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Estimates of Weyl sums over subsequences of natural numbers

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Contents

1. Introduction

The method of trigonometric sums is one of the few general methods which allows to solve a large class of different problems in number theory and its applications.

A trigonometric sum is a sum of the form

$$S_P = \sum_{x=1}^{P} e^{2\pi i f(x)}$$

where the summation is over some set of natural numbers and $f(x)$ is an arbitrary function which takes real values for natural numbers.

The most important trigonometric sums are those with $f(x)$ – a polynomial of degree $n \in \mathbb{N}$. Those sums are called Weyl sums.

The central question in this theory is to get as effective as possible upper bounds for the absolute value of Weyl sums.

Under some conditions on the coefficients of the polynomial $f(x)$ the first non-trivial estimates where derived by H. Weyl [1] and I.M. Winogradov [2], who gave the bounds

$$|S_P| \leq \quad CP^{1-\frac{\gamma}{2^{n-1}}} \quad \text{and}$$

$$|S_P| \leq \quad CP^{1-\frac{\gamma}{n^2 \ln n}}, \quad \text{respectively}$$

($\gamma > 0, C > 0$ – are constants).

In [3] and [4] L.D. Pustyl'nikov found effective bounds not depending on n for almost all polynomials of large enough degree:

$$|S_P| \leq C(n) P^{1-\rho}$$

for all $\rho \in (0, \frac{1}{2})$ and $n > n_0(\rho)$. Moreover, he considered an approximation process which allows to get effective bounds for all polynomials of large enough degree. These investigations were continued in [5]. All these estimates are made for Weyl sums over infinite or finite intervals of consecutive natural numbers. But a lot of problems in number theory are connected with subsets of natural numbers as for instance the prime numbers and therefore they demand estiamtions of Weyl sums over subsets of natural numbers. This will be the aim of this paper. Namely, for a given subsequence $M : m_1 < m_2 < m_3 < \ldots$ of the naturals and $Q \in \mathbb{N}$ we are looking for upper bounds of sums of the type

$$S^M_{Q,P} = \sum_{k=Q+1}^{Q+P} e^{2\pi i f(m_k)}$$

with $f(x)$ – a polynomial of degree n. We are able to derive estimations corresponding to those of L.D. Pustyl'nikov. A special example of such subsequences is the sequence of primes.

The proof of these estimates consists of two parts.

The first part (§2) starts with a slight modified version of Mordell's method used in [6] and then follows exactly the number theoretical part in [3].

The second part (§4) uses a "non-invariant ergodic theory" of translations of the n-dimensional torus developed in §3.

In §5 we give some applications to the law of the distribution of the fractional part of a polynomial.

2. Definitions and notations

(1) T^n denotes the n-dimensional torus considered as $T^n = \{\alpha_1, \dots, \alpha_n | 0 \leq \alpha_1 < 1, \dots, 0 \leq \alpha_n \leq 1\}$.

(2) mes is the usual Lebesque (Haar) measure on T^n.

(3) For a real number β we write $\{\beta\}$ and $[\beta]$ for the fractional part and the integer part of β, respectively.

(4) For natural $n \geq m > 0 \binom{n}{m}$ denotes the binomial coefficient.

(5) Let $n > 1$ be natural and α_n be a fixed real number. The transformation $A_{n-1} = A_{n-1}(\alpha_n) : T^{n-1} \to T^{n-1}$ is defined by

$$A_{n-1} : \alpha = (\alpha_1, \dots, \alpha_{n-1}) \to \alpha' = (\alpha'_1, \dots, \alpha'_{n-1})$$

where

$$\alpha'_s = \sum_{\nu=0}^{n-s} \binom{s+\nu}{\nu} \alpha_{s+\nu} \mod 1, \quad 1 \leq s \leq n-1. \tag{1}$$

This transformation is a skew translation of T^{n-1} and hence invertible. The inverse is given by

$$\alpha_s = \sum_{\nu=0}^{n-s-1} \binom{s+\nu}{\nu} (-1)^\nu \alpha'_{s+\nu} + \binom{n}{n-s} (-1)^{n-s} \alpha'_n \mod 1. \tag{2}$$

(6) We call a subset of a complete metric space residual if it contains a countable intersection of open and dense sets. The complement of a residual set is called meager.

(7) For a Borel set B its characteristic function is denoted by χ_B.

3. The number theoretical part

In this chapter we use a modified version of Mordell's lemma used in [6]. After this we can completely follow the number theoretical part in [3]. For reasons of completenes we will repeat the main arguments used there.

Lemma 1. *(Mordell) Consider the simultaneous diophantine equations:*

$$\begin{gathered} x_1 + \dots + x_n = y_1 + \dots + y_n \\ - \\ x_1^n + \dots + x_n^n = y_1^n + \dots + y_n^n \end{gathered} \tag{3}$$

where $x_i, y_i \quad (i = 1, \dots, n)$ can take values in $\{m_1, \dots, m_P\}$ independently. Then the number of solutions of (3) does not exceed $n!P^n$.

The proof is essentially contained in [6]. $\square$

Lemma 2. *For $f(x) = \alpha_n x^n + \cdots + \alpha_1 x$ and $y \in \mathbb{N}$ we set*

$$\begin{aligned} f(x+y) - f(y) &= \alpha_n x^n + Y_{n-1}(y)x^{n-1} + \cdots + Y_1(y)x \\ Y(y) &= (Y_1(y), \ldots, Y_{n-1}(y)) \end{aligned} \tag{4}$$

then

$$Y(y) = \overline{A}_{n-1}^{y}(\alpha_1, \ldots, \alpha_{n-1})$$

where $\overline{A}_{n-1}^{y}$ is the y th power of the lift of the transformation A_{n-1} to $\mathbb{R}^{n-1}$.

Proof. By induction (see f.i. [3]). $\square$

Let Ψ_n be a complex valued function with $|\Psi_n(x)| \leq 1$. For $P \in \mathbb{N}$ we set

$$\begin{aligned} \tilde{S}^M_{n,P} &= \tilde{S}^M_{n,P}(\alpha) = \\ &= \sum_{k=1}^{P} \Psi_n(m_k) e^{2\pi i(\alpha_1 m_k + \cdots + \alpha_n m_k^n)} \end{aligned}$$

Lemma 3.

$$\int_0^1 \cdots \int_0^1 \left| \tilde{S}^M_{n,P} \right|^{2n} d\alpha_1 \ldots d\alpha_n < n! P^n. \tag{5}$$

Proof. Writing $\overline{\Psi}_n$ for the complex conjugate of Ψ_n and $N_\nu \quad (\nu = 1, \ldots, n)$ for the expression $x_1^\nu + \cdots + x_n^\nu - y_1^\nu - \cdots - y_n^\nu$ we rewrite

$$\int_0^1 \cdots \int_0^1 \left| \tilde{S}^M_{n,P} \right|^{2n} d\alpha_1 \ldots d\alpha_n =$$

$$= \int_0^1 \cdots \int_0^1 \left(\sum_{x_i, y_i} \Psi_n(x_1) \ldots \Psi_n(x_n) \overline{\Psi}_n(y_1) \ldots \overline{\Psi}_n(y_n) e^{2\pi i(\alpha_1 N_1 + \cdots + \alpha_n N_n)} \right) d\alpha_1 \ldots d\alpha_n =$$

$$= \sum_{x_i, y_i} \Psi_n(x_1) \ldots \Psi_n(x_n) \overline{\Psi}_1(y_1) \ldots \overline{\Psi}_n(y_n) \int_0^1 \cdots \int_0^1 e^{2\pi i(\alpha_1 N_1 + \cdots + \alpha_n N_n)} d\alpha_1 \ldots d\alpha_n$$

where the summation runs over all x_i, y_i out of the set $\{m_1, \ldots, m_P\}$. Now the integral on the right hand side of the equations is vanishing if at least one of the $N_i \quad (i = 1, \ldots, n)$ is non–zero and equals 1 if all $N_i = 0$. Hence the integral can't exceed the number of solutions of (3). But $|\Psi_n| \leq 1$. This proves (5). $\square$

Lemma 4. *For given $0 < \rho < \frac{1}{2}$ and $P \in \mathbb{N}$ we can find an open set Π_P with the properties*

$$i) \qquad mes\,(\Pi_P) \leq \frac{1}{P^{n(1-2\rho)}} \qquad and \tag{6}$$

$$ii) \qquad \left|\tilde{S}^M_{n,P}(\alpha)\right| \leq (n!)^{\frac{1}{2n}} P^{1-\rho} \qquad for\ \alpha \in T^n \backslash \Pi_P. \tag{7}$$

Proof. Let $\Pi_P \subset T^n$ be the open set such that

$$\left|\tilde{S}^M_{n,P}(\alpha)\right| > (n!)^{\frac{1}{2n}} P^{1-\rho} \text{ for } \alpha \in \Pi_P. \tag{8}$$

or

$$\left|\tilde{S}^M_{n,P}(\alpha)\right|^{2n} > n! P^{2n(1-\rho)} \tag{9}$$

By application of lemma 3 we can derive

$$n! P^{2n(1-\rho)} \text{mes}\,(\Pi_P) \leq n! P^n. \tag{10}$$

This together with (8) gives

$$\text{mes}\,(\Pi_P) \leq \frac{1}{P^{n(1-2\rho)}}.$$

□

Theorem 1. *Let $\mathbb{Q} \in \mathbb{N}$, ρ be a real number in $(0, \frac{1}{2})$ and $f(x) = a_1 x + \ldots a_n x^n$ a polynomial of degree $n > [1 + \frac{2}{1-2\rho}]$. Then for $\hat{\Pi}_Q = \bigcup\limits_{P=Q}^{\infty} \Pi_P$ the following properties hold:*

$$i) \qquad mes\,(\hat{\Pi}_Q) \leq \frac{2}{Q^{n(1-2\rho)-1}} < 1 \tag{11}$$

$$ii) \qquad |\tilde{S}^M_{n,P}(\alpha)| \leq (n!)^{\frac{1}{2n}} P^{1-\rho}\ for\ all\ P \geq Q\ if\ \alpha \in T^n \backslash \Pi_Q. \tag{12}$$

Proof. By virtue of lemma 4

$$\text{mes}\,(\hat{\Pi}_Q) \leq \sum_{P=Q}^{\infty} \frac{1}{P^{n(1-2\rho)}} \leq \int_Q^{\infty} (x-1)^{-n(1-2\rho)} dx.$$

The assumption of the theorem implies that $n(1-2\rho) > 2$ and consequently

$$\text{mes}\,(\hat{\Pi}_Q) < \frac{2}{Q^{n(1-2\rho)-1}} < 1.$$

The second statement follows immediately from the definition of $\hat{\Pi}_Q$ and Π_P. □

4. Pseudo–ergodicity

The main idea in [3] to obtain the bounds on Weyl sums is to use the ergodicity of the transformation $A_{n-1} = A_{n-1}(\alpha_n)$ as long as the main coefficient α_n is irrational. This attempt is not succesful for our purposes because, as we will see in §5, we have to take in account only iterates $A_{n-1}^{m_k}$ of A_{n-1} which are indicized by the subsequence M and not the entire natural series. Moreover, we can't apply ergodic theory to some iterate of A_{n-1} not knowing whether M contains an infinite arithmetic sequence. The aim of this chapter is to overcome these difficulties by mean of some "pseudo ergodic" properties, which we will desribe below. In contrast to usual ergodicity we are not dealing with invariant sets but with sets consisting of entire orbits of points under $A_{n-1}^{m_k}, m_k \in M$.

Definition 1. Let $E : X \to X$ be an invertible map of a topological space with invariant Borel–probability–measure μ. We say E is **pseudo-ergodic** with respect to the increasing subsequence of the natural numbers $M = \{m_1, m_2, \dots\}$ if for each Borel set $B \subset X$ of positive measure the set of points $x \in X$ for which

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_B(E^{m_k}(x)) = \mu(B) \tag{13}$$

has full measure (property PE for the set B).

Remark. (1) The notion of pseudo-ergodicity is stronger than the usual notion of ergodicity (It is the same if $m_k = k, \quad k = 1, 2, \dots$).

(2) Let $\tilde{B} = \bigcup_{k=1}^{\infty} E^{m_k}(B)$. By the invariance of μ we have

$$\begin{aligned}\int_B \chi_{\tilde{B}}(E^{m_k}(x))d\mu = \int_{E^{m_k}(B)} \chi_{\tilde{B}}(x)d\mu &= \mu(\tilde{B} \cap E^{m_k}(B)) \\ =&\mu(E^{m_k}(B)) = \mu(B).\end{aligned} \tag{14}$$

If E is pseudo-ergodic this implies

$$\begin{aligned}\mu(\tilde{B})\mu(B) = \int_B \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_{\tilde{B}}(E^{m_k}(x))d\mu = \\ \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \int_B \chi_{\tilde{B}}(E^{m_k}(x))d\mu = \\ \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \mu(B) = \mu(B).\end{aligned} \tag{15}$$

Hence, for all Borel sets B of positive measure

$$\mu(\tilde{B}) = 1.$$

In the following we will show that given M for almost all $\alpha \in [0,1)$ the transformation $A_{n-1}(\alpha)$ is pseudo-ergodic w.r.t.M.

Definition 2. The pseudo-ergodicity set $PE(M) \subset [0,1)$ of the subsequence M with respect to the transformation $A_{n-1}(\alpha)$ is defined as

$$PE(M) = \left\{\alpha \in [0,1) | A_{n-1}(\alpha) : T^{n-1} \to T^{n-1} \text{ is pseudo-ergodic w.r.t.} M\right\}.$$

For $k \in \mathbb{N}$ let us consider the following mapping $a_k(\alpha)$ of the circle $S^1 = \mathbb{R}/\mathbb{Z}$:

$$a_k(\alpha) = k\alpha \; (\text{mod } 1).$$

This mapping assigns to each $\alpha \in [0,1)$ the image of the origin projected to the first coordinate under a skew rotation A of the torus T^{n-1} of the form

$$A(\alpha_1, \dots, \alpha_{n-1}) = (\alpha_1 + k_1\alpha, \dots, \alpha_{n-1} + l_{n-1,1}\alpha_1 + \cdots + l_{n-1,n-2}\alpha_{n-2} + k_{n-1}\alpha)(\text{mod } 1)$$
$$l_{i,j} \in \mathbb{Z} \quad i = 2, \dots, n-1; \quad j = 1, \dots, n-2 \tag{16}$$

Proposition 1. *Let $U = [\xi_1, \xi_2] \subset S^1$ and a sequence of mappings $\{a_{k_i}\}$ be given. If the conditions*

$$\lim_{i \to \infty} k_i = \infty \tag{17}$$

are satisfied then the set

$$N(U) = \{\alpha \in [0,1) | a_{k_i} \notin U \text{ for all } i\} \tag{18}$$

is meager and has zero Lebesque measure.

The proof relies on the following lemma.

Lemma 5. *Let U as in proposition 1, $L = [\alpha_1, \beta_1] \cup \cdots \cup [\alpha_r, \beta_r]$ be a finite union of intervals in S^1. Then for $\varepsilon > 0$ there is a number $K = K(\varepsilon, L)$ such that for all a_k with*

$$k > K$$

and

$$N^{\underline{k}}(U, L) = \{\alpha \in [0,1) | \alpha \in L \text{ and } a_k(\alpha) \notin U\} \tag{19}$$

$$mes\ (N^{\underline{k}}(U, L)) \leq (1 + \varepsilon) mes\ (L)(1 - mes\ (U)) \tag{20}$$

holds. Moreover $N^k(U, L)$ again is a finite union of intervals with diameters not larger than k^{-1}.

Proof. First we fix $\varepsilon > 0, U$ and L. Let $\delta > 0$ be specified later.

Let ξ_M be the center of the complement of the interval $[\xi_1, \xi_2]$. By elementary properties of the map $a_l : \alpha \to l\alpha(\mod 1)$ the preimages of ξ_M^i under a_l are exactly l points in S^1 where neighbors have distance $\frac{1}{l}$. Moreover, the full preimage of the complements of $[\xi_1, \xi_2]$ are l intervals of length $\frac{1-\text{mes }(\xi_1,\xi_2)}{l}$ centered at the preimages of ξ_M^i.

From this extremely uniform distribution we see that for a given interval $[\alpha, \beta]$

$$\left[\frac{\text{mes }(\alpha,\beta)}{l}\right] \leq \#\left\{a_l^{-1}(\xi_M) \in [\alpha,\beta]\right\} \leq \left[\frac{\text{mes }(\alpha,\beta)}{l}\right] + 1.$$

Hence, if

$$k > \frac{1}{\{\min\limits_{1\leq i\leq r} \text{mes }(\alpha_i, \beta_i)\} \cdot \delta} \tag{21}$$

then

$$(1-\delta)\text{mes }(L) \leq \frac{\#\{a_k^{-1}(\xi_M) \in L\}}{k} \leq (1+\delta)\text{mes }(L). \tag{22}$$

The set N^k is contained in the union

$$\tilde{N}^k = \bigcup_{\tau \in a_k^{-1}(\xi_M)\cap L} \left(\tau - \frac{1-\text{mes }(\xi_1,\xi_2)}{2k}, \tau + \frac{1-\text{mes }(\xi_1,\xi_2)}{2k}\right).$$

By the above estimates

$$\begin{aligned}(1-\delta)\text{mes }(L)(1-\text{mes }(\xi_1,\xi_2)) \;&\leq\; \text{mes }(\tilde{N}^k) = \sum_{\tau\in a_k^{-1}(\xi_M)\cap L} \frac{1-\text{mes }(\xi_1,\xi_2)}{k} \leq \\ &\leq\; (1+\delta)\text{mes }(L)(1-\text{mes }(\xi_1,\xi_2)).\end{aligned}$$

□

Proof of the proposition: We fix U and a sequence $\{a_{k_i}\}$ subject to the assumptions of the proposition. We will inductively construct a nested sequence J_r, $r = 1, 2, \ldots$, of unions of intervals fulfilling

(1) $J_r = \bigcup\limits_{k=1}^{m(r)} I_k^{(r)}$, where $I_k^{(r)}$ are intervals in S^1
(2) $J_{r+1} \not\subseteq J_r$
(3) $\lim\limits_{r\to\infty} \text{mes }(J_r) = 0$
(4) $N(U) \subset \bigcap\limits_{r=1}^{\infty} J_r$.

The statement of the proposition follows immediately from 1.–4.

We set $J_1 = S^1$ and fix $0 < \varepsilon < \frac{\mu(U)}{1-\mu(U)}$. Let K be the number defined in the lemma applied to $L = S^1$.

By the assumption we can find an $i_2 \in \mathbb{N}$ such that

$$k_{i_2} > K.$$

The lemma then tells us the set $J_2 = N^{k_{i_2}}(U, S^1)$ is a union of intervals in S^1 satisfying

$$\begin{aligned} \text{mes}\,(J_2) &\leq (1+\varepsilon)(1-\text{mes}\,(U))\text{mes}\,(J_1) \\ &\leq (1+\varepsilon)(1-\text{mes}\,(U)) \end{aligned}$$

Let us assume we have already constructed $J_1 \supset \cdots \supset J_{r-1}$ with the properties

1. J_l is the finite union of intervals

3.' mes $(J_{l+1}) \leq (1+\varepsilon)(1-\text{mes}\,(U))\text{mes}\,(J_l)$

4.' $N^{k_{i_l}}(U, J_l) = J_{l-1}$

Again, applying the lemma to the set J_{r-1} we find a number K_r. To this number K_r there is a mapping $a_{k_{i_r}}$ in the sequence for which

$$k_{i_r} > K_r.$$

Then the lemma states that for $J_r = N^{k_{i_r}}(U, J_{r-1})$

$$\text{mes}\,(J_r) \leq (1+\varepsilon)(1-\text{mes}\,(U))\text{mes}\,(J_{r-1}).$$

So far we have constructed a sequence $\{J_r\}$ having properties 1., 2., 3.', 4.'. Since

$$N(U) \subset \bigcap_{r=1}^{\infty} N^{k_{i_r}}(U, J_{r-1}) = \bigcap_{r=1}^{\infty} J_r$$

property 4. holds.
By virtue of 3.' and $0 < \varepsilon < \frac{\text{mes}\,(U)}{1-\text{mes}\,(U)}$ we have

$$\begin{aligned} \text{mes}\,(N(U)) &\leq \text{mes}\,(J_r) \leq (1+\varepsilon)(1-\text{mes}\,(U))\text{mes}\,(J_{r-1}) \\ &\leq ((1+\varepsilon)(1-\text{mes}\,(U)))^{r-1}\text{mes}\,(J_1) \\ &\leq ((1+\varepsilon)(1-\text{mes}\,(U))^{r-1} \to 0 \;\; (r \to \infty) \end{aligned}$$

This completes the proof of the proposition. □

Theorem 2. *Let $M = \{m_k\}$ be a subsequence of the natural series. Then $PE(M)$ is a residual set of full measure.*

Proof. The proof consists of 5 parts. In the first part we derive an expression of the orbit of a point under $A(\alpha_n)$. In the second part we apply proposition 1. Part (3) and (4) – which will be proved below – give property PE for elementary open sets (paralleliples) and in (5) we derive PE for all Borel sets of positive measure as long as α_n is in a residual set of full measure.

(1) In this step we use a recursive formula for the M-orbit of a point under A to get explicit expressions for this M-orbit.

Let $(\alpha_1,\dots,\alpha_{n-1}) \in T^{n-1}$ and $\alpha_n \in S^1$ and write $(\alpha_1^{(k)},\dots,\alpha_{n-1}^{(k)})$ for $A^k(\alpha_n)$ $(\alpha_1,\dots,\alpha_{n-1})$, $k = 0,1,\dots$. Then from (1) we have

$$\begin{aligned}
\alpha_{n-1}^{(k+1)} &= \alpha_{n-1}^{(k)} + \binom{n}{1}\alpha_n \\
\alpha_{n-2}^{(k+1)} &= \alpha_{n-2}^{(k)} + \binom{n-1}{1}\alpha_{n-1}^{(k)} + \binom{n}{2}\alpha_n \\
&\cdots\cdots \\
\alpha_1^{(k+1)} &= \alpha_1^{(k)} + \binom{2}{1}\alpha_2^{(k)} + \cdots + \binom{n}{n-1}\alpha_n
\end{aligned} \tag{23}$$

with initial conditions

$$\alpha_j^{(0)} = \alpha_j \qquad j = 1,\dots,n-1.$$

By inductive insertion the solution from $\alpha_{n-j}^{(k)}$ into $\alpha_{n-j-1}^{(k)}$, $j = 1,\dots,n-1$ we derive:

$$\alpha_{n-1}^{(k)} = \alpha_{n-1} + k \cdot n \cdot \alpha_n$$

and

$$\alpha_{n-j}^{(k)} = \alpha_{n-j} + P_{n-j+1}^{1}(k)\alpha_{n-j+1} + \cdots + P_{n-1}^{j-1}(k)\alpha_{n-1} + P_n^j(k)\alpha_n$$

where $P_s^r(k)$ $(s = 1,\dots,n;\ r = 1,\dots,s-1)$ are polynomials with integer coefficients of degree r.

(2) By (1) the projection π_{n-1} of $A^{m_k}(\alpha)(0)$ to the $(n-1)$–st coordinate has the form

$$\alpha^{(k)} = \pi_{n-1}(A^{m_k}(\alpha))0)) = m_k \cdot n \cdot \alpha.$$

Therefore, proposition 1 implies that for all intervals U in S^1 the set $N(U)$ for the sequence $\{\pi_{n-1}(A^{m_k}(\alpha)(0))\}$ is meager and has zero measure. Choosing a countable base $\{U_i\}_1^\infty$ of the topology consisting of intervals we see that the set

$$N = \left\{\alpha \in S^1 | \alpha^{(k)} \text{ is not dense in } S^1\right\} \subset \bigcup_{i=1}^{\infty} N(U_i)$$

is meager and has zero measure. This means the set

$$D = \left\{\alpha \in S^1 | \alpha^{(k)} \text{ is dense }\right\}$$

is residual and has full measure.

(3) If $\alpha \in D$ we can find a subsequence $m_{k_j} = l_j$ of m_k such that for all $h, N \in \mathbb{N}$

$$\left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h \alpha^{(k_j)}}\right| < \frac{c(h)}{N} \tag{24}$$

for some constant c depending only on h.

(4) If $\alpha \in D$ and l_j is the subsequence from (3) then the sequence $\{A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1})\}$ is uniformly distributed (with respect to mes $^{(n-1)}$) on T^{n-1}

for mes $^{(n-1)}$ - a.e. $(\alpha_1,\dots,\alpha_{n-1})\in T^{n-1}$, i.e. for a.e. $(\alpha_1,\dots,\alpha_{n-1})$ and all parallelepipeds $\Pi=(\xi_1^1,\xi_2^1)\times\cdots\times(\xi_1^{n-1},\xi_2^{n-1})$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\chi_\Pi(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))=\mu(\Pi), \tag{25}$$

where χ_Π is the characteristic function of Π.

(5) Here we want to show that (13) holds for any set of positive measure. This immediately implies the theorem. First let U be an arbitrary open set and ε be fixed. For $M\in\mathbb{N}$ we can decompose U into

$$U=\bigcup_{s=1}^{r(M)}\Pi_s\cup V,\qquad \text{where mes}^{(n-1)}(V)<\frac{\varepsilon}{M}\text{ and }\Pi_s\text{ are disjoint parallelepipeds.}$$

by (4) for a.e. $(\alpha_1,\dots,\alpha_{n-1})$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\chi_U(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))\geq\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\chi_{\cup\Pi_s}(A^{l_j}(\alpha_1,\dots,\alpha_{n-1}D)$$

$$=\mu(\bigcup_{s=1}^{r}\Pi_s)\geq\mu(U)-\frac{\varepsilon}{M}.$$

On the other hand, by the invariance of mes $^{(n-1)}$

$$\lim_{N\to\infty}\int_{T^{n-1}}\frac{1}{N}\sum_{k=1}^{N}\chi_V(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))d\text{mes}^{(n-1)}(\alpha_1,\dots,\alpha_{n-1})=$$

$$=\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\int\chi_V(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1})d\text{mes}^{(n-1)}(\alpha_1,\dots,\alpha_{n-1})=$$

$$=\lim\frac{1}{N}\sum_{k=1}^{N}\text{mes}^{(n-1)}(V)=\text{mes}^{(n-1)}(V)<\frac{\varepsilon}{M}.$$

By Chebyshev's inequality

$$\text{mes}^{(n-1)}\{(\alpha_1,\dots,\alpha_{n-1})|\lim\frac{1}{N}\sum_{k=1}^{N}\chi_V(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))>$$

$$>M\cdot\text{mes}^{(n-1)}(V)\}\leq\frac{1}{M}.$$

Therefore, for $(\alpha_1,\dots,\alpha_{n-1})$ in a set of measure $1-\frac{1}{M}$

$$\begin{aligned}\lim_{N\to\infty}\sum_{k=1}^{N}\chi_U(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))&\leq\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\chi_{\cup\Pi_s}(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))+\\&+\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\chi_V(A^{l_j}(\alpha)(\alpha_1,\dots,\alpha_{n-1}))\leq\\&\leq\text{mes}^{(n-1)}(\bigcup_{s=1}^{r}\Pi_s)+M\text{mes}^{(n-1)}(V)<\\&<\text{mes}^{(n-1)}(U)+\varepsilon.\end{aligned}\tag{26}$$

Because M and ε were arbitrary we have (13) to hold for all open sets and hence for all sets ($\chi_{T^{n-1}\setminus U} = 1 - \chi_U$). For an arbitrary set B of positive measure using the regularity of mes $^{(n-1)}$ we can find for any $\delta > 0$ open sets U_δ and closed sets K_δ with

$$U_\delta \supset B \supset F_\delta$$

$$\text{mes}^{\,n-1}(U_\delta) - \delta < \text{mes}^{\,n-1}(B) < \text{mes}^{\,n-1}(F_\delta) + \delta.$$

Since (13) holds for all U_δ and F_δ and

$$\chi_{U_\delta} \geq \chi_B \geq \chi_{F_\delta}$$

(13) is proved for all sets B of positive measure.

Proof of (3)
Let β be irrational. We consider the sequence

$$b_j = j \cdot \beta(\mod 1)\; j = 1, 2, \ldots$$

then for $h \in \mathbb{Z}\setminus\{0\}$

$$\left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h b_j}\right| = \left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h j\beta}\right| = \frac{1}{N}\frac{|e^{2\pi i h N\beta} - 1|}{|e^{2\pi i h\beta} - 1|} \leq \frac{\tilde{C}(h)}{N}. \tag{27}$$

The Taylor expansion of the exponential function gives

$$\left|e^{2\pi i h(\beta+\theta)} - e^{2\pi i h\beta}\right| \leq 2\pi h\theta + \bar{o}(\theta). \tag{28}$$

Since $\alpha^{(k)}$ is dense in S^1 we can find a subsequence $\alpha^{(k_j)}$ satisfying

$$\left|b_j - \alpha^{(k_j)}\right| \leq \frac{1}{2^j} \qquad j = 1, 2, \ldots. \tag{29}$$

Combining (27), (28) and (29) we get

$$\begin{aligned} \left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h\alpha^{(k_j)}}\right| &= \left|\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h\beta_j} + \left(\frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h\alpha^{(k_j)}} - \frac{1}{N}\sum_{j=1}^{N} e^{2\pi i h\beta_j}\right)\right| \leq \\ &\leq \frac{\tilde{C}(h)}{N} + \frac{1}{N}\sum_{j=1}^{N}\left(2\pi h\frac{1}{2^j} + \bar{o}(\frac{1}{2^j})\right) \leq \frac{c(h)}{N} \end{aligned} \tag{30}$$

□

Proof of (5)
The proof is an application of the multidimensional Weyl criterion on T^{n-1} : $\{A^{l_j}(\alpha_n)(\alpha_1, \ldots, \alpha_{n-1})\}_{j=1}^{\infty}$ is uniformly distributed in T^{n-1} iff for all integer vectors $\underline{h} = (h_1, \ldots, h_{n-1}) \neq (0, \ldots, 0)$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\exp\left\{2\pi i(\underline{h}, A^{l_j}(\alpha_n)(\alpha_1, \ldots, \alpha_{n-1}))\right\} = 0 \tag{31}$$

where $(\underline{h}, \underline{s}) = h_1 s_1 + \cdots + h_{n-1} s_{n-1}$. For given $\underline{h} \neq (0, \ldots, 0)$ we consider the sums

$$S(N, Q, \alpha_1, \ldots, \alpha_{n-1}) = \frac{1}{N} \sum_{j=Q}^{N+Q-1} \exp\{2\pi i (\underline{h}, A^{l_j}(\alpha_n)(\alpha_1, \ldots, \alpha_{n-1}))\} \quad Q \in \mathbb{N}. \tag{32}$$

Using the expressions from (1) for $A^{l_j}(\alpha_n)(\alpha_1, \ldots, \alpha_{n-1})$ we can proceed

$$\begin{aligned}
&\int\limits_{T^{n-1}} \left|S(N, Q, \alpha_1, \ldots, \alpha_{n-1})\right|^2 d\alpha_1, \ldots, d\alpha_{n-1} = \\
&= \frac{1}{N^2} \sum_{k,j=Q}^{N+Q-1} \int\limits_{T^{n-1}} \exp\Big\{2\pi i \Big(\underline{h} A^{l_k}(\alpha_n)(\alpha_1, \ldots, \alpha_{n-1}) - \\
&\quad - A^{l_j}(\alpha_n)(\alpha_1, \ldots, \alpha_{n-1})\Big)\Big\} d\alpha_1, \ldots, \alpha_{n-1} = \\
&= \frac{1}{N^2} \sum_{k,j=Q}^{N+Q-1} \int\limits_{T^{n-1}} \exp\Big\{2\pi i \Big[\sum_{r=1}^{n-1} h_r \Big(\alpha_{n-r} + \sum_{s=1}^{r} P^s_{n-r+s}(l_k)\alpha_{n-r+s}\Big) - \\
&\quad - \sum_{r=1}^{n-1} h_r \Big(\alpha_{n-r} + \sum_{s=1}^{r} P^s_{n-r+s}(l_j)\alpha_{n-r+s}\Big)\Big]\Big\} d\alpha_1, \ldots, d\alpha_{n-1} = \\
&= \frac{1}{N^2} \sum_{k,l=Q}^{N+Q-1} \exp\Big\{2\pi i \sum_{r=1}^{n-1} h_r \left(P^r_n(l_k)\alpha_n - P^r_n(l_j)\alpha_n\right)\Big\} \times \\
&\quad \times \int\limits_{T^{n-1}} \exp\Big\{2\pi i \sum_{s=1}^{r-1} h_r \left(P^s_{n-r+s}(l_k) - P^s_{n-r+s}(l_j)\right) \alpha_{n-r+s}\Big\} d\alpha_1, \ldots, d\alpha_{n-1}
\end{aligned} \tag{33}$$

The integral on the right-hand-side takes value 1 if

$$\sum_{s=1}^{r-1} h_r \left(P^s_{n-r+s}(l_k) - P^s_{n-r+s}(l_j)\right) = 0, \tag{34}$$

simultaneously for all $r = 1, \ldots, n-1$. Otherwise its value is zero.

Lets fix $\underline{h}$ and assume that at least for one $r \in [2, \ldots, n-1]$ $h_r \neq 0$. Since all the P^s_{n-r+s} are polynomials, we can find a natural $Q = Q(h)$ such that (34) holds (under the above assumption) if and only if $k = j$ (If we choose Q large then all $l_k, = Q, Q+1, \ldots, N+Q$, are large). Consequently, in this case

$$\int\limits_{T^{n-1}} \left|S(N, Q, \alpha_1, \ldots, \alpha_{n-1})\right|^2 d\alpha_1, \ldots, d\alpha_{n-1} = \frac{1}{N}.$$

In the opposite case $h_2 = h_3 = \cdots = h_{n-1} = 0$ and

$$\begin{aligned}
&\int\limits_{T^{n-1}} \left|S(N, 1, \alpha_1, \ldots, \alpha_{n-1})\right|^2 d\alpha_1, \ldots, d\alpha_{n-1} = \\
&= \frac{1}{N^2} \sum_{k,l=1}^{N} \exp\left\{2\pi i h_1 (P^1_n(l_k)\alpha_n - P^1_n(l_j)\alpha_n)\right\} = \\
&= \frac{1}{N^2} \sum_{k,j=1}^{N} \exp\left\{2\pi i h_1 \alpha_n^{(l_k)} - 2\pi i h_1 \alpha_n^{(l_j)}\right\} = \left(\frac{1}{N} \sum_{j=1}^{N} e^{2\pi i h_1 \alpha_n^{(l_j)}}\right) \overline{\left(\frac{1}{N} \sum_{j01}^{N} e^{2\pi i h_1 \alpha_n^{(l_j)}}\right)}.
\end{aligned} \tag{35}$$

where the bar stands for complex conjugation. By virtue of (3) the integral can be estimated by

$$\int_{T^{n-1}} \left|S(N,1,\alpha_1,\dots,\alpha_{n-1})\right|^2 d\alpha_1,\dots,d\alpha_{n-1} \le \left(\frac{c(h)}{N}\right)^2 \text{ if } \alpha_n \in D.$$

This implies that for all $\underline{h} \neq (0,\dots,0)$ there is an Q such that

$$\sum_{N=1}^{\infty} \int_{T^{n-1}} \left|S(N^2,Q,\alpha_1,\dots,\alpha_{n-1})\right|^2 d\alpha_1,\dots,d\alpha_{n-1} < \infty.$$

Hence, $\int_{T^{n-1}} \sum_{N=1}^{\infty} |S(N^2,Q,\alpha_1,\dots,\alpha_{n-1})|^2 d\alpha_1,\dots,d\alpha_{n-1} < \infty$. This yields

$$\sum_{N=1}^{\infty} \left|S(N^2,Q,\alpha_1,\dots,\alpha_{n-1})\right|^2 < \infty \text{ for mes }^{(n-1)} - a.e.(\alpha_1,\dots,\alpha_{n-1}) \text{ and } \alpha_n \in D.$$

Therefore

$$\lim_{N\to\infty} S(N^2,Q,\alpha_1,\dots,\alpha_{n-1}) = 0 \text{ for mes }^{(n-1)} a.e.(\alpha_1,\dots,\alpha_{n-1}) \text{ and } \alpha_n \in D.$$

Now if $M^2 \le N < (M+1)^2$ then

$$|S(N,Q,\alpha_1,\dots,\alpha_{n-1})| \le |S(M^2,Q,\alpha_1,\dots,\alpha_{n-1})| + \frac{2M}{N} \le$$

$$\le |S(M^2,Q,\alpha_1,\dots,\alpha_{n-1})| + \frac{2}{\sqrt{N}}.$$

Remarking that

$$\lim_{N\to\infty} S(N,1,\alpha_1,\dots,\alpha_{n-1}) = \lim_{N\to\infty} S(N,Q,\alpha_1,\dots,\alpha_{n-1})$$

(these infinite Česaro means do not depend on the beginning.) We have shown that if $\alpha \in D$ for given $\underline{h}$ (31) holds for a.e. $(\alpha_1,\dots,\alpha_{n-1})$. From the countability of integer vectors $\underline{h}$ follows (31) for a.e. $(\alpha_1,\dots,\alpha_{n-1})$ independent of $\underline{h}$. □

5. Estimation of Weyl sums

We are now ready to state and prove the main theorems.

Theorem 3. *Let $M = \{m_k\}$ be a subsequence of the natural numbers,*

$$a_{n+1} \in PE(M), \quad 0 < \rho < \frac{1}{2}, \quad n > n_0 = n_0(\rho) = \left[1 + \frac{4}{1-2\rho}\right].$$

Then there is a set $B^M \subset T^n$ of full measure such that for $(a_1,\dots,a_n) \in B^M$ there is a $Q \in \mathbb{N}$ with

$$\left|S^M_{Q,P}\right| = \left|\sum_{k=Q+1}^{Q+P} e^{2\pi i(a_1 m_k + \dots + a_{n+1} m_k^{n+1})}\right| < (n!)^{\frac{1}{2n}} P^{1-\rho} \textit{ for all } P > Q.$$

Proof. Let us consider the following sequences $M_1 = M, M_2 = \{m_k - m_1\}_{k=2}, \ldots, M_i = \{m_k - m_{i-1}\}_{k=i}^{\infty}, \ldots$ and let $\hat{\Pi}_i$ be the set $\hat{\Pi}_i = \bigcup_{P=i+1}^{\infty} \Pi_P$ from theorem 1 for the sequence M_i and the function $\Psi_n(x) = e^{2\pi i a_{n+1} x^{n+1}}$. Then the set $\tilde{\Pi} = \bigcup_{i=1}^{\infty} \hat{\Pi}_i$ has the following properties

$$\begin{aligned} 1. \quad \text{mes } \left(\tilde{\Pi}\right) &\leq \sum_{i=1}^{\infty} \text{mes } (\hat{\Pi}_i) \leq \sum_{i=1}^{\infty} \frac{2}{i^{n(1-2\rho)-1}} \leq \\ &\leq \frac{2}{2^{n(1-2\rho)-2}} < 1 \end{aligned} \tag{36}$$

(here we used $n > [1 + \frac{4}{1-2\rho}]$ and theorem 1.)
and

$$\begin{aligned} 2. \quad \left|\tilde{S}_{n,P}^{M_i}\right| &= \left|\sum_{k=1}^{P} e^{2\pi i a_{n+1}(m_{k+i} - m_i)^{n+1}} \cdot e^{2\pi i (a_1(m_{i+k}-m_i) + \ldots + a_n(m_{i+k}-m_i)^n}\right| \\ &\leq (n!)^{\frac{1}{2n}} P^{1-\rho} \qquad \text{for all } P > i \text{ and } (a_1, \ldots, a_n) \in T^n \backslash \tilde{\Pi}. \end{aligned} \tag{37}$$

Because mes $(T^n \backslash \tilde{\Pi}) > 0$, by virtue of the definition of $PE(M)$ the set

$$B^M = \left\{(a_1, \ldots, a_n) \in T^n \middle| \exists k \ \text{ such that } A^{m_k}(a_1, \ldots, a_n) \in T^n \backslash \tilde{\Pi}\right\}$$

has full measure.

Now for $(a_1, \ldots, a_n) \in B^M$, Q such that $A^{m_Q}(a_1, \ldots, a_n) \in T^n \backslash \tilde{\Pi}$ we consider the sum

$$S'_{Q,P} = \sum_{k=1}^{P} e^{2\pi i (f((m_{k+Q} - m_Q) + m_Q) - f(m_Q))}$$

with $f(x) = a_1 x + \ldots + a_{n+1} x^{n+1}$.

Then

$$\begin{aligned} &\left|S'_{Q,P}\right| = \left|S^M_{Q,P}\right| \qquad \text{and} \\ &S'_{Q,P} = \sum_{k=1}^{P} \Psi_n(m_{k+Q} - m_Q) e^{2\pi i (Y_1(m_Q)(m_{k+Q} - m_Q) + \ldots + Y_n(m_Q)(m_{k+Q} - m_Q)^n)}. \end{aligned} \tag{38}$$

where $Y_i(y)$ is defined by (4). This means by (37)

$$\left|S^M_{Q,P}\right| = \left|S'_{Q,P}\right| = \left|\tilde{S}^{M_Q}_{n,P}\right| \leq (n!)^{\frac{1}{2n}} P^{1-\rho} \text{ for all } P > Q.$$

This completes the proof of the theorem. $\square$

We will now state two theorems which are conclusions of the above considerations and theorem 3. Their proofs are analogous to those in [3], [4] and therefore omitted.

Theorem 4. *Assume that the assumptions of theorem 3 hold. Let us write the vector* $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in T^{n-1}$ *as*

$$\alpha = \alpha' + \hat{A}^{m_k}_{n-1} \beta \qquad \beta \in T^{n-1} \backslash \Pi_{n-1}.$$

If $|\alpha'_j| \le \frac{q^{-j-\rho}}{2\pi(n-1)}$ *then for* $n > n_0(\rho), \alpha_n \in PE(M)$, *and* $2 \le P \le q$

$$\left|\sum_{k=1}^{P} e^{2\pi i(\alpha_1, m_k + \cdots + \alpha_n m_k^n)}\right| \le \left(m_k + (n-1)!\right)^{\frac{1}{2(n-1)}} + 1\Big) P^{1-\rho}$$

Remark. This theorem points to an approximation process for the coefficients of the polynomial where the degree of the approximation determines the estimation of the corresponding Weyl sum.

The next theorem concerns the remainder term in the law of distributions of the fractional part of a polynomial.

Theorem 5. *Assume the conditions of theorem 3 to hold. Let* $0 < \sigma \le 1, D_{Q,P}(\sigma)$ *be the number of integers* $Q \le k \le P$ *satisfying* $\{\alpha_1 m_k + \cdots + \alpha_n m_k^n\} < \sigma, \alpha_n \in PE(M)$, $\quad 0 < \rho < \frac{1}{2}$ *and* $n > n_0$. *Then for almost all* $(\alpha_1, \ldots, \alpha_{n-1}) \in T^{n-1}$ *there is a* $Q = Q(\alpha_1, \ldots, \alpha_n, n) \in \mathbb{N}$ *such that*

$$D_{Q,P}(\sigma) = P\sigma + \lambda_P(\sigma)$$

with

$$|\lambda_P(\sigma)| \le cP^{1-\rho}, \quad c = c(n, \rho).$$

6. Concluding remarks

One of the most interesting applications of the above results are the one to the sequence of the primes. Then theorem 3 gives estimation of Weyl sums over primes and may lead to bounds on the number of solutions to diophantine equations in prime numbers.

Theorem 5 states that for almost all polynomials their values at the prime numbers are uniformly distributed modulo 1 in $[0, 1)$ and the discrepancy λ_P satisfies $|\lambda_p| \le cP^{1-\rho}$.

In [5] it was shown that these estimations are valid only for a large set in the sense of Lebesque measure. Indeed it can be shown that for given α_n the set of coefficients $(\alpha_1, \ldots, \alpha_{n-1})$ for which the above estimations are violated is residual in T^{n-1}.

Finally we want to formulate a conjecture and give heuristic arguments for it to hold. We believe that these arguments can be made rigorous by a cautious and more detailed analysis of the proofs in this paper.

Let us consider a sequence $\{m_k\}$ which has a certain density property:

There is a $\varepsilon > 0$ such that

$$m_k^\varepsilon < k. \tag{39}$$

Notice that the primes and all polynomial sequences are subject to this condition. If we can carry out the construction in the proofs of lemma 5 and proposition 1 for all numbers in the sequence $\{m_k\}$ rather than for a subsequence $\{m_{k_i}\}$ we have that J_k is contained in approximately

$$\text{mes}\,(J_{k-1}) \cdot m_k$$

intervals of length $\frac{1-\text{mes}\,(U)}{m_k}$. Together with (22) this indicates that the exponential growth rate of the number of intervals should be not greater than that of the term

$$(1 - \text{mes}\,(U))^k \cdot m_k \leq (1 - \text{mes}\,(U))^k \cdot k^{\frac{1}{\epsilon}}$$

which tends to zero. The reason why we can estimate only the exponential growth rate is the influence of "boundary" effects which we could neglect in the proofs by considering sparse subsequences.

These heuristics lead to the following conjecture.

Conjecture 1. Let M satisfy (36). Then

$$\dim_H(S^1 \setminus PE(M)) = 0.$$

We think that even a stronger conjecture may be true.

Conjecture 2. Let M satisfy (36). Then the set $S^1 \setminus PE(M)$ is countable.

If M is the entire natural series then $S^1 \setminus PE(M)$ are the rational numbers and the conjectures are true.

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