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Internal oscillations of solitons in two-dimensional NLS equation with nonlocal nonlinearity

N.N. Rosanov [a], A.G. Vladimirov [b], D.V. Skryabin [c,*], W.J. Firth [c]

[a] *Research Institute for Laser Physics, 199034, St. Petersburg, Russia*
[b] *Physics Faculty, St. Petersburg State University, 198904, St. Petersburg, Russia*
[c] *Department of Physics and Applied Physics, University of Strathclyde, Glasgow, G4 0NG, UK*

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Abstract

We develop asymptotic theory to find small eigenfrequencies of two-dimensional bright solitons in the nonlinear Schrödinger equation with weak nonlocality. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Nonlocality of nonlinear response in wave propagation problems is an important factor in many physical contexts. Among examples which have attracted significant recent attention one could mention nonlocal nonlinearities appearing in the modelling of propagation of cold atomic beams in the presence of optically induced dipole–dipole interaction [1,2], nonlocality of interaction of ultracold atoms in atomic Bose–Einstein condensation [3] and nonlinear propagation of light in photorefractive materials [4,5].

Propagation of matter waves with two-body collisions, and of the slowly varying envelope of an electromagnetic wave in a medium with a weak nonlocal nonlinearity, can be described by the same basic model equation—the nonlocal nonlinear Schrödinger (NLS) equation [6]:

$$i\frac{\partial \Phi(t,\mathbf{r})}{\partial t} + \nabla^2\Phi(t,\mathbf{r}) - \Phi(t,\mathbf{r})\int \left|\Phi(t,\mathbf{r}')\right|^2 U(\mathbf{r}-\mathbf{r}')\,d\mathbf{r}' = 0. \tag{1}$$

If effects of nonlocality can be neglected one replaces potential function $U(\mathbf{r}-\mathbf{r}')$ with Dirac-delta function $\delta(\mathbf{r}-\mathbf{r}')$ and Eq. (1) is replaced with standard NLS:

$$i\frac{\partial \Phi(t,\mathbf{r})}{\partial t} + \nabla^2\Phi(t,\mathbf{r}) - U_0\Phi(t,\mathbf{r})\left|\Phi(t,\mathbf{r})\right|^2 = 0, \tag{2}$$

where $U_0 = \int U(\mathbf{r})\,d\mathbf{r}$ is the zero-order moment of the interaction potential U. Considering applications of

* Corresponding author. Current address: Department of Physics, University of Bath, Bath BA2 7AY, UK.
E-mail address: d.v.skryabin@bath.ac.uk (D.V. Skryabin).

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two-dimensional NLS to the beam propagation problem an independent variable t should be interpreted as space coordinate measured along the propagation direction.

Depending on the context of the problem, the Laplace operator in Eqs. (1), (2) can be either one-, two-, or three-dimensional. Effects of nonlocality on existence and interaction properties of one-dimensional bright and dark solitons in NLS have been recently analyzed in [5]. Two- and three-dimensional cases have been considered in [7–9]. A well-known phenomenon in two- and three-dimensional NLS (2) with the cubic nonlinearity is critical or, respectively, exponential collapse of bright solitons for $U_0 < 0$. Taking into account higher-order nonlinearities [10] or nonlocality of the nonlinear interaction can suppress collapse [7–9].

Suppression of an instability of an equilibrium solution in Hamiltonian systems generally implies appearance of new frequencies in its spectrum. This is because the complex frequencies of modes responsible for instability become purely real and can then be associated with long-lived oscillations of the equilibrium under consideration. This scenario can apply to solitary wave instabilities [12]. The aim of this Letter is to present the first analytical calculations of eigenfrequencies emerging due to suppression of the collapse of two-dimensional solitary waves in the NLS equation with weak nonlocality.

2. Approximation of weakly nonlocal interaction

To proceed further we assume that nonlinearity is only weakly nonlocal and therefore to calculate integral in Eq. (1) we can decompose $|\Phi|^2$ in a Taylor series:

$$\begin{aligned}\left|\Phi(\mathbf{r}')\right|^2 &= \left|\Phi\left(\mathbf{r}+(\mathbf{r}'-\mathbf{r})\right)\right|^2 \\ &= \left|\Phi(\mathbf{r})\right|^2 + \left\{(\mathbf{r}'-\mathbf{r})\cdot\nabla\right\}\left|\Phi(\mathbf{r})\right|^2 \\ &\quad + \frac{1}{2}\left\{(\mathbf{r}'-\mathbf{r})\cdot\nabla\right\}^2\left|\Phi(\mathbf{r})\right|^2 + \cdots. \end{aligned} \tag{3}$$

We now fix the dimension of the Laplacian to two, i.e., $\mathbf{r} = \mathbf{i}_x x + \mathbf{i}_y y$, $\nabla = \mathbf{i}_x\partial_x + \mathbf{i}_y\partial_y$ and assume the potential function U is cylindrically symmetric, i.e., $U(\mathbf{r}-\mathbf{r}') = U(|\mathbf{r}-\mathbf{r}'|)$. Under this assumption nontrivial contributions to the integral come only from the first and third terms in Eq. (3). Substituting (3) into (1) we find an NLS equation with weakly nonlocal nonlinearity:

$$i\frac{\partial\Phi}{\partial t} + \nabla^2\Phi - U_0\Phi|\Phi|^2 - U_2\Phi\nabla^2|\Phi|^2 = 0. \tag{4}$$

Here $U_2 = (\pi/2)\int r^3\,dr\,U(r)$ is the second-order moment of the interaction potential U. Let us also assume that at sufficiently large distances $r > r_c$ the nonlinear interaction is attractive, i.e., $U(r) < 0$, while at smaller distances it changes to repulsion, i.e., $U(r) > 0$ for $r < r_c$. Then

$$\begin{aligned} U_2 &= \frac{\pi}{2}\int_0^\infty U(r)r^3\,dr \\ &= \frac{\pi}{2}\left(\int_0^{r_c} U(r)r^3\,dr + \int_{r_c}^\infty U(r)r^3\,dr\right) \\ &< 2\pi r_c^2\left(\int_0^{r_c} U(r)r\,dr + \int_{r_c}^\infty U(r)r\,dr\right) \\ &= r_c^2 U_0. \end{aligned} \tag{5}$$

Therefore $U_2 < 0$ providing that $U_0 < 0$. The same estimate can be done for a three-dimensional potential describing, for example, van der Waals like forces acting between atoms.

Fixing for the rest of the Letter $U_0 < 0$, which ensures existence of bright solitary solutions, and introducing the scaling $\tilde{\Phi} = \sqrt{|U_0|}\Phi$, we reduce Eq. (4) to the form

$$i\frac{\partial\tilde{\Phi}}{\partial t} + \nabla^2\tilde{\Phi} + \tilde{\Phi}\left|\tilde{\Phi}\right|^2 + s\tilde{\Phi}\nabla^2\left|\tilde{\Phi}\right|^2 = 0. \tag{6}$$

The parameter $s = U_2/U_0$ characterizing nonlocality of the nonlinearity could also be scaled away. However, it is more convenient for us to keep it in the equation explicitly, and use it as a small parameter in the subsequent derivations.

3. Solitary solution

Stationary cylindrically symmetric solitary solution of Eq. (9) is sought in the form

$$\tilde{\Phi}(t,\mathbf{r}) = A(r)\exp(i\kappa t). \tag{7}$$

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Here $\kappa > 0$ is the nonlinear frequency shift. The amplitude A is a real function determined from the ordinary differential equation

$$\left[\nabla_r^2 - \kappa + A^2 + s\left(\nabla_r^2 A^2\right)\right]A = 0,$$
$$\nabla_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \tag{8}$$

with the requirement of $dA/dr = 0$ at $r = 0$, $A \to 0$ at $r \to \infty$ and $A(r) > 0$. For $s\nabla_r^2 A^2$ small compare to A^2 we can use a perturbation approach to solve for A in the form

$$A(r) = A_0(r) + sA_2(r) + \cdots. \tag{9}$$

In the zero order ($s = 0$) we have the CGT (Chiao–Garmire–Tawnes) soliton known from the paraxial theory of self-focusing of optical radiation [13]

$$A_0 = \sqrt{\kappa}\,F_0(\rho), \quad \rho = r\sqrt{\kappa}, \tag{10}$$

where $F_0(\rho)$ solves

$$\left(\nabla_\rho^2 - 1 + F_0^2\right)F_0 = 0,$$
$$\nabla_\rho^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}. \tag{11}$$

Note that Eq. (6) conserves the integral $P = \int dx \times dy\,|\tilde{\Phi}|^2$, the "number of particles". κ parameterizes the family of solitary solutions with different width $w \propto \kappa^{-1/2}$. If $s = 0$, then one can show that $P = P_0$ does not depend on κ, i.e., $\partial_\kappa P_0 = 0$, where

$$P_0 = 2\pi \int_0^\infty A_0^2 r\,dr = 2\pi \int_0^\infty F_0^2 \rho\,d\rho = 2\pi\,11.701. \tag{12}$$

The lowest-order nonlocal correction to the CGT soliton is

$$A_2 = s\kappa^{3/2} F_2(\rho), \tag{13}$$

where $F_2(\rho)$ is the solution of the linear inhomogeneous ordinary differential equation

$$\left(\nabla_\rho^2 - 1 + 3F_0^2\right)F_2 = -F_0 \nabla_\rho^2 F_0^2. \tag{14}$$

Numerically computed transverse profiles of the functions $F_{0,2}(\rho)$ are shown in Fig. 1.

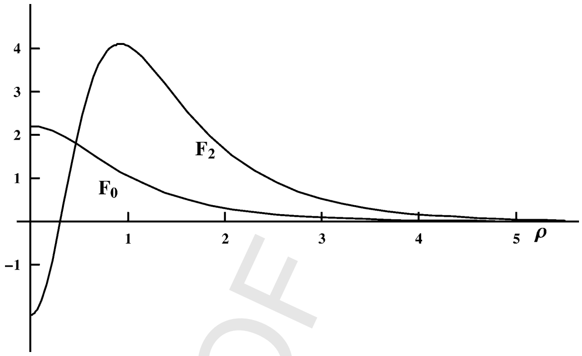

Fig. 1. Radial profiles of the functions F_0 and F_2 corresponding respectively to the CGT soliton and the lowest-order correction due to weak nonlocality.

Nonlocality induces a nontrivial dependence of P on κ:

$$P = 2\pi \int_0^\infty \left(A_0(r) + sA_2(r) + \cdots\right)^2 r\,dr$$
$$= P_0 + 4\pi s p_1 \kappa + O\left(s^2\right), \tag{15}$$

where

$$p_1 = \int_0^\infty F_0(\rho)F_2(\rho)\rho\,d\rho = 5.05889. \tag{16}$$

The limit of local nonlinearity corresponds $s \to 0$ or/and $\kappa \to 0$. According to Eqs. (15) and (16), for $s > 0$ the density increases with κ, and therefore the Vakhitov–Kolokolov stability criterion [10]

$$\frac{\partial P}{\partial \kappa} = 4\pi s p_1 > 0 \tag{17}$$

is satisfied. However, this criterion was not so far rigorously proved for an arbitrary nonlinearity and, therefore, needs to be checked on a case by case basis. This can be done by calculating the modification of the spectrum of the CGT soliton under the action of small nonlocal effects. For eigenvalues deviating from zero this can be done in closed analytical form.

4. Soliton linear stability and internal modes

To find frequencies of the internal modes emerging from zero for s small, we consider a perturbed station-

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ary soliton for $s \neq 0$:

$$\tilde{\Phi} = \left[A(r) + \delta\Phi(t,x,y)\right]\exp(i\kappa t). \tag{18}$$

The linearized equation (6), $\partial_t \delta\mathbf{\Phi} = \hat{\mathbf{M}}\delta\mathbf{\Phi}$, for 2-vector $\delta\mathbf{\Phi} = (\mathrm{Re}\,\delta\Phi, \mathrm{Im}\,\delta\Phi)^{\mathrm{T}}$, has eigensolutions of the form

$$\delta\mathbf{\Phi} = \mathbf{\Psi}(r)\exp(\lambda t). \tag{19}$$

Substituting (19) into this equation we get the following linear eigenvalue problem:

$$\hat{\mathbf{M}}\mathbf{\Psi} = \lambda\mathbf{\Psi}, \quad \hat{\mathbf{M}} = \begin{pmatrix} 0 & -\hat{L} \\ \hat{N} & 0 \end{pmatrix}, \tag{20}$$

where

$$\hat{L} = \nabla^2 - \kappa + A^2 + 2s\left[A\left(\nabla^2 A\right) + |\nabla A|^2\right], \tag{21}$$

$$\begin{aligned} \hat{N} = {} & \nabla^2 - \kappa + 3A^2 \\ & + 2s\left[A^2\nabla^2 + 2A(\nabla A \cdot \nabla) \right. \\ & \left. + 2A\left(\nabla^2 A\right) + |\nabla A|^2\right] \end{aligned} \tag{22}$$

are second-order differential operators. The linear operator $\hat{\mathbf{M}}$ has a double zero eigenvalue with geometrical multiplicity 1:

$$\hat{\mathbf{M}}\mathbf{U}_0 = 0, \qquad \hat{\mathbf{M}}\mathbf{U}_1 = \mathbf{U}_0, \tag{23}$$

where the neutral mode $\mathbf{U}_0 = (0, A)^{\mathrm{T}}$ with generalized eigenvector $\mathbf{U}_1 = (\partial_\kappa A, 0)^{\mathrm{T}}$ corresponds to the symmetry of (6) with respect to a phase shift. For $s = 0$, the linear operator $\hat{\mathbf{M}}$ is transformed into $\hat{\mathbf{M}}^{(0)}$ having two additional zero eigenvalues responsible for critical collapse [11]. The second and third generalized eigenvectors associated with these eigenvalues are $\mathbf{U}_2 = (0, -(1/8\kappa)A_0 r^2)^{\mathrm{T}}$ and $\mathbf{U}_3$:

$$\hat{\mathbf{M}}^{(0)}\mathbf{U}_2 = \mathbf{U}_1^{(0)}, \qquad \hat{\mathbf{M}}^{(0)}\mathbf{U}_3 = \mathbf{U}_2. \tag{24}$$

Here $\mathbf{U}_1^{(0)}$ is the vector $\mathbf{U}_1$ evaluated at $s = 0$. The vector $\mathbf{U}_3$ can be found only numerically.

Taking into account the nonlocal perturbation ($s \neq 0$), the four-fold degenerate zero eigenvalue $\lambda^4 = 0$ of $\hat{\mathbf{M}}^{(0)}$ splits into a two-fold degenerate zero eigenvalue and two nonzero eigenvalues with opposite signs. The latter eigenvalues can be found by a perturbation approach similar to used in [11,14–16]. Multiplying (20) by the neutral eigenmode $\mathbf{V}_0 = (A, 0)^{\mathrm{T}}$ of the adjoint operator $\hat{\mathbf{M}}^{\dagger}$, $\hat{\mathbf{M}}^{\dagger}\mathbf{V}_0 = 0$, and using the relation $\langle\mathbf{V}_0, \hat{\mathbf{M}}\mathbf{\Psi}\rangle = \langle\hat{\mathbf{M}}^{\dagger}\mathbf{V}_0, \mathbf{\Psi}\rangle = 0$, we get

$$\lambda\langle\mathbf{V}_0, \mathbf{\Psi}\rangle = 0. \tag{25}$$

Next, assuming that λ is small, $\lambda^2 \sim s$, we seek the eigenfunction $\mathbf{\Psi}$ in the form

$$\mathbf{\Psi} = \mathbf{U}_0 + \lambda\mathbf{U}_1 + \lambda^2\mathbf{U}_2 + \lambda^3\mathbf{U}_3 + O\left(|\lambda|^4\right). \tag{26}$$

Substituting (26) into (25) and taking into account that $\langle\mathbf{V}_0, \mathbf{U}_2\rangle = 0$, we find that

$$\lambda^2\langle\mathbf{V}_0, \mathbf{U}_1\rangle + \lambda^4\left\langle\mathbf{V}_0^{(0)}, \mathbf{U}_3\right\rangle = O\left(|\lambda|^5\right), \tag{27}$$

where $\langle\mathbf{V}_0, \mathbf{U}_1\rangle = \partial_\kappa P$ and $\langle\mathbf{V}_0^{(0)}, \mathbf{U}_3\rangle = \langle\mathbf{V}_1^{(0)}, \mathbf{U}_2\rangle = (2\pi/16\kappa^3)p_3$, $p_3 = \int_0^\infty F_0^2(\rho)\rho\,d\rho = 2.211$. Thus we find following nonzero roots:

$$\lambda^2 = -36.60 s\kappa^3, \tag{28}$$

which explicitly shows that the solitary solution would be exponentially unstable for $s < 0$ ($U_2 > 0$). However, for $s > 0$ (see (5)) the soliton is stable and eigenfrequencies $\omega = \pm|\lambda| \simeq \pm 6.05\kappa^{3/2}$ correspond to the internal modes, which if excited by initial perturbations do not decay in the present approximation. More exactly, decay of these perturbations is expected to be slow (nonexponential) and happens due to transfer of energy from the discrete part of the spectrum into the continuum, which can be described by methods developed in [12,17].

The discrete spectrum of the CGT soliton is known to consist only of the neutral modes [18]. Therefore our theory describes all possible modifications of the discrete spectrum. Preliminary numerical studies of the spectrum of $\hat{\mathbf{M}}$ indicate that no discrete eigenvalues split from continuum for small s.

5. Discussion and summary

In physical units Eq. (1) applied in the context of Bose–Einstein condensate and after making approximation (3) takes form

$$\begin{aligned} & i\hbar\frac{\partial\Phi}{\partial t} + \frac{\hbar^2}{2m}\nabla^2\Phi \\ & \quad - \frac{4\pi\hbar^2 a}{m}\left(\Phi|\Phi|^2 + s\Phi\nabla^2|\Phi|^2\right) = 0, \end{aligned} \tag{29}$$

where m is the mass of an atom, a is the two-body scattering length and s is the positive dimensional nonlocality constant. Assuming that $\Phi = \psi(x,y,z) \times e^{ikz-\omega t}$, where ψ is the amplitude slowly varying along z, $k = mv/\hbar$ is the atomic wavenumber, $\omega =$

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$mv^2/(2\hbar)$ and v is the atomic velocity, we derive $2D$ nonlocal NLS equation

$$i2k\partial_z\psi + \left(\partial_x^2 + \partial_y^2\right)\psi - 8\pi a\left(\psi|\psi|^2 + s\psi\left(\partial_x^2 + \partial_y^2\right)|\psi|^2\right) = 0. \tag{30}$$

Multiplying this equation by the characteristic width of the atomic beam, one can easily make it dimensionless, and afterwards the theory developed above can be applied, providing a is negative, i.e., the interatomic interaction is attractive. Atomic condensate with $a < 0$ was experimentally achieved in ^{7}Li [19] and beam-like propagation of the condensate was observed in [20]. Though the latter experiments were performed for atoms with $a > 0$, we believe that they can be reproduced for ^{7}Li, thereby prospects for experimental observation of the solitons described in this Letter look realistic.

In summary, we have developed a theory of collapse suppression of two-dimensional bright solitons in the NLS equation with weak nonlocality and found an analytic expression for the eigenfrequency of the internal modes bifurcating from zero due to nonlocal effects.

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