Optics and Spectroscopy, Vol. 89, No. 5, 2000, pp. 731–736. Translated from Optika i Spektroskopiya, Vol. 89, No. 5, 2000, pp. 795–800.

PHYSICAL AND QUANTUM OPTICS

Stability and Oscillations of Two-Dimensional Solitons Described by the Perturbed Nonlinear Schrödinger Equation

A. G. Vladimirov* and N. N. Rozanov**

**Institute of Physics, St. Petersburg State University, Peterhof, St. Petersburg, 198904 Russia*

***Research Institute of Laser Physics, St. Petersburg, 199034 Russia*

Received April 26, 2000

Abstract—A perturbation theory for determining the stability characteristics of spatial optical solitons with a 2D transverse profile in a transparent medium with a weak saturation of nonlinear refractive index is developed. For Kerr nonlinearity, a new solution of linearized equations for weak soliton perturbations is found. Using this solution, an expression for the stability characteristic is deduced, which, in the case of unstable solitons, determines their decay length and, in the case of stable solitons, shows the presence of perturbations with anomalously weak damping (internal modes) and determines their oscillation period.

1. INTRODUCTION

Spatial, time, and spatial-time solitons, for which diffraction and/or dispersion spread is counterbalanced by nonlinear focusing, are of considerable interest. In particular, they offer promise for information transfer and processing. Of critical importance is the question of soliton stability. In the case of paraxial solitons in a transparent medium with a saturable nonlinearity of refraction index, stability can be determined by the Vakhitov–Kolokolov criterion [1] (see also [2–4]). However, a complete answer to the question of stability should be quantitative rather than qualitative. For instance, even unstable structures that possess metastability are almost indistinguishable from stable structures, provided that their decay length is sufficiently large. On the other hand, if the perturbations of a soliton, which may even be stable, decay only on very large lengths, this soliton cannot be realized in practice. The presence of long-lived perturbations (internal modes) of stable optical solitons of different types was noted in [5–11].

The aim of this paper is to find a qualitative stability characteristic of spatial (two-dimensional) conservative optical solitons that determines not only their stability or instability, but also the decay length for unstable structures and the presence of oscillations of the internal modes of stable solitons and their period. To demonstrate the approach, we study a medium with weak saturation.

In Section 2, we present general expressions for solitons in a medium with a nonlinear refractive index and then introduce a perturbation theory for determining stationary solitons themselves. The zero approximation of this theory describes the degenerate self-channeling mode in a Kerr medium, which is described by the nonlinear Schrödinger equation (NSE) [12]. In Section 3, we analyze linearized equations for small deviations from a stationary soliton. We use there a new solution of the linearized equations for perturbations of a degenerate soliton of the NSE. This solution corresponds to the fourfold degeneracy of their zero eigenvalue. In Section 4, we build on the basis of this solution a perturbation theory for the stability characteristic of a perturbed soliton. The main results are discussed in the conclusion (Section 5).

2. INITIAL RELATIONS

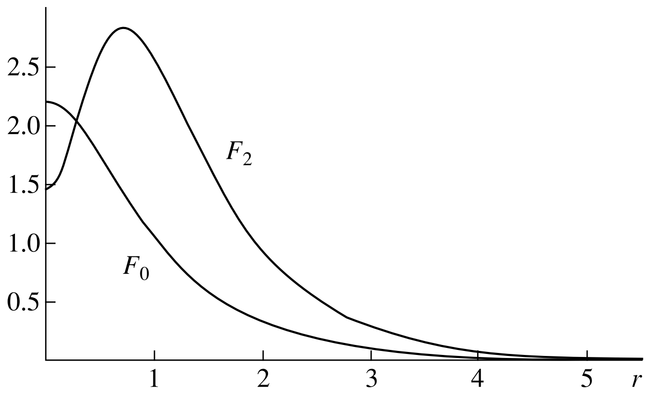
We start with the standard paraxial equation for the envelope of the electric field E of a sufficiently wide (in comparison with the light wavelength) beam of coherent polarized radiation traveling predominantly along the z-axis (we use dimensionless variables)

$$i\frac{\partial E}{\partial z} + \Delta_\perp E + |E|^2 E + \delta D = 0, \qquad (2.1)$$

where $\Delta_\perp = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the transverse Hamiltonian describing diffraction, and x and y are transverse coordinates. The last term on the left-hand side of (2.1) takes into account a small deviation of nonlinearity from Kerr nonlinearity; for $\delta D = 0$, (2.1) changes to the NSE. In what follows, we study a saturable nonlinearity with characteristic saturation intensity I_s. In this case, we have for the intensities $I = |E|^2 \ll |I_s|$

$$\delta D = -\frac{1}{I_s}|E|^4 E. \qquad (2.2)$$

This dependence is typical of many nonlinear optical media and, generally speaking, the parameter I_s may be negative. Kerr nonlinearity corresponds to the limit



Profiles of universal functions $F_0(r)$ and $F_2(r)$.

$I_s \longrightarrow \infty$. In the general case, and specifically in a case in which one takes into account spatial dispersion, the correction term may have a nonlocal dependence on the electric field [13].

A stationary soliton is characterized by an unchanged transverse field profile

$$E(x, y, z) = A_s(\rho)\exp(i\Gamma z), \tag{2.3}$$

where $\Gamma > 0$ is the propagation constant, and the soliton spectrum over Γ is continuous. Expression (2.3) corresponds to the fundamental soliton with the axially symmetric field distribution ($\rho = \sqrt{x^2 + y^2}$). $A_s(\rho)$ is a real and positive function. For a fixed Γ, the transverse field profile is determined as the solution of the equation

$$LA_s = 0, \quad L = \Delta_\perp - \Gamma + A_s^2 - \frac{1}{I_s}A_s^4 \tag{2.4}$$

that is finite on the interval $0 < \rho < \infty$. Because of axial soliton symmetry, Eq. (2.4) takes the form

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \Gamma + A_s^2 - \frac{1}{I_s}A_s^4\right]A_s = 0. \tag{2.5}$$

Note that even solitary solutions of the NSE [Eq. (2.5) for $I_s = \infty$]

$$L_0 A_{s0} = 0, \quad L_0 = \Delta_\perp - \Gamma + A_{s0}^2 \tag{2.6}$$

or

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \Gamma + A_{s0}^2\right]A_{s0} = 0 \tag{2.7}$$

cannot be expressed in terms of elementary functions and are determined only numerically. One can easily see from (2.7) that the profiles of the family of solitons of the NSE with parameter Γ are expressed in the form

$$A_{s0} = \sqrt{\Gamma}F_0(\sqrt{\Gamma}\rho) \tag{2.8}$$

in terms of the universal (and finite for $0 < r < \infty$) function $F_0(r)$, $r = \rho\sqrt{\Gamma}$, which, in turn, satisfies the equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 1 + F_0^2\right]F_0 = 0. \tag{2.9}$$

Because the function F_0 (see figure) is well known, solitons of perturbed Eq. (2.5) in the limit of high saturation intensity considered here should be sought for by the perturbation theory method. For this purpose, we set

$$A_s(\rho) = A_{s0}(\rho) + \mu^2 A_{s2}(\rho) + \dots, \tag{2.10}$$

where we introduced a small parameter $\mu^2 = 1/|I_s|$, so that $1/I_s = s\mu^2$ and $s = \mathrm{sgn} I_s = \pm 1$. The correction of the nearest approximation to soliton form A_{s2} is determined by Eq. (2.4) [or (2.5)], which is linearized with respect to this correction:

$$N_0 A_{s2} = sA_{s0}^5, \tag{2.11}$$

$$N_0 = L_0 + 2A_{s0}^2. \tag{2.12}$$

The homogeneous equation corresponding to (2.11), i.e.,

$$N_0 A = 0, \tag{2.13}$$

has among its localized solutions only solutions corresponding to a small soliton shift along the x- and y-axes [1]:

$$\begin{aligned} A_s &= \frac{\partial A_{s0}}{\partial x} = \frac{dA_{s0}}{d\rho}\cos\varphi, \\ A_y &= \frac{\partial A_{s0}}{\partial y} = \frac{dA_{s0}}{d\rho}\sin\varphi. \end{aligned} \tag{2.14}$$

Let us introduce the scalar product of functions of transverse coordinates

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^\infty d\rho\rho\int_0^{2\pi} d\varphi u v. \tag{2.15}$$

One can easily see that the radiation power for solitons of the NSE

$$P = P_0 = \langle A_{s0}, A_{s0}\rangle = 11.701/2\pi = 1.862 \tag{2.16}$$

is independent of the spectral parameter Γ, i.e., $dP_0/d\Gamma = 0$. Moreover, because of the angular dependence, both functions (2.14) are orthogonal to the right-hand side of Eq. (2.11), from which follows the solvability of this equation. Thus, the axially symmetric version of linear Eq. (2.11)

$$\left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \Gamma + 3A_{s0}^2\right]A_{s2} = sA_{s0}^5, \tag{2.17}$$

which is of interest for us, has a unique solution. As in the case of an unperturbed soliton of the NSE, a change to the universal function F_2 enables one to write out, in the explicit form, the dependence on the parameter Γ:

$$A_{s2} = s\Gamma^{3/2}F_2(\sqrt{\Gamma}\rho), \tag{2.18}$$

where the function $F_2(r)$ is a finite solution of the inhomogeneous linear ordinary differential equation

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 1 + 3F_0^2\right]F_2 = F_0^5. \tag{2.19}$$

The plot obtained for the function $F_2(r)$ by the numerical solution of (2.19) is presented in the figure.

3. LINEARIZED EQUATIONS

Consider the development of small field deviations from the stationary soliton determined by Eq. (2.4) [or (2.5)]. For this purpose, we set

$$E = [A_s + \delta E]\exp(i\Gamma z). \tag{3.1}$$

The correction δE, depending on transverse and radial coordinates, is assumed to be small. The linearization of (2.1) in δE gives the master equation

$$i\frac{\partial \delta E}{\partial z} + L\delta E + (A_s^2 - 2s\mu^2 A_s^4)(\delta E + \delta E^*) = 0. \tag{3.2}$$

For real and imaginary parts of a perturbation ($\delta E = \delta E_r + i\delta E_i$), Eq. (3.2) is written in the form

$$\frac{\partial \delta E_r}{\partial z} + L\delta E_i = 0, \quad \frac{\partial \delta E_i}{\partial z} - N\delta E_r = 0. \tag{3.3}$$

The last equation contains the operator

$$N = L + 2A_s^2 - 4s\mu^2 A_s^4. \tag{3.4}$$

One can see from the comparison with (2.12) that $N \longrightarrow N_0$ when $\mu \longrightarrow 0$. Equation (3.3) can be written in the matrix form

$$\frac{\partial}{\partial z}\delta\mathbf{E} = \boldsymbol{M}\delta\mathbf{E}, \tag{3.5}$$

where we introduced the real two-dimensional vector and the real (operator) 2×2 matrix

$$\delta\mathbf{E} = \begin{pmatrix}\delta E_r \\ \delta E_i\end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 0 & -L \\ N & 0\end{pmatrix}. \tag{3.6}$$

We shall also need the following well-known solutions of Eqs. (3.3) or (3.5). First, from the symmetry of the initial Eq. (2.1), with respect to the soliton phase shift, follows the presence of the solution

$$\delta E_r = 0, \quad \delta E_i = A_s. \tag{3.7}$$

The corresponding vector

$$\delta\mathbf{E}_\varphi = \begin{pmatrix} 0 \\ A_s\end{pmatrix} \tag{3.8}$$

is an eigenvector of matrix M with a zero eigenvalue

$$\boldsymbol{M}\delta\mathbf{E}_\varphi = 0. \tag{3.9}$$

The second solution of the linearized equations corresponds to the shift of the spectral parameter Γ. We have for it

$$\delta E_r = \frac{\partial A_s}{\partial \Gamma}, \quad \delta E_i = A_s z. \tag{3.10}$$

The vector

$$\delta\mathbf{E}_\Gamma = \begin{pmatrix}\dfrac{\partial A_s}{\partial \Gamma} \\ 0\end{pmatrix} \tag{3.11}$$

is a root vector rather than an eigenvector:

$$\boldsymbol{M}\delta\mathbf{E}_\Gamma = \delta\mathbf{E}_\varphi, \quad \boldsymbol{M}^2\delta\mathbf{E}_\Gamma = 0. \tag{3.12}$$

Now we set

$$\boldsymbol{M} = \boldsymbol{M}_0 + \mu^2\boldsymbol{M}_2 + \mu^4\boldsymbol{M}_4 + \ldots, \tag{3.13}$$

where

$$\boldsymbol{M}_0 = \begin{pmatrix} 0 & -L_0 \\ N_0 & 0\end{pmatrix}, \quad \boldsymbol{M}_2 = \begin{pmatrix} 0 & -L_2 \\ N_2 & 0\end{pmatrix}, \tag{3.14}$$
$$L_2 = 2A_{s0}A_{s2} - sA_{s0}^4, \quad N_2 = 6A_{s0}A_{s2} - 5sA_{s0}^4.$$

In a similar way, one can represent eigenvectors and root vectors of perturbations:

$$\delta\mathbf{E}_\Gamma = \delta\mathbf{E}_{\Gamma 0} + \mu^2\delta\mathbf{E}_{\Gamma 2} + \mu^4\delta\mathbf{E}_{\Gamma 4} + \ldots. \tag{3.15}$$

We shall also need the relations

$$\boldsymbol{M}_0\delta\mathbf{E}_{\Gamma 0} = \delta\mathbf{E}_{\varphi 0}, \quad \delta\mathbf{E}_{\Gamma 2} = \begin{pmatrix}\dfrac{\partial A_{s2}}{\partial \Gamma} \\ 0\end{pmatrix}, \tag{3.16}$$
$$\delta\mathbf{E}_{\varphi 0} = \begin{pmatrix} 0 \\ A_{s0}\end{pmatrix}.$$

In the zero approximation to which the Kerr nonlinearity corresponds ($\mu = 0$, NSE), linearized Eq. (3.3) is replaced with

$$\frac{\partial \delta E_r}{\partial z} + L_0\delta E_i = 0, \quad \frac{\partial \delta E_i}{\partial z} - N_0\delta E_r = 0. \tag{3.17}$$

This case is characterized by the maximum degeneracy of the spectrum and has the following two solutions of the linearized equations. From the invariance of quasi-optic Eq. (2.1) under the focusing transformation, which has been found by Talanov [14], follows the solution [15]

$$\delta E_r = \frac{\partial A_s}{\partial \Gamma}z, \quad \delta E_i = \frac{1}{2\Gamma}\left(-\frac{\rho^2}{4} + \Gamma z^2\right)A_s. \tag{3.18}$$

The corresponding vector

$$\delta \mathbf{E}_f = \begin{pmatrix} 0 \\ -\frac{1}{8\Gamma} A_{s0} \rho^2 \end{pmatrix} \quad (3.19)$$

is of the root kind as well:

$$\boldsymbol{M}_0 \delta \mathbf{E}_f = \delta \mathbf{E}_{\Gamma 0}, \quad \boldsymbol{M}_0^3 \delta \mathbf{E}_f = 0. \quad (3.20)$$

The solution of linearized solutions (3.17) of the form [16]

$$\delta E_r = a(\rho) + b(\rho) z^2, \ \delta E_i = c(\rho) z + d(\rho) z^3 \quad (3.21)$$

is likely to be of the greatest interest for further analysis.

Substitution of (3.21) in (3.17) gives

$$d = A_{s0}, \quad c = -\frac{3}{4\Gamma} A_{s0} \rho^2, \quad b = 3\frac{\partial A_{s0}}{\partial \Gamma}. \quad (3.22)$$

The function $a(\rho)$ is determined by the equation

$$N_0 a = -\frac{3}{4\Gamma} A_{s0} \rho^2. \quad (3.23)$$

In view of the fact that solutions of the homogeneous equation $N_0 a = 0$ [see (2.14)] are orthogonal to the right-hand side of (3.23) because of angular dependence, this equation can be solved. Thus, the function $a(\rho)$ is found as a uniquely bounded axially symmetric solution of the nonhomogeneous linear ordinary differential equation

$$\frac{d^2 a}{d\rho^2} + \frac{1}{\rho}\frac{da}{d\rho} - \Gamma a + 3A_{s0}^2 a = -\frac{3}{4\Gamma} A_{s0} \rho^2. \quad (3.24)$$

We shall not need the concrete form of this function, and a detailed discussion of the sense of additional symmetry requires a separate analysis. The perturbation vector found above and corresponding to additional symmetry is of the root kind as well:

$$\boldsymbol{M}_0 \delta \mathbf{E}_a = \delta \mathbf{E}_f, \quad \boldsymbol{M}_0^4 \delta \mathbf{E}_a = 0,$$

$$\delta \mathbf{E}_a = \begin{pmatrix} \frac{1}{6} a(\rho) \\ 0 \end{pmatrix}. \quad (3.25)$$

Our results suggest that the zero approximation of the axially symmetric linearized equation for Schrödinger solitons has a fourfold multiplicity (the fact that the multiplicity is not greater follows from the comparison of the results presented below with the results of Vakhitov and Kolokolov [1]; see also [16]).

4. STABILITY AND OSCILLATIONS OF PERTURBED SOLITONS

Following [17], we can now find the eigenvalue of the operator matrix M that changes to the zero value when $\mu \longrightarrow 0$ (the saturation intensity $I_s \longrightarrow \infty$). To be specific, we shall see that the removal of symmetry corresponding to root vectors (3.19) and (3.25) (due to the inclusion of a weak nonlinearity saturation in the analysis) causes a split of the fourfold zero eigenvalue into the twofold zero value and two nonzero eigenvalues (with different signs).

The characteristic solution of linear Eq. (3.5) will be sought in the form

$$\delta \mathbf{E} = \boldsymbol{\Psi}(\rho) \exp(\mu\gamma z). \quad (4.1)$$

Here, we introduced the desired eigenvalue

$$\mu\gamma = \mu\gamma_1 + \mu^2\gamma_2 + \mu^3\gamma_3 + \dots \quad (4.2)$$

and the eigenvector $\boldsymbol{\Psi}$ satisfying the equation

$$\boldsymbol{M}\boldsymbol{\Psi} = \mu\gamma\boldsymbol{\Psi}, \quad (4.3)$$

which follows from (3.5). The expansion of the eigenvector $\boldsymbol{\Psi}$ will be represented in the form

$$\begin{aligned} \boldsymbol{\Psi} &= \delta \mathbf{E}_\varphi + \mu\gamma\delta \mathbf{E}_\Gamma + \mu^2 \boldsymbol{\Psi}_2 \\ &+ \mu^3 \boldsymbol{\Psi}_3 + \mu^4 \boldsymbol{\Psi}_4 + \dots , \end{aligned} \quad (4.4)$$

which changes to the eigenvector $\delta \mathbf{E}_{\varphi 0}$ (3.16) when $\mu \longrightarrow 0$. Substituting expansions of the corresponding quantities in powers of the small parameter μ into the right- and left-hand sides and equating terms with the same powers of this parameter, we find in the second order in μ [for the choice made in (4.4), equations of lower orders are satisfied automatically]

$$\boldsymbol{M}_0 \boldsymbol{\Psi}_2 = \gamma_1^2 \delta \mathbf{E}_{\Gamma 0}. \quad (4.5)$$

Because of (3.20), this relation gives

$$\boldsymbol{\Psi}_2 = \gamma_1^2 \delta \mathbf{E}_f. \quad (4.6)$$

In the third order, we have

$$\begin{aligned} \boldsymbol{M}_0 \boldsymbol{\Psi}_3 &= 2\gamma_1\gamma_2 \delta \mathbf{E}_{\Gamma 0} + \gamma_1 \boldsymbol{\Psi}_2 \\ &= 2\gamma_1\gamma_2 \delta \mathbf{E}_{\Gamma 0} + \gamma_1^3 \delta \mathbf{E}_f, \end{aligned} \quad (4.7)$$

so that

$$\boldsymbol{\Psi}_3 = 2\gamma_1\gamma_2 \delta \mathbf{E}_f + \gamma_1^3 \boldsymbol{\Psi}_a. \quad (4.8)$$

Finally, we obtain in the fourth order

$$\begin{aligned} \boldsymbol{M}_0 \boldsymbol{\Psi}_4 + \boldsymbol{M}_2 \boldsymbol{\Psi}_2 &= \gamma_1^2 \delta \mathbf{E}_{\Gamma 2} \\ &+ (\gamma_2^2 + 2\gamma_1\gamma_3) \delta \mathbf{E}_{\Gamma 0} + \gamma_1 \boldsymbol{\Psi}_3 + \gamma_2 \boldsymbol{\Psi}_2 \end{aligned} \quad (4.9)$$

which, because of (4.6) and (4.8), takes the form

$$\begin{aligned} \boldsymbol{M}_0 \boldsymbol{\Psi}_4 + \gamma_1^2 \boldsymbol{M}_2 \delta \mathbf{E}_f &= \gamma_1^2 \delta \mathbf{E}_{\Gamma 2} \\ &+ (\gamma_2^2 + 2\gamma_1\gamma_3) \delta \mathbf{E}_{\Gamma 0} + 3\gamma_1^2\gamma_2 \delta \mathbf{E}_f + \gamma_1^4 \boldsymbol{\Psi}_a. \end{aligned} \quad (4.10)$$

Let us introduce the conjugate operator matrix $\boldsymbol{M}_0^\dagger$ (the transposed matrix) and the vectors $\boldsymbol{\Psi}_{\varphi 0}^\dagger$ and $\boldsymbol{\Psi}_{\Gamma 0}^\dagger$

according to the relations

$$\boldsymbol{M}_0^\dagger = \begin{pmatrix} 0 & N_0 \\ -L_0 & 0 \end{pmatrix}, \quad \boldsymbol{M}_0^\dagger \delta\mathbf{E}_{\varphi 0}^\dagger = 0, \qquad (4.11)$$
$$\boldsymbol{M}_0^\dagger \delta\mathbf{E}_{\Gamma 0}^\dagger = \delta\mathbf{E}_{\varphi 0}^\dagger.$$

In the explicit form, we have

$$\delta\mathbf{E}_{\varphi 0}^\dagger = \begin{pmatrix} A_{s0} \\ 0 \end{pmatrix}, \quad \delta\mathbf{E}_{\Gamma 0}^\dagger = \begin{pmatrix} 0 \\ \dfrac{\partial A_{s0}}{\partial \Gamma} \end{pmatrix}. \qquad (4.12)$$

One can easily verify that the properties of conjugate operators provide the fulfillment of the identity

$$\langle \mathbf{U}, \boldsymbol{M}_0, \mathbf{V} \rangle = \langle \boldsymbol{M}_0^\dagger \mathbf{U}, \mathbf{V} \rangle \qquad (4.13)$$

for arbitrary vectors **U** and **V**. Because of this, the matrix elements satisfy the relations

$$\langle \delta\mathbf{E}_{\varphi 0}^\dagger, \boldsymbol{M}_0 \boldsymbol{\Psi}_4 \rangle = \langle \boldsymbol{M}_0^\dagger \delta\mathbf{E}_{\varphi 0}^\dagger, \boldsymbol{\Psi}_4 \rangle = 0, \qquad (4.14)$$

$$m_{0,a} = \langle \delta\mathbf{E}_{\varphi 0}^\dagger, \boldsymbol{\Psi}_a \rangle = \langle \boldsymbol{M}_0^\dagger \delta\mathbf{E}_{\Gamma 0}^\dagger, \boldsymbol{\Psi}_a \rangle$$
$$= \langle \delta\mathbf{E}_{\Gamma 0}^\dagger, \boldsymbol{M}_0 \boldsymbol{\Psi}_a \rangle = \langle \delta\mathbf{E}_{\Gamma 0}^\dagger, \delta\mathbf{E}_f \rangle$$
$$= \left\langle \begin{pmatrix} 0 \\ \dfrac{\partial A_{s0}}{\partial \Gamma} \end{pmatrix}, \begin{pmatrix} 0 \\ -\dfrac{1}{8\Gamma} A_{s0} \rho^2 \end{pmatrix} \right\rangle \qquad (4.15)$$
$$= -\frac{1}{8\Gamma} \int_0^\infty A_{s0} \frac{\partial A_{s0}}{\partial \Gamma} \rho^3 d\rho,$$

$$m_{0,\Gamma} = \langle \delta\mathbf{E}_{\varphi 0}^\dagger, \delta\mathbf{E}_{\Gamma 2} \rangle = \int_0^\infty A_{s0} \frac{\partial A_{s2}}{\partial \Gamma} \rho d\rho, \qquad (4.16)$$

$$\langle \delta\mathbf{E}_{\varphi 0}^\dagger, \delta\mathbf{E}_f \rangle = \left\langle \begin{pmatrix} A_{s0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\dfrac{1}{8\Gamma} A_{s0} \rho^2 \end{pmatrix} \right\rangle = 0. \qquad (4.17)$$

We now equate the scalar products of the vector $\delta\mathbf{E}_{\varphi 0}^\dagger$ on the left- and right-hand sides of Eq. (4.10). Taking into account the above relations, we obtain an equation for determining the eigenvalue in the lowest approximation

$$\gamma_1^2 [(m_{0,f} - m_{0,\Gamma}) - m_{0,a} \gamma_1^2] = 0, \qquad (4.18)$$

where

$$m_{0,f} = \langle \delta\mathbf{E}_{\varphi 0}^\dagger, \boldsymbol{M}_2 \delta\mathbf{E}_f \rangle. \qquad (4.19)$$

5. NORMALIZED FUNCTIONS AND DISCUSSION OF RESULTS

First of all, we note that the radiation power for the stationary soliton is given by the expression

$$P = \int_0^\infty (A_{s0} + \mu^2 A_{s2} + \ldots)^2 \rho d\rho$$
$$= P_0 + 2s\mu^2 \Gamma p_1 + \ldots, \qquad (5.1)$$

where the critical power for self-focusing (in a Kerr medium) P_0 is introduced by relation (2.16), and

$$p_1 = \int_0^\infty F_0(r) F_2(r) r dr = 3.771. \qquad (5.2)$$

It is substantial that the sgn of the quantity $dP/d\Gamma = 2s\mu^2 p_1$, which enters into the Vakhitov–Kolokolov criterion [1], is determined by the sign of s (stability is observed for $s > 0$).[1]

Now we turn to the quantitative analysis of soliton stability. As follows from (4.18), in the case where one takes into account the nonlinearity saturation, the eigenvalue γ_1 retains a twofold zero value (symmetry with respect to the phase shift and the propagation constant), and a nonzero value (two roots with opposite signs) splits off from it. Let us transform the matrix elements determining the last value by separating out the dependence on the propagation constant Γ, which is due to the change in universal functions F_0 and F_2.

$$m_{0,\Gamma} = \frac{s}{2}(3p_1 + p_4), \quad m_{0,f} = \frac{1}{8} s p_2,$$
$$m_{0,a} = \frac{1}{16\Gamma^3} p_3, \qquad (5.3)$$

where

$$p_2 = \int_0^\infty [2F_0^3(r) F_2(r) - F_0^6(r)] r^3 dr = 3.668,$$
$$p_3 = \int_0^\infty F_0^2(r) r^3 dr = 2.211, \qquad (5.4)$$
$$p_4 = \int_0^\infty F_0(r) \frac{dF_2(r)}{dr} r^2 dr = -2.854.$$

Thus, we finally obtain, for the nonzero root of equation (4.18),

$$\gamma_1^2 = -2s\Gamma^3 \frac{4(3p_1 + p_4) - p_2}{p_3} = -27.28 s \Gamma^3 \qquad (5.5)$$

or

$$(\mu\gamma_1)^2 = -27.28 \frac{1}{I_s} \Gamma^3. \qquad (5.6)$$

According to the Vakhitov–Kolokolov criterion, (5.5) and (5.6) show that a stationary soliton is stable for $s < 0$ ($I_s < 0$) and unstable for $s > 0$ ($I_s > 0$). In the

---

[1] Note that one can also express in terms of this derivative the quantity entering in (4.18): $m_{0,f} - m_{0,\Gamma} = -\dfrac{1}{2}\dfrac{d\rho}{d\Gamma}$. Using this result, one can support the qualitative priority of the Vakhitov–Kolokolov criterion of soliton stability.

latter case, from (5.5) and (5.6), there follows the presence of undamped oscillations of weak perturbations (internal modes) with the characteristic longitudinal frequency $\mu|\gamma_1| \sim \Gamma^{3/2}$. In actual conditions, stationary solitons are formed (this is accompanied by oscillations) because these oscillations damp due to the presence of terms that are nonlinear in amplitude [10, 11]. But the oscillation damping is nonexponential and rather weak, especially for solitons with small shifts in their propagation constant (the limit corresponds to $\Gamma \longrightarrow 0$).

In summary, we developed in this paper a version of the consistent perturbation theory for determining the stability and internal modes of the solitons described by the perturbed nonlinear Schrödinger equation. Although we analyzed an example of soliton perturbation caused by nonlinearity saturation, the method here proposed can be extended to other cases, and in particular, to the case in which nonparaxial soliton characteristics are taken into account [18–24].

ACKNOWLEDGMENTS

This work was supported by the International Science and Technology Center (project no. 666), the Russian Foundation for Basis Research (project no. 98-02-18202), the Ministry of Higher Education of Russia (projects no. 2-98-13 and 2-98-25), and INTAS (grant no. 1997-581).

REFERENCES

1. N. G. Vakhitov and A. A. Kolokolov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **16**, 1020 (1973).
2. E. A. Kuznetsov, A. M. Rubenchik, and V. E. Zakharov, Phys. Rep. **142** (3), 103 (1986).
3. V. E. Zakharov and E. A. Kuznetsov, Zh. Éksp. Teor. Fiz. **113**, 1892 (1998) [JETP **86**, 1035 (1998)].
4. E. A. Kuznetsov, Zh. Éksp. Teor. Fiz. **116**, 299 (1999) [JETP **89**, 163 (1999)].
5. V. E. Zakharov, V. V. Sobolev, and V. S. Synakh, Zh. Éksp. Teor. Fiz. **60**, 136 (1971) [Sov. Phys. JETP **33**, 77 (1971)].
6. A. W. Snyder, S. Hewlett, and D. J. Mitchell, Phys. Rev. E **51**, 6297 (1995).
7. D. E. Pelinovsky, V. V. Afanasjev, and Yu. S. Kivshar, Phys. Rev. E **53**, 1940 (1996).
8. C. Etrich, U. Peshel, F. Lederer, *et al.*, Phys. Rev. E **54**, 4321 (1996).
9. B. A. Malomed, P. Drummond, H. He, *et al.*, Phys. Rev. E **56** (4), 4725 (1997).
10. D. E. Pelinovsky, Yu. S. Kivshar, and V. V. Afanasjev, Physica D (Amsterdam) **116**, 121 (1998).
11. N. N. Rozanov, P. I. Krepostnov, V. O. Popov, and D. A. Kirsanov, Opt. Zh. **67** (4), 28 (2000) [J. Opt. Technol. **67**, 322 (2000)].
12. S. N. Vlasov and V. I. Talanov, *Self-Focusing of Waves* (Inst. Prikl. Fiz. Ross. Akad. Nauk, Nizhni Novgorod, 1997).
13. S. K. Turitsyn, Teor. Mat. Fiz. **64** (2), 226 (1985).
14. V. I. Talanov, Pis'ma Zh. Éksp. Teor. Fiz. **11**, 303 (1970) [JETP Lett. **11**, 199 (1970)].
15. E. A. Kuznetsov and S. K. Turitsyn, Phys. Lett. A **112** (6–7), 273 (1985).
16. V. E. Zakharov and A. M. Rubenchik, Zh. Éksp. Teor. Fiz. **65**, 997 (1973).
17. V. M. Malkin and E. G. Shapiro, Physica D (Amsterdam) **53**, 25 (1991).
18. D. I. Abakarov, A. A. Akopov, and S. I. Pekar, Zh. Éksp. Teor. Fiz. **52**, 463 (1967) [Sov. Phys. JETP **25**, 303 (1967)].
19. V. M. Eleonskiĭ and V. P. Silin, Pis'ma Zh. Éksp. Teor. Fiz. **13**, 167 (1971) [JETP Lett. **13**, 117 (1971)].
20. V. M. Eleonskiĭ, L. G. Oganes'yants, and V. P. Silin, Zh. Éksp. Teor. Fiz. **62**, 81 (1972) [Sov. Phys. JETP **35**, 44 (1972)].
21. V. M. Eleonskiĭ, L. G. Oganes'yants, and V. P. Silin, Zh. Éksp. Teor. Fiz. **63**, 532 (1972) [Sov. Phys. JETP **36**, 282 (1972)].
22. D. A. Kirsanov and N. N. Rozanov, Opt. Spektrosk. **87**, 423 (1999) [Opt. Spectrosc. **87**, 390 (1999)].
23. V. E. Semenov, N. N. Rozanov, and N. V. Vysotina, Zh. Éksp. Teor. Fiz. **116**, 458 (1999) [JETP **89**, 243 (1999)].
24. N. N. Rozanov, Opt. Spektrosk. **89** (2000) (in press) [Opt. Spectrosc. **89** (2000) (in press)].

Translated by A. Kirkin