Vortex Induced Rotation of Clusters of Localized States in the Complex Ginzburg-Landau Equation

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We report existence of a qualitatively distinct class of spiral waves in the two-dimensional cubic-quintic complex Ginzburg-Landau equation. These are stable clusters of localized states rotating around a central vortex core emerging due to interference of the tails of the individual states involved. We also develop an asymptotic theory allowing calculation of the angular frequency and stability analysis of the rotating clusters.

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Emergence of a coherent phase in physical systems is typically associated with a phase transition. For example, in condensed matter these are transitions to superfluid and superconducting states, and in optics it is a transition from the incoherent photon emission to coherent generation in laser systems. There are also other ways of looking at the emergence of phase degrees of freedom. In particular, Hopf bifurcation is generic in nonlinear systems of any nature [1] and it is universally followed by the emergence of a periodic state, which is parametrized by a certain initial phase. The dynamics in the vicinity of all the above transitions can be described by the ubiquitous complex generalization of the Ginzburg-Landau equation (CGLE) [1,2],

$$\partial_t \psi - (i + d)\nabla^2 \psi = c_0 \psi + \psi f(|\psi|^2), \qquad (1)$$

where $d > 0$ is the diffusion constant and c_0 measures distance from the threshold.

If the bifurcation at $c_0 = 0$ is supercritical, then it suffices in many cases to use the cubic CGLE, i.e., $f = (c_1 + ic_2)|\psi|^2$, [1]. The two-dimensional, $\nabla = \mathbf{j}_x \partial_x + \mathbf{j}_y \partial_y$, cubic CGLE admits well-known and extensively studied types of solutions with quantized circulation [1,2]. They are often referred to as vortices or spiral waves and have the following general form: $\psi = A(r) \times e^{iM\theta - i\Omega t}$, where $r = |\mathbf{r}| = |\mathbf{j}_x x + \mathbf{j}_y y|$, $\theta = \arg(x + iy)$, and $M = \pm 1, \pm 2, \ldots$ is the topological charge. Reψ and Imψ vanish simultaneously at the vortex core: $r = 0$. Ω/M is the frequency of rotation of the phase flow around the core. These and other similar types of rotating solutions are well known in many specific practical models such as, e.g., optical Maxwell-Bloch equations [3] and reaction-diffusion systems [1] broadly used in biological and chemical applications, which are reducible to the GL model.

If the bifurcation at $c_0 = 0$ is subcritical, as, e.g., in lasers with a saturable absorber [4], then the normal form of f in (1) is $f = (c_1 + ic_2)|\psi|^2 - (c_3 + ic_4)|\psi|^4$ [1]. Remarkably, this form of f admits existence, for $c_0 < 0$, and stability of the two dimensional (2D) spatially localized states (LSs) seating on a zero background [5,6]. One should expect that, in this case, a strong enough initial excitation of the system can lead to the emergence of a set of LSs. Thereby long term evolution of the system will be determined by the laws of the interaction of these structures, which have been studied previously in 1D and 2D geometries (see, e.g., [4,6–12]).

The constant phase of a solitary LS is arbitrary due to invariance of Eq. (1) under the phase shift: $\psi \to \psi e^{i\varphi}$. However, the relative phase was found to play a paramount role in their pairwise interaction [4,7–10]. The theory of interaction of dissipative LSs in the somewhat simple situation without phase degrees of freedom has been recently explored in detail in the framework of models describing externally pumped optical cavities; see [13] and [14], respectively, for 1D and 2D cases. In the Hamiltonian limit, $d = c_{0,1,3} = 0$, Eq. (1) conserves a number of integrals of motion including momentum. Therefore the effective center of mass of the initially resting group, or in other words *cluster*, of the interacting LSs must remain steady in time. The interaction induced motion can be only relative in this case. For example, two π-out-of-phase LSs will repel each other, moving along the line connecting their centers [10]. In contrast, non-Hamiltonian terms present in Eq. (1) allow spontaneous motion of structures emerging from initial conditions with zero momentum [1]. In particular, this motion can arise through interaction of LSs [4,8,9].

In this work, we describe qualitatively new dynamical behavior that appears in the cubic-quintic CGLE as a result of the interplay between the quantized circular phase flow and the binding of the LSs through the spatial oscillations of their tails. In particular, we consider clusters of N two-dimensional LSs arranged in a way that total phase $\varphi = \arg\psi$ changes by 2π when making a closed loop around the geometrical center of the cluster, i.e., the nth LS has phase $\varphi_n = 2(n - 1)\pi/N$ with $n = 1, \ldots, N$. We calculate frequency of rotation and determine stability properties of such clusters focusing our attention on the simplest rotating cluster with $N = 3$.

A LS positioned at the origin of the coordinate system is a solution of Eq. (1) having the form $\psi = \psi^{(0)}(r) \times e^{-i\nu^{(0)}t + i\varphi}$, where $\nu^{(0)}$ is the frequency shift [5]. For r large enough nonlinear terms in Eq. (1) can be neglected, which gives asymptotic behavior of the tails of the LS: $\psi^{(0)} \simeq bK_0(kr)$, where $k = k_r + ik_i = \sqrt{(ic_0 - \nu^{(0)})/(1 - id)}$, $k_r > 0$, b is a constant which can be determined numerically, and K_0 is the zeroth-order modified Bessel function.

We start our analysis of the rotational dynamics with the description of the results of numerical simulations of Eq. (1) with initial conditions in the form $\psi(t = 0) = \sum_{n=1}^{N} \psi^{(0)}(\mathbf{r} - \mathbf{R}_n)e^{i\varphi_n}$. Here $\varphi_n = 2(n - 1)\pi/N$ are the LS phases and $\mathbf{R}_n = \mathbf{j}_x X_n + \mathbf{j}_y Y_n$ are their positions. As in the case of vortices, we expect that rotational dynamics can be created by the presence of the phase singularity of the field, i.e., of a point with $\mathrm{Re}\psi = \mathrm{Im}\psi = 0$, in the center of the structure. In the simplest case—$N = 2$, $Y_{1,2} = 0$, and $X_1 = -X_2$—elementary symmetry considerations show that the lines $\mathrm{Re}\psi = 0$ and $\mathrm{Im}\psi = 0$ simply coincide along the line $x = 0$. Thus, no circular phase flow is created and we observe only formation of stable bound states of out-of-phase LSs for a discrete set of the separation distances. These bound states exist and are stable over a broad range of parameters, similar to the 1D case [4]. The phase flow can be easily introduced by taking initial phases of the LSs in the form $\varphi_n + \alpha_{nx}x + \alpha_{ny}y$ with $\alpha_{1x,1y} = -\alpha_{2x,2y}$. This creates opposite tilts of the lines $\mathrm{Re}\psi = 0$, $\mathrm{Im}\psi = 0$ introducing a vortex located between the LSs. However, stationary bound states of two LSs with a point vortex between them do not exist. In typical evolutions of such initial conditions, the LSs begin to rotate around the vortex, but centrifugal forces quickly become dominant and the LSs are pushed away from each other. If the initial tilt is less than some critical value, then the stationary bound state is quickly restored. Using notion of the centrifugal force, we have implicitly assumed that the LSs under consideration can be considered as quasiparticles having some sort of effective mass. This fact is well known in the Hamiltonian limit, $d, c_{0,1,3} \to 0$ (see, e.g., [4,10]), and will be put into a more formal context below.

The next example we examine is the possible existence of $N = 3$ rotating clusters of LSs forming an equilateral triangle. Plotting transverse profiles of the real and the imaginary parts of the linear superposition of three LSs with phases $\varphi_n = 2\pi(n - 1)/3$, $n = 1, 2, 3$, positioned in the vertices of the triangle straightforwardly reveals existence of a unit vortex in the center. For certain values of the separation distance, we have found that these initial conditions converge to periodic solutions corresponding to the triangular cluster spiralling around the vortex core for the duration of tens of periods with no noticeable changes (see Fig. 1(a) and [15]). The phase difference $2\pi/3$ between the neighboring LSs remains unchanged in the course of the spiralling.

Introducing small deviations of the relative phases and positions of LSs forming spiralling solution shown in

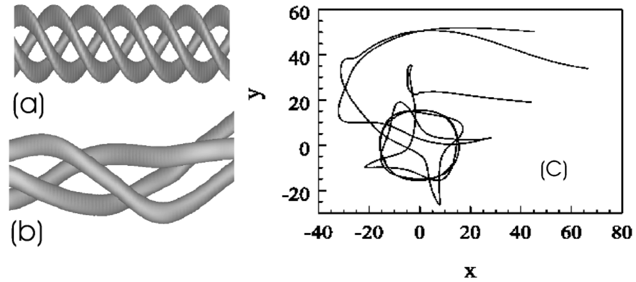

FIG. 1. (a) Stable rotation of the triangular cluster: $d = 0.03$. (b) Onset of the instability, $d = 0.01$, predicted by the presence of the eigenvalue with positive real part shown in Fig. 2(b). Integration time (horizontal axis) is (a) 18 000 and (b) 4000 time units; $c_0 = -0.38$, $c_1 = 1$, $c_{2,4} = 0$, $c_3 = 0.46$. (c) Trajectories of the peaks of the individual LSs showing transformation of the rotating triangle into the moving one. Parameters as for (b). Integration time $t = 32\,000$.

Fig. 1(a), we have observed that the deviations decay sufficiently quickly and perfect spiralling is restored, which indicates its dynamical stability. Stable rotation was always observed to happen in the direction of increase of phase around the vortex. We therefore conclude that for a certain set of the separation distances the binding force due to oscillatory tails of the interacting LSs can be strong enough to hold a cluster of three LSs against breakup due to centrifugal force created by the rotation.

The presence of the vortex at the center of the rotating triangle can be rigorously proven using transparent symmetry considerations. Since the three interacting LSs are identical the structure shown in Fig. 1(a) is invariant under rotation by $2\pi/3$ in the (x, y) plane accompanied by phase shift $2\pi/3$. Therefore, at any given moment of time the total field ψ obeys the relation $\psi(x, y) = \psi(x\cos\frac{2\pi}{3} - y\sin\frac{2\pi}{3}, y\cos\frac{2\pi}{3} + x\sin\frac{2\pi}{3})e^{i2\pi/3}$. In particular, for $x = y = 0$ we obtain $\psi(0, 0) = \psi(0, 0)e^{i2\pi/3}$, which implies that ψ must vanish at the origin, while its phase $\varphi = \arg(\psi)$ changes by 2π when making a roundtrip along a contour encircling the origin. It is now obvious that the rotating triangles must have a central vortex with unit charge. Note that, throughout this paper, we study only structures with topological charge $+1$. As well as the obvious existence of rotating clusters with charge -1, one can also find clusters with other integer charges and study their stability using the approach described below.

After having established existence of the rotating clusters with quantized circulation numerically, we turn our attention to their analytical understanding. This will allow us to gain deeper insight into the physical mechanisms involved in this phenomenon. Our theoretical approach originates from and further develops the perturbation technique of the seminal papers [10,12]. The essence of the method is an assumption that overlap of the tails of the LSs induces adiabatic evolution of their phases and positions, which is governed by the equations to be determined.

Assuming that to leading approximation a cluster of LSs can be represented as a linear superposition of N

noninteracting LSs, we write the solution of Eq. (1) in the form

$$\psi = e^{-i\nu t}\left[\left(\sum_{n=1}^{N}\psi_n^{(0)} + \psi_n^{(1)}\right) + \psi^{(2)} + O(\epsilon^{3/2})\right]. \tag{2}$$

Here $\nu = \nu^{(0)} + \nu^{(1)} + \nu^{(2)} + O(\epsilon^{3/2})$, $\nu^{(m)} = O(\epsilon^{m/2})$, $\epsilon \ll 1$ is a dummy parameter, and $\psi_n^{(0)} = e^{-i\varphi_n(t)} \times \psi^{(0)}[\mathbf{r}_n(t)]$ with $\mathbf{r}_n = \mathbf{r} - \mathbf{R}_n$. $\psi_n^{(1)} = O(\epsilon^{1/2})$ is a function localized in the vicinity of the nth LS, which describes radially nonsymmetric corrections related to the LSs motion. Also, $\psi^{(2)} = O(\epsilon)$ is a function describing deviation of the true cluster solution from the linear superposition of LSs due to overlap of their tails. In what follows, we choose d as a primary control parameter and assume that $d = O(\epsilon^{1/2})$.

Making assumptions about order of smallness of the interaction induced velocity, $\partial_t \mathbf{R}_n = O(\epsilon^{1/2})$, acceleration, $\partial_{tt}\mathbf{R}_n = O(\epsilon)$, and derivative of the phase, $\partial_t\varphi_n = O(\epsilon)$, substituting Eq. (2) into Eq. (1) and equating terms of the same order in ϵ, we obtain a recurrent system of equations, which is quite cumbersome to be presented in the Letter format. Therefore we restrict ourselves to the qualitative description of the essence of our calculations.

At order $\epsilon^{1/2}$, one needs to solve N *independent* inhomogeneous problems for differential operators $\hat{\mathcal{L}}_n$ describing spectral stability of the nth LS for $d = 0$. These problems can be resolved providing that the corresponding right-hand sides are orthogonal to the neutral modes, i.e., the eigenmodes corresponding to zero eigenvalues, of the operators $\hat{\mathcal{L}}_n^{\dagger}$. Here $\hat{\mathcal{L}}_n^{\dagger}$ is the linear operator adjoint to $\hat{\mathcal{L}}_n$. The spectra of $\hat{\mathcal{L}}_n$ and $\hat{\mathcal{L}}_n^{\dagger}$ are identical, but their eigenmodes are not. Applying infinitesimal phase rotation and transverse translations to the LSs, one can generate phase and translational neutral eigenmodes of $\hat{\mathcal{L}}_n$. No equivalent procedure is known, however, for the corresponding eigenmodes of $\hat{\mathcal{L}}_n^{\dagger}$, and they can be found only numerically. Note that there is only one zero eigenvalue corresponding to the phase related neutral mode and two pairs of zero eigenvalues corresponding to x and y translations. Each of the eigenvalues corresponding to the translational neutral modes is doubly degenerate. This is because for $d = 0$ Eq. (1) acquires an extra symmetry group, which is the Galilean transformation: $\psi(\mathbf{r}) \rightarrow \psi(\mathbf{r} - \mathbf{v}t)e^{(i/2)\mathbf{v}\cdot\mathbf{r}-(1/4)|\mathbf{v}|^2 t}$. Solvability conditions at order $\epsilon^{1/2}$ can be satisfied by choosing the value of $\nu^{(1)}$ only.

Because the overlap of the tails of the LSs is assumed to be $O(\epsilon)$, it means that the corresponding equation for $\psi^{(2)}$ cannot be split into independent parts corresponding to the different LSs. Solvability conditions at this order have been applied to the operator $\sum_{n=1}^{N}\hat{\mathcal{L}}_n$. These conditions not only fix the value of $\nu^{(2)}$, but result in a set of coupled dynamical equations governing evolution of the coordinates and phases of the interacting LSs:

$$m\partial_{tt}\mathbf{R}_n + \gamma^{(r)}\partial_t\mathbf{R}_n = \nabla_{R_n}\sum_{l\neq n}^{N} G_{nl}^{(r)} + O(\epsilon^{3/2}), \tag{3}$$

$$\gamma^{(\varphi)}\partial_t\varphi_n - Q|\partial_t\mathbf{R}_n|^2 = \sum_{l\neq n}^{N} G_{nl}^{(\varphi)} + O(\epsilon^{3/2}). \tag{4}$$

Here $G_{nl}^{(r,\varphi)} = 4\pi\,\mathrm{Im}\{e^{i\varphi_{nl} - i\arg(k)/2 + \Phi^{(r,\varphi)}}K_0(kR_{nl})\}$ are functions describing coupling between the LSs, $\Phi^{(r,\varphi)}$ are constants which can be found from the analysis of the behavior of the tails of the neutral modes of $\hat{\mathcal{L}}_n$ and $\hat{\mathcal{L}}_n^{\dagger}$, $\varphi_{nl} = \varphi_n - \varphi_l$, $R_{nl} = |\mathbf{R}_n - \mathbf{R}_l|$, and $\nabla R_n \equiv \mathbf{j}_x\partial_{X_n} + \mathbf{j}_y\partial_{Y_n}$. Also $m = O(1)$ is the constant characterizing effective mass of a LS, and $\gamma^{(\varphi)} = O(1)$ and $\gamma^{(r)} = O(\epsilon^{1/2})$ are the phase relaxation and friction coefficients, which can be calculated only numerically as scalar products of the neutral eigenmodes of $\hat{\mathcal{L}}_n$ and $\hat{\mathcal{L}}_n^{\dagger}$. The term $Q|\partial_t\mathbf{R}_n|^2$ describes the frequency shift of an individual LS due to its motion. The existence of a frequency shift proportional to the square of the LS velocity can be inferred from the Galilean transformation. Note that all coefficients in Eqs. (3) and (4) are real and $\gamma^{(r)}|_{d=0} = 0$. For $d = O(1)$, we get $\gamma^{(r)} = O(1)$, $\partial_t\mathbf{R}_n = O(\epsilon)$. In this case, the terms proportional to m and Q have a higher order of smallness than the other ones and can be neglected.

For the rotating cluster shown in Fig. 1(a), we have $k_r R_{nl} \simeq 10 \gg 1$; therefore we can with a good accuracy replace Bessel functions with their asymptotics and find the solutions of Eqs. (3) and (4) corresponding to the rotating equilateral triangle in the closed analytical form:

$$\begin{aligned}
\varphi_{nJ} &= 2\pi(n-1)/3 + \tilde{\nu}_J t, \\
X_{nJ} &= (\rho_J + \delta_J)/\sqrt{3}\sin[\Omega_J t + 2\pi(n-1)/3], \\
Y_{nJ} &= (\rho_J + \delta_J)/\sqrt{3}\cos[\Omega_J t + 2\pi(n-1)/3].
\end{aligned} \tag{5}$$

Here $n = 1, 2, 3$ labels the LSs, and $\tilde{\nu}_J = (-1)^J \times e^{-k_r\rho_J}(\gamma^{(\varphi)}\sqrt{\rho_J})^{-1}\sin(\Phi^{(\varphi)} - \Phi^{(r)} + \delta_J) + 3Qe^{-2k_r\rho_J} \times \cos^2\delta_J(4\rho_J m^2\gamma^{(r)2})^{-1}$ is the constant frequency shift induced by the interaction. Index $J = 1, 2, 3, 4, \ldots$ is a positive integer determining the discrete set of the equilibrium distances $\rho_J + \delta_J$ between the LSs in the cluster. Here the ρ_J are associated directly with the set of maxima and minima of the function $G_{nl}^{(r)}$. Further, Ω_J is the rotation frequency of the cluster characterized by equilibrium distances $\rho_J + \delta_J$:

$$\Omega_J = \tilde{\Omega}_J\cos(k_i\delta_J), \qquad \tilde{\Omega}_J = (-1)^J\frac{3e^{-k_r\rho_J}}{2\gamma^{(r)}\rho_J^{3/2}}, \tag{6}$$

and $\tilde{\Omega}_J$ is the rotation frequency calculated for $m = Q = 0$. When inertia is present, i.e., $m \neq 0$, a centrifugal force appears and leads to a small increase of the equilibrium distances between the LSs in the rotating cluster by an amount δ_J. Stationary values of ρ_J and δ_J are obtained solving two coupled equations: $k_i(\rho_J - \delta_J) = \pi J - \Phi^{(r)}$ and $\tilde{\Omega}_J = \gamma^{(r)}\sin(k_i\delta_J)\sqrt{\gamma^{(r)} + m^2\tilde{\Omega}_J^2}$. Thus, see also (6),

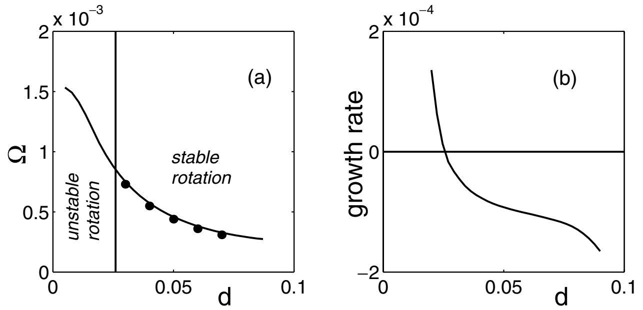

FIG. 2. Frequency Ω (a) and real part of the critical eigenvalue (b) of the rotating triangular cluster calculated from Eqs. (3)–(6) for $J = 4$. Dots show frequency found by the numerical modeling of Eq. (1). Other parameters as Fig. 1.

δ_J (as well as the centrifugal force itself) increases with decreasing effective friction $\gamma^{(r)} = O(d)$. Let us stress that frequency of rotation is uniquely determined through the system parameters. This is in contrast to the spiralling structures known in the Hamiltonian systems [12,16,17], where frequency of rotation is a continuously tunable parameter determined by the value of the angular momentum integral stored in the initial conditions.

Frequency of rotation decreases with increasing ρ_J [see (6)]. This indicates that stronger interaction induces faster rotation. The rotating triangle shown in Fig. 1(a) corresponds to $J = 4$ and has $\rho_4 + \delta_4 \simeq 26.7$, which is the closest stable distance we have been able to identify. Comparison of the values of Ω_4 given by Eq. (5) with those obtained by direct numerical modeling of Eq. (1) is presented in Fig. 2(a). It reveals excellent agreement between the two. For larger J stable triangles also exist, but they rotate so slowly that their stability cannot be practically studied by means of direct numerical modeling of Eq. (1). Even values of J correspond to triangles rotating in the direction of increasing of the total phase around the cluster center. These triangles can be either dynamically stable or not. Triangles with J odd rotate in the direction of phase decrease and are always unstable. The complete set of the stability conditions, for the triangular clusters with even J, can be derived by linearizing Eqs. (3) and (4) near solution (5). However, it is an extremely cumbersome procedure, which was practically accomplished using computer algebra. Therefore we prefer to show dependence of the critical eigenvalue governing stability of the rotating triangular cluster against diffusion parameter d in Fig. 2(b). One can see that for d small enough, i.e., when δ_4 and centrifugal forces are relatively large, the cluster destabilizes. This result was found to be in excellent agreement with direct numerical modeling of Eq. (1). The onset of the instability leading to the destruction of the regular spiral structure is shown in Fig. 1(b). Note that the instability does not affect significantly the distance between the LSs, and the final outcome of this process is transformation of the rotating equilateral triangle into a uniformly moving isosceles one. Simulation of Eq. (1) visualizing this process can be downloaded from [15], and trajectories calculated from Eqs. (4) and (5) are shown in Fig. 1(c). Both models demonstrate very close dynamical behavior, thereby supporting once more the validity of our asymptotic approach. Existence of a wide variety of other restless clusters can be derived from Eqs. (3) and (4). Further studies of those will be the subject of future research.

In conclusion, we have provided numerical and analytical evidence for the existence of a distinct class of stable spiral waves in the 2D cubic-quintic CGLE, which are formed due to balance between interaction forces binding three LSs together and a centrifugal force. The latter force appears due to rotation induced by the quantized vortex formed through the interference of the tails of the LSs.

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Note added.—After submission of this manuscript, Ref. [17] describing rotating clusters with central vortices in the nonlinear Schrödinger equation with saturable type of nonlinearity has been published. This corresponds to the Hamiltonian limit of the model considered above.

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