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Threshold to liquefaction in granular materials as a formation of strong wave discontinuity in poroelastic media

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ABSTRACT: We consider a one-dimensional problem of propagation of acoustic waves in a nonlinear poroelastic saurated material. Stress-strain relations in the skeleton are described by Signorini-type constitutitve equations. Material parameters depend on the current porosity. The governing set of equations describes changes of extension of the skeleton, and of the mass density of the fluid, partial velocities of the skeleton and of the fluid and a porosity. We rely on a second order approximation. Relations of the critical time to an initial porosity and to an initial amplitude are discussed. The connection to the threshold of liquefaction in granular materials is indicated.

1 INTRODUCTION, EVOLUTION OF THE AMPLITUDE OF ACOUSTIC WAVES

Threshold from the compact granular structure to liquefaction is usually attributed to a critical behavior in the strain-stress relations (Wood 1990). This is not always the case and the liquefaction may be dynamical in nature. The idea of attributing liquefaction of granular materials to the critical growth of the amplitude of acoustic waves has been described by Osinov (Osinov 1998). He relied in his considerations on the one component model in which the evolution of stresses is described by the nonlinear law of hypoplasticity. He has proven the existence of a critical distance of longitudinal waves after which the continuous solutions no longer exist. This distance depends on the slope and curvature of the stress-strain relation in the monotone loading. Consequently, Osinov could also incorporate an influence of the fluid component which changes effective material parameters of the one-component model.

Similar argument can be made in a two-component model provided it is hyperbolic and nonlinear. In this work we use the standard argument related to the existence of a critical time for weak discontinuity waves.

The liquefaction related to rapid changes of porosity were also investigated in (Wilhelm and Wilmanski 2002) in which, instead of the full nonlinearity, only the momentum source has been extended on a nonlinear influence of the gradient of porosity.

For the quasilinear hyperbolic system of equations the following argument leads to the existence of critical amplitudes. Let us consider an arbitrary set of the first order hyperbolic partial differential equations

$$\frac{\partial u_A}{\partial t} + A_{AB}\frac{\partial u_B}{\partial x} = B_A, \quad A,B = 1,...,N, \quad (1)$$

where A_{AB}, B_A are differentiable functions of u_A and the unknown fields of variables (x,t) are defined on $\Re \times \Re$. Hyperbolicity of this system means that the eigenvalue problem

$$\left(A_{AB} - \lambda\delta_{AB}\right) r_B = 0, \quad (2)$$

possesses solely real eigenvalues λ and the right eigenvectors r_B span the space of solutions. Eigenvalues λ describe speeds of propagation in characteristic directions given in turn by the eigenvectors.

We consider the problem in which the fields are continuously differentiable everywhere except of a singular orientable surface S on which the fields are continuous and their first derivatives may possess finite discontinuities. S is called the surface of **weak discontinuity**. On such a surface the following **compatibility conditions** hold

$$\begin{aligned}
&[\![u_A]\!] = 0 \quad \Rightarrow \quad \left[\!\!\left[\frac{\partial u_A}{\partial t}\right]\!\!\right] = -c\left[\!\!\left[\frac{\partial u_A}{\partial x}\right]\!\!\right], \\
&\left[\!\!\left[\frac{\partial^2 u_A}{\partial x \partial t}\right]\!\!\right] = \frac{d}{dt}\left[\!\!\left[\frac{\partial u_A}{\partial x}\right]\!\!\right] - c\left[\!\!\left[\frac{\partial^2 u_A}{\partial x^2}\right]\!\!\right], \text{ etc.,} \\
&[\![\ldots]\!] = (\ldots)^+ - (\ldots)^-, \quad (3)
\end{aligned}$$

where c is the speed of the surface S, $\frac{d}{dt}$ is the derivative in the direction of propagation (along the characteristic), and $(\ldots)^+, (\ldots)^-$ denote one-sided limits on this surface.

We construct the jumps on S in the set (1). It follows

$$[\![u_A]\!] = 0 \;\Rightarrow\; \left[\!\!\left[\frac{\partial u_A}{\partial t}\right]\!\!\right] + A_{AB}\left[\!\!\left[\frac{\partial u_B}{\partial x}\right]\!\!\right] = 0,$$

i.e.

$$(A_{AB} - c\delta_{AB})\left[\!\!\left[\frac{\partial u_B}{\partial x}\right]\!\!\right] = 0. \tag{4}$$

Hence the speeds of propagation of S coincide with the eigenvalues of the matrix A_{AB}, i.e. they are characteristic, and the jumps $\left[\!\left[\frac{\partial u_B}{\partial x}\right]\!\right]$ are right eigenvectors of this matrix. The surface of weak discontinuities propagates in a characteristic direction. The jumps

$$\left[\!\!\left[\frac{\partial u_B}{\partial x}\right]\!\!\right] = \mathsf{A} r_B, \quad r_A r_A = 1, \tag{5}$$

are called the amplitudes of discontinuity. It is seen that the **amplitude** A determines all discontinuities once the eigenvalue problem has been solved.

By differentiation of the set (1) with respect to the variable x, it can be easily shown (e.g. (Wilmanski 1998b)) that the equation for the evolution of the amplitude A can be written in the following form

$$\frac{d\mathsf{A}}{dt} + \alpha_1 \mathsf{A} + \alpha_2 \mathsf{A}^2 = 0, \tag{6}$$

where

$$\begin{aligned}\alpha_1 &:= \left[l_A \frac{\partial A_{AB}}{\partial u_C} r_C \left.\frac{\partial u_B}{\partial x}\right|^+ + \right.\\ &\quad + l_A \frac{\partial A_{AB}}{\partial u_C} r_B \left.\frac{\partial u_C}{\partial x}\right|^+ - \\ &\quad \left. - l_A \frac{\partial B_A}{\partial u_C} r_C + l_A \frac{dr_A}{dt}\right] \frac{1}{r_D l_D}, \\ \alpha_2 &:= -l_A \frac{\partial A_{AB}}{\partial u_C} r_B r_C \frac{1}{r_D l_D},\end{aligned} \tag{7}$$

$\ldots|^+$ denotes the limit on the positive side of the surface, and l_A denotes the left eigenvector of A_{AB}. Certainly, both coefficients are functions of the fields u_A. However, due to the continuity of u_A across the surface S they can be evaluated on its positive side (ahead of the wave if S is the wave front) where their values are usually known. Consequently, equation (6) is the Bernoulli equation for the amplitude of discontinuity with known coefficients. Its integration yields

$$\frac{1}{\mathsf{A}} = \left[\frac{1}{\mathsf{A}_0} + \int_0^t \alpha_2 \mathrm{e}^{-\int_0^\eta \alpha_1 ds} d\eta\right] \mathrm{e}^{\int_0^t \alpha_1 d\eta}, \tag{8}$$

where A_0 is the initial value of the amplitude of discontinuity.

It is shown in the theory of hyperbolic systems that solutions possessing weak discontinuities may only exist for a finite time (e.g. (Lax 1964)). At the instant of time, t_c, the so-called critical time, the solution becomes singular on the surface S and this singularity is strong, i.e. the amplitude of the weak discontinuity A grows to infinity. The critical time can be easily found by means of relation (8). Namely, this expression must go to zero at t_c, i.e.

$$\frac{1}{\mathsf{A}_0} + \int_0^{t_c} \alpha_2 \mathrm{e}^{-\int_0^\eta \alpha_1 ds} d\eta = 0. \tag{9}$$

This relation determines t_c.

The above presented procedure shall be applied to field equations for poroelastic saturated materials. It is easily seen that a nontrivial solution for the critical time does not exist for linear systems in which $\alpha_2 \equiv 0$, i.e. if we expect the creation of strong discontinuities we have to describe the system by means of a **nonlinear** set of equations.

2 GOVERNING EQUATIONS

The two-component model of porous saturated material is based on the following choice of fields

$$\left\{\rho^F, \rho^S, \mathbf{v}^F, \mathbf{v}^S, \mathbf{e}^S, n\right\} \tag{10}$$

where ρ^F, ρ^S are current partial mass densities of the fluid and skeleton, respectively, $\mathbf{v}^F, \mathbf{v}^S$ are the velocities of components, $\mathbf{e}^S$ is the Almansi-Hamel strain tensor of the skeleton, and n denotes the porosity.

We assume that material parameters of the model depend on the current porosity n. Hence, the model must be nonlinear. For the purpose of this work we limit attention to processes in which solely the deformation of the skeleton is large. We neglect nonlinearities related to the deformation of the fluid component and to the dependence of the permeabilty coefficient from porosity. The latter can be easily incorporated, it yields a correction in the source term and, simultaneously, does not lead to qualitative changes of the results. The case of a nonlinear fluid (e.g. gaseous) in a linear skeleton has been considered elsewhere (Wilmanski 1998a). Certainly, these assumptions are related to the fact that the liquefaction yields primarily considerable deformations of the skeleton. There exists an experimental evidence (e.g. (Wilhelm and Wilmanski 2002)) that compressibility of the skeleton reduces very considerably near the threshold to liquefaction while, of course, compressibility of fluids like water remains unchanged. Simultaneously, we leave out relaxation processes of porosity. This means that the porosity reduces to its equilibrium changes (Wilmanski 2004) and it can be written in the form

$$n = n_0\left(1 + \delta\,\mathrm{tr}\,\mathbf{e}^S\right), \tag{11}$$

where δ is the material parameter. The non-equilibrium contribution to porosity following from the relative motion could contribute but it can be shown (Wilmanski 2004) that this contribution is very small indeed.

We consider the process of transition from the poroelastic region with compact skeleton to a fluidized state as related to a sudden change of porosity.

For basic geometric quantities we make the following **assumptions**

1) the deformation of the skeleton, described by the Almansi-Hamel deformation tensor $\mathbf{e}^S$ is small of the second order, i.e.,

$$\begin{aligned} &\max\left\{\sum_{i,j}\left|\lambda_e^{(i)}\lambda_e^{(j)}\right|\right\} \ll 1, \\ &\det\left(\mathbf{e}^S - \lambda_e^{(i)}\mathbf{1}\right) = 0, \end{aligned} \tag{12}$$

2) volume changes of the fluid ε are small, i.e.

$$|\varepsilon| \ll 1, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}, \tag{13}$$

where ρ^F is the current mass density of the fluid, and ρ_0^F its initial value.

As the mass density of the skeleton ρ^S must satisfy the mass balance equation, we have

$$\rho^S = \rho_0^S\left(\det\mathbf{B}^S\right)^{-\frac{1}{2}}, \quad \mathbf{B}^{S-1} := \mathbf{1} - 2\mathbf{e}^S, \tag{14}$$

and the above assumption yields

$$\begin{aligned} &\rho^S = \rho_0^S\sqrt{1 - 2I - 4II}, \quad I := \operatorname{tr}\mathbf{e}^S, \\ &II := \tfrac{1}{2}\left(I^2 - \operatorname{tr}\mathbf{e}^{S2}\right). \end{aligned} \tag{15}$$

Contributions of the third invariant $III = \det\mathbf{e}^S$ are of the third order in eigenvalues $\lambda_e^{(i)}$. Consequently, the mass density of the skeleton is not an independent field.

Bearing the above considerations in mind we seek field equations for the following **fields**

$$\left\{\varepsilon, \mathbf{e}^S, \mathbf{v}^S, \mathbf{v}^F\right\}. \tag{16}$$

Field equations follow from the mass conservation for the fluid, partial momentum balance equations for the skeleton and for the fluid. They can be written in the form

$$\frac{\partial\varepsilon}{\partial t} + \operatorname{div}(\varepsilon - 1)\mathbf{v}^F = 0, \tag{17}$$

$$\rho^S\left(\frac{\partial\mathbf{v}^S}{\partial t} + \mathbf{L}^S\mathbf{v}^S\right) = \operatorname{div}\mathbf{T}^S + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

$$\rho^F\left(\frac{\partial\mathbf{v}^F}{\partial t} + \mathbf{L}^F\mathbf{v}^F\right) = -\operatorname{grad}p^F - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

where $\mathbf{T}^S$ denotes the partial stress tensor in the skeleton,

$$\mathbf{L}^S := \operatorname{grad}\mathbf{v}^S, \quad \mathbf{L}^F := \operatorname{grad}\mathbf{v}^F, \tag{18}$$

and the partial stress tensor $\mathbf{T}^F$ for the fluid component reduces to the pressure $p^F = -\frac{1}{3}\operatorname{tr}\mathbf{T}^F$.

In addition, we have the kinematic compatibility equation

$$\begin{aligned} \frac{\partial\mathbf{e}^S}{\partial t} + \mathbf{v}^S\cdot\operatorname{grad}\mathbf{e}^S = \tfrac{1}{2}\left(\mathbf{L}^S + \mathbf{L}^{ST}\right) - \\ - \left(\mathbf{L}^{ST}\mathbf{e}^S + \mathbf{e}^S\mathbf{L}^S\right). \end{aligned} \tag{19}$$

We close the system by constitutive relations which are based on the Signorini constitutive relations for the second order model (e.g. (Albers and Wilmanski 1999))

$$\begin{aligned} \mathbf{T}^S = \mathbf{T}_0^S + \left(\lambda^S I + \tfrac{1}{2}\left(\lambda^S + \mu^S\right)I^2\right)\mathbf{1} + \\ +2\left(\mu^S - \left(\lambda^S + \mu^S\right)I\right)\mathbf{e}^S, \end{aligned}$$

$$p^F = p_0^F - \rho_0^F\kappa\varepsilon, \tag{20}$$

where the material parameters λ^S, μ^S, κ are functions of the current porosity n. $\mathbf{T}_0^S, p_0^F$ are initial (constant) partial stress in the skeleton and the initial partial pressure in the fluid, respectively.

Approximations presented above yield the following **constitutive relations** (**simple mixture model of the second order**)

$$\begin{aligned} \mathbf{T}^S = \mathbf{T}_0^S + \lambda_0^S I\mathbf{1} + 2\mu_0^S\mathbf{e}^S + \\ + \left(\delta\left.\frac{\partial\lambda^S}{\partial n}\right|_0 n_0 I^2 + \tfrac{1}{2}\left(\lambda_0^S + \mu_0^S\right)I^2\right)\mathbf{1} + \\ +2\left(\delta\left.\frac{\partial\mu^S}{\partial n}\right|_0 n_0 I - \left(\lambda_0^S + \mu_0^S\right)I\right)\mathbf{e}^S, \end{aligned} \tag{21}$$

$$p^F = p_0^F - \rho_0^F\kappa_0\varepsilon - \rho_0^F\delta\left.\frac{\partial\kappa}{\partial n}\right|_0 n_0 I\varepsilon, \tag{22}$$

where the constant δ describes equilibrium changes of porosity (see: (11)) and the index 0 indicates evaluation at the porosity n_0. Simultaneously,

$$\rho^S \approx \rho_0^S\left(1 - I + \tfrac{1}{2}\left(I^2 + 4II\right)\right). \tag{23}$$

This completes the set of relations for the second order model under considerations.

3 ONE-DIMENSIONAL MODEL

We proceed to consider a one-dimensional problem with the longitudinal motion in the $\mathbf{e}_x$-direction. Then

$$\begin{aligned} &\mathbf{v}^S = v^S\mathbf{e}_x, \quad \mathbf{v}^F = v^F\mathbf{e}_x, \\ &\mathbf{e}^S = e^S\mathbf{e}_x\otimes\mathbf{e}_x, \quad |\mathbf{e}_x| = 1, \end{aligned} \tag{24}$$

and

$$I = e^S, \quad II = 0, \quad \mathbf{L}^S = \frac{\partial v^S}{\partial x}\mathbf{e}_x \otimes \mathbf{e}_x, \\ \mathbf{L}^F = \frac{\partial v^F}{\partial x}\mathbf{e}_x \otimes \mathbf{e}_x. \tag{25}$$

Hence

$$\rho^S = \rho_0^S \left(1 - e^S + \tfrac{1}{2} e^{S2}\right), \\ \rho^F = \rho_0^F \left(1 - \varepsilon\right), \quad n = n_0 \left(1 + \delta e^S\right). \tag{26}$$

The partial stresses reduce in this case to the following nontrivial contributions

$$\sigma^S = T_x^S = \sigma_0^S + \left(\lambda^S + 2\mu^S\right) e^S - \\ - \tfrac{3}{2}\left(\lambda_0^S + \mu_0^S\right) e^{S2} + O\left(\left|e^S\right|^3\right) \tag{27}$$

$$p^F = p_0^F - \rho_0^F \kappa \varepsilon,$$

with

$$\lambda^S + 2\mu^S = \lambda_0^S + 2\mu_0^S + \tag{28}$$

$$+\delta n_0 \frac{\partial}{\partial n}\left(\lambda^S + 2\mu^S\right)\Big|_0 e^S + O\left(\left|e^S\right|^2\right),$$

$$\kappa = \kappa_0 + \delta n_0 \frac{\partial \kappa}{\partial n}\Big|_0 e^S + O\left(\left|e^S\right|^2\right).$$

As before the index 0 denotes evaluation of the quantity in the initial state $\varepsilon = 0$, $e^S = 0$, $v^S = 0$, $v^F = 0$, $n = n_0$.

Substitution of these relations in the governing set of equations yields the following set of equations in the second order approximation

$$\frac{\partial u_A}{\partial t} + A_{AB}\frac{\partial u_B}{\partial x} = B_A, \quad A, B = 1, ..., 4, \tag{29}$$

where

$$[u_A]^T = \left[\begin{array}{cccc} \varepsilon & v^F & v^S & e^S \end{array}\right]^T, \tag{30}$$

and

$$[A_{AB}] = \left[\begin{array}{cccc} v^F & \varepsilon - 1 & 0 & 0 \\ -c_{P2}^2 (1+\varepsilon) - L^F e^S & v^F & 0 & -L^F\varepsilon \\ 0 & 0 & v^S & -c_{P1}^2 - L^S e^S \\ 0 & 0 & -\left(1 - 2e^S\right) & v^S \end{array}\right], \tag{31}$$

with

$$\begin{aligned} c_{P1}^2 &\equiv \frac{\lambda_0^S + 2\mu_0^S}{\rho_0^S}, \quad c_{P2}^2 \equiv \kappa_0, \\ c_S^2 &\equiv \frac{\mu_0^S}{\rho_0^S}, \quad L^F \equiv \delta \left.\frac{\partial \kappa}{\partial n}\right|_0 n_0, \\ L^S &\equiv \frac{2\delta n_0}{\rho_0^S} \left.\frac{\partial(\lambda + 2\mu)}{\partial n}\right|_0 - \\ &\qquad - \left(2c_{P1}^2 - 3c_S^2\right). \end{aligned} \tag{32}$$

The vector on the right hand side of relation (29) is given by

$$[B_A]^T = \left[\begin{array}{cccc} 0 & -\frac{\pi}{\rho_0^F}(1+\varepsilon)\left(v^F - v^S\right) & \frac{\pi}{\rho_0^S}\left(1 + e^S\right)\left(v^F - v^S\right) & 0 \end{array}\right]^T. \tag{33}$$

We write the above set in the dimensionless form. For the normalization we choose the speed of the P1-wave c_{P1} and the unit of time related to a characteristic frequency of the model, i.e. $2\rho_0^S/\pi$. Then the dimensionless quantities are as follows

$$\begin{aligned} x' &= \frac{x\pi}{2\rho_0^S c_{P1}}, \quad t' = \frac{t\pi}{2\rho_0^S}, \\ \varepsilon' &= \varepsilon, \quad e^{S\prime} = e^S, \\ v^{F\prime} &= \frac{v^F}{c_{P1}}, \quad v^{S\prime} = \frac{v^S}{c_{P1}}, \\ c_f &= \frac{c_{P2}}{c_{P1}}, \quad c_s = \frac{c_S}{c_{P1}}, \\ L^{F\prime} &= \frac{l^F}{c_{P1}^2} = \delta n_0 \frac{\partial}{\partial n} \left.\frac{\rho_0^S \kappa}{\lambda_0^S + 2\mu_0^S}\right|_0, \\ L^{S\prime} &= \frac{l^S}{c_{P1}^2} = 2\delta n_0 \frac{\partial}{\partial n} \left.\frac{\lambda^S + 2\mu^S}{\lambda_0^S + 2\mu_0^S}\right|_0 - \\ &\qquad - \left(2 - 3c_s^2\right). \end{aligned} \tag{34}$$

The matrices transform as follows

$$[A'_{AB}] = \left[\begin{array}{cccc} v^{F\prime} & \varepsilon - 1 & 0 & 0 \\ -c_f^2 (1+\varepsilon) - L^{F\prime} e^S & v^{F\prime} & 0 & -L^{F\prime}\varepsilon \\ 0 & 0 & v^{S\prime} & -1 - L^{S\prime} e^S \\ 0 & 0 & -(1 - 2e^s) & v^{S\prime} \end{array}\right], \tag{35}$$

$$[B'_A]^T = \left[\begin{array}{cccc} 0 & -2(1+\varepsilon)\left(v^{F\prime} - v^{S\prime}\right) & 2\left(1 + e^S\right)\left(v^{F\prime} - v^{S\prime}\right) & 0 \end{array}\right]^T. \tag{36}$$

We investigate the propagation of a wave in the medium whose undisturbed state is natural, i.e. in front of the wave the region is undisturbed

$$\varepsilon^{+}=0, \quad v^{S+}=0, \quad v^{F+}=0, \quad e^{S+}=0. \tag{37}$$

As the fields are continuous across the surface S the matrix A'_{AB} must be also continuous. Consequently, it is sufficient to evaluate its properties on the positive side of the surface. We denote this matrix by A'^{+}_{AB}.

The solution of the eigenvalue problem for this matrix yields the existence of P1-waves which propagate in both directions of the x-axis as well as slow P2-waves following the P1-waves. The latter enter the regions disturbed already by P1-waves. Consequently, we cannot use the relations presented above in the case of slow waves. However, the analysis of these waves yields the conclusion that they are very strongly attenuated. Consequently, they do not yield a creation of strong discontinuities. For this reason we limit the attention to P1-waves. The corresponding eigenvalue and eigenvectors are as follows

$$\begin{aligned} &\lambda'=1, \quad r'_A=\begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T, \\ &l'_A=\begin{bmatrix} 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T. \end{aligned} \tag{38}$$

Now we can easily find the coefficients of the Bernoulli equation for the amplitude A. We have

$$\begin{aligned} \alpha'_1 &= -l'_A \frac{\partial B'_A}{\partial u'_C} r'_C \frac{1}{r'_D l'_D}\Bigg| = 1, \\ \alpha'_2 &= -l'_A \frac{\partial A'_{AB}}{\partial u'_C} r'_B r'_C \frac{1}{r'_D l'_D}\Bigg| = \\ &= -\frac{1}{\sqrt{2}}\left(2-\frac{1}{2}L^{S\prime}\right). \end{aligned} \tag{39}$$

Consequently, we obtain the following solution for the evolution of the amplitude of discontinuity A along the characteristic of the P1-wave

$$\mathsf{A}=e^{-t'}\left[\frac{1}{\mathsf{A}_0}-\frac{1}{\sqrt{2}}\left(2-\frac{1}{2}L^{S\prime}\right)\left(1-e^{-t'}\right)\right]^{-1}. \tag{40}$$

It is essential to observe that $L^{S\prime}$ is negative. Namely, it can be written in the form

$$\begin{aligned} L^{S\prime} = 2\delta n_0 \frac{1}{\lambda_0^S+2\mu_0^S}\frac{\partial\left(\lambda^S+2\mu^S\right)}{\partial n} - \\ -\frac{1+2\nu}{2(1-\nu)}, \end{aligned} \tag{41}$$

where ν denotes Poisson's number. As argued in (Wilmanski 2004), the first contribution is negative and, hence, the whole expression is negative as well.

Consequently, a positive initial amplitude of discontinuity A_0 may yield a singularity of the above solution which appears in the critical time t'_c given by the relation

$$t'_c = -\ln\left[1-\frac{\sqrt{2}}{\mathsf{A}_0\left(2-\frac{1}{2}L^{S\prime}\right)}\right]. \tag{42}$$

Obviously this yields the condition on the minimum value of the amplitude for which the singularity must appear, namely

$$\mathsf{A}_0 > \frac{\sqrt{2}}{\left(2-\frac{1}{2}L^{S\prime}\right)}. \tag{43}$$

Let us check the physical consequences described by the amplitude A. According to relations (5) and (34) we have

$$\begin{aligned} &\left[\!\left[\frac{\partial e^S}{\partial x'}\right]\!\right] = -\left.\frac{\partial e^S}{\partial x'}\right|^{-} = \mathsf{A}r_4 = \frac{1}{\sqrt{2}}\mathsf{A} \\ &\quad\Rightarrow \left.\frac{\partial e^S}{\partial x}\right|^{-} = -\frac{\pi}{2\sqrt{2}\rho_0^S c_{P1}}\mathsf{A}, \\ &\left[\!\left[\frac{\partial v'^S}{\partial x'}\right]\!\right] = -\left.\frac{\partial v'^S}{\partial x'}\right|^{-} = \mathsf{A}r_3 = -\frac{1}{\sqrt{2}}\mathsf{A} \\ &\quad\Rightarrow \left.\frac{\partial e^S}{\partial t}\right|^{-} = \frac{\pi}{2\sqrt{2}\rho_0^S}\mathsf{A}. \end{aligned} \tag{44}$$

Hence, for the existence of the critical time, it is necessary that initially the gradient of deformation should be negative. As the deformation ahead of the wave is zero $e^{S+}=0$, this means that the infinite growth of the amplitude can appear only in **tension** $e^S \geq 0$ behind the wave front. Simultaneously, the deformation grows in time. This means, of course, that the porosity grows as well and, after reaching the critical time, the system ceases to exist as the poroelastic material. It reaches the state of **liquefaction**. We return to this phenomenon further in this work.

4 A NUMERICAL EXAMPLE

We illustrate the above considerations by a numerical example. We choose $K_s = 48\times 10^9$ Pa, and $K_f = 2.25\times 10^9$ Pa for the true compressibility coefficients of the grains and the fluid, respectively.

Now, the material parameters $\lambda^S, \mu^S, \kappa, \delta$ are calculated by means of Gassmann relations (Wilmanski 2004). Simultaneously, we choose constant Poisson's ratio as $\nu = 0.25$ which seems to be a reasonable approximation for many soils.

We proceed to illustrate the behavior of the critical time t'_c given by (42). Obviously, it depends on two variables – porosity n and initial amplitude A_0. In Fig. 1, we show the dependence on n for a chosen initial amplitude $\mathsf{A}_0=\sqrt{2}/\left(2-\frac{1}{2}L^{S\prime}(n=0.55)\right)$. As $\left|L^{S\prime}\right|$

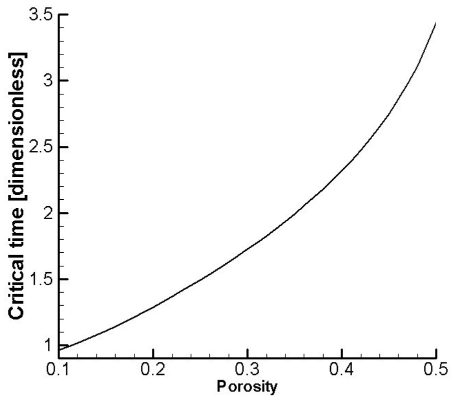

Figure 1: Critical time t_c' (dimensionless) for the initial amplitude $\mathsf{A}_0 = \sqrt{2}/\left(2 - \frac{1}{2}L^{S\prime}\left(n = 0.55\right)\right)$ as a function of porosity.

decays with growing n this choice satisfies condition (43). It is seen that the critical time t_c' becomes longer for larger porosities.

In Fig. 2, we illustrate the behavior of the critical time as a function of the initial amplitude for a chosen value of porosity. In the example, we choose $n = 0.25$. As already indicated the critical time does not exist if the amplitude is too small. In our example the threshold value of the amplitude is 0.3133. For higher values of the amplitude, the critical time decays very rapidly.

It is easy to check that the above numerical results correspond quantitatively very well with results obtained by Osinov (Osinov 1998). Bearing relation $(34)_2$ in mind, we find approximately

$$x_c \approx \frac{2\rho_0^S c_{P1}}{\pi} t_c', \tag{45}$$

where x_c is the critical distance of Osinov. For the typical data (e.g. (Wilmanski and Albers 2003)) $\rho_0^S = 2500$ kg/m^3, $c_{P1} = 2500$ m/s, $\pi = 10^6$ kg/m^3/s, we have $x_c \approx 12.5 t_c'$ m. This agrees with both theoretical and experimental data for Karlsruhe sand reported in Fig. 7 of the work (Osinov 1998).

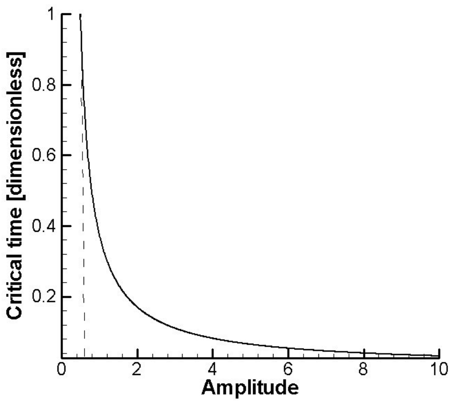

Figure 2: Critical time t_c' (dimensionless) for the porosity $n = 0.25$ as a function of the initial amplitude A_0.

5 CONCLUDING REMARKS

The above qualitative analysis demonstrates a potential of the description of threshold to liquefaction in granular materials by means of the analysis of the growth of strong discontinuities in acoustic waves. This method seems to be natural as the most common liquefaction process in nature is accompanying the propagation of acoustic waves created by earthquakes.

We have shown that liquefaction can appear if the wave yields the extension of the skeleton on the front of the wave. The creation of the strong discontinuity is related to changes of porosity rather than to an increment of pore pressure. However, it is obvious that by means of micro-macro relations, changes of porosity can be uniquely related to changes of pore pressure. In this way, we can transform the present results to the classical arguments explaining liquefaction in saturated granular materials.

Results of this work show that the threshold of liquefaction can be theoretically explained by models accounting for large changes of porosity yielding nonlinearity of field equations in either the above presented form or in the form of a gradient contribution to the momentum source, (Wilhelm and Wilmanski 2002), as well as by a model whose stress-strain relation is described by a hypoplastic law, (Osinov 1998). Most likely in reality all these mechanisms play a role and it requires experiments to select ranges of domination of one of them.

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