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Discretization of frequencies in delay coupled chaotic oscillators

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Abstract

We study the dynamics of two mutually coupled oscillators with a time delayed coupling. Due to the delay, the allowed frequencies of the oscillators are shown to be discretized. The phenomenon is observed in the case when the delay is much larger than the characteristic period of the solitary uncoupled oscillator.

The goal of this paper is to study the influence of delay on the dynamics of coupled oscillators. In particular, we report a new phenomenon, "discretization of frequencies", which arises due to a delay in the coupling. We show that this effect persists also in the case when the oscillators are chaotic. The main motivation for this study comes from [1], where similar effects have been observed in a system of coupled semiconductor lasers.

Dynamical properties of instantaneously coupled oscillators have been the subject of extensive research during the last decades [2, 3, 4, 5]. Many new collective phenomena have been discovered and understood such as complete synchronization [6], generalized [7], phase [8], and lag [9] synchronization, clustering [10], etc. At the same time, the study of coupled systems appears to be important for many practical applications such as laser dynamics [11, 12], biology [13], neurophysiology [14], chemistry [15], and others.

It is evident that a delay in the coupling is common, since coupled subsystems are usually located discretely in space. There are also evidences that the delay can change the dynamics significantly [16]. As soon as the delay becomes comparable with the period of oscillations of the solitary system, a correct modeling should take it into account. The resulting systems of coupled oscillators with delay possess new features and exhibit new phenomena, e.g. anticipated synchronization [17]. Moreover, such models are more complicated objects to study [18, 19] and determining properties of delay coupled systems is still a challenging problem.

In this paper we consider the well studied paradigm of Rössler oscillators, which are bidirectionally coupled

$$\begin{aligned} x'(t) &= f_{\omega_1}(x(t)) + ky(t-\tau), \\ y'(t) &= f_{\omega_2}(y(t)) + kx(t-\tau), \end{aligned} \tag{1}$$

where $x, y \in R^3$ are vectors, $f_\omega(x) = (-\omega x_2 - x_3, \omega x_1 + a x_2, b + x_3(x_1 - c))^T$, $\tau > 0$ is the delay time of the coupling, k is the coupling strength. Note that similar coupling configuration appears in a system describing two optically coupled semiconductor lasers [1, 17, 12].

In order to distinguish between periodic and chaotic cases, we use c as the control parameter, which determines regularity of the solitary system. Figure 1 shows how the largest

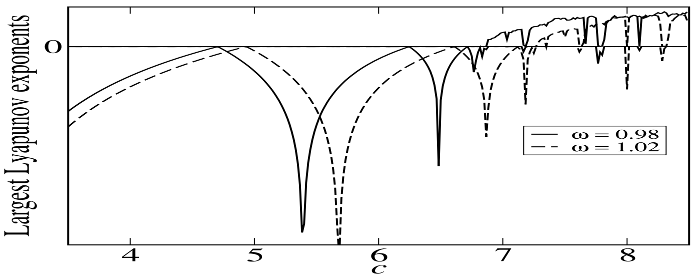

Figure 1: Dependence of the largest Lyapunov exponents on c for the solitary Rössler system. $a = 0.15$ and $b = 0.4$. The solid line corresponds to $\omega = 0.98$ and the dashed one to $\omega = 1.02$.

Lyapunov exponents of the solitary system depend on c with fixed $a = 0.15$, $b = 0.4$, and for two different values of omega $\omega = 0.98$ and $\omega = 1.02$. One can observe a period doubling route to chaos. In what follows, $c = 4$ will correspond to periodic, $c = 8.5$ to chaotic, and $c = 7$ to the mixed (where one uncoupled oscillator is periodic and the other is chaotic) regime.

For the considered parameter values, one can introduce phases of the oscillators in a simple manner [2] $\varphi_1 = \arctan(x_2/x_1)$, $\varphi_2 = \arctan(y_2/y_1)$. The mean observed frequencies of the oscillators are $\Omega_i = \lim_{t\to\infty} \varphi_i/t$, $i = 1, 2$. Synchronization properties of instantaneously ($\tau = 0$) mutually coupled systems are well studied for both periodic and chaotic systems [2]. In the both cases one observes synchronization regions, which correspond to the case $\Omega_1 = \Omega_2$. These regions have the form of cones in the parameter space detuning - coupling strength, i.e. $\Delta\omega$ and k. Figure 2 illustrates the dependence of Ω_i and $\Omega_2 - \Omega_1$ on $\Delta\omega$ for fixed $k = 0.005$ and $\tau = 0$. We observe the "classical" synchronization plateau and a smooth dependence of the frequencies on the control parameter. In Fig. 5 (left panel) we compute the corresponding Lyapunov exponents, which indicate, that the dynamics for $c = 4$ remains regular for all values of $\Delta\omega$ while for $c = 7$ and $c = 8.5$ it is chaotic. One can also note that transition to the phase synchronization occurs at the moment when the second Lyapunov exponent approach zero value, cf. [2].

The new effect of the delay, which we would like to report here is illustrated in Figs. 3,4. We plot there the same quantities as in Fig. 2 but for delay coupled oscillators. Instead of the smooth behavior of the frequencies with changing $\Delta\omega$, we observe a "quantization" effect when some preferable values of frequencies appear, which destroy the previously smooth dependence on the parameters. Ω_i undergo jumps of the magnitude π/τ with varying $\Delta\omega$. As illustrated in Fig. 3, the allowed values of the frequencies and the jumps are closely related to the roundtrip frequency $\omega_f = \pi/\tau$. From this point of view, one can interpret this phenomenon as the resonances to the multiples of the roundtrip frequency. Fig. 5 (right panel) shows largest Lyapunov exponents for the case with delay. One can

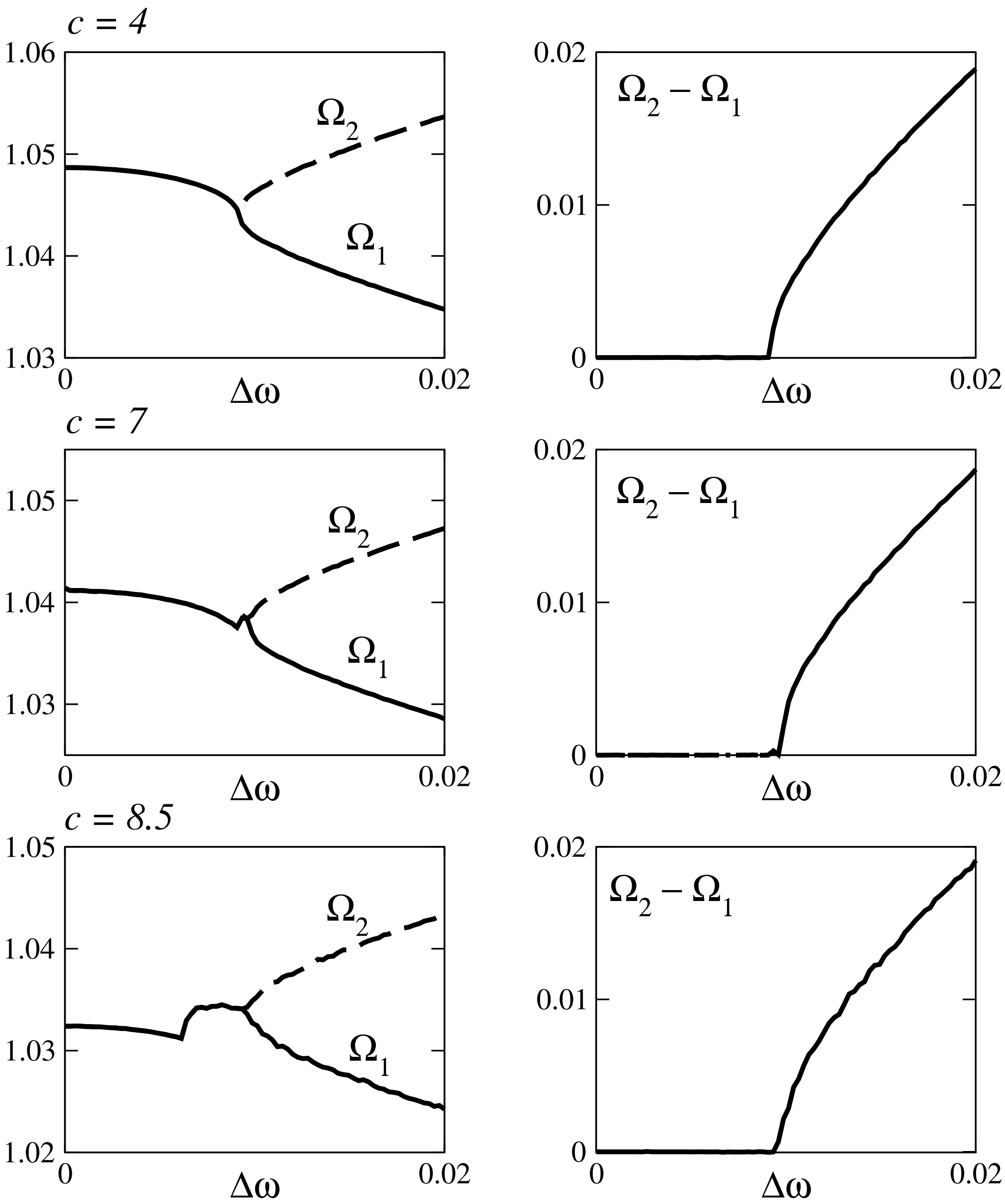

Figure 2: Mean frequencies Ω_1 and Ω_2 (left panel) and their difference $\Omega_2 - \Omega_1$ (right panel) for instantaneously coupled systems. $k = 0.005, \tau = 0$. Different rows correspond to different values of c, as indicated in the figure. $c = 8.5$ stands for the chaotic, $c = 4$ for the regular, and $c = 7$ for the mixed case.

note that the case $c = 4$ still corresponds to a regular dynamics and $c = 7$ and $c = 8.5$ to a chaotic. Hence, the observed phenomenon takes place for chaotic oscillators as well. Note that in the chaotic case there are many (at least more than 10) positive Lyapunov exponents which behave similarly to each other. This phenomenon is in agreement with recent results on delay systems with large delay [19, 22]. In our simulations we choose $\tau = 3000$. We were not able to observe the discretization phenomenon for small values of τ, which are comparable with the characteristic period of the Rössler oscillator $\omega_R \approx 1$, i.e. we have $\pi/\tau < \Delta\omega \ll \omega_R$. Considering coupled Kuramoto system in the last part of the paper, we will provide additional arguments in the favor of large delay.

Inspecting the Lyapunov exponents in Fig. 5, we note, that the phase synchronization transition is no longer correlated with the second largest Lyapunov exponent. Instead, one could expect its correlation with the first negative Lyapunov exponent. We do not monitor negative Lyapunov exponents for $c = 7$ and $c = 8.5$ here, since, this would involve calculation of a large number of Lyapunov exponents and give unreliable results. This problem is a consequence of the large value of the delay.

In the following we would like to present some additional arguments showing that the described phenomenon is generic. Let us introduce an artificial parameter k_1 such that system (1) admits the form

$$\begin{aligned} x'(t) &= f_{\omega_1}(x(t)) + ky(t) + k_1(y(t-\tau) - y(t)), \\ y'(t) &= f_{\omega_2}(y(t)) + kx(t) + k_1(x(t-\tau) - x(t)). \end{aligned} \tag{2}$$

System (2) coinsides with (1) if $k = k_1$ while at $k_1 = 0$ it has instantaneous coupling. Therefore, increasing the parameter k_1 from 0 to k, the case with instantaneous coupling is transformed to the delayed one. In a short form, (2) can be written as

$$z' = F(z) + Kz(t) + K_1[z(t-\tau) - z(t)], \tag{3}$$

where $z = (x, y)^T$, $K = \begin{pmatrix} 0 & kI_3 \\ kI_3 & 0 \end{pmatrix}$ and $K_1 = \begin{pmatrix} 0 & k_1 I_3 \\ k_1 I_3 & 0 \end{pmatrix}$, I_3 is 3×3 unit matrix. Our main observation is that system (3) can be considered as the instantaneously coupled system $z' = F(z) + Kz(t)$ under the action of the feedback term $K_1[z(t-\tau) - z(t)]$. As follows from [20], this term, under some conditions, enhances spectral properties of the solutions, e.g. stabilizes periodic solutions with a period close to fractions of τ, for which the feedback term vanishes. Roughly speaking, such a feedback induces a filtering of frequencies, which are close to multiples of $2\pi/\tau$. Following this idea, one may consider (3), and hence (1) as well, as an instantaneously coupled system, which undergoes the influence of the delayed feedback. As a result, frequencies of the instantaneous system (see Fig. 2) are "filtered" through the delayed feedback term and one observes an enhancing of those frequencies, which are multiples of $2\pi/\tau$ (see Fig. 3).

Figure 6 shows evolution of the phase difference $\varphi_1 - \varphi_2$ for $c = 4$ and $c = 8.5$. In particular, in Fig. 6a orbits A, B, and C where computed for three different values of detuning, which correspond to three minimal allowed frequencies $\Omega_2 - \Omega_1$, cf. also points A, B, and C in

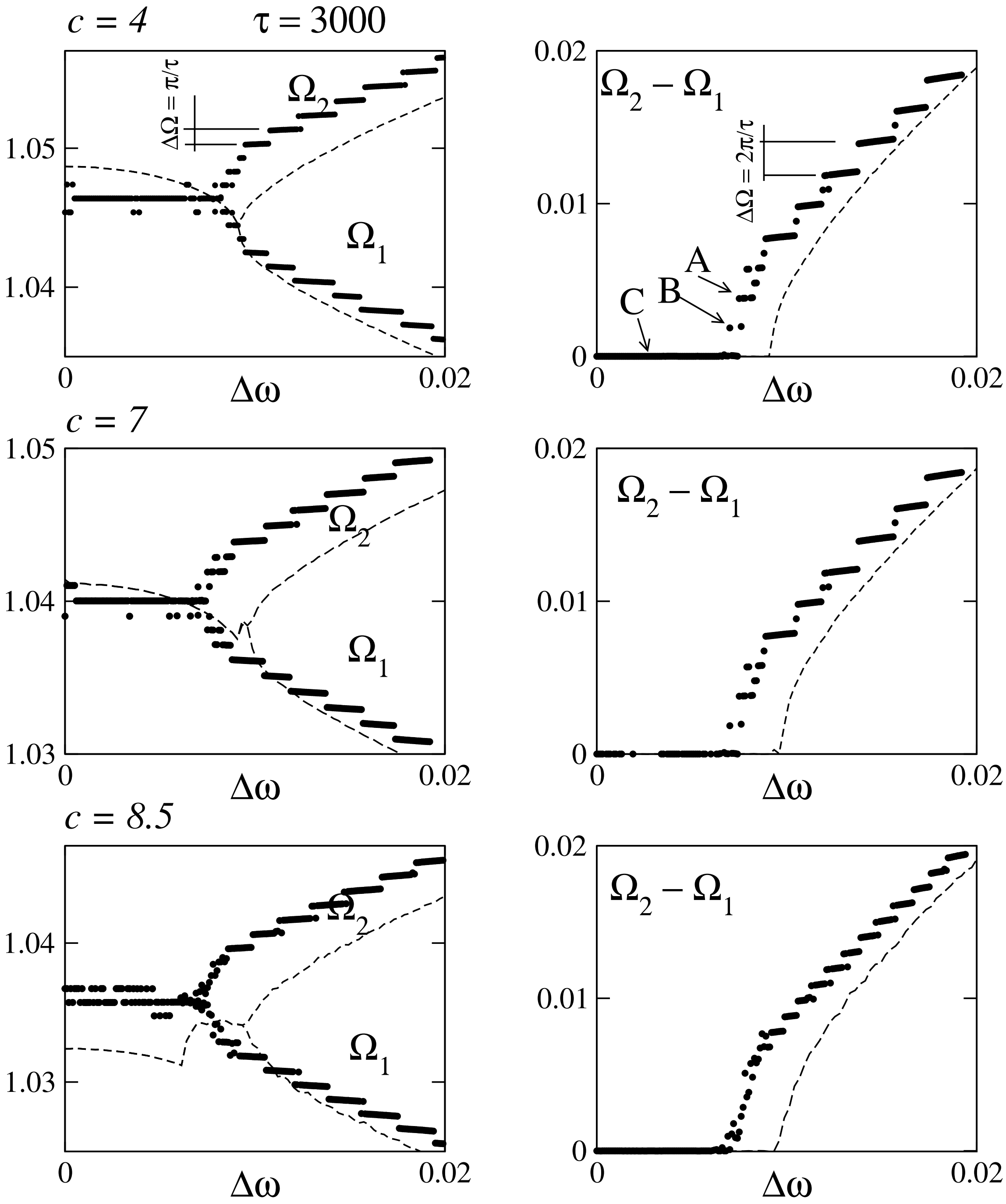

Figure 3: Ω_1 and Ω_2 (left panel) and $\Omega_2 - \Omega_1$ (right panel) for delay coupled systems. $k = 0.005, \tau = 3000$. Different rows correspond to different values of c, as indicated in the figure. Dashed lines – delay-free case.

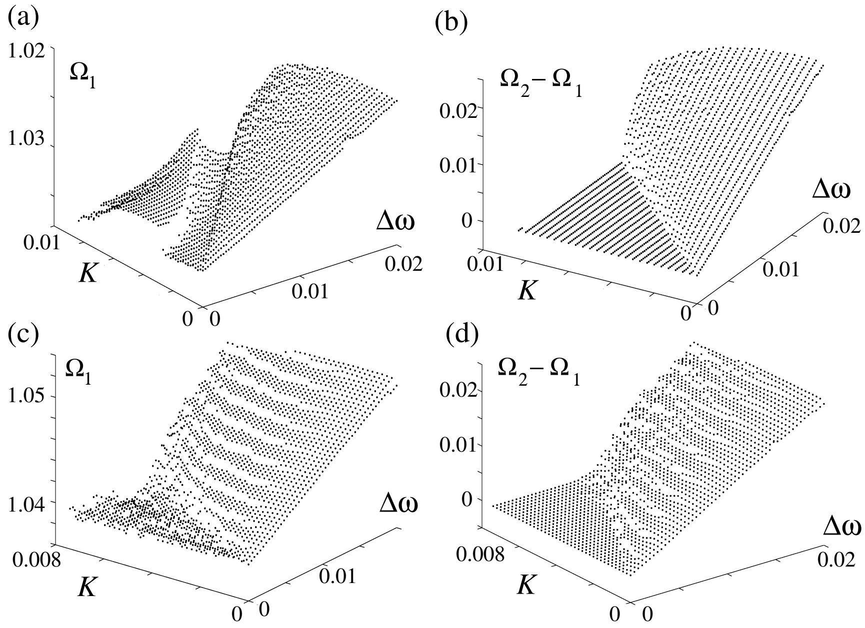

Figure 4: Mean frequencies of two delay coupled Rössler oscillators as a function of coupling k and detuning $\Delta\omega$. (a) frequency of the first oscillator Ω_1 and (b) frequency difference $\Omega_2 - \Omega_2$ for the instantaneously coupled systems. (c) and (d): the same for $\tau = 3000$.

Fig. 3. It is interesting to note that phase slips in the both nonsynchronous cases A and B occur with the same rate, but in the case A these slips have the magnitude 2π, while in the case B the magnitude of the slips is π. Figure 7 illustrates this in more details.

Our observations suggest that in the case with delay, transition to the phase synchronization differs from those that occurs in the instantaneous case. In particular, scaling properties of the intervals between phase slips can be different. We will report the scaling results elsewhere.

Finally, we would like to present additional analytical arguments, which are based on the analysis of the Kuramoto model with delay

$$\begin{aligned} \psi_1'(t) &= \omega_1 - k\sin(\psi_1(t) - \psi_2(t-\tau)), \\ \psi_2'(t) &= \omega_2 - k\sin(\psi_2(t) - \psi_1(t-\tau)). \end{aligned} \tag{4}$$

Within the locking region, this system is known [21] to exhibit a series of synchronized solutions of the form $\psi_{1,2} = \Omega t \pm \alpha/2$, where Ω and α are constants. Stability and existence of such solutions have been studied in [21]. Unfortunately, we can not apply their results here in order to support our calculations, since the described phenomenon goes beyond the simple "constant frequency" solutions. Instead, we would like to show that all possible Hopf bifurcations of these elementary solutions lead to the modulation frequencies restricted to the values $\omega_{H1} = \frac{\pi}{\tau} + \frac{2\pi n}{\tau}$ or $\omega_{H2} = \frac{2\pi n}{\tau}$, where $n = 0, \pm 1, \pm 2, \ldots$.

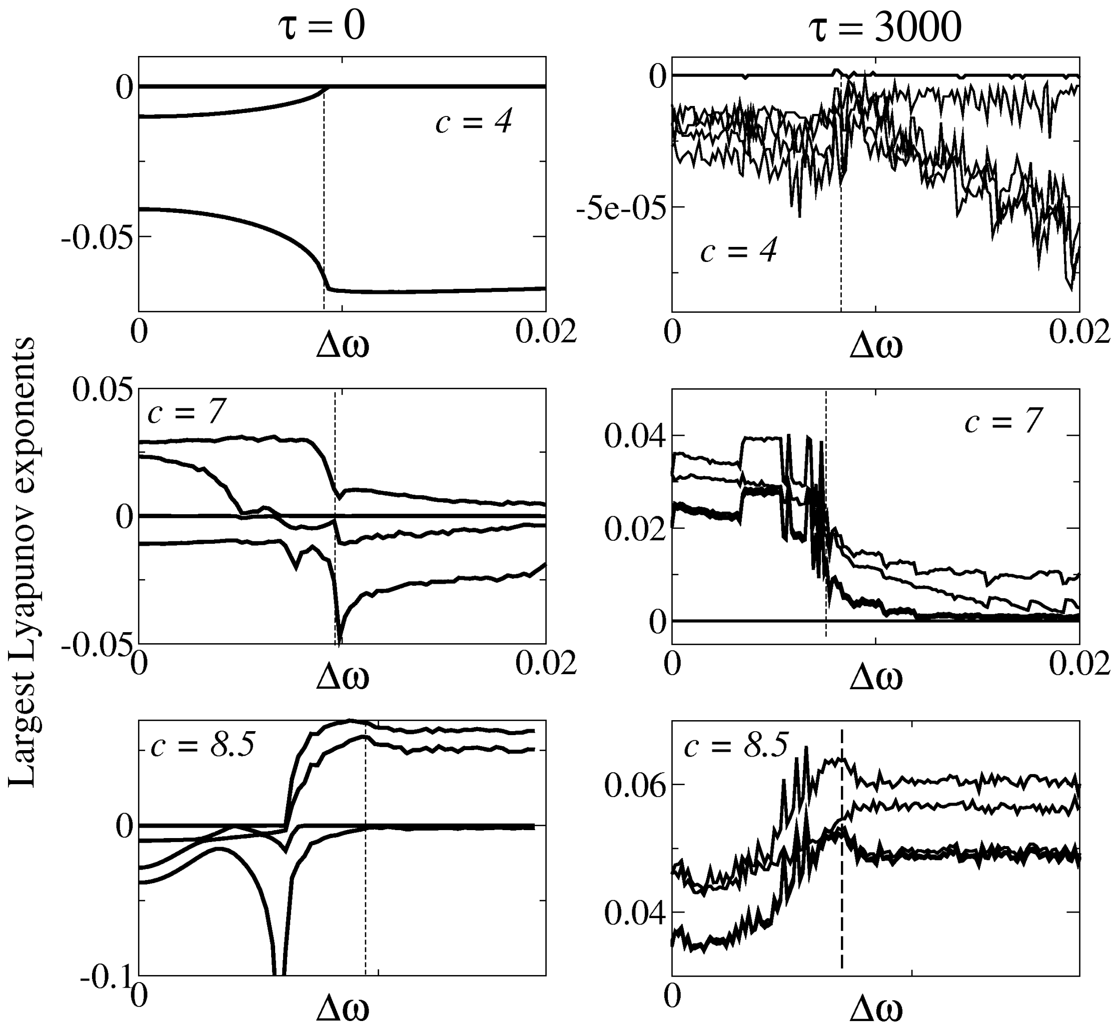

Figure 5: Largest Lyapunov exponents as functions of $\Delta\omega$. The left panel corresponds to the instantaneous coupling and the right one to the delayed case with $\tau = 3000$. The dashed vertical lines mark the parameter value at which the phase synchronization transition happens.

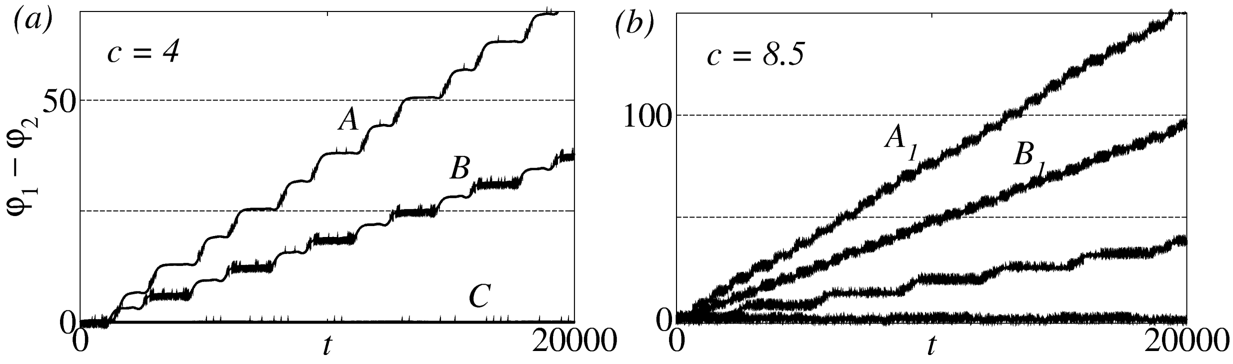

Figure 6: Evolution of phase differences of the delay coupled oscillators. Orbit C corresponds to the phase synchronized case, B to the first minimal allowed frequency difference $\Omega_2 - \Omega_1$, and A to the second one.

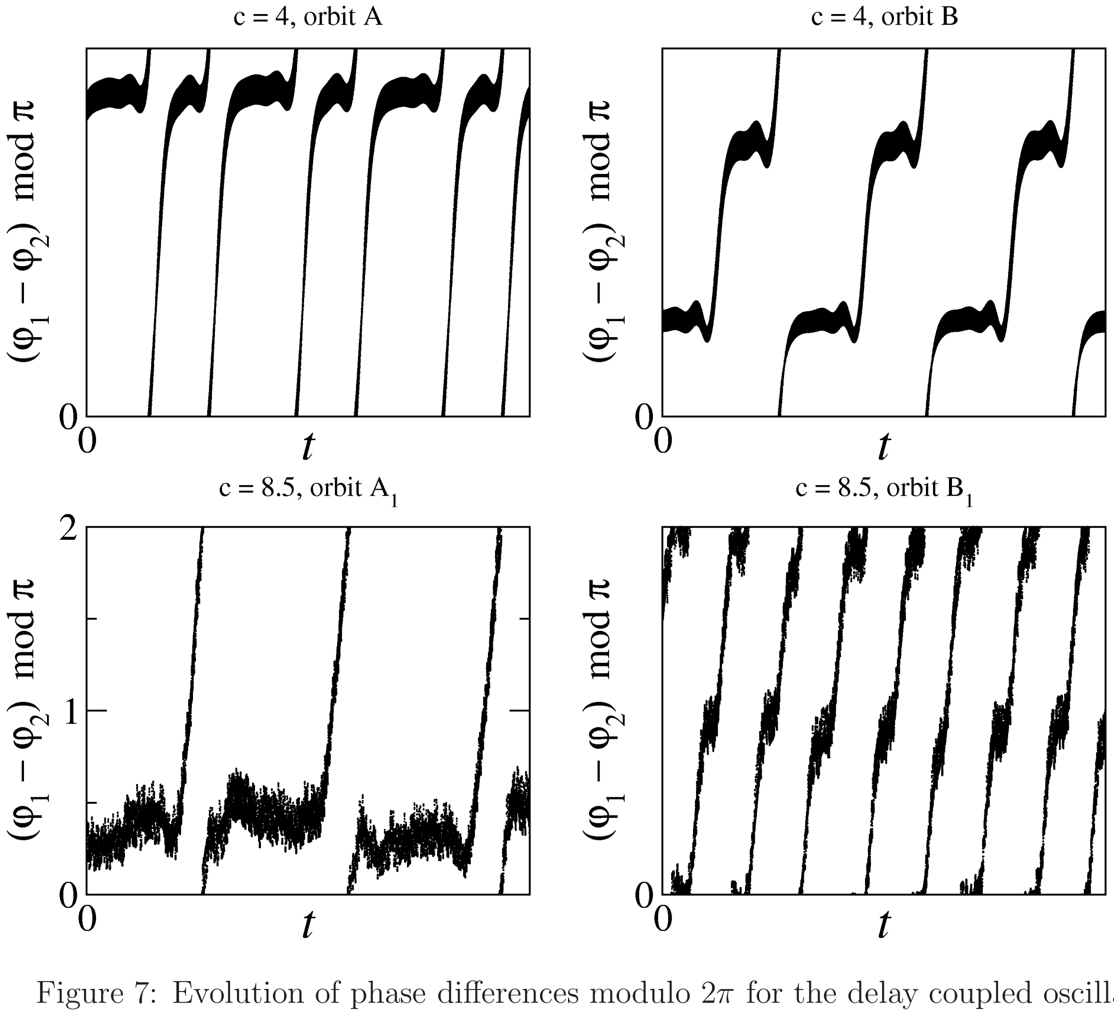

Figure 7: Evolution of phase differences modulo 2π for the delay coupled oscillators.

Assuming that $\psi^0_{1,2} = \Omega t \pm \alpha/2$ is a solution of (4) with some given Ω and α, the linearized system, which determines stability of $\psi^0_{1,2}$ reads

$$\begin{aligned} \xi_1'(t) &= k\cos(\alpha+\Omega\tau)\xi_1(t) - k\cos(\alpha+\Omega\tau)\xi_2(t-\tau), \\ \xi_2'(t) &= k\cos(\alpha-\Omega\tau)\xi_2(t) - k\cos(\alpha-\Omega\tau)\xi_1(t-\tau). \end{aligned} \tag{5}$$

As follows from the asymptotic technique developed in [19, 22], possible imaginary parts of critical eigenvalues are approaching asymptotically the values $\omega_H = \mathrm{Arg}\,(\mu)/\tau + 2\pi n/\tau$ as τ becomes large. Here Arg $(\cdot)$ denote the argument of a complex number and μ is a zero the following equation

$$\det \begin{bmatrix} k\mu\cos(\alpha+\Omega\tau) & -k\cos(\alpha+\Omega\tau) \\ -k\cos(\alpha-\Omega\tau) & k\mu\cos(\alpha-\Omega\tau) \end{bmatrix} = 0. \tag{6}$$

From (6) we have $\mu = \pm 1$. Therefore, the only possible modulation frequencies, which appear at Hopf bifurcations, are ω_{H1} or ω_{H2}. Collecting them together, we obtain the set $\omega_H = \frac{\pi}{\tau} + \frac{\pi n}{\tau}$, which coinsides with the detected numerically available frequencies in Fig. 3 for Rössler systems. This is an analytical evidence, that there are preferable frequencies in the model, which manifest itself as the "frequency discretization phenomenon. A key condition for the application of asymptotic analysis from [19] is the assumption that the delay is large $\tau >> \omega_i$.

To summarize, we report a phenomenon of frequency discretization in systems of coupled regular or chaotic oscillators. A large delay is shown to be essential for its appearance.

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References

[1] H.-J. Wünsche, S. Bauer, J. Kreissl, O. Ushakov, N. Korneev, F. Henneberger, E. Wille, H. Erzgräber, M. Peil, W. Elsäßer, et al., sobmitted to Phys. Rev. Lett.

[2] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization. A universal concept in nonlinear sciences* (Cambridge University Press, 2001).

[3] E. Mosekilde, Y. Maistrenko, and D. Postnov, *Chaotic synchronization. Application to living systems.* (World Scientific, 2002).

[4] C.-W. Wu, *Synchronization in coupled chaotic circuits and systems* (World Sci., 2002).

[5] E. Ott, *Chaos in dynamical systems (Second edition)* (Cambridge University press, 2002).

[6] H. Fujisaka et al., Prog. Theor. Phys. **69**, 32 (1983). L. M. Pecora et al., Phys. Rev. Lett. **64**, 821 (1990). L. Pecora et al., Chaos **7**, 520 (1997). S. Yanchuk et al., Chaos **13**, 388 (2003). S. Yanchuk et al., Physica D **154**, 26 (2001a).

[7] N. F. Rulkov, M. M. Sushchik, L. S. Tsimring, and H. D. I. Abarbanel, Phys. Rev. E **51**, 980 (1995).

[8] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[9] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).

[10] K. Kaneko, Physica D **41**, 137 (1990). S. Yanchuk, Y. Maistrenko, and E. Mosekilde, Math. Comp. Simul. **54**, 491 (2001b).

[11] R. Roy and K. S. Thornburg, Phys. Rev. Lett. **72**, 2009 (1994).

[12] S. Yanchuk, K. R. Schneider, and L. Recke, Phys. Rev. E **69**, 056221 (2004).

[13] H. Bohr, K. S. Jensen, T. Petersen, B. Rathjen, E. Mosekilde, and N.-H. Holstein-Rathlou, Parallel Comput. **12**, 113 (1989); R. C. Elson, H. I. Selverston, R. Huerta, N. R. Rulkov, M. I. Rabinovich, and H. D. I. Abarbanel, Phys. Rev. Lett. **81**, 5692 (1998); E. Mosekilde, B. Lading, S. Yanchuk, and Y. Maistrenko, Biosystems **63**, 3 (2001).

[14] P. A. Tass, T. Fieseler, J. Dammers, K. Dolan, P. Morosan, M. Majtanik, F. Boers, A. Muren, K. Zilles, and G. R. Fink, Phys. Rev. Lett. **90**, 088101 (2003).

[15] I. Z. Kiss, Y. Zhai, and J. L. Hudson, Phys. Rev. Lett. **88**, 238301 (pages 4) (2002).

[16] T. Heil, I. Fischer, W. Elsäßer, J. Mulet, and C. R. Mirasso, Phys. Rev. Lett. **86**, 795 (2001).

[17] A. Locquet, C. Masoller, and C. R. Mirasso, Phys. Rev. E **65**, 056205 (2002).

[18] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations* (Springer-Verlag, 1993).

[19] S. Yanchuk, Math. Meth. Appl. Sci. **28**, 363 (2005).

[20] K. Pyragas, Phys. Lett. A **170**, 421 (1992).

[21] H. G. Schuster and P. Wagner, Prog. Theor. Phys. **81**, 939 (1989).

[22] S. Yanchuk and M. Wolfrum, submitted.