Synchronization of weakly stable oscillators and semiconductor laser arrays

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Abstract. – We study the synchronization properties of an array of nonidentical globally coupled limit cycle oscillators. Above a critical coupling strength, some oscillators undergo a self-pulsing instability. We study analytically the synchronization conditions below and above this instability threshold, thus removing the usual restriction of limit cycle stability. Self-pulsing decreases the order parameter and synchronization degradation can be reduced by delaying the coupling among the oscillators. Semiconductor lasers coupled by an external mirror are used as a convenient realization of that model.

Physics, chemistry, and biology offer many examples of large systems of weakly interacting limit cycle oscillators [1]. If the limit cycles are stable, nearly identical, and the coupling is weak enough, the dynamics of a system of N coupled oscillators can be described in an N-dimensional phase space whose coordinates are the phases of the individual oscillators. When studying collective behaviors, the reduction to a phase model constitutes a great simplification. In particular, the Kuramoto phase equations result from such a reduction [2]. Since these equations are universal and allow analytical description, they are good reference models to study the transition to synchronization. Arrays of Josephson junctions [3], neural networks in the brain [4], rhythmic applause [5], flavor evolution of neutrinos [6] and, to some extent, arrays of coupled semiconductor lasers [7,8] have been successfully modelled by the Kuramoto equations. Recently several generalizations of this model were proposed, which include amplitude dynamics [9], time delay [7, 10–13] and inertia [14]. Phase synchronization was also described analytically for the Winfree mean-field model [15].

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In many cases, however, the main assumption underlining these phase models is the stability of the limit cycles. If the coupling strength is comparable to the dynamical decay rates that characterize the stability of the individual limit cycles, additional degrees of freedom can be excited and the limit cycles may be destabilized. The Kuramoto model and its variations become inadequate in this case. In this letter, we analyze such a situation analytically and show that a third-order equation can be derived as a generalization of the Kuramoto equation for the case of nonidentical oscillators. We study this equation analytically and show that beyond the threshold where some of the oscillators become unstable, the order parameter decreases with a 3/2 scaling power law near threshold. Below the threshold, the situation was studied in [12,13] but led to results expressed in terms of unsolved integrals. We show how to solve them, which leads eventually to a complete analytic discussion. This is achieved in terms of an infinite set of eigenfunctions which are the analog of the external cavity modes in lasers with feedback.

To express the problem in terms amenable to an explicit formulation, we consider as an example the synchronization properties of an array of semiconductor lasers globally coupled through an optical feedback produced by an external mirror. Aside from its fundamental interest, this subject is also of technological importance: if the lasers are locked in phase, a coherent high power output can be concentrated in a single-lobe far field pattern. Each laser in the array is a limit cycle oscillator for the single-mode electric field with a single natural (*i.e.*, optical) frequency. Taken separately, the lasers are only weakly stable: The eigenvalues of the equations linearized around the constant intensity solution (or cw regime) have real parts which are strictly negative but very close to zero. In order to study synchronization in the array, we use coupled third-order delayed phase equations that are an extension of the Kuramoto phase model. We show that for small values of the coupling strength, the dependence of the modulus of the complex order parameter [2] is the same as for the usual Kuramoto phase model with a time delay. As the coupling strength is increased, some lasers can be destabilized via a Hopf bifurcation which may result in a degradation of the synchronization.

Consider an array of N coupled semiconductor lasers modelled by coupled Lang-Kobayashi equations [16] for the dimensionless electric-field envelope E and excess free carrier density Z. For the j-th laser, these equations are

$$\frac{\mathrm{d}E_j}{\mathrm{d}t} = i\delta_j E_j + (1 + i\alpha)\frac{Z_j E_j}{\tau_\mathrm{p}} + i\frac{\kappa e^{-i\omega t_\mathrm{D}}}{N}\sum_{n=1}^{N} E_n(t - t_\mathrm{D}),\tag{1}$$

$$\frac{\mathrm{d}Z_j}{\mathrm{d}t} = \frac{1}{\tau_\mathrm{c}}\big[P_j - Z_j - (1 + 2Z_j)|E_j|^2\big],\tag{2}$$

where $\omega = N^{-1}\sum_j \omega_j$ is the average frequency in the array, $\delta_j = \omega_j - \omega$, α is the linewidth enhancement factor, $\tau_\mathrm{p} \simeq 10^{-12}\,\mathrm{s}$ is the photon lifetime, κ is the feedback rate, t_D is the external cavity round-trip time, ωt_D is the mean optical phase-shift between emitted and feedback fields, and $\tau_\mathrm{c} \simeq 10^{-9}\,\mathrm{s}$ is the free carrier lifetime. Finally, P_j is the excess pump parameter which is proportional to the injection current above threshold. In previous studies, synchronization in an array of solid-state lasers characterized by different natural frequencies was analyzed, but with the assumption that the population inversions Z_j can be adiabatically eliminated [17].

The j-th solitary laser exhibits a low-frequency dynamics characterized by a relaxation oscillation frequency $\Omega_{\mathrm{R},j}$ that lies in the gigahertz range. Assuming that $|1 - P_j/P| \ll 1$, where $P = N^{-1}\sum_j P_j$, we have $\Omega_{\mathrm{R},j} \approx \Omega_\mathrm{R} = \sqrt{2P/\tau_\mathrm{c}\tau_\mathrm{p}}$. The coupling can be achieved by an external mirror placed at the Talbot distance from the array, which is of the order of $1\,\mathrm{mm}$ [18]. In this case, $\Omega_\mathrm{R} t_\mathrm{D} \ll 1$ and the approximation $E_n(t - t_\mathrm{D}) \simeq E_n(t)$ can be

used in (1). If, however, the distance between the array and the feedback mirror is such that $\Omega_{\mathrm{R}} t_{\mathrm{D}} = O(1)$, the time delay t_{D} in the argument of $E_n(t - t_{\mathrm{D}})$ cannot be neglected [7, 8].

To study the synchronization properties of the array, we decompose the electric field as $E_j = |E_j| e^{i\Phi_j}$. It is also convenient to rescale the time $s = \Omega_{\mathrm{R}} t$ $(s_{\mathrm{D}} = \Omega_{\mathrm{R}} t_{\mathrm{D}})$ and introduce the complex Kuramoto order parameter

$$\sigma(s) e^{i\xi(s)} = \frac{1}{N} \sum_n e^{i[\Phi_n(s) - \omega t_{\mathrm{D}}]}. \tag{3}$$

For semiconductor lasers with a large α factor, eqs. (1) and (2) can be asymptotically approximated by the third-order phase equations [8]

$$\frac{\mathrm{d}^3 \Phi_j}{\mathrm{d}s^3} + \varepsilon \frac{\mathrm{d}^2 \Phi_j}{\mathrm{d}s^2} + (1 + 2\varepsilon\Omega_j) \frac{\mathrm{d}\Phi_j}{\mathrm{d}s} = \varepsilon \Delta_j + \varepsilon K \sigma(s - s_{\mathrm{D}}) \sin\left[\xi(s - s_{\mathrm{D}}) - \Phi_j(s)\right], \tag{4}$$

where $\varepsilon = (2P+1)\sqrt{\tau_{\mathrm{p}}/2P\tau_{\mathrm{c}}}$, $\Omega_j = (P_j/P - 1)/\varepsilon$, $\Delta_j = \delta_j \tau_{\mathrm{c}}/(1 + 2P)$, and $K = \alpha\kappa\tau_{\mathrm{c}}/(1 + 2P)$. For semiconductor and other class-B lasers we have $\tau_{\mathrm{p}}/\tau_{\mathrm{c}} \ll 1$. Therefore, for moderate pump strengths, $P = O(1)$, weak enough coupling $\kappa\tau_{\mathrm{c}} = O(1)$, and small parameter dispersion over the array, $P_j/P - 1 = O(\varepsilon)$, $\delta_j \tau_{\mathrm{c}} = O(1)$, we obtain $\varepsilon \ll 1$ and $\Omega_j, \Delta_j, K = O(1)$. The frequencies Δ_j are randomly distributed with Lorentzian probability $g(\Delta) = \Gamma/[\pi(\Delta^2 + \Gamma^2)]$. We note that neglecting $\mathrm{d}^2\Phi_j/\mathrm{d}s^2$ and $\mathrm{d}^3\Phi_j/\mathrm{d}s^3$ in (4) yields the delayed Kuramoto model [12, 13], while neglecting $\mathrm{d}^3\Phi_j/\mathrm{d}s^3$ only yields the system used to study the effect of inertia on the synchronization properties of globally coupled phase oscillators [14]. The system of equations (4) is a generic model for an array of weakly stable globally coupled oscillators.

In the limit $N \to \infty$, the absolute value of the order parameter, σ, vanishes if the lasers are completely desynchronized, and tends to unity as they approach perfect synchronization. Our aim is to determine the dependence of σ on the coupling strength K and on Γ, the width of the natural frequency distribution. We restrict our analysis to the solutions of (4) with time-independent σ. This includes the branch of partially synchronized solutions emerging at the synchronization threshold where σ becomes nonzero [2, 13].

For typical low-power semiconductor lasers, we have $\varepsilon \ll 1$. This is related to the weak stability of the cw regime. Setting $\varepsilon = 0$ in eq. (4) yields the solution $\Phi_j = \phi_j + \rho_j \sin(s + \theta_j)$ with time-independent ϕ_j, ρ_j, and θ_j. For small but finite ε, we therefore seek a solution of the form

$$\Phi_j(s) = \phi_j(\tau) + \rho_j(\tau) \sin\left[s + \theta_j(\tau)\right], \tag{5}$$

where ϕ_j, ρ_j, and θ_j depend on the slow time $\tau = \varepsilon s$ $(\tau_{\mathrm{D}} = \varepsilon s_{\mathrm{D}})$. A similar ansatz was recently used in [19]. If $\rho_j = 0$ for all j in eq. (5), all the lasers in the array operate in a cw regime which is the stable limit cycle oscillator regime. As the coupling strength reaches a critical value, a fraction of the phase-locked lasers acquires a nonzero ρ_j. The number of destabilized lasers depends on the laser parameters and on the natural frequency distribution. As a result, their intensities and phases oscillate at a frequency close to the relaxation oscillation frequency Ω_{R}. This primary partially synchronized cw state is then destabilized by a secondary branch of partially synchronized solutions, and so on. Various phase configurations are possible for the relaxation oscillation phases θ_j, each giving rise to a secondary branch. We shall describe only the branch characterized by completely desynchronized phases θ_j. The existence of such a solution is inferred from the analysis of identical lasers. If the time delay is small on the relaxation oscillation time scale, $s_{\mathrm{D}} \ll 1$, the Hopf bifurcation always produces antiphase oscillations of the laser intensities [7, 8]. We focus on the most commonly observed antiphase regime, the so-called splay state regime characterized by $\theta_j = \theta_0 + 2jk\pi/N$ with

integer k. In the continuous limit, $N \to \infty$, the relaxation phases constrained by this relation are homogeneously distributed over the entire interval $(0, 2\pi)$, independently of ρ_j and ϕ_j. This suggests to neglect harmonic corrections of the order parameter on the time s. In the large-N limit, we therefore seek solutions of the form

$$\sigma e^{i\xi} = \frac{1}{N} \sum_n e^{i[\phi_n + \rho_n \sin(s + \theta_n) - \omega t_D]} \simeq \frac{1}{N} \sum_n e^{i(\phi_n - \omega t_D)} J_0(\rho_n), \tag{6}$$

where σ is time-independent and ξ varies linearly in time, $\xi = \nu\tau$. The parameters σ and ν are still to be determined. Substituting (5) into eq. (4) and using the averaging procedure, we obtain the following equations for the slow time evolution of ϕ_j, ρ_j, and θ_j:

$$\frac{d\phi_j}{d\tau} = \Delta_j + K\sigma \sin \left[\nu(\tau - \tau_D) - \phi_j \right] J_0(\rho_j), \tag{7}$$

$$\frac{d\rho_j}{d\tau} = -\frac{1}{2}\rho_j + K\sigma \cos \left[\nu(\tau - \tau_D) - \phi_j \right] J_1(\rho_j), \tag{8}$$

$$\frac{d\theta_j}{d\tau} = \Omega_j. \tag{9}$$

Equation (9) is consistent with the assumption that the relaxation oscillations phases θ_j are unlocked since, in general, the frequencies Ω_j differ for different oscillators.

We first consider the domain $K\sigma < 1$ where all lasers are stable. The solution $\rho_j = 0$ of eq. (8) is linearly stable for all ϕ_j and relaxation oscillations decay on the τ time-scale: $\rho_j(\tau) \to 0$. Equations (7) to (9) then reduce to the delayed Kuramoto model

$$\frac{d\phi_j}{d\tau} = \Delta_j + K\sigma \sin \left[\nu(\tau - \tau_D) - \phi_j \right]. \tag{10}$$

The solution $\sigma = \sigma(K)$ can be derived in two steps using the approach described in [2]: first, eq. (10) is solved as a function of σ and ν; second, that solution ϕ_j is used to determine σ and ν from (6). Introducing the phase deviations $\psi_j = \phi_j - \nu\tau + \nu\tau_D$, let $P(\psi, \Delta)$ be the probability for an oscillator with natural frequency Δ to have its phase between ψ and $\psi + d\psi$. Using (6) with $\rho_n = 0$ and taking the continuum limit $N \to \infty$, the order parameter becomes

$$\sigma e^{i(\omega t_D + \nu\tau_D)} = \int_{-\infty}^{\infty} \int_0^{2\pi} g(\Delta) P(\psi, \Delta) e^{i\psi} \, d\Delta \, d\psi. \tag{11}$$

The form of $P(\psi, \Delta)$ in (11) is well known [13], while the analytical expression for the value of the integral is not. Performing the integration over ψ and then over Δ in the complex plane, we find

$$\sigma e^{i(\omega t_D + \nu\tau_D)} = \sqrt{1 + \left(\frac{\Gamma + i\nu}{K\sigma} \right)^2} - \frac{\Gamma + i\nu}{K\sigma}. \tag{12}$$

Solving (12) leads to

$$\sigma^2 = 1 - 2\Gamma / \left[K\cos(\nu\tau_D + \omega t_D) \right], \tag{13}$$

$$\nu = -K\sin(\nu\tau_D + \omega t_D) + \Gamma \tan(\nu\tau_D + \omega t_D), \tag{14}$$

with $\cos(\nu\tau_D + \omega t_D) > 0$. These equations determine an infinite number of branches of partially coherent solutions that emerge from the fully incoherent state. Letting $\sigma \to 0$ reproduces the linear stability results for that incoherent state [12], so that (13) and (14) complete the analysis
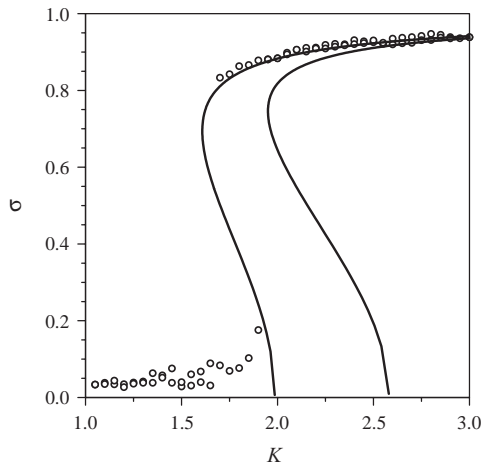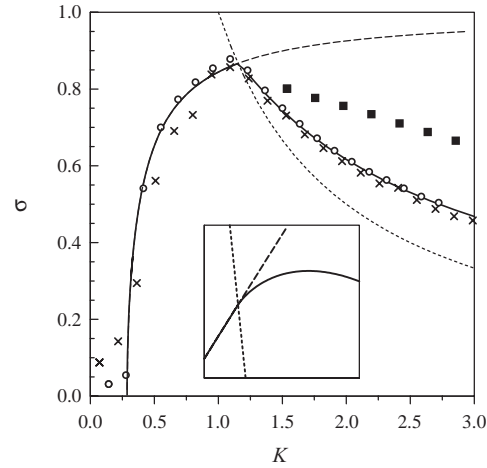
Fig. 1

Fig. 2

Fig. 1 – Branches of partially synchronized solutions obtained by solving eq. (12) with $\Gamma = 0.143$, $\mathrm{mod}(\omega t_\mathrm{D}, 2\pi) = 2.9$, and $\tau_\mathrm{D} = 1.5$. Dots represent the time-average of σ obtained by numerical simulations of eq. (10) with 300 elements.

Fig. 2 – Order parameter σ as a function of K with $\Gamma = 0.143$, $\mathrm{mod}(\omega t_\mathrm{D}, 2\pi) = 0.1$, $\Omega_j = 0$ for all j, and $\varepsilon = 0.09$. This corresponds to the laser parameters: $\tau_\mathrm{p} = 10^{-12}\,\mathrm{s}$, $\tau_\mathrm{c} = 10^{-9}\,\mathrm{s}$, and $P = 3$. Circles are the averaged values of σ obtained by numerical integration of the phase equations (4) with $s_\mathrm{D} = 0$. Crosses are the averages of σ obtained with the original laser equations (1)-(2). Full line: solution of eqs. (16) and (17). Dashed line: unstable primary branch of partially synchronized solutions. Dotted line: $\sigma = 1/K$. Inset shows the enlarged vicinity of the secondary instability point $K\sigma = 1$. Finally, squares are the averaged values of σ obtained by numerical integration of the phase equations (4) with $s_\mathrm{D} = 11.5$, corresponding to $\tau_\mathrm{D} \simeq 1$.

of these authors. The multiplicity of the synchronized solutions directly follows from the time delay in the coupling. In the short-delay limit, $\tau_\mathrm{D} \ll 1$, (12) leads to the result obtained for arrays of solid-state lasers well above lasing threshold [17]. Physically, the multiple solutions of (12) can be considered as the external cavity modes (ECM) of the partially coherent array. They generalize the ECM of the single semiconductor laser with external feedback [20]. The two branches of solutions of eqs. (13) and (14) shown in fig. 1 correspond to two different ECM. Multiple branches of solutions of eqs. (10) similar to those presented in fig. 1 were found numerically with $g(\Delta)$ being a Gaussian distribution [13]. Finally, the ECM of an array of identical lasers [7] are simply obtained by letting $\Gamma \to 0$ in (13) and (14).

Next, we analyze the domain $K\sigma > 1$, where part of the oscillators undergo a Hopf bifurcation leading to undamped pulsations of their intensities $|E_j|^2$. We focus on the limit $\tau_\mathrm{D} \ll 1$. Equation (8) indicates that the phases of the locked oscillators with $K\sigma \cos(\nu\tau - \phi_j) > 1$ acquire an oscillating component with finite amplitude ρ_j. These oscillators move periodically away from the center of mass of the synchronized cluster on the $[0, 2\pi)$ interval. As a result of this dynamical dispersion, the modulus σ of the order parameter (6) decreases. Moreover, it follows from (7) that the destabilized oscillators experience an effective coupling strength that is reduced by a factor $J_0(\rho_j)$. This, in turn, degrades their synchronization. On

this secondary branch of partially synchronized solutions we find in the long-time limit

$$\sin\psi = \frac{\Delta - \nu}{K\sigma J_0(\rho)}\,, \qquad \cos\psi = \frac{\rho}{2K\sigma J_1(\rho)}\,, \tag{15}$$

for the oscillators such that $\Delta \in [\Delta_-, \Delta_+]$, where $\Delta_\pm = \nu \pm \sqrt{K^2\sigma^2 - 1}$. Eliminating ψ from (15) determines ρ as a function of Δ and K. By contrast, $e^{i\psi}$ becomes $e^{i\psi}|_0 \equiv [1 - (\Delta - \nu)^2/K^2\sigma^2]^{1/2} + i(\Delta - \nu)/K\sigma$ on the primary branch that is the solution of eq. (12). Using (15), the implicit relation between σ and K becomes

$$\sigma e^{i\omega t_\mathrm{D}} = \sqrt{1 + \left(\frac{\Gamma + i\nu}{K\sigma}\right)^2} - \frac{\Gamma + i\nu}{K\sigma} + \mathcal{R}_1, \tag{16}$$

where

$$\mathcal{R}_1 = \int_{\Delta_-}^{\Delta_+} g(\Delta)\big[e^{i\psi}J_0(\rho) - e^{i\psi}|_0\big]\,\mathrm{d}\Delta. \tag{17}$$

In the vicinity of the instability $K\sigma = 1$, we introduce a small parameter $\epsilon \equiv K\sigma - 1$, so that $\rho \ll 1$ and $\Delta_\pm \ll 1$. Therefore (17) becomes

$$\mathcal{R}_1 = -\frac{2^{7/2}\Gamma\epsilon^{3/2}}{15\pi(\nu^2 + \Gamma^2)} + O\big(\epsilon^{5/2}\big). \tag{18}$$

Note that it follows from (16) and (18) that, unlike the usual $1/2$ scaling, the solution bifurcating at $K\sigma = 1$ scales as $\delta\sigma \propto \delta K^{3/2}$, where $\delta\sigma$ and δK are small deviations from the instability threshold (see inset in fig. 2). Equation (16) has been solved numerically. The result is shown in fig. 2 and compared with a direct numerical simulation of eqs. (1), (2), and (4) with 100 and with 1000 elements. For $s_\mathrm{D} = 0$, the equations have been integrated until $s = 3800$ for each value of K in order to discard the transient behavior. Then, σ was averaged over the time interval $\Delta s = 200$ and plotted against K. During this time interval, σ was almost time independent on the whole range of coupling strengths notwithstanding unavoidable fluctuations of the order $1/\sqrt{N}$. The figure indicates that $\sigma(K)$ is maximum at $K\sigma \simeq 1$. Substituting this value in eq. (16), the maximum attainable coherence is $\sigma_{\max} \simeq \sqrt{1 + \Gamma^2 \sec^2(\omega t_\mathrm{D})} - \Gamma\sec(\omega t_\mathrm{D})$, with $\sec(\omega t_\mathrm{D}) > 0$. The maximal possible value of the coherence parameter is thus reached for a finite coupling strength and is less than unity. Except for the numerical example shown in fig. 2, we have not performed any study of the stability of the secondary branch bifurcating from the partially coherent state (12). This could be the subject of a separate investigation. Note, however, that the stability problem is still not completely solved even for the partially coherent solution of the usual Kuramoto model without delay [2, 21].

A possible way to improve the synchronization performance of the array is to exploit the time-delay τ_D in the global coupling. For identical oscillators, large delays produce new bifurcations leading to periodic solutions of the type (5), but with all relaxation phases θ_j equal [7, 8]. Due to these instabilities, a higher level of synchronization can be maintained. Like the ECM, the number of branches of "in-phase" self-pulsing solutions is infinite and their density increases with the time delay. Therefore, if the delay is sufficiently large, the system is liable to operate on one of these self-pulsing states. Numerical simulations with nonidentical oscillators confirm this interpretation: the dynamic degradation of synchronization reported in this letter can be substantially reduced by using the delayed coupling with $\tau_\mathrm{D} = O(1)$. The time-averaged absolute values of the order parameter calculated for $\tau_\mathrm{D} \simeq 1$ are shown by squares in fig. 2. One can see that above the self-pulsing instability threshold the synchronization level is increased in the presence of delay.

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