the L^2-norm. See e.g. Chapter 1 of [Wal86] for background on white noises.

First we consider test functions consist only of finitely many steps: Let

$$\varphi(s,x) = \sum_{k=1}^n \mathbf{1}_{\cup_{l=1}^{m(k)} [r_l^k, t_l^k]}(s) \mathbf{1}_{A_k}(x) = \sum_{k=1}^n \mathbf{1}_{A_k}(x) \sum_{l=1}^{m(k)} \mathbf{1}_{[r_l^k, t_l^k]}(s), \tag{3.8}$$

where $A_1, \dots, A_n \subset \mathbb{R}^3$ are disjoint (say, bounded parallelepipeds) and $r_1^k < t_1^k \le r_2^k < t_2^k \le \dots \le r_{m(k)}^k < t_{m(k)}^k$. Let

$$Z_t^{N,k} = N^{1/4} \sum_{x \in \mathbb{Z}^3/\sqrt{N}} \lambda\left(A_k \cap (x + \Lambda_N)\right) \tilde{N}_{Nt}^{\sqrt{N}x,+}, \quad k = 1, \dots, n.$$

Then $(Z_t^N)_{0 \le t \le T} = (Z_t^{N,1}, \dots, Z_t^{N,n})_{0 \le t \le T}$ is an $\mathbb{R}^n$-valued martingale. The assumptions of Proposition A.1 are fulfilled since:

(i) We observe for $k \neq l$ that

$$\langle Z^{N,k}, Z^{N,l} \rangle_t = N^{1/2} \sum_{x \in \mathbb{Z}^3/\sqrt{N}} \lambda\left(A_k \cap (x+\Lambda_N)\right) \lambda\left(A_l \cap (x+\Lambda_N)\right) \int_0^{Nt} \frac{\rho}{2} \xi_s(\sqrt{N}x)\, ds \xrightarrow[N\to\infty]{P} 0,$$

since $\mathbb{E}[\langle Z^{N,k}, Z^{N,l} \rangle_t] \le \rho\vartheta t/(2N^{3/2}) \times \#\{x \in \mathbb{Z}^3/\sqrt{N} : \mathrm{dist}(x, A_l), \mathrm{dist}(x, A_k) \le N^{-1/2}\} = O(N^{-1/2})$.

For $k = l$ we calculate

$$\langle Z^{N,k}, Z^{N,k} \rangle_t = N^{1/2} \sum_{x \in \mathbb{Z}^3/\sqrt{N}} \Big(\lambda(A_k \cap (x+\Lambda_N)) \Big)^2 \int_0^{Nt} \frac{\rho}{2} \xi_s(\sqrt{N}x)\, ds \xrightarrow[N\to\infty]{P} \frac{1}{2}\vartheta\rho\lambda(A_k) t$$

due to Theorem 3.

(ii) We observe that $Z^{N,k}$ has jumps of size $N^{-5/4}$, such that condition (ii) of Proposition A.1 is obviously fulfilled.

By Proposition A.1 we can conclude

$$(Z^{N,1},\dots,Z^{N,n}) \underset{N\to\infty}{\longrightarrow} (Z^1,\dots,Z^n), \tag{3.9}$$

where $\{Z^k\}$ are independent Brownian motions with variance parameter $\frac{1}{2}\vartheta\rho\lambda(A_k)$.

For φ defined in (3.8) we obtain

$$\langle Y_{N,T},\varphi\rangle = \sum_{k=1}^{n}\sum_{l=1}^{m(k)}\left(Z^{N,k}_{t^k_l} - Z^{N,k}_{r^k_l}\right) \underset{N\to\infty}{\longrightarrow} \sum_{k=1}^{n}\sum_{l=1}^{m(k)}\left(Z^k_{t^k_l} - Z^k_{r^k_l}\right).$$

The limit is a sum of independent normal random variables by (3.9). Therefore the limit is normal with variance $\sum_{k=1}^n\sum_{l=1}^{m(k)}\frac{1}{2}\vartheta\rho\lambda(A_k)(t^k_l - r^k_l)$ and hence

$$\langle Y_{N,T},\varphi\rangle \underset{N\to\infty}{\longrightarrow} \langle Y_T,\varphi\rangle.$$

Then we can extend the convergence statement to all $\varphi\in L^2_{\frac{1}{2}\vartheta\rho}([0,T]\times\mathbb{R}^3)$, since the functions of the form (3.8) are dense in $L^2_{\frac{1}{2}\vartheta\rho}([0,T]\times\mathbb{R}^3)$ and since (for $\varphi:\mathbb{R}^3\to\mathbb{R}$ such that φ^2 is Riemann integrable)

$$\lim_{N\to\infty}\mathbb{E}\left[\langle Y_{N,T},\varphi\rangle^2\right] = \frac{1}{2}\vartheta\rho\int_{\mathbb{R}^3}\int_0^T\varphi(s,z)^2\,ds\,dz = \|\varphi\|_2^2.$$

The last assertion can be seen by the following calculation

$$\begin{aligned}\mathbb{E}[\langle Y_{N,T},\varphi\rangle^2] &= N^{1/2}\sum_{x\in\mathbb{Z}^3/\sqrt{N}}\int_0^T\left[\int_{x+\Lambda_N}\varphi(s,z)\,dz\right]^2 N\frac{1}{2}\vartheta\rho\,ds\\ &= \frac{1}{2}\vartheta\rho\sum_{x\in\mathbb{Z}^3/\sqrt{N}}\int_0^T\left[\int_{x+\Lambda_N}\varphi(s,z)\,dz\right]\left[N^{3/2}\int_{x+\Lambda_N}\varphi(s,w)\,dw\right]ds,\end{aligned}$$

which is a Riemannian sum for $\frac{1}{2}\vartheta\int_0^T\int_{\mathbb{R}^3}\varphi^2(s,z)\,dz\,ds$. This completes the proof of the assertion.

Part 2: Now we show that $\langle Y_{N,T}, v_{N,T}-v_T\rangle \overset{P}{\underset{N\to\infty}{\longrightarrow}} 0$, in fact we will show that

$$\mathbb{E}\left[\langle Y_{N,T}, v_{N,T}-v_T\rangle^2\right] \underset{N\to\infty}{\longrightarrow} 0. \tag{3.10}$$

We have

$$
\begin{aligned}
&(\frac{1}{2}\vartheta\rho)^{-1} \times \mathbb{E}\left[\langle Y_{N,T}, v_{N,T} - v_T\rangle^2\right] \\
&\quad = N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \left(\int_{x+\Lambda_N} dz \big(v_{N,T}(s,z) - v_T(s,z)\big)\right)^2 \\
&\quad = N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \left(\int_0^{T-s} dr \int_{x+\Lambda_N} dz \left(N^{3/2} a_{Nr}(\sqrt{N}x, 0) - p_r(z,0)\right)\right)^2 \\
&\quad = N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \left(\int_0^{\varepsilon} dr \ \ldots + \int_{\varepsilon}^{T-s} dr \ \ldots \right)^2 \\
&\quad \le 2N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \Big\{ \Big[\int_0^{\varepsilon} dr \ \ldots\Big]^2 + \Big[\int_{\varepsilon}^{T-s} dr \ \ldots\Big]^2 \Big\}.
\end{aligned}
$$

Now note that

$$
\begin{aligned}
&N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \Big[\int_0^{\varepsilon} dr \int_{x+\Lambda_N} dz\, N^{3/2} a_{Nr}(\sqrt{N}x, 0)\Big]^2 \\
&\quad = N^{3/2} T \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \Big[\frac{1}{N}\int_0^{N\varepsilon} dr\, a_r(\sqrt{N}x, 0)\Big]^2 = N^{-1/2} T ||u_{N\varepsilon}(\cdot, 0)||_2^2 \le C\sqrt{\varepsilon},
\end{aligned}
$$

where we use for the last estimate that

$$
||u_t(\cdot, 0)||_2^2 \;=\; \sum_{x\in\mathbb{Z}^d} \int_0^t dr \int_0^t ds\, a_r(x,0) a_s(x,0) = 2\int_0^t dr \int_r^t ds\, a_{r+s}(0,0) \sim \text{Const.} \times \sqrt{t}
$$

by (2.3). Similarly

$$
\begin{aligned}
&N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \Big[\int_0^{\varepsilon} dr \int_{x+\Lambda_N} dz\, p_r(z,0)\Big]^2 = N^{3/2} T \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \Big[\int_0^{\varepsilon} dr \int_{x+\Lambda_N} dz\, p_r(z,0)\Big]^2 \\
&\quad \le N^{3/2} T \sum_{x\in\mathbb{Z}^3/\sqrt{N}} |\Lambda_N| \int_{x+\Lambda_N} dz \left(\int_0^{\varepsilon} dr\, p_r(z,0)\right)^2 = T\int_{\mathbb{R}^3} dz \int_0^{\varepsilon} dr \int_0^{\varepsilon} ds\, p_r(z,0) p_s(z,0) \\
&\quad = 2T \int_0^{\varepsilon} dr \int_r^{\varepsilon} ds\, p_{r+s}(0,0) \le C\sqrt{\varepsilon},
\end{aligned}
$$

where we used the Cauchy-Schwarz inequality.

In order to treat the remaining term we use that (see e.g. Prop. A.2)

$$
\big|N^{3/2} a_{Nr}(\sqrt{N}x, 0) - p_r(z,0)\big| \le C_{\varepsilon} \frac{1}{1 + |z|^2/r}\, \psi(N) \tag{3.11}
$$

uniformly in N, $r \in [\varepsilon, T]$, $x \in \mathbb{Z}^3/\sqrt{N}$, $z \in x + \Lambda_N$, where $\psi(N) \to 0$ as $N \to \infty$. (Note that this requires only a second moment assumption on a) This yields

$$\begin{aligned}
&N^{3/2} \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \int_0^T ds \Big[\int_\varepsilon^{T-s} dr \int_{x+\Lambda_N} dz \, \big(N^{3/2} a_{Nr}(\sqrt{N}x,0) - p_r(z,0)\big)\Big]^2 \\
&\quad \le N^{3/2}\psi(N)^2 T \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \Big(\int_\varepsilon^T dr \int_{x+\Lambda_N} dz \, \frac{C_\varepsilon}{1+|z|^2/r}\Big)^2 \\
&\quad \le N^{3/2}\psi(N)^2 T \sum_{x\in\mathbb{Z}^3/\sqrt{N}} \Big(\int_{x+\Lambda_N} dz \, \frac{C_\varepsilon T}{1+|z|^2/T}\Big)^2 \\
&\quad \le N^{3/2}\psi(N)^2 T^3 C_\varepsilon^2 \sum_{x\in\mathbb{Z}^3/\sqrt{N}} |\Lambda_N| \int_{x+\Lambda_N} dz \, \big(1+|z|^2/T\big)^{-2} \\
&\quad = C_\varepsilon^2 T^3 \psi(N)^2 \int_{\mathbb{R}^3} dz \, \big(1+|z|^2/T\big)^{-2} \underset{N\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Combining we see that $\limsup_{N\to\infty} \mathbb{E}\left[\langle Y_{N,T}, v_{N,T} - v_T\rangle^2\right] \le C\sqrt{\varepsilon}$, now let $\varepsilon \to 0$ to obtain (3.10).

Thus we have shown that $Z_2(N,T)$ converges to a Gaussian limit. $Z_3(N,T)$ can be treated completely analogously, and as it involves only integrals with respect to $(\tilde{N}_t^{x,-})$, $x \in \mathbb{Z}^3$, and the martingales $\tilde{N}^{x,-}$ and $\tilde{N}^{x,+}$ are all pairwise orthogonal, we see that $Z_2(N,T)$ and $Z_3(N,T)$ converge jointly to (independent) Gaussian processes. Thus $(N^{-3/4} M_{NT}^{NT})$ converges as $N \to \infty$ to a Gaussian process.

Finally, a remark on the *joint* convergence of U^{NT} and M^{NT} when starting from the invariant distribution Λ_ϑ is in order: Note that $U_{NT}^{NT}(\xi_0)$ depends only on the initial condition, whereas M^{NT} is a function of the driving martingales $\tilde{N}^{x,\pm}$, $x \in \mathbb{Z}^3$. Scrutinising the proof the reader will find that even conditional on $\xi_0 = \eta$, M^{NT} will converge to the same Gaussian process, as long as η is such that $\mathscr{L}(\xi_t|\xi_0 = \eta) \Rightarrow \Lambda_\vartheta$ as $t \to \infty$. This is the more careful (but also more lengthy) argument, we hope to have convinced our reader nonetheless.

□

4 Moment computations for branching random walks

In this section, we express space-time moments of critical binary branching random walk as sums of certain integrals over tree-indexed random walks. The trees appearing in our expression for an n-th moment have a natural interpretation as the possible ancestral relations between n sampled particles in the branching population. Our computation is very much in the spirit of [Dyn88], Section 2, where a similar program is carried out for super-Brownian motion, the continuous relative of BRW.

Let time points $0 =: t_0 < t_1 < t_2 < \cdots < t_n$, non-negative test functions $f_1, \dots, f_n : \mathbb{Z}^d \to \mathbb{R}_+$, and coefficients $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}_+^n$ be given. Our first aim is to find an expression for $\mathbb{E}^{(r,\delta_x)}\left[\prod_{i=1}^n \langle f_i, \xi_{t_i}\rangle\right]$, where $\mathbb{E}^{(r,\delta_x)}$ refers to expectation with respect to the ξ-process starting at time $r \ge 0$ with exactly one particle at $x \in \mathbb{Z}^d$. In order to do this we put

$$w(r,x,\alpha) := \mathbb{E}^{(r,\delta_x)}\left[\exp\Big(-\sum_{i=1}^n \mathbf{1}_{\{r<t_i\}} \alpha_i \langle f_i, \xi_{t_i}\rangle\Big)\right],$$

and for $\Lambda \subset \{1, \ldots, n\}$ put

$$w_\Lambda(r,x) := (-1)^{|\Lambda|} \frac{\partial^{|\Lambda|}}{\partial \alpha_\Lambda} w(r,x,\alpha)\Big|_{\alpha=0}.$$

Note that w_Λ implicitly depends on the t_i and f_i, but we will consider them as fixed in the following. Note that

$$w_\Lambda(r,x) = \mathbb{E}^{(r,\delta_x)}\left[\prod_{i\in\Lambda} \langle f_i, \xi_{t_i}\rangle\right]$$

if $r < t_i$ for all $i \in \Lambda$. The next lemma shows how to compute w_Λ recursively:

Lemma 4.1 *Let $1 \le i_1 < \cdots < i_k \le n$, $\Lambda = \{i_1, \ldots, i_k\}$. For $x \in \mathbb{Z}^d$, $r \le t_{i_1}$ we have*

$$\begin{aligned} w_\Lambda(r,x) &= K_\Lambda(r,x) + \rho \int_r^{t_{i_1}} ds \sum_{y\in\mathbb{Z}^d} a_{s-r}(x,y) \sum_{\substack{\{M_1,M_2\} \\ M_1\cup M_2=\Lambda,\ disj.,\ both \neq \emptyset}} w_{M_1}(s,y) w_{M_2}(s,y) \qquad (4.1) \\ &\quad + \sum_{j=2}^{k} \sum_{x_1,\ldots,x_{j-1}\in\mathbb{Z}^d} a_{t_{i_1}-r}(x,x_1) \times \prod_{\ell=2}^{j-1} a_{t_{i_\ell}-t_{i_{\ell-1}}}(x_{\ell-1},x_\ell) \times \prod_{m=1}^{j-1} f_{i_m}(x_m) \\ &\quad \times \rho \int_{t_{i_{j-1}}}^{t_{i_j}} ds \sum_{y\in\mathbb{Z}^d} a_{s-t_{i_{j-1}}}(x_{j-1},y) \sum_{\substack{\{M_1,M_2\} \\ M_1\cup M_2=\{i_j,\ldots,i_k\},\ disj.,\ both\neq\emptyset}} w_{M_1}(s,y) w_{M_2}(s,y), \end{aligned}$$

with $K_\Lambda(r,x) := \mathbb{E}^{(r,x)}\big[\prod_{i\in\Lambda} f_i(W_{t_i})\big]$, where $(W_t)_{t\ge 0}$ is a random walk with kernel a.

Proof Let us write

$$w_\Lambda(r,x,\alpha) := \mathbb{E}^{(r,\delta_x)}\left[\exp\Big(-\sum_{i\in\Lambda} \mathbf{1}_{\{r<t_i\}} \alpha_i \langle f_i, \xi_{t_i}\rangle\Big)\right].$$

First of all we prove for $r < t_{i_1}$

$$\begin{aligned} & w_\Lambda(r,x,\alpha) \qquad (4.2) \\ &= S_{t_{i_1}-r}\big(e^{-\alpha_{i_1} f_{i_1}} S_{t_{i_2}-t_{i_1}}\big(e^{-\alpha_{i_2} f_{i_2}} S_{t_{i_3}-t_{i_2}}\big(\cdots S_{t_{i_k}-t_{i_{k-1}}} e^{-\alpha_{i_k} f_{i_k}}\big)\cdots\big)\big)(x) \\ &\quad + \rho \int_r^{t_{i_1}} ds\, S_{s-r}\Big(\frac{1}{2}\big(1 - w(s,\cdot,\alpha)\big)^2\Big)(x) \\ &\quad + \sum_{j=2}^{k} \sum_{x_1\in\mathbb{Z}^d} a_{t_{i_1}-r}(x,x_1) \sum_{x_2\in\mathbb{Z}^d} a_{t_{i_2}-t_{i_1}}(x_1,x_2) \cdots \sum_{x_{j-1}\in\mathbb{Z}^d} a_{t_{i_{j-1}}-t_{i_{j-2}}}(x_{j-2},x_{j-1}) \\ &\qquad \exp\Big(-\sum_{m=1}^{j-1} \alpha_{i_m} f_{i_m}(x_m)\Big) \quad \rho \int_{t_{i_{j-1}}}^{t_{i_j}} ds\, S_{s-t_{i_{j-1}}}\Big(\frac{1}{2}\big(1 - w_\Lambda(s,\cdot,\alpha)\big)^2\Big)(x_{j-1}). \end{aligned}$$

The starting point is the well-known fact that $\phi(r,x) := \mathbb{E}^{(r,\delta_x)} \exp(-\langle f, \xi_t\rangle)$ (where $r \le t$) solves

$$\phi(r,x) = S_{t-r}\big(e^{-f}\big)(x) + \rho \int_r^t ds\, S_{s-r}\Big(\frac{1}{2}\big(1 - \phi(s,\cdot)\big)^2\Big)(x). \qquad (4.3)$$

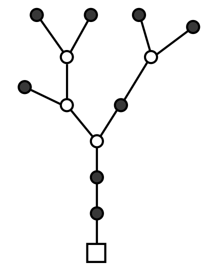


Figure 1: An example, the special nodes are shaded

Now we iterate this. For $r \in [t_{i_{k-1}}, t_{i_k})$, claim (4.2) is true by (4.3). If we assume it to be true for all $r' \in [t_{i_\ell}, t_{i_{\ell+1}})$ and consider some $r \in [t_{i_{\ell-1}}, t_{i_\ell})$ we find by stopping at time t_{i_ℓ}

$$\begin{aligned} w_\Lambda(r,x,\alpha) &= \mathbb{E}^{(r,\delta_x)}\Big[e^{-\alpha_{i_\ell}\langle f_{i_\ell},\xi_{t_i}\rangle}\mathbb{E}^{(t_{i_\ell},\xi_{t_{i_\ell}})}\Big[\exp\Big(-\sum_{j=\ell+1}^{k}\alpha_{i_j}\langle f_{i_j},\xi_{t_{i_j}}\rangle\Big)\Big]\Big] \\ &= \mathbb{E}^{(r,\delta_x)}\Big[\exp\Big(\big\langle -\alpha_{i_\ell}f_{i_\ell}(\cdot)+\log w_\Lambda(t_{i_\ell},\cdot,\alpha)\,,\xi_{t_{i_\ell}}\big\rangle\Big)\Big]\,. \end{aligned}$$

Hence again by (4.3)

$$w_\Lambda(r,x,\alpha) = S_{t_{i_\ell}-r}\left(e^{-\alpha_{i_\ell}f_{i_\ell}(\cdot)}w_\Lambda(t_{i_\ell},\cdot,\alpha)\right)(x) + \rho\int_r^{t_{i_\ell}} ds\, S_{s-r}\Big(\frac{1}{2}\big(1-w_\Lambda(s,\cdot,\alpha)\big)^2\Big)(x).$$

This completes the proof of (4.2). We obtain the assertion by differentiating (4.2) with respect to α_i, $i \in \Lambda$ and evaluating at $\alpha = (0,\cdots,0)$. □

4.1 Trees and bookkeeping

For the bookkeeping of terms appearing in the computation of space-time moments of BRW we will need finite, rooted, unordered trees τ in which each node has at most two successors and the root has exactly one child. For a vertex $v \neq$ root, we denote by $\overleftarrow{v}$ its predecessor. We write $v \prec v'$ if v' is a (direct or indirect) descendant of v. We define the degree of a node v as the number of its direct descendants. Let V denote the set of all nodes except for the root. The set $\tilde{V}$ of leaves and inner nodes of degree one plays a special role (see Figure 1), and each node $v \in \tilde{V}$ carries a mark $\varphi(v) \in \mathbb{N}$. We only consider such marked trees τ which have the property that $v \prec v'$ implies $\varphi(v) \leq \varphi(v')$ and that any $m \in \mathbb{N}$ appears at most once as a mark. For a subset $\Lambda \subset \{1,\ldots,n\}$, let $\mathbb{T}_\Lambda$ be the set of all such trees τ where the set of marks is the given Λ.

For given $\Lambda \subset \{i,i+1,\ldots,n\}$, $\tau \in \mathbb{T}_\Lambda$, $r < t_i$, and $x \in \mathbb{Z}^d$ we put

$$\begin{aligned} &S(\tau,\Lambda;r,x) := \qquad (4.4) \\ &\quad\Big(\rho^{\#\text{nodes of degree 2 in }\tau}\Big)\Big(\prod_{v\in V\setminus\tilde{V}}\int_{\mathbb{R}_+} ds_v\Big)\Big\{\mathbf{1}_{\{\forall\, v\prec v':\, s_v\leq s_{v'}\}} \\ &\qquad\qquad \sum_{\{x_v\in\mathbb{Z}^d: v\in V\}}\prod_{v\in V} a_{s_v - s_{\overleftarrow{v}}}(x_{\overleftarrow{v}},x_v)\prod_{\tilde{v}\in\tilde{V}} f_{\varphi(\tilde{v})}(x_{\tilde{v}})\Big\} \end{aligned}$$

where we implicitly understand that $s_{\tilde{v}} = t_{\varphi(\tilde{v})}$ for all $\tilde{v} \in \tilde{V}$, and $s_{\text{root}} = r$, $x_{\text{root}} = x$.

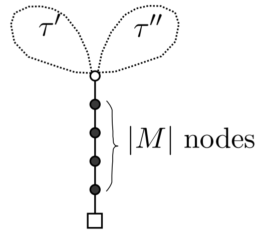


Figure 2: Concatenation of trees

Proposition 4.2 *For $i \leq n$, $\Lambda \subset \{i, i+1, \ldots, n\}$ and $r < t_i$ we have*

$$w_\Lambda(r,x) = \sum_{\tau \in \mathbb{T}_\Lambda} S(\tau, \Lambda; r, x). \tag{4.5}$$

Proof Let us assume that $\Lambda = \{i_1, i_2, \ldots, i_{|\Lambda|}\}$ with $i_1 < \cdots < i_{|\Lambda|}$. For $\ell < |\Lambda|$ let $\mathbb{T}_{\Lambda,\ell}$ be the set of all trees marked with Λ in which the first branching occurs at height ℓ, that is which look as follows: the root is followed by a string of $\ell - 1$ nodes of degree one (necessarily marked with $i_1, \ldots, i_{\ell-1}$), then there is an inner node with two subtrees. $\mathbb{T}_{\Lambda,|\Lambda|}$ is the set consisting just of one tree in $\mathbb{T}_\Lambda$ which has no branching points at all.

Note that for $\Lambda' \cap \Lambda'' = \emptyset$, marked trees $\tau' \in \mathbb{T}_{\Lambda'}$, $\tau'' \in \mathbb{T}_{\Lambda''}$ and $M \subset \mathbb{N}$ with $m \in M, i \in \Lambda' \cup \Lambda'' \Rightarrow m < i$ we obtain a (unique) tree $\tau \in \mathbb{T}_{M \cup \Lambda' \cup \Lambda''}$ by the following prescription: A string of $|M|$ nodes of degree one, marked by the elements of M, is followed by a split node, which is obtained by identifying the roots of τ' and τ'' (see Figure 2). We write $\tau = \tau' *_M \tau''$. On the other hand, for each $\ell < |\Lambda|$, any $\tau \in \mathbb{T}_{\Lambda,\ell}$ can be constructed in this way (with $M = \{i_1, \ldots, i_{\ell-1}\}$ and $M = \emptyset$ for $\ell = 1$).

We prove the proposition by induction on $|\Lambda|$. For $\Lambda = \{j\}$ $(i \leq j \leq n)$, there is only one $\tau \in \mathbb{T}_\Lambda$: the tree consisting of the root followed by a leaf which is marked with j. We have

$$w_{\{j\}}(r,x) = \mathbb{E}^{(r,x)}[f_j(W_{t_j})] = \sum_{y \in \mathbb{Z}^d} a_{t_j - r}(x,y) f_j(y) = \sum_{\tau \in \mathbb{T}_{\{j\}}} S(\tau, \Lambda; r, x)$$

in this case.

Now assume that the claim is true for all $\Lambda' \subset \{1, \ldots, n\}$ with $|\Lambda'| \leq k-1$, and let $\Lambda = \{i_1, \ldots, i_k\}$ with k elements $i_1 < \cdots < i_k$ be given. For $1 \leq \ell < |\Lambda|$ we can, according to the observation above, decompose

$$\sum_{\tau \in \mathbb{T}_{\Lambda,\ell}} S(\tau, \Lambda; r, x) = \sum_{\substack{\{\Lambda',\Lambda''\} \\ \Lambda' \cup \Lambda'' = \{i_\ell, i_{\ell+1}, \ldots, i_k\} \\ \text{disj., both} \neq \emptyset}} \sum_{\tau' \in \mathbb{T}_{\Lambda'}, \tau'' \in \mathbb{T}_{\Lambda''}} S\big(\tau' *_{\{i_1,\ldots,i_{\ell-1}\}} \tau'', \Lambda; r, x\big).$$

Furthermore, we have for $\ell > 1$

$$\begin{aligned} & S(\tau' *_{\{i_1,\ldots,i_{\ell-1}\}} \tau'', \Lambda; r, x) \\ = \; & \rho \sum_{x_1 \in \mathbb{Z}^d} a_{t_{i_1} - r}(x, x_1) \sum_{x_2 \in \mathbb{Z}^d} a_{t_{i_2} - t_{i_1}}(x_1, x_2) \cdots \sum_{x_{\ell-1} \in \mathbb{Z}^d} a_{t_{i_{\ell-1}} - t_{i_{\ell-2}}}(x_{\ell-2}, x_{\ell-1}) \\ & \qquad \prod_{j=1}^{\ell-1} f_{i_j}(x_j) \int_{t_{i_{\ell-1}}}^{t_{i_\ell}} ds \sum_{y \in \mathbb{Z}^d} a_{s - t_{i_{\ell-1}}}(x_{\ell-1}, y) S(\tau', \Lambda'; s, y) S(\tau'', \Lambda''; s, y), \end{aligned}$$

for $\ell = 1$

$$S(\tau' *_{\{\}} \tau'', \Lambda; r, x) = \rho \int_r^{t_{i_1}} ds \sum_{y \in \mathbb{Z}^d} a_{s-r}(x,y) S(\tau', \Lambda'; s, y) S(\tau'', \Lambda''; s, y),$$

and for the one tree $\tau \in \mathbb{T}_{\Lambda, |\Lambda|}$

$$S(\tau, \Lambda; r, x) = K_\Lambda(r, x).$$

Adding up these equations and using the induction hypothesis we see that the r.h.s. of (4.5) solves (4.1), and thus the claim is proved. □

Remark 4.3 A similar representation can be developed in the general case when the t_i are not necessarily pairwise distinct. This would require an appropriate set of trees where the nodes can carry multiple marks in order to accommodate situations when at a time $t_i = t_{i+1} = \cdots = t_{i+k}$ a single particle has to be counted several times. We have refrained from making this explicit as we will not need that generality in the following.

Now we consider the case of Poisson initial configurations.

Proposition 4.4 *The multi-space-time moment of the BRW started in a Poisson point process has the following form*

$$\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\prod_{j \in \Lambda} \langle f_j, \xi_{t_j} \rangle\right] = \sum_{k=1}^{n} \sum_{\substack{\{\Lambda_1, \ldots, \Lambda_k\} \\ \text{partition of } \Lambda}} \vartheta^k \prod_{i=1}^{k} \left(\sum_{x_i \in \mathbb{Z}^d} \sum_{\tau \in \mathbb{T}_{\Lambda_i}} S(\tau, \Lambda_i; 0, x_i) \right).$$

Proof We assume that ξ_0 is a Poisson point process on $\mathbb{Z}^d$ with constant intensity $\vartheta > 0$. Let us write $\xi_0 = \sum_i \delta_{Y_i}$, and let $(\xi_t^{(i)})$ be the family founded by δ_{Y_i} at time 0. We have

$$\begin{aligned}
\mathbb{E}\Big[\prod_{j \in \Lambda} \langle f_j, \xi_{t_j} \rangle\Big] &= \mathbb{E}\Big[\prod_{j \in \Lambda} \langle f_j, \textstyle\sum_i \xi_{t_j}^{(i)} \rangle\Big] = \sum_{(i_j) \in \mathbb{N}^\Lambda} \mathbb{E}\Big[\prod_{j \in \Lambda} \langle f_j, \xi_{t_j}^{(i_j)} \rangle\Big] \\
&= \sum_{k=1}^{n} \sum_{\substack{\{\Lambda_1, \ldots, \Lambda_k\} \\ \text{part. of } \Lambda}} \sum_{\substack{\ell_1, \ldots, \ell_k = 1, \\ \text{pairw. diff.}}}^{\infty} \mathbb{E}\Big[\prod_{i=1}^{k} \prod_{j \in \Lambda_i} \langle f_j, \xi_{t_j}^{(\ell_i)} \rangle\Big] \\
&= \sum_{k=1}^{n} \sum_{\substack{\{\Lambda_1, \ldots, \Lambda_k\} \\ \text{part. of } \Lambda}} \sum_{\substack{\ell_1, \ldots, \ell_k = 1, \\ \text{pairw. diff.}}}^{\infty} \mathbb{E}\Big[\prod_{i=1}^{k} w_{\Lambda_i}(0, Y_{\ell_i})\Big] \\
&= \sum_{k=1}^{n} \sum_{\substack{\{\Lambda_1, \ldots, \Lambda_k\} \\ \text{part. of } \Lambda}} \mathbb{E}\Big[\int \cdots \int \Big(\prod_{i=1}^{k} w_{\Lambda_i}(0, x_i)\Big) \big(\xi_0 - \delta_{x_1} - \cdots - \delta_{x_{k-1}}\big)(dx_k) \cdots \\
&\qquad\qquad \big(\xi_0 - \delta_{x_1}\big)(dx_2) \xi_0(dx_1)\Big].
\end{aligned}$$

For a Poisson point process η with intensity measure ν the k-th factorial moment measure is just $\nu^{\otimes k}$, see e.g. [DVJ88] Example 7.4(a), p. 227f, so

$$\mathbb{E}\left[\int \cdots \int \left(\prod_{i=1}^k g_i(x_i)\right)\big(\eta - \delta_{x_1} - \cdots - \delta_{x_{k-1}}\big)(dx_k)\cdots\big(\eta-\delta_{x_1}\big)(dx_2)\eta_0(dx_1)\right]$$
$$= \prod_{i=1}^k \int g_i(x)\nu(dx).$$

The result follows from Proposition 4.2. □

As an easy consequence we obtain the well-known second moment formulas for branching random walk:

Corollary 4.5 *For $u \le v$, $x, y \in \mathbb{Z}^d$ we have*

$$\begin{aligned}\mathbb{E}^{\mathcal{H}(\vartheta)}[\xi_u(x)\xi_v(y)] &= \vartheta^2 + \vartheta a_{v-u}(x,y) + \frac{\vartheta\rho}{2}\int_{v-u}^{v+u} a_r(x,y)\,dr,\\ \mathbb{E}^{\Lambda_\vartheta}[\xi_u(x)\xi_v(y)] &= \vartheta^2 + \vartheta a_{v-u}(x,y) + \frac{\vartheta\rho}{2}\int_{v-u}^{\infty} a_r(x,y)\,dr.\end{aligned}$$

Proof We apply Proposition 4.4 with $\Lambda = \{1,2\}$, $f_1 = \mathbf{1}_{\{x\}}$, $f_2 = \mathbf{1}_{\{y\}}$ and $t_1 = u$, $t_2 = v$ and we use (4.4) to obtain

$$\begin{aligned}\mathbb{E}^{\mathcal{H}(\vartheta)}[\xi_u(x)\xi_v(y)] &= \vartheta^2 + \vartheta\sum_{z\in\mathbb{Z}^d} a_u(z,x)a_{v-u}(x,y)\\ &\quad + \vartheta\rho\sum_{z\in\mathbb{Z}^d}\int_0^u ds \sum_{w\in\mathbb{Z}^d} a_s(z,w)a_{u-s}(w,x)a_{v-s}(w,y)\\ &= \vartheta^2 + \vartheta a_{v-u}(x,y) + \frac{\vartheta\rho}{2}\int_{v-u}^{u+v} dr\, a_r(x,y).\end{aligned}$$

For the second equation note that

$$\mathbb{E}^{\Lambda_\vartheta}[\xi_u(x)\xi_v(y)] = \lim_{T\to\infty}\mathbb{E}^{\mathcal{H}(\vartheta)}[\xi_{T+u}(x)\xi_{T+v}(y)].$$

□

4.2 Covariance computation

In this subsection we compute the covariance of the limit of the renormalised occupation time.

Proposition 4.6 *The variance of the limit of the renormalised occupation time is*

$$\mathbb{E}^{\mu}\left[X_s^N X_t^N\right] \underset{N\to\infty}{\longrightarrow}$$
$$\begin{cases} \frac{\sqrt{2}}{3\pi^{3/2}}(\det Q)^{-1/2}\vartheta\rho\left[t^{3/2}+s^{3/2}-|t-s|^{3/2}\right], & d=3, \mu=\Lambda_\vartheta,\\ \frac{2\sqrt{2}}{3\pi^{3/2}}(\det Q)^{-1/2}\vartheta\rho\left[t^{3/2}+s^{3/2}-\frac{1}{2}|t-s|^{3/2}-\frac{1}{2}(t+s)^{3/2}\right], & d=3, \mu=\mathcal{H}(\vartheta),\\ (2\pi)^{-2}(\det Q)^{-1/2}\vartheta\rho\times(s\wedge t), & d=4, \mu\in\{\Lambda_\vartheta,\mathcal{H}(\vartheta)\},\\ \left[2\int_0^\infty du\, a_u(0,0)+\rho\int_0^\infty du\, u a_u(0,0)\right]\vartheta\times(s\wedge t), & d\ge 5, \mu\in\{\Lambda_\vartheta,\mathcal{H}(\vartheta)\}.\end{cases}$$

Proof The proof is split up into different cases. We assume $s \le t$ throughout. Let us first consider the situation $\mathscr{L}(\xi_0) = \mathcal{H}(\vartheta)$. By Corollary 4.5 we have

$$\begin{aligned}
\mathbb{E}^{\mathcal{H}(\vartheta)}\left[X_s^N X_t^N\right] &= \frac{1}{h_d(N)^2}\int_0^{Ns} du \int_0^{Nt} dv\, \mathrm{Cov}^{\mathcal{H}(\vartheta)}(\xi_u(0), \xi_v(0)) \\
&= \frac{\vartheta}{h_d(N)^2}\int_0^{Ns} du \int_0^{Nt} dv\, a_{|v-u|}(0,0) \\
&\quad + \frac{\vartheta\rho}{2h_d(N)^2}\int_0^{Ns} du \int_0^{Nt} dv \int_{|v-u|}^{v+u} dr\, a_r(0,0) =: I_1 + I_2.
\end{aligned}$$

Case 1: Let $d = 3$. We have $0 \le I_1 \le \vartheta N^{-3/2}(Ns)\int_0^\infty a_r(0,0)\,dr = O(N^{-1/2})$, so that this term is asymptotically negligible. Fix $\varepsilon > 0$ for the moment. By (2.3), we can find $K > 0$ such that $a_r(0,0) \le (1+\varepsilon)c_3 r^{-3/2}$ for $r \ge K$, where $c_3 = (2\pi)^{-3/2}(\det Q)^{-1/2}$. Thus we can bound I_2 by

$$\frac{(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K} du \int_{u+K}^{Ns} dv \int_{v-u}^{v+u} \frac{dr}{r^{3/2}} + \frac{(1+\varepsilon)\vartheta\rho c_3}{2N^{3/2}}\int_0^{Ns-K} du \int_{Ns}^{Nt} dv \int_{v-u}^{v+u} \frac{dr}{r^{3/2}} + O(N^{-1/2}) \tag{4.6}$$

The first term in (4.6) is equal to

$$\begin{aligned}
&\frac{2(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K} du \int_{u+K}^{Ns} dv \Big\{\frac{1}{(v-u)^{1/2}} - \frac{1}{(v+u)^{1/2}}\Big\} \\
&= \frac{4(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K} du \Big\{\Big[(v-u)^{1/2} - (v+u)^{1/2}\Big]_{v=u+K}^{v=Ns}\Big\} \\
&= \frac{4(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K} du \Big\{(Ns-u)^{1/2} - K^{1/2} - (Ns+u)^{1/2} + (2u+K)^{1/2}\Big\} \\
&= \frac{4(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\Big[-\frac{2}{3}(Ns-u)^{3/2} - K^{1/2}u - \frac{2}{3}(Ns+u)^{3/2} + \frac{1}{3}(2u+K)^{3/2}\Big]_{u=0}^{u=Ns-K} \\
&\longrightarrow \frac{4}{3}(1+\varepsilon)\vartheta\rho c_3(4 - 2^{3/2})s^{3/2} \quad \text{as } N \to \infty.
\end{aligned}$$

Analogously, the second term in (4.6) is equal to

$$\begin{aligned}
&\frac{2(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K} du \Big\{\Big[(v-u)^{1/2} - (v+u)^{1/2}\Big]_{v=Ns}^{v=Nt}\Big\} \\
&= \frac{2(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K} du \Big\{(Nt-u)^{1/2} - (Ns-u)^{1/2} - (Nt+u)^{1/2} + (Ns+u)^{1/2}\Big\} \\
&= \frac{2(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\Big[-\frac{2}{3}(Nt-u)^{3/2} + \frac{2}{3}(Ns-u)^{3/2} - \frac{2}{3}(Nt+u)^{3/2} + \frac{2}{3}(Ns+u)^{3/2}\Big]_{u=0}^{u=Ns-K} \\
&\longrightarrow \frac{4}{3}(1+\varepsilon)\vartheta\rho c_3\Big(2t^{3/2} - (t-s)^{3/2} - (t+s)^{3/2} - (2-2^{3/2})s^{3/2}\Big) \quad \text{as } N \to \infty.
\end{aligned}$$

Combining these terms and letting $\varepsilon \to 0$ we see that

$$\limsup_{N\to\infty} I_2 \le \frac{8}{3}c_3\vartheta\rho\big(t^{3/2} + s^{3/2} - \frac{1}{2}(t-s)^{3/2} - \frac{1}{2}(t+s)^{3/2}\big).$$

$\liminf_{N\to\infty} I_2$ can be analogously bounded from below, concluding the proof in this case.

Case 2: Let $d = 4$. We have $0 \le I_1 \le \vartheta(N\log N)^{-1}(Ns)\int_0^\infty a_r(0,0)\,dr = O(1/\log N)$, so that this term is again asymptotically negligible. Arguing as in case 1 we can now bound I_2

from above by

$$\frac{(1+\varepsilon)\vartheta\rho c_4}{N\log N}\int_0^{Ns-K}du\int_{u+K}^{Ns}dv\int_{v-u}^{v+u}\frac{dr}{r^2}+\frac{(1+\varepsilon)\vartheta\rho c_4}{2N\log N}\int_0^{Ns-K}du\int_{Ns}^{Nt}dv\int_{v-u}^{v+u}\frac{dr}{r^2}+O(1/\log N), \tag{4.7}$$

where $c_4=(2\pi)^{-2}(\det Q)^{-1/2}$. The first term in (4.7) is

$$\begin{aligned}
&\frac{(1+\varepsilon)\vartheta\rho c_4}{N\log N}\int_0^{Ns-K}du\int_{u+K}^{Ns}dv\Big(\frac{1}{v-u}-\frac{1}{v+u}\Big)\\
&\quad=\frac{(1+\varepsilon)\vartheta\rho c_4}{N\log N}\int_0^{Ns-K}du\Big(\log(Ns-u)-\log K-\log(Ns+u)+\log(2u+K)\Big)\\
&\quad=\frac{(1+\varepsilon)\vartheta\rho c_4}{N\log N}\Big[-(Ns-u)\log(Ns-u)-u-u\log K\\
&\qquad\qquad -(Ns+u)\log(Ns+u)+u+(u+K/2)\log(2u+K)-u\Big]_{u=0}^{u=Ns-K}\\
&\quad\longrightarrow(1+\varepsilon)\vartheta\rho c_4 s\quad\text{as } N\to\infty.
\end{aligned}$$

Now the second term in (4.7) is bounded above by

$$\begin{aligned}
&\frac{(1+\varepsilon)\vartheta\rho c_4}{2N\log N}\int_0^{Ns-K}du\int_{Ns}^{Nt}\frac{dv}{v-u}\\
&\quad=\frac{(1+\varepsilon)\vartheta\rho c_4}{2N\log N}\int_0^{Ns-K}du\big\{\log(Nt-u)-\log(Ns-u)\big\}\\
&\quad=\frac{(1+\varepsilon)\vartheta\rho c_4}{2N\log N}\Big[-(Nt-u)\log(Nt-u)-u+(Ns-u)\log(Ns-u)+u\Big]_{u=0}^{u=Ns-K}\\
&\quad=\frac{(1+\varepsilon)\vartheta\rho c_4}{2N\log N}\big\{Nt\log(Nt)-(N(t-s)+K)\log(N(t-s)+K)+K\log K-Ns\log(Ns)\big\}\\
&\quad\longrightarrow 0\quad\text{as } N\to\infty.
\end{aligned}$$

Thus letting $\varepsilon\to 0$ we see that $\limsup_{N\to\infty}I_2\le\vartheta\rho c_4 s$ in this case. Again, $\liminf_{N\to\infty}I_2$ can be bounded analogously, completing this part of the proof.

Case 3: Let $d\ge 5$. We have

$$\begin{aligned}
I_1&=\frac{2\vartheta}{N}\int_0^{Ns}du\int_u^{Ns}dv\,a_{v-u}(0,0)+\frac{\vartheta}{N}\int_0^{Ns}du\int_{Ns}^{Nt}dv\,a_{v-u}(0,0)\\
&=\frac{2\vartheta}{N}\int_0^{Ns}du\int_0^{\infty}dr\,a_r(0,0)+O\Big(N^{-1}\int_0^{Ns}du\int_{Ns}^{\infty}dv\,a_{v-u}(0,0)\Big)\\
&=2\vartheta s\int_0^{\infty}dr\,a_r(0,0)+O(1/N).
\end{aligned}$$

We decompose I_2 as

$$I_2=\frac{\vartheta\rho}{N}\int_0^{Ns}du\int_u^{Ns}dv\int_{v-u}^{v+u}dr\,a_r(0,0)+\frac{\vartheta\rho}{2N}\int_0^{Ns}du\int_{Ns}^{Nt}dv\int_{v-u}^{v+u}dr\,a_r(0,0). \tag{4.8}$$

The second term in (4.8) is bounded by

$$\begin{aligned}
&CN^{-1}\int_{Ns}^{Nt}dv\int_{v-Ns}^{\infty}dr\,a_r(0,0)+CN^{-1}\int_0^{Ns-1}du\int_{Ns}^{Nt}dv\int_{v-u}^{\infty}dr\,a_r(0,0)\\
&\quad=O(N^{-1})+O(N^{2-d/2})
\end{aligned}$$

by (2.3), while the first term is

$$\vartheta\rho\int_0^s du\int_0^{N(s-u)}dv\int_v^{v+2Nu}dr\,a_r(0,0)$$
$$\underset{N\to\infty}{\longrightarrow}\vartheta\rho s\int_0^\infty dv\int_v^\infty dr\,a_r(0,0)=\vartheta\rho s\int_0^\infty du\,ua_u(0,0).$$

This completes the case $d\geq 5$.

Now let us consider the situation $\mathscr{L}(\xi_0)=\Lambda_\vartheta$. Then we have

$$\begin{aligned}\mathbb{E}^{\Lambda_\vartheta}\left[X_s^N X_t^N\right] &= \frac{\vartheta}{h_d(N)^2}\int_0^{Ns}du\int_0^{Nt}dv\,a_{|v-u|}(0,0)\\ &\quad+\frac{\vartheta\rho}{2h_d(N)^2}\int_0^{Ns}du\int_0^{Nt}dv\int_{|v-u|}^\infty dr\,a_r(0,0)=:I_1+I_2'.\end{aligned}$$

The computations for $d=4$ and $d\geq 5$ are entirely analogous to those above, and will be omitted. Let us briefly comment on the case $d=3$ in this situation. I_1 is again negligible, and choosing K large enough we can now bound I_2' from above by

$$\begin{aligned}&\frac{2(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K}du\int_{u+K}^{Ns}\frac{dv}{(v-u)^{1/2}}+\frac{(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K}du\int_{Ns}^{Nt}\frac{dv}{(v-u)^{1/2}}+O(N^{-1/2})\\ &=\frac{2(1+\varepsilon)\vartheta\rho c_3}{N^{3/2}}\int_0^{Ns-K}du\big[(Ns-u)^{1/2}+(Nt-u)^{1/2}\big]+O(N^{-1/2})\\ &\underset{N\to\infty}{\longrightarrow}\frac{4}{3}(1+\varepsilon)\vartheta\rho c_3\big(t^{3/2}+s^{3/2}-(t-s)^{3/2}\big).\end{aligned}$$

We conclude the proof as above. □

4.3 Fourth centred multi-time moments

First of all we establish a general formula for centred multi-time moments.

Lemma 4.7 *Let* $0<t_1<\cdots<t_n$.

$$\begin{aligned}&\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\prod_{j=1}^n\left(\xi_{t_j}(0)-\vartheta\right)\right]\\ &=\sum_{j=1}^{n-1}\vartheta^j\sum_{m=n+1-j}^{n}\sum_{\substack{M\subset\{1,\dots,n\}\\|M|=m}}(-1)^{n-m}\sum_{\substack{\{\Lambda_1,\dots,\Lambda_{m+j-n}\}\\ \text{partition of } M}}\prod_{i=1}^{m+j-n}\Big(\sum_{x_i}w_{\Lambda_i}(0,x_i)\Big).\end{aligned}\tag{4.9}$$

Proof We have

$$\begin{aligned}
&\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\prod_{j=1}^{n}\left(\xi_{t_j}(0)-\vartheta\right)\right] \\
&= (-\vartheta)^n + \sum_{\emptyset\neq M\subset\{1,\dots,n\}} (-\vartheta)^{n-|M|}\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\prod_{i\in M}\xi_{t_i}(0)\right] \\
&= (-\vartheta)^n + \sum_{\emptyset\neq M\subset\{1,\dots,n\}} (-\vartheta)^{n-|M|}\sum_{k=1}^{|M|}\sum_{\substack{\{\Lambda_1,\dots,\Lambda_k\}\\ \text{partition of } M}}\vartheta^k\prod_{i=1}^{k}\Big(\sum_{x_i} w_{\Lambda_i}(0,x_i)\Big),
\end{aligned}$$

where we applied Proposition 4.4 for $f_i = \mathbf{1}_{\{0\}}$. For all other parameters fixed we can consider this as a polynomial in ϑ. There is no constant term. To obtain ϑ^n we must partition any given $M \subset \{1,\dots,n\}$ into $k = |M|$ subsets, i.e. into $|M|$ singletons. Thus the coefficient of ϑ^n is

$$(-1)^n + \sum_{\emptyset\neq M\subset\{1,\dots,n\}} (-1)^{n-|M|}\sum_{x_1,\dots,x_{|M|}}\prod_{i\in M} w_{\{i\}}(0,x_i) = \sum_{M\subset\{1,\dots,n\}}(-1)^{n-|M|} = (1-1)^n = 0,$$

because $\sum_x w_{\{i\}}(0,x) = \sum_x a_{t_i}(x,0) = 1$ for each i.

In order to obtain the term for ϑ^j (for $j \in \{1,\dots,n-1\}$) in (4.9) we have to partition M into $k = |M| + j - n$ subsets. Hence the coefficient of ϑ^j is

$$\sum_{m=n+1-j}^{n}\ \sum_{\substack{M\subset\{1,\dots,n\}\\ |M|=m}} (-1)^{n-m} \sum_{\substack{\{\Lambda_1,\dots,\Lambda_{m+j-n}\}\\ \text{partition of } M}}\ \prod_{i=1}^{m+j-n}\Big(\sum_{x_i} w_{\Lambda_i}(0,x_i)\Big). \tag{4.10}$$

This completes the proof. □

Let us specialise now to $n = 4$, and let $t_1 < \cdots < t_4$ be fixed for the moment.

Lemma 4.8 *Let $t_1 < \cdots < t_4$. Then*

$$\begin{aligned}
&\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\prod_{j=1}^{4}\left(\xi_{t_j}(0)-\vartheta\right)\right] \\
&= \vartheta\sum_{x\in\mathbb{Z}^d} w_{\{1,2,3,4\}}(0,x) + \vartheta^2 \sum_{\substack{\{\Lambda_1,\Lambda_2\}\\ \text{partition of } \{1,2,3,4\}\\ |\Lambda_1|=|\Lambda_2|=2}}\Big(\sum_{x_1\in\mathbb{Z}^d} w_{\Lambda_1}(0,x_1)\Big)\Big(\sum_{x_2\in\mathbb{Z}^d} w_{\Lambda_2}(0,x_2)\Big).
\end{aligned}$$

Proof We see from (4.9) that the coefficient of ϑ^1 is

$$\sum_{x\in\mathbb{Z}^d} w_{\{1,2,3,4\}}(0,x)$$

and the coefficient of ϑ^2 is

$$
\begin{aligned}
& - \sum_{\substack{M\subset\{1,2,3,4\}\\ |M|=3}} \sum_{x\in\mathbb{Z}^d} w_M(0,x) \quad + \sum_{\substack{\{\Lambda_1,\Lambda_2\}\\ \text{partition of } \{1,2,3,4\}}} \Big(\sum_{x_1\in\mathbb{Z}^d} w_{\Lambda_1}(0,x_1)\Big)\Big(\sum_{x_2\in\mathbb{Z}^d} w_{\Lambda_2}(0,x_2)\Big) \\
&= \sum_{\substack{\{\Lambda_1,\Lambda_2\}\\ \text{partition of } \{1,2,3,4\}\\ |\Lambda_1|=|\Lambda_2|=2}} \Big(\sum_{x_1\in\mathbb{Z}^d} w_{\Lambda_1}(0,x_1)\Big)\Big(\sum_{x_2\in\mathbb{Z}^d} w_{\Lambda_2}(0,x_2)\Big)
\end{aligned}
$$

because the terms with $|M|=3$ are cancelled by partitions in the second sum which have one singleton (we use again that $\sum_x w_{\{i\}}(0,x)=1$). The coefficient of ϑ^3 vanishes: In (4.10) we have to sum m from 2 to 4. The term for $m=2$ contains the sum over the following subsets (which are then only trivially partitioned) with a $+$-sign:

$$\{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\}.$$

For $m=3$ we have to sum over the subsets $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, $\{2,3,4\}$, giving the following list of partitions (each with a $-$-sign):

$$
\begin{aligned}
&\big\{\{1,2\},\{3\}\big\},\big\{\{1,3\},\{2\}\big\},\big\{\{2,3\},\{1\}\big\},\ \big\{\{1,2\},\{4\}\big\},\big\{\{1,4\},\{2\}\big\},\big\{\{2,4\},\{1\}\big\},\\
&\big\{\{1,3\},\{4\}\big\},\big\{\{1,4\},\{3\}\big\},\big\{\{3,4\},\{1\}\big\},\ \big\{\{2,3\},\{4\}\big\},\big\{\{2,4\},\{3\}\big\},\big\{\{3,4\},\{2\}\big\}.
\end{aligned}
$$

For $m=4$ we have only the full set $M=\{1,2,3,4\}$ which can be partitioned into three subsets in one of the following ways:

$$
\begin{aligned}
&\big\{\{1\},\{2\},\{3,4\}\big\},\big\{\{1\},\{2,3\},\{4\}\big\},\big\{\{1,4\},\{2\},\{3\}\big\},\\
&\big\{\{1,2\},\{3\},\{4\}\big\},\big\{\{1\},\{2,4\},\{3\}\big\},\big\{\{1,3\},\{2\},\{4\}\big\},
\end{aligned}
$$

and the corresponding terms have a $+$-sign. Using the fact that singleton partitions do not contribute because $\sum_x w_{\{i\}}(0,x)=1$ we see that for each of the six possible subsets M with 2 elements, $\sum_x w_M(0,x)$ is counted twice with a $+$- and twice with a $-$-sign.

Putting all this together we obtain the assertion. □

Now we are prepared to estimate the fourth moments of the increments of the centred occupation time for $d\ge 4$. Recall that $g(x)=\mathcal{G}\mathbf{1}(x)=\int_0^\infty a_t(0,x)\,dt$ with $\mathcal{G}f(x)=\int_0^\infty S_tf(x)\,dt$. Let $g^*(x):=\mathcal{G}(\mathcal{G}\mathbf{1})(x)=\int_0^\infty ta_t(0,x)\,dt$. It is well known that $g(x)<\infty$ for $d\ge 3$ and $g^*(x)<\infty$ for $d\ge 5$. Furthermore define $g_t(x):=\int_0^t a_s(0,x)\,ds$ and $g_t^*(x):=\int_0^t sa_s(0,x)\,ds$. Note that $g(x)\le g(0)$ for all $x\in\mathbb{Z}^d$.

Lemma 4.9 *For each $t_0>0$ there is a $C=C(t_0,\vartheta,d)$ such that for all $0\le s\le t\le t_0$, $\mu\in\{\mathcal{H}(\vartheta),\Lambda_\vartheta\}$ we have*

$$\mathbb{E}^\mu\left[\left(\int_{Ns}^{Nt}(\xi_u(0)-\vartheta)\,du\right)^4\right] \le \begin{cases} CN^2(\log N)^2(t-s)^2, & d=4,\\ CN^2(t-s)^2, & d\ge 5.\end{cases}$$

Proof It suffices to show that

$$\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\left(\int_{T+Ns}^{T+Nt}(\xi_u(0)-\vartheta)\,du\right)^4\right] \le \begin{cases} CN^2(\log N)^2(t-s)^2, & d=4,\\ CN^2(t-s)^2, & d\ge 5,\end{cases} \tag{4.11}$$

holds uniformly in N and $T \geq 0$. Let $0 \leq s \leq t \leq t_0$ and abbreviate $\delta := t - s$. By Lemma 4.8 we get

$$\begin{aligned}
&\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\left(\int_{T+Ns}^{T+Nt}(\xi_u(0)-\vartheta)\,du\right)^4\right] \\
&= 4!\int_{T+Ns}^{T+Nt}dt_1\int_{t_1}^{T+Nt}dt_2\int_{t_2}^{T+Nt}dt_3\int_{t_3}^{T+Nt}dt_4\,\mathbb{E}^{\mathcal{H}(\vartheta)}\left[\prod_{j=1}^{4}(\xi_{t_j}(0)-\vartheta)\right] \\
&= 4!\left[\vartheta\sum_{x\in\mathbb{Z}^d}\int_{T+Ns}^{T+Nt}dt_1\ldots\int_{t_3}^{T+Nt}dt_4\,w_{\{1,2,3,4\}}(0,x)\right. &(4.12)\\
&\quad\left.+\vartheta^2\sum_{\substack{\{\Lambda_1,\Lambda_2\}\\ \text{partition of }\{1,2,3,4\}\\ |\Lambda_1|=|\Lambda_2|=2}}\int_{T+Ns}^{T+Nt}dt_1\ldots\int_{t_3}^{T+Nt}dt_4\left(\sum_{x_1\in\mathbb{Z}^d}w_{\Lambda_1}(0,x_1)\right)\left(\sum_{x_2\in\mathbb{Z}^d}w_{\Lambda_2}(0,x_2)\right)\right]. &(4.13)
\end{aligned}$$

Note that all the terms on the r.h.s. are positive, also recall that the $w_\Lambda(\cdot,\cdot)$ depend on $t_1,\ldots,t_4$, even though this dependence is not explicitly stated.

Let us first treat the (easier) term in line (4.13) without the constant factor $4!\vartheta^2$. For a given $\Lambda = \{i,j\} \subset \{1,2,3,4\}$, with $i < j$, say, $w_\Lambda(0,x)$ is a sum of two terms representing the two different possible ancestral trees of two related particles (see Appendix C, case $n = 2$). The term for tree 1 can be estimated as follows:

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}\int_{T+Ns}^{T+Nt}dt_i\int_{t_i}^{T+Nt}dt_j\,a_{t_i}(x,0)a_{t_j-t_i}(0,0)\\
&\quad=\int_0^{N\delta}dt_i\int_{t_i}^{N\delta}dt_j\,a_{t_j-t_i}(0,0)\;\leq\;g(0)N\delta.
\end{aligned}$$

Tree 2 yields the following expression:

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d}\int_{T+Ns}^{T+Nt}dt_i\int_{t_i}^{T+Nt}dt_j\int_0^{t_i}du\sum_{y\in\mathbb{Z}^d}a_u(x,y)a_{t_i-u}(y,0)a_{t_j-u}(y,0)\\
&=\int_{T+Ns}^{T+Nt}dt_i\int_{t_i}^{T+Nt}dt_j\int_0^{t_i}du\,a_{t_i+t_j-2u}(0,0)\\
&=\frac{1}{2}\int_0^{N\delta}dt_i\int_{t_i}^{N\delta}dt_j\int_{t_j-t_i}^{t_j+t_i+2T+2Ns}du\,a_u(0,0)\\
&\leq\frac{1}{2}\int_0^{N\delta}dt_i\int_0^{N\delta-t_i}dv\int_v^\infty du\,a_u(0,0)\;\leq\;\frac{1}{2}\int_0^{N\delta}dt_i\int_0^{Nt_0}dv\int_v^\infty du\,a_u(0,0).
\end{aligned}$$

Now $a_u(0,0) \leq C|1\wedge u|^{-d/2}$ by (2.3), hence we see that this is bounded by $CN\delta$ for $d \geq 5$ and by $CN(\log N)\delta$ for $d = 4$. Combining we see that each of the terms appearing in (4.13) obeys a bound as in (4.11).

Now let us turn to the term in (4.12) without the constant factor $4!\vartheta$. Recall that $w_{\{1,2,3,4\}}(0,x)$ represents a sum over 52 different types of trees given in Appendix C (case $n = 4$), each of which represents a possible ancestral structure among four chosen individuals. Let us decompose this sum according to the number of splitting nodes of the tree.

0) There is only one tree without any splitting node, namely tree 1. The corresponding term is

$$\int_{T+Ns}^{T+Nt} dt_1 \int_{t_1}^{T+Nt} dt_2 \int_{t_2}^{T+Nt} dt_3 \int_{t_3}^{T+Nt} dt_4 \sum_{x\in\mathbb{Z}^d} a_{t_1}(x,0)a_{t_2-t_1}(0,0)a_{t_3-t_2}(0,0)a_{t_4-t_3}(0,0)$$
$$= \int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4\, a_{t_2-t_1}(0,0)a_{t_3-t_2}(0,0)a_{t_4-t_3}(0,0) \le \frac{1}{2} g(0)^2 N^2\delta^2$$

by simply integrating out t_4 and then t_3 and estimating $a_{t_2-t_1}(0,0) \le 1$.

1) There are 11 trees with one splitting node (trees 2, 3, 5, 7, 9, 17, 21, 29, 33, 41, 45).

First of all we consider trees in which the root is followed by the splitting node (trees 9, 17, 21, 29, 33, 41, 45). Each of the corresponding terms can be estimated as follows. Let 1 and $i \in \{2,3,4\}$ be the (labels of the) direct successors of the splitting node. We perform the sum over x and then over y. For fixed $T + Ns \le t_1 \le t_i \le T + Nt$ let us integrate out t_j, $j \ne 1, i$. Each such integral yields at most a factor $g(0)$, thus the term can be estimated as

$$g(0)^2 \int_{T+Ns}^{T+Nt} dt_1 \int_{t_1}^{T+Nt} dt_i \int_0^{t_1} ds\, a_{t_1+t_i-2s}(0,0) \le \frac{1}{2} g(0)^3 N^2\delta^2.$$

Now let us consider trees in which the root is followed by the node labelled by 1 (trees 2, 3, 5, 7). In case of tree 2 we perform the sum over x and we estimate the integral over t_4 by $g(0)$. After that we perform the sum over y, thus the term for tree 2 can be estimated by

$$g(0) \int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3\, a_{t_2-t_1}(0,0)(t_3-t_2)a_{t_3-t_2}(0,0)$$
$$\le g(0)N\delta \int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3\, a_{t_2-t_1}(0,0)a_{t_3-t_2}(0,0) \le g(0)^3 N^2\delta^2.$$

In case of trees 3, 5 and 7 the integrals over t_3 and t_4 can be estimated by $g(0)$. We perform the sum over x and then over y and we end up with

$$g(0)^2 \int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2\, (t_2-t_1)a_{t_2-t_1}(0,0) \le g(0)^2 N\delta \int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2\, a_{t_2-t_1}(0,0) \le g(0)^3 N^2\delta^2.$$

2) There are 25 trees with two splitting nodes (trees 4, 6, 8, 10, 11, 13, 15, 18, 19, 22, 23, 25, 27, 30, 31, 34, 35, 37, 39, 42, 43, 46, 47, 49, 51).

a) Consider tree 4 (trees 6, 8 can be treated analogously). We perform the sum over x, we estimate $a_{s_2-s_1}(y_1,y_2) \le 1$ and then we perform the sum over y_1 and y_2. Thus we get the estimate

$$\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{t_1}^{t_2} ds_1 \int_{s_1}^{t_3} ds_2\, a_{t_2-t_1}(0,0)a_{t_3+t_4-2s_2}(0,0)$$
$$= \frac{1}{2}\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{t_1}^{t_2} ds_1 \int_{t_4-t_3}^{t_3+t_4-2s_1} du\, a_{t_2-t_1}(0,0)a_u(0,0)$$
$$\le \frac{1}{2}\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2\, (t_2-t_1)a_{t_2-t_1}(0,0) \int_0^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{t_4-t_3}^{t_3+t_4} du\, a_u(0,0)$$
$$\le \frac{1}{2}N^2\delta^2 \int_0^{Nt_0} dv\, v a_v(0,0) \int_0^{Nt_0} dr \int_r^{3Nt_0} du\, a_u(0,0).$$

Using again $a_u(0,0) \le C|1 \wedge u|^{-d/2}$ we see that this is bounded by $CN^2\delta^2$ for $d \ge 5$ and by $CN^2(\log N)^2\delta^2$ for $d = 4$.

b) Consider tree 10 (trees 22, 34, 46 can be treated analogously). We estimate the integral over t_4 by $g(0)$, we perform the sum over x and then over y_1 and y_2 to get

$$g(0)\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{-T-Ns}^{t_1} ds_1\,(t_3-t_2)a_{t_1+t_2-2s_1}(0,0)a_{t_3-t_2}(0,0).$$

Now we estimate the integral over s_1 by $g(0)$, hence we obtain

$$g(0)^2\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3\,(t_3-t_2)a_{t_3-t_2}(0,0) \le g(0)^2\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_0^{Nt_0} dt_3\, t_3 a_{t_3}(0,0).$$

As $a_u(0,0) \le C|1 \wedge u|^{-d/2}$, this is bounded by $CN^2\delta^2$ for $d \ge 5$ and by $CN^2(\log N)\delta^2$ for $d = 4$.

c) Consider tree 11 (trees 13, 15, 23, 25, 27, 35, 37, 39, 47, 49, 51 can be treated analogously). We perform the sum over x and then over y_1 and we estimate the integral over t_4 by $g(0)$. Then we change the order of integration of s_1 and s_2, estimate the integral over s_1 by $g(0)$ and perform the sum over y_2 to get

$$\begin{aligned}
& g(0)^2\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{-T-Ns}^{t_2} ds_2\, a_{t_2+t_3-2s_2}(0,0) \\
= \;& \frac{1}{2}g(0)^2\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3-t_2}^{t_2+t_3+2T+2Ns} du\, a_u(0,0) \\
\le \;& \frac{1}{2}g(0)^2\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_0^{Nt_0} dv \int_v^{\infty} du\, a_u(0,0).
\end{aligned}$$

Again we use (2.3) to see that this is bounded by $CN^2\delta^2$ for $d \ge 5$ and by $CN^2(\log N)\delta^2$ for $d = 4$.

d) Consider tree 18 (trees 19, 30, 31, 42, 43 can be treated analogously). We perform the sum over x and then over y_1 and we estimate the integral over t_2 by $g(0)$. Then we change the order of integration of s_1 and s_2, estimate the integral over s_1 by $g(0)$ and perform the sum over y_2 to get

$$g(0)^2\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{-T-Ns}^{t_3} ds_2\, a_{t_3+t_4-2s_2}(0,0)$$

As in c) this is bounded by $CN^2\delta^2$ for $d \ge 5$ and by $CN^2(\log N)\delta^2$ for $d = 4$.

3) There are 15 trees with three splitting nodes (12, 14, 16, 20, 24, 26, 28, 32, 36, 38, 40, 44, 48, 50, 52). Up to permutation of labels, there are two different types, represented e.g. by tree 12 and tree 20, respectively.

a) Consider tree 12 (trees 14, 16, 24, 26, 28, 36, 38, 40, 48, 50, 52 can be treated analogously). We perform the sum over x and then over y_1 and get

$$\begin{aligned}
&\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{-T-Ns}^{t_1} ds_1 \int_{s_1}^{t_2} ds_2 \int_{s_2}^{t_3} ds_3 \sum_{y_2,y_3\in\mathbb{Z}^d} \\
&\qquad a_{t_1+s_2-2s_1}(0,y_2)a_{t_2-s_2}(y_2,0)a_{s_3-s_2}(y_2,y_3)a_{t_3-s_3}(y_3,0)a_{t_4-s_3}(y_3,0).
\end{aligned}$$

We need to distinguish $d \geq 5$ and $d = 4$.

First consider $d \geq 5$. Here we estimate the integral over t_4 by $g(0)$, then we perform the sum over y_3. After that we estimate the integral over t_3 by $g^*(0)$ and then we perform the sum over y_2 to arrive at the estimate

$$g(0)g^*(0)\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{-T-Ns}^{t_1} ds_1 (t_2 - s_1) a_{t_1+t_2-2s_1}(0,0) \leq \frac{1}{4} g(0)g^*(0)^2 N^2\delta^2.$$

Now let $d = 4$. We change the order of integration of s_1 and s_2 to estimate the integral over s_1 by $\frac{1}{2}g(y_2)$, thus

$$\frac{1}{2}\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{-T-Ns}^{t_2} ds_2 \int_{s_2}^{t_3} ds_3 \sum_{y_2,y_3\in\mathbb{Z}^4} g(y_2)$$
$$a_{t_2-s_2}(y_2,0)a_{s_3-s_2}(y_2,y_3)a_{t_3-s_3}(y_3,0)a_{t_4-s_3}(y_3,0).$$

Now we change the order of integration of t_2 and t_3 and then the order of t_2 and s_2 to estimate the integral over t_2 by $g(y_2)$, then we change the order of integration of s_2 and s_3 to estimate the integral over s_2 by $g(y_3 - y_2)$, thus

$$\frac{1}{2}\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{-T-Ns}^{t_3} ds_3 \sum_{y_2,y_3\in\mathbb{Z}^4} g(y_2)^2 g(y_3-y_2) a_{t_3-s_3}(y_3,0)a_{t_4-s_3}(y_3,0).$$

Then we estimate the integral over t_4 and then the integral over s_3 by $g(y_3)$, thus we get the estimate

$$\frac{1}{4}N^2\delta^2 \sum_{y_2,y_3\in\mathbb{Z}^d} g(y_2)^2 g(y_3-y_2) g(y_3)^2$$

Finally note that $\lim_{||x||\to\infty} g(x)||x||^2/\log||x|| = 0$ (see [Law94], and recall that the Green's function of a discrete random walk and its continuous-time analogue agree), hence for any $\varepsilon > 0$ we have $g(x) \leq C||x||^{-2+\varepsilon}$. Thus we can estimate

$$\begin{aligned}\sum_{y,z\in\mathbb{Z}^4} g(y)^2 g(z-y) g(z)^2 \;&\leq\; C\int_1^\infty du \int_{u+1}^\infty dv\, u^3 v^3 u^{-4+2\varepsilon}(v-u)^{-2+\varepsilon} v^{-4+2\varepsilon} \\ &= C\int_1^\infty \frac{du}{u^{1-2\varepsilon}} \int_{u+1}^\infty dv\, \frac{1}{v^{1-2\varepsilon}}(v-u)^{-2+\varepsilon} \leq C\int_1^\infty \frac{du}{u^{2-4\varepsilon}}\int_{u+1}^\infty \frac{dv}{(v-u)^{2-\varepsilon}} < \infty\end{aligned}$$

if $\varepsilon < 1/4$. We end up with an estimate $C'N^2\delta^2$.

b) Consider tree 20 (trees 32 and 44 can be treated analogously). We perform the sum over x and then over y_1. After that we change the order of integration of s_1, s_2 and s_3 to estimate the integral over s_1 by $g(0)$. Then we perform the sums over y_2 and y_3 and we end up with the estimate

$$\begin{aligned} g(0)&\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{-T-Ns}^{t_1} ds_2 \int_{-T-Ns}^{t_3} ds_3\, a_{t_1+t_2-2s_2}(0,0)a_{t_3+t_4-2s_3}(0,0) \\ \leq\; &\frac{1}{4}g(0)\int_0^{N\delta} dt_1 \int_{t_1}^{N\delta} dt_2 \int_{t_2}^{N\delta} dt_3 \int_{t_3}^{N\delta} dt_4 \int_{t_2-t_1}^\infty du \int_{t_4-t_3}^\infty dv\, a_u(0,0)a_v(0,0) \\ \leq\; &\frac{1}{4}g(0)\int_0^{N\delta} dt_1 \int_0^{Nt_0} dt_2 \int_{t_2}^\infty du\, a_u(0,0) \int_0^{N\delta} dt_3 \int_0^{Nt_0} dt_4 \int_{t_4}^\infty dv\, a_v(0,0)\end{aligned}$$

By (2.3) this is bounded by $CN^2\delta^2$ for $d \geq 5$ and by $CN^2(\log N)^2\delta^2$ for $d = 4$.

□

5 Proof of Theorem 1

Here we complete the proof of Theorem 1. In view of Proposition 3.1 it suffices to check that the sequence X^N, $N \in \mathbb{N}$, is tight (e.g. in the space of all continuous processes, equipped with the norm of locally uniform convergence). In order to do so we use the well-known criterion on moments of increments, stating that a sequence of processes X^N is tight (and furthermore, any limit point has continuous paths) if there exist $\alpha, \beta > 0$ such that for each $t_0 > 0$

$$\mathbb{E}\big[(X_t^N - X_s^N)^\alpha\big] \le C(t-s)^{1+\beta} \tag{5.1}$$

holds uniformly in N and $0 \le s < t \le t_0$ (see e.g. [Kal97] Corollary 14.9).

For $d \ge 4$, with $\alpha = 4$ and $\beta = 1$, this is the content of Lemma 4.9. In the case $d = 3$ it turns out that second moments ($\alpha = 2$, $\beta = 1/2$ in (5.1)) suffice. The corresponding estimate is provided in Lemma 5.1 below. □

Lemma 5.1 *Let $d = 3$ and $\mu \in \{\mathcal{H}(\vartheta), \Lambda_\vartheta\}$. For each $t_0 \ge 0$ there exists a constant $C = C(t_0, \vartheta)$ such that*

$$\mathbb{E}^\mu\big[(X_t^N - X_s^N)^2\big] \le C(t-s)^{3/2} \quad \forall\, 0 \le s \le t \le t_0 \tag{5.2}$$

holds uniformly in N.

Proof Note that for any initial distribution μ we have

$$0 \le \mathbb{E}^\mu\big[(X_t^N - X_s^N)^2\big] = \frac{1}{N^{3/2}} \int_{Ns}^{Nt} du \int_{Ns}^{Nt} dv\, \mathrm{Cov}^\mu(\xi_u(0), \xi_v(0)),$$

thus we see from Corollary 4.5 that $\mathbb{E}^{\mathcal{H}(\vartheta)}\big[(X_t^N - X_s^N)^2\big] \le \mathbb{E}^{\Lambda_\vartheta}\big[(X_t^N - X_s^N)^2\big]$ and it is hence sufficient to consider the stationary initial distribution Λ_ϑ. By stationarity, we can assume without loss of generality that $s = 0$, $t \le t_0$. Put $\varphi(r) := \mathrm{Cov}^{\Lambda_\vartheta}(\xi_r(0), \xi_0(0))$. We have $0 \le \varphi(r) \le C(1 \wedge r^{-1/2})$ by Corollary 4.5 and (2.3). This allows to estimate

$$\begin{aligned}
&\mathbb{E}^{\Lambda_\vartheta}\big[(X_t^N - X_0^N)^2\big] \\
&\quad = \frac{2}{N^{3/2}} \int_0^{Nt} du \int_u^{Nt} dv\, \varphi(v-u) = 2N^{1/2} \int_0^t du \int_u^t dv\, \varphi(N(v-u)) \le 2tN^{1/2} \int_0^t dw\, \varphi(Nw) \\
&\quad \le 2CN^{1/2} \left\{ t^2 \mathbf{1}(Nt \le 1) + t\mathbf{1}(Nt > 1) \Big[\frac{1}{N} + \int_{1/N}^t ds\, \frac{1}{\sqrt{Ns}}\Big] \right\} \\
&\quad = 2Ct^{3/2} \Bigg\{ (Nt)^{1/2}\mathbf{1}(Nt \le 1) + (Nt)^{-1/2}\mathbf{1}(Nt > 1) \\
&\qquad\qquad + (t/N)^{-1/2}\mathbf{1}(Nt > 1)\Big(2\sqrt{t/N} - \frac{2}{N}\Big) \Bigg\} \\
&\quad \le 6Ct^{3/2}.
\end{aligned}$$

□

A Auxiliary results

Proposition A.1 (Rebolledo) *Let $(Z_t^N)_{t \ge 0}$ with $Z_t^N = (Z_t^{N,1}, \dots, Z_t^{N,n})$ be a sequence of $\mathbb{R}^n$-valued martingales with $\mathbb{E}[Z_t^{N,k}]^2 < \infty$ and which fulfils the following assumptions* *(i)* $\langle Z^{N,k}, Z^{N,l}\rangle_t \underset{N\to\infty}{\longrightarrow} \sigma_k \delta_{k,l} t$ *in probability.*

(ii) $\max_{t\le T} \Delta Z_t^{N,k} \le c_N$ *with* $c_N \underset{N\to\infty}{\longrightarrow} 0$ *(where* $\Delta Z_t := Z_t - Z_{t-}$*).*

Then the process Z^N *converges in distribution to an n-dimensional Brownian motion where the k-th component has variance parameter* σ_k*.*

This is Proposition II.1 in [Reb80].

Proposition A.2 (Local CLT) *Under the assumption of finite moment of order s*

$$\sup_{x\in\mathbb{Z}^d} \left(\left(\frac{|x|}{\sqrt{t}}\right)^s + 1\right) \left| a_t(0,x) - p_t(0,x)\Big[1 + \sum_{k=1}^{s-2} t^{-k/2} P_k\left(\frac{x}{\sqrt{t}}\right)\Big]\right| = o(t^{-(d+s-2)/2}),$$

where P_k *is a polynomial of degree* $3k$ *and*

$$p_t(0,x) = (2\pi t)^{-d/2} (\det Q)^{-1/2} \exp\left(-\frac{x^T Q^{-1} x}{2t}\right).$$

The local CLT for discrete time random walks can be found in [BR76] as Corollary 22.3. From that one can derive the corresponding result for continuous time. This can be done similarly to [AN72] page 113, where a result on the Galton-Watson process is transferred from discrete time to continuous time.

B A particular case of spatial renormalisation of the equilibrium in $d = 3$

Lemma B.1 *Assume* $\mathscr{L}(\xi_0) = \Lambda_\vartheta$*. Then the process* $\left(N^{-3/4} U_0^{NT}(\xi_0)\right)_{T\ge 0}$ *converges to a Gaussian process in the sense of finite dimensional distributions as* $N \to \infty$*.*

Proof It suffices to show that for any $m \in \mathbb{N}$, $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and $T_1, \dots, T_m \ge 0$ the distributions of the random variables $N^{-3/4}\big(\alpha_1 U_0^{NT_1}(\xi_0) + \dots + \alpha_m U_0^{NT_m}(\xi_0)\big)$ have a Gaussian limit. We use the decomposition of ξ_0 under Λ_ϑ into equilibrium clans $\xi^{k,l}$ as in [Zäh02], (3.14). Put $Y_{k,l}^{(N)} := N^{-3/4}\langle \sum_{n=1}^m \alpha_n u_{NT_n}, \xi^{k,l}\rangle$. In order to invoke the CLT on p.$\tilde{1}$7 in [Zäh02] it remains to check condition (3.5) ((3.3) and (3.4) there follow from our covariance computations), i.e. that

$$\lim_{N\to\infty} \mathbb{E}\Big[\sum_k \sum_{l=1}^{N_k} |Y_{k,l}^{(N)}|^3\Big] = 0. \tag{B.3}$$

Note that

$$\begin{aligned} |Y_{k,l}^{(N)}|^3 &\le N^{-9/4} \sum_{x,y,z} \sum_{n_1,n_2,n_3=1}^m |\alpha_{n_1}\alpha_{n_2}\alpha_{n_3}| u_{NT_{n_1}}(x) u_{NT_{n_2}}(y) u_{NT_{n_3}}(z) \xi^{k,l}(x)\xi^{k,l}(y)\xi^{k,l}(z) \\ &\le N^{-9/4} m^3 \alpha^3 \sum_{x,y,z} u_{NT}(x) u_{NT}(y) u_{NT}(z) \xi^{k,l}(x)\xi^{k,l}(y)\xi^{k,l}(z) \end{aligned}$$

with $\alpha := \max_{1\le i\le m} |\alpha_i|$, $T := \max_{1\le i\le m} T_i$, so that it suffices to check (B.3) in the case $m = 1$. This is the content of Lemma B.2, see below. □

Lemma B.2 *Let $\{\xi^{k,l}, k \in \mathbb{N}, 1 \le l \le N_k\}$ be the decomposition of ξ_0 under Λ_ϑ into equilibrium clans as given in [Zäh02], (3.14), $T \ge 0$. Then*

$$\mathbb{E}\Big[\sum_k \sum_{l=1}^{N_k} |Y_{k,l}^{(N)}|^3\Big] \le C(\rho,\vartheta)(T+1)^2 N^{-1/4}.$$

Proof We have

$$\begin{aligned}\mathbb{E}\Big[\sum_k \sum_{l=1}^{N_k} |Y_{k,l}^{(N)}|^3\Big] &= N^{-9/4}\mathbb{E}\Big[\sum_k \sum_{l=1}^{N_k} \sum_{x,y,z} u_{NT}(x)u_{NT}(y)u_{NT}(z)\xi^{k,l}(x)\xi^{k,l}(y)\xi^{k,l}(z)\Big] \\ &= N^{-9/4}\sum_k \theta_k \int \sum_{x,y,z} u_{NT}(x)u_{NT}(y)u_{NT}(z)\xi(x)\xi(y)\xi(z)\tilde{P}_{0,k}(d\xi) \\ &= N^{-9/4}\vartheta \sum_x u_{NT}(x) \int \sum_{y,z} u_{NT}(y)u_{NT}(z)\xi(y)\xi(z)Q^{(x)}(d\xi). \qquad \text{(B.4)}\end{aligned}$$

Using Lemma B.3 and Lemma B.4 we can rewrite this as

$$\begin{aligned}&\mathbb{E}\Big[\sum_k \sum_{l=1}^{N_k} |Y_{k,l}^{(N)}|^3\Big] = N^{-9/4}\vartheta \sum_x u_{NT}(x) \sum_{y,z} u_{NT}(y+x)u_{NT}(z+x) \int \xi(y)\xi(z)Q^{(0)}(d\xi) \\ &= N^{-9/4}\vartheta \sum_{x,y,z} u_{NT}(x)u_{NT}(y+x)u_{NT}(z+x) \\ &\quad \Big\{\delta_0(y)\delta_0(z) + \rho\delta_0(y)\int_0^\infty a_{2s}(0,z)\,ds + \rho\delta_0(z)\int_0^\infty a_{2s}(0,y)\,ds + \delta_{yz}\int_0^\infty a_{2s}(0,y)\,ds \\ &\quad + 2\rho^2 \int_0^\infty ds \int_s^\infty dt \sum_w a_s(0,w)a_s(w,y)a_{2t-s}(w,z) \\ &\quad + \rho\int_0^\infty ds \int_0^s du \sum_w a_{2s-u}(0,w)a_u(w,y)a_u(w,z)\Big\} \\ &\le N^{-9/4}\vartheta \sum_x u_{NT}(x)^3 + 2N^{-9/4}\vartheta\rho\Big(\textstyle\int_0^\infty a_{2s}(0,0)\,ds\Big)\sum_x u_{NT}(x)^2 \sum_z u_{NT}(z+x) \\ &\quad + N^{-9/4}\vartheta\Big(\textstyle\int_0^\infty a_{2s}(0,0)\,ds\Big)\sum_x u_{NT}(x)\sum_y u_{NT}(x+y)^2 \\ &\quad + 2N^{-9/4}\vartheta\rho^2 \sum_{x,y,z} u_{NT}(x)u_{NT}(y+x)u_{NT}(z+x)\int_0^\infty ds \int_s^\infty dt \sum_w a_s(0,w)a_s(w,y)a_{2t-s}(w,z) \\ &\quad + N^{-9/4}\vartheta\rho \sum_{x,y,z} u_{NT}(x)u_{NT}(y+x)u_{NT}(z+x)\int_0^\infty ds \int_0^s du \sum_w a_{2s-u}(0,w)a_u(w,y)a_u(w,z) \\ &=: S_1 + S_2 + S_3 + S_4 + S_5.\end{aligned}$$

Noting that

$$\sum_x u_{NT}(x)^2 = \int_0^{NT} ds \int_0^{NT} dt \sum_x a_s(x,0)a_t(x,0) = \int_0^{NT} ds \int_0^{NT} dt\, a_{s+t}(0,0) \sim C\sqrt{NT}$$

we obtain

$$S_1, S_2, S_3 \le \text{Const.} \times N^{-9/4} \times NT \sum_x u_{NT}(x)^2 = O(T^{3/2}N^{-3/4}).$$

Using symmetry of a we find

$$\frac{S_4}{2\vartheta\rho^2} = N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \Big\{ \sum_{x,y,z} a_{t_1}(0,-x)a_{t_2}(-x,y)a_{t_3}(-x,z) \int_0^{\infty} ds \int_s^{\infty} dt \sum_w a_s(0,w)a_s(w,y)a_{2t-s}(w,z)\Big\}$$

$$= N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \int_0^{\infty} ds \frac{1}{2}\int_s^{\infty} dt \sum_{x',w} a_{t_1}(0,x')a_{t_2+s}(x',w)a_{t_3+t}(x',w)a_s(w,0).$$

As $a_{t_2+s}(x',w) = a_{t_2+s}(0,w-x') \le a_{t_2+s}(0,0) \le C(1+t_2+s)^{-3/2}$ we can estimate

$$\frac{S_4}{2\vartheta\rho^2} \le \frac{C}{2}N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \int_0^{\infty} ds\,(1+t_2+s)^{-3/2}\int_0^{\infty} dt\, a_{t_1+t_3+t+s}(0,0)$$

$$\le C^2 N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \int_0^{\infty} ds\,(1+t_2+s)^{-3/2}(1+t_1+t_3+s)^{-1/2}$$

$$\le C^2 N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_3 \frac{1}{\sqrt{1+t_1+t_3}}\int_0^{NT} dt_2 \int_0^{\infty} \frac{ds}{(1+t_2+s)^{3/2}}$$

$$= 2C^2 N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_3 \frac{1}{\sqrt{1+t_1+t_3}}\int_0^{NT} \frac{dt_2}{\sqrt{1+t_2}} \le C'T^2N^{-1/4}.$$

We treat S_5 similarly:

$$\frac{S_5}{\vartheta\rho} = N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \Big\{ \sum_{x,y,z} a_{t_1}(0,-x)a_{t_2}(-x,y)a_{t_3}(-x,z) \int_0^{\infty} ds \int_0^{s} du \sum_w a_{2s-u}(0,w)a_u(w,y)a_u(w,z)\Big\}$$

$$= N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \int_0^{\infty} ds \int_0^{s} du \sum_{x',w} a_{t_1}(0,x')a_{t_2+u}(x',w)a_{t_3+u}(x',w)a_{2s-u}(0,w)$$

$$\le N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \int_0^{\infty} ds \int_0^{s} du\, C(1+u+t_2)^{-3/2}a_{t_1+t_3+2s}(0,0)$$

$$\le 2CN^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \frac{1}{\sqrt{1+t_2}}\int_0^{\infty} ds\, C(1+t_1+t_3+s)^{-3/2}$$

$$= 4C^2N^{-9/4}\int_0^{NT} dt_1 \int_0^{NT} dt_2 \int_0^{NT} dt_3 \frac{1}{\sqrt{1+t_2}}\frac{1}{\sqrt{1+t_1+t_3}} \le C'T^2N^{-1/4}.$$

The proof is completed by combining these estimates. □

Lemma B.3

$$\int \xi(y)\xi(z)Q^{(0)}(d\xi) = \delta_0(y)\delta_0(z) + \rho\delta_0(y)\int_0^{\infty} a_{2s}(0,z)\,ds + \rho\delta_0(z)\int_0^{\infty} a_{2s}(0,y)\,ds + \delta_{yz}\int_0^{\infty} a_{2s}(0,y)\,ds + 2\rho^2\int_0^{\infty} ds \int_s^{\infty} dt \sum_v a_s(0,v)a_s(v,y)a_{2t-s}(v,z) + \rho\int_0^{\infty} ds \int_0^{s} du \sum_w a_{2s-u}(0,w)a_u(w,y)a_u(w,z).$$

Proof We use the well-known representation of ξ under $Q^{(0)}$ as $\delta_0 + \sum_{i=1}^{\infty} \xi^{(i)}$, where $\xi^{(i)}$ are branching random walks founded at the time points $(T_1, T_2, \dots)$ of a Poisson process with rate ρ along the backbone path Y which follows a^T-motion starting from 0. Thus

$$\int \xi(y)\xi(z)Q^{(0)}(d\xi) = \mathbb{E}\left[\left(\delta_0(y) + \textstyle\sum_i \xi^{(i)}(y)\right)\left(\delta_0(z) + \textstyle\sum_i \xi^{(i)}(z)\right)\right]$$
$$= \delta_0(y)\delta_0(z) + \delta_0(y)\mathbb{E}\left[\textstyle\sum_i \xi^{(i)}(z)\right] + \delta_0(z)\mathbb{E}\left[\textstyle\sum_i \xi^{(i)}(y)\right] + \mathbb{E}\left[\textstyle\sum_{i,j} \xi^{(i)}(y)\xi^{(j)}(z)\right] \tag{B.5}$$

Now

$$\mathbb{E}\left[\textstyle\sum_i \xi^{(i)}(y)\right] = \rho \int_0^\infty \hat{a}_{2s}(0,y), \quad \mathbb{E}\left[\textstyle\sum_i \xi^{(i)}(z)\right] = \rho \int_0^\infty \hat{a}_{2s}(0,z), \tag{B.6}$$

furthermore

$$\mathbb{E}\left[\textstyle\sum_{i,j} \xi^{(i)}(y)\xi^{(j)}(z)\middle|(Y_s), T_1, T_2, \dots\right]$$
$$= \sum_{i \neq j} a_{T_i}(Y_{T_i}, y)a_{T_j}(Y_{T_j}, z) + \sum_i \left\{\delta_{yz} a_{T_i}(Y_{T_i}, y) + \rho \int_0^{T_i} ds \sum_w a_{T_i - s}(Y_{T_i}, w)a_s(w,y)a_s(w,z)\right\}.$$

Note that $\mathbb{E}\Big[\sum_{i\neq j} f(T_i)g(T_j)\Big] = \rho \int_0^\infty f(s)\,ds \times \rho \int_0^\infty g(t)\,dt$ (for suitable $f,\ g$), so

$$\mathbb{E}\left[\textstyle\sum_{i,j} \xi^{(i)}(y)\xi^{(j)}(z)\middle|(Y_s)\right] = \rho^2 \int_0^\infty a_s(Y_s, y)\,ds \int_0^\infty a_t(Y_t, z)\,dt + \delta_{yz}\int_0^\infty a_s(Y_s, y) ds$$
$$+ \rho \int_0^\infty ds \sum_w \int_0^s du\, a_{s-u}(Y_s, w)a_u(w,y)a_u(w,z).$$

This yields

$$\mathbb{E}\left[\textstyle\sum_{i,j} \xi^{(i)}(y)\xi^{(j)}(z)\right]$$
$$= 2\rho^2 \int_0^\infty ds \int_s^\infty dt \sum_{v,w} a_s^T(0,v)a_{t-s}^T(v,w)a_s(v,y)a_t(w,z)$$
$$+ \delta_{yz}\int_0^\infty \hat{a}_{2s}(0,y)\,ds + \rho\int_0^\infty ds \int_0^s du \sum_{v,w} a_s^T(0,v)a_{s-u}(v,w)a_u(w,y)a_u(w,z). \tag{B.7}$$

The claim follows by combining (B.5), (B.6) and (B.7) and the assumed symmetry of $a = \hat{a}$. □

Lemma B.4 *For symmetric a we have $a_t(0,x) \leq a_t(0,0)$ for all $t \geq 0$, $x \in \mathbb{Z}^d$.*

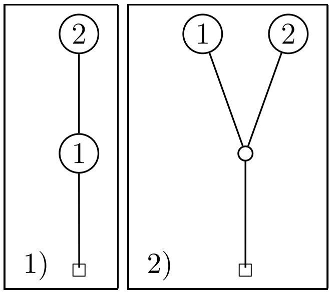
C The trees in $\mathbb{T}_{\{1,\dots,n\}}$, $n = 2,3,4$

Recall that by Proposition 4.2

$$w_{\{1,\dots,n\}}(r,x) = \sum_{\tau \in \mathbb{T}_{\{1,\dots,n\}}} S\Big(\tau, \{1,\dots,n\}; r, x\Big).$$

Here we list all trees in $\mathbb{T}_{\{1,\dots,n\}}$ and some of the terms $S\Big(\tau, \{1,\dots,n\}; r, x\Big)$ for $n = 2,3,4$.

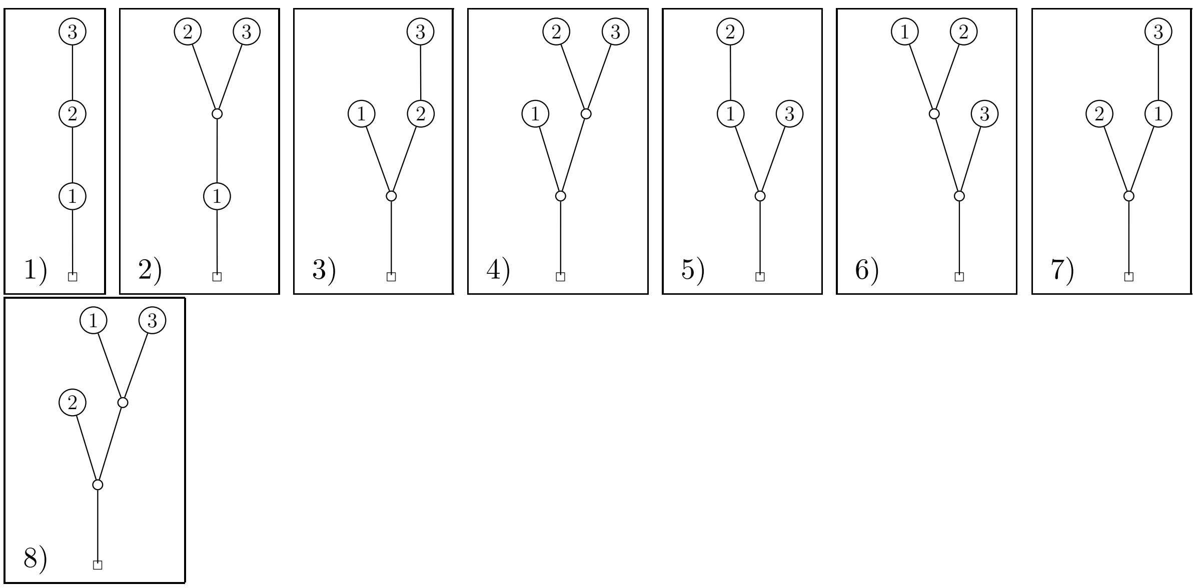
Case $n = 2$**:**



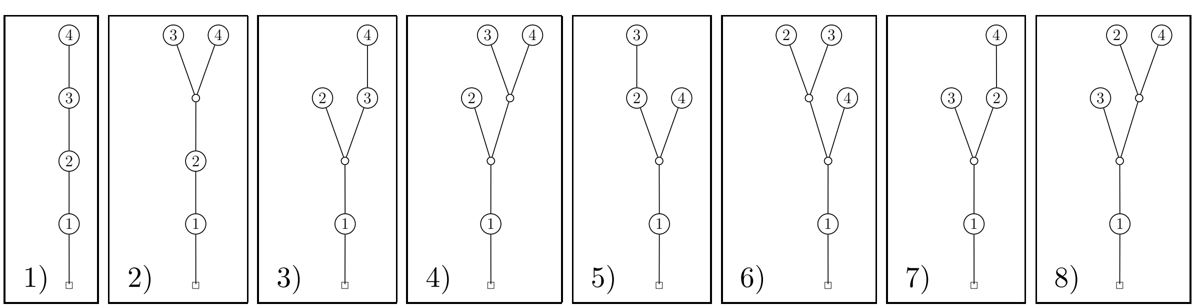
1) $\sum_{z_1} a_{t_1}(x, z_1) f_1(z_1) \sum_{z_2} a_{t_2 - t_1}(z_1, z_2) f_2(z_2)$

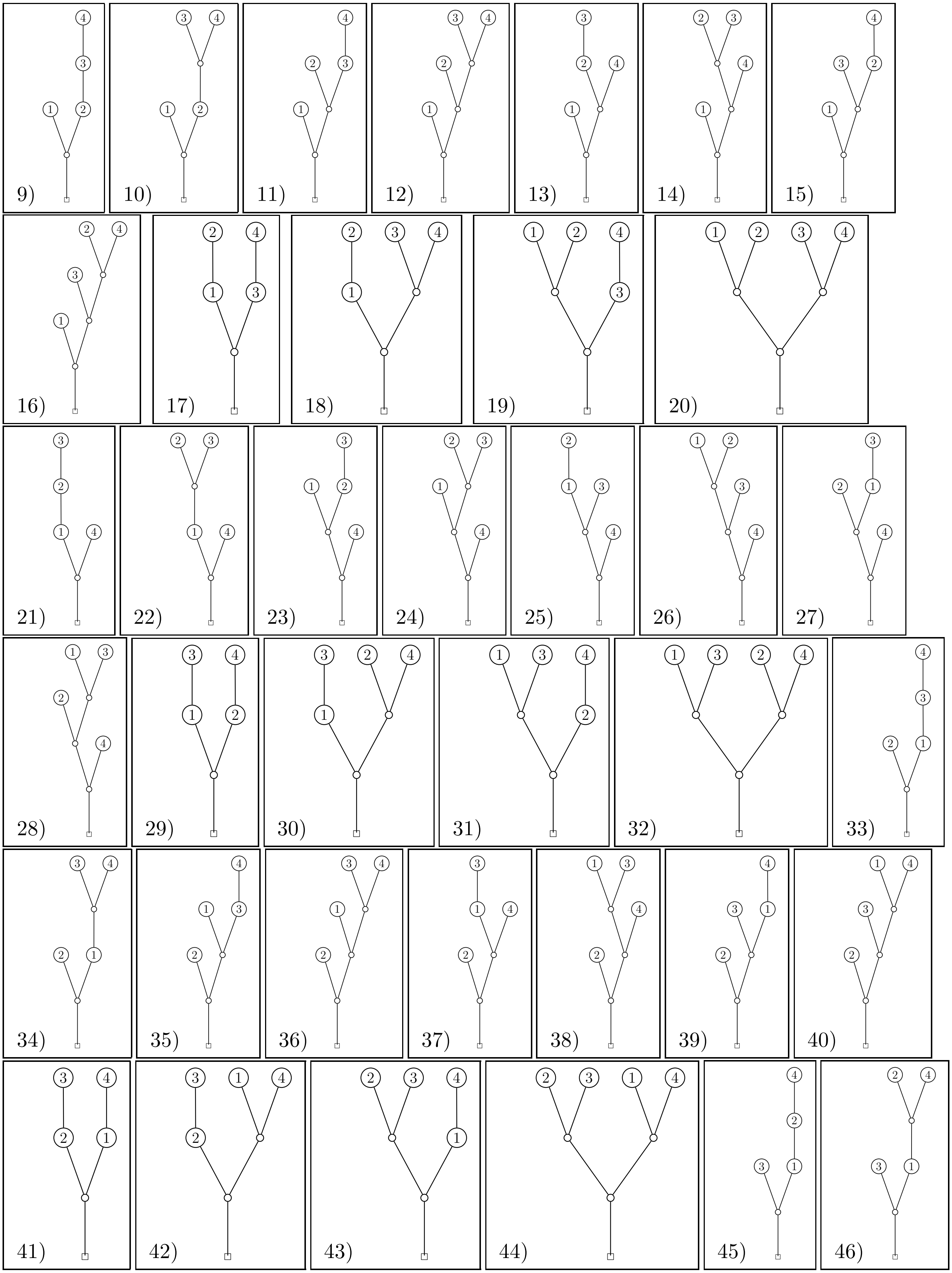
2) $\rho \int_0^{t_1} ds_1 \sum_{y_1} a_{s_1}(x, y_1) \sum_{z_1} a_{t_1 - s_1}(y_1, z_1) f_1(z_1) \sum_{z_2} a_{t_2 - s_1}(y_1, z_2) f_2(z_2)$

Case $n = 3$**:**



Case $n = 4$**:**

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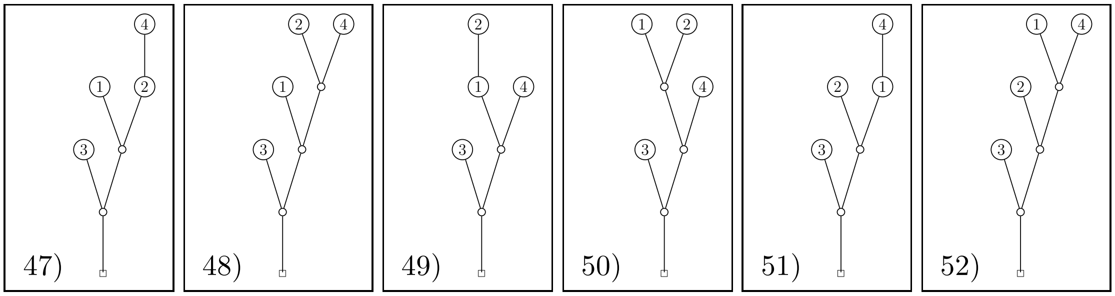

Some selected terms

2) $\rho \sum_{z_1} a_{t_1}(x,z_1) f_1(z_1) \sum_{z_2} a_{t_2-t_1}(z_1,z_2) f_2(z_2) \int_{t_2}^{t_3} ds_1 \sum_{y_1} a_{s_1-t_2}(z_2,y_1) \sum_{z_3} a_{t_3-s_1}(y_1,z_3) f_3(z_3)$
$\sum_{z_4} a_{t_4-s_1}(y_1,z_4) f_4(z_4)$

8) $\rho^2 \sum_{z_1} a_{t_1}(x,z_1) f_1(z_1) \int_{t_1}^{t_3} ds_1 \sum_{y_1} a_{s_1-t_1}(z_1,y_1) \sum_{z_3} a_{t_3-s_1}(y_1,z_3) f_3(z_3) \int_{s_1}^{t_2} ds_2$
$\sum_{y_2} a_{s_2-s_1}(y_1,y_2) \sum_{z_2} a_{t_2-s_2}(y_2,z_2) f_2(z_2) \sum_{z_4} a_{t_4-s_2}(y_2,z_4) f_4(z_4)$

20) $\rho^3 \int_0^{t_1} ds_1 \sum_{y_1} a_{s_1}(x,y_1) \int_{s_1}^{t_1} ds_2 \sum_{y_2} a_{s_2-s_1}(y_1,y_2) \sum_{z_1} a_{t_1-s_2}(y_2,z_1) f_1(z_1)$
$\sum_{z_2} a_{t_2-s_2}(y_2,z_2) f_2(z_2) \int_{s_1}^{t_3} ds_3 \sum_{y_3} a_{s_3-s_1}(y_1,y_3) \sum_{z_3} a_{t_3-s_3},(y_3,z_3) f_3(z_3)$
$\sum_{z_4} a_{t_4-s_3}(y_3,z_4) f_4(z_4)$

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